Chapter 2

Waves as Particles and Particles as Waves

(The Limits of Classical Physics)

Experimental results required concepts totally incompatible with classical physics.

The new concepts are
- the particle properties of radiation
- the wave properties of matter
- the quantization of physical quantities.

1. Black-body Radiation – See Mattews, Dicke and Wittke

Body heat $\Rightarrow$ it is seen to radiate

In equilibrium the light emitted over the whole spectrum of frequencies ν with a spectral distribution depend on ν and T

$\underbrace{\lambda}_{\text{wavelength}} \nu = \underbrace{c}_{\text{velocity}}$ $\Rightarrow$ we can either use λ or ν as variable

$E(\lambda, T)\, d\lambda$ = energy emitted per unit time in radiation with wavelength in the interval λ and $\lambda + d\lambda$ from a unit area of surface at absolute temperature T.

1859 Kirchhoff's law of radiation

$A(\lambda, T)$ = fraction of incident radiation λ that is absorbed by the body

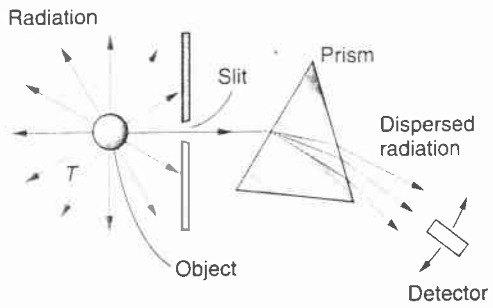
$\dfrac{E(\lambda, T)}{A(\lambda, T)}$ is same for all bodies

$A \equiv 1 \rightarrow$ blackbody

$\Rightarrow$ $E(\lambda, T)$ is a universal function independent of the detailed structure of the black body

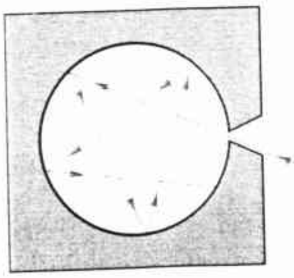
It is evident that the universal properties of the thermal radiation emitted by black bodies makes them of particular theoretical interest.

Experimental set up



Radiation emitted by the object at temperature T that passes through the slit is dispersed according to its wavelength. The prism shown would be an appropriate device for that part of the emitted radiation in the visible region. In other spectral regions other types of devices or wavelength-sensitive detectors would be used.

Source for blackbody radiation.



A small hole in the wall of a cavity approximating an ideal blackbody. Radiation entering the hole has little chance of leaving before it is completely absorbed within the cavity.

Radiation from a small hole in an enclosure heated to temperature T

Radiation incident upon the hole from the outside the cavity and is reflected back and forth by the wall of the cavity, eventually being absorbed by these walls.

If the hole is very small, compared with the area of the inner surface of the cavity, a negligible amount of the incident radiation will be reflected through the hole

$\Rightarrow$ all the radiation incident upon the hole is absorbed

$\Rightarrow$ $A = 1$ and the hole must have the properties of a black body.

Assume the walls of the cavity are uniformly heated to a temperature T

$\Rightarrow$ the wall will emit thermal radiation which will fill the cavity, the small fraction of this radiation incident from inside upon the hole will pass through the hole.

$\Rightarrow$ the hole will act as an emitter of thermal radiation

$\downarrow$

$A = 1$

$\Rightarrow$ the radiation emitted by the hole must have a black-body spectrum.

But the hole is merely sampling the thermal radiation present inside the cavity $\Rightarrow$ radiation in the cavity must have a black-body spectrum characteristic of the temperature T

$\downarrow$

problem is to understand the distribution of radiation inside a cavity whose walls are at a temperature T

Kirchhoff: radiation in the cavity must be isotropic and homogeneous

$u(\lambda, T)\, d\lambda$ = energy density inside the cavity, with wavelength between λ and $\lambda + d\lambda$, at temperature T.

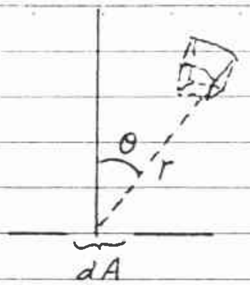
Relation between $E(\lambda, T)$ and $u(\lambda, T)$

$$u(\lambda, T) = \frac{4E(\lambda, T)}{c}$$

$\downarrow$

theoretical interest

Proof:



Dimensional check

$$RHS = \frac{\text{energy}/\text{area}\cdot\text{time}}{\text{length}/\text{time}} = \frac{\text{energy}}{\text{area}\cdot\text{length}} \longrightarrow \text{energy density}$$

The shaded volume element $dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$

$$= r^2 dr \, d\Omega$$

The energy contained in the volume element dV

$$= u(\lambda, T) \, r^2 dr \sin\theta \, d\theta \, d\phi$$

The radiation is isotropic, the fraction passing through dA

$$= \frac{dA \cos\theta}{4\pi r^2}$$

The amount of energy emerging through dA from the volume element

$$= \frac{dA \cos\theta}{4\pi r^2} \cdot r^2 dr \sin\theta \, d\theta \, d\phi \; u(\lambda, T)$$

The amount of energy emerging per unit area from the volume element

$$= \frac{\cos\theta \sin\theta}{4\pi} \, d\theta \, d\phi \, dr \; u(\lambda, T)$$

In time interval Δt only r between 0 and $c\Delta t$ can contribut

$\Rightarrow$ The amount of energy emerging per unit area in time Δt

$$= u(\lambda, T) \int_0^{\pi/2} \int_0^{2\pi} \int_0^{c\Delta t} \frac{\cos\theta \sin\theta}{4\pi} \, d\theta \, d\phi \, dr$$

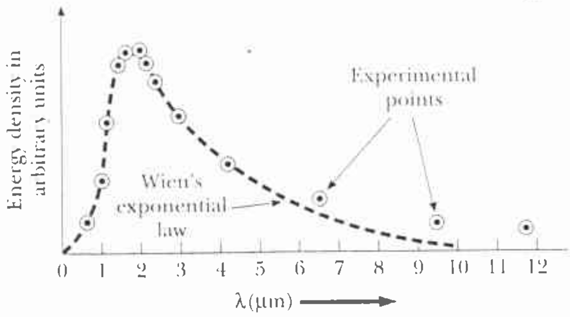
$$= u(\lambda, T) \frac{1}{4} c \Delta t$$

The amount of energy emerging per unit area per unit time

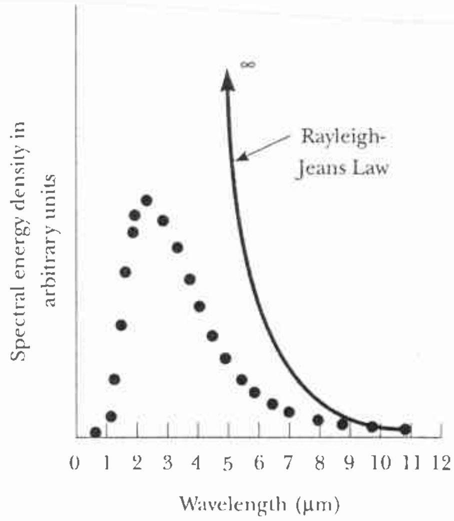
$$\equiv E(\lambda, T) = \frac{1}{4} c \, u(\lambda, T)$$

$$\Rightarrow \quad u(\lambda, T) = \frac{4E(\lambda, T)}{c}$$

1894 Wien's displacement law



Discrepancy between Wien's law and experimental data for a black body at 1500 K.



The failure of the classical Rayleigh-Jeans law (Equation 2.18) to fit the observed spectrum of a black body heated to 1000 K.

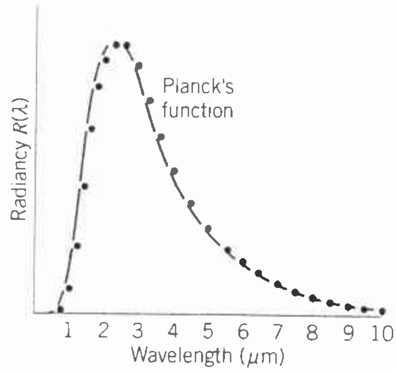


FIGURE 3.17 Planck's function fits the observed data perfectly.

$$u(\lambda, T) = \lambda^{-5} f(\lambda T)$$

function of a single variable λT

$\Rightarrow \quad \dfrac{u(\lambda, T)}{T^5}$ is a universal function of λT.

$u(\nu, T)\, d\nu = u(\lambda, T)\, d\lambda$ change of variable

$$\Rightarrow \quad u(\nu, T) = u(\lambda, T) \left| \frac{d\lambda}{d\nu} \right|$$

$$\lambda \nu = c, \quad \lambda = \frac{c}{\nu} \quad \Rightarrow \quad d\lambda = -\frac{c}{\nu^2} d\nu$$

$$\Rightarrow \quad u(\nu, T) = \frac{c}{\nu^2} u(\lambda, T)$$

$$= \frac{c}{\nu^2} \cdot \frac{1}{\lambda^5} f(\lambda T)$$

$$= \frac{c}{\nu^2} \cdot \frac{1}{(\frac{c}{\nu})^5} f(\lambda T)$$

$$= \nu^3 g\left(\frac{\nu}{T}\right)$$

Comments:

(i) Given the spectral distribution of blackbody radiation at one temperature, the distribution at any other temperature can be found with the help of the expression given above.

$u(\lambda, T_0)$ is known for all λ at T_0

Want to find $u(\lambda_1, T_1)$

$$u(\lambda_1, T_1) = \lambda_1^{-5} f(\lambda_1 T_1)$$

Find a λ_0 such that $\lambda_0 T_0 = \lambda_1 T_1 \Rightarrow \lambda_0 = \dfrac{\lambda_1 T_1}{T_0}$

$$u(\lambda_0, T_0) = \lambda_0^{-5} f(\lambda_0 T_0)$$

$$\frac{u(\lambda_1, T_1)}{u(\lambda_0, T_0)} = \left(\frac{\lambda_0}{\lambda_1}\right)^5$$

$$u(\lambda_1, T_1) = \underbrace{u\left(\frac{\lambda_1 T_1}{T_0}, T_0\right)}_{\text{known}} \left(\frac{T_1}{T_0}\right)^5$$

(ii) $\lambda_{max}(T)$: the wavelength at which the energy density has its maximum value at temperature T

$$\Rightarrow \quad \lambda_{max} = \frac{b}{T} \leftarrow \text{universal constant}$$

$$u(\lambda, T) = \lambda^{-5} f(\lambda T)$$

$$\frac{\partial u}{\partial \lambda} = \lambda^{-5} f'(\lambda T) T - 5\lambda^{-6} f(\lambda T)$$

′ refers to derivative with respect to λT

$$= \lambda^{-6} [\underbrace{\lambda T f'(\lambda T) - 5 f(\lambda T)}_{\text{function of } \lambda T}]$$

Maximum occur at $\frac{\partial u}{\partial \lambda} = 0$

$\Rightarrow$ at $\lambda T = b = $ constant

$\Rightarrow \lambda_{max} = \frac{b}{T}$

The problem is to specify $g(\frac{\nu}{T})$

Wien's model $\rightarrow g(\frac{\nu}{T}) = c e^{-\beta\nu/T}$

c, β are adjustable parameters
works for high frequency.

1900 Rayleigh - Jeans law

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3} kT$$

$$u(\nu, T) d\nu = [\frac{8\pi\nu^2}{c^3} V d\nu \; kT] / V$$

kT → average energy per mode, from equipartition law

$\frac{8\pi\nu^2}{c^3} V d\nu$ → number of modes (i.e., degree of freedom) for EM radiation with frequency in the interval ν and $\nu + d\nu$ inside a cavity of volume V

Thus, in classical physics, Rayleigh - Jean law is "inevitable"

$$u(\nu, T) = \nu^3 \frac{8\pi}{c^3} \frac{1}{\nu} kT$$

$$= \nu^3 g(\frac{\nu}{T})$$

$$g(\frac{\nu}{T}) = \frac{8\pi k}{c^3} \cdot (\frac{\nu}{T})^{-1}$$

The above equation works well for ν small (i.e., low frequency

Number of modes with frequency between ν and $d\nu$

$$= \frac{8\pi\nu^2}{c^3} d\nu \cdot V$$ See Eisberg and Resnick P. 8

Electromagatic wave (standing) in one dimension box ($x: 0 \to a$)

$$E(x,t) = E_0 \sin kx \sin \omega t$$

Boundary condition $E = 0$ at $x = 0$ and a

$$ka = n\pi \leftrightarrow \frac{2\pi}{\lambda} = \frac{n\pi}{a} \qquad n = \text{integers (positive)}$$

$$\nu = \frac{c}{\lambda} = \frac{cn}{2a}$$

Generalize to three dimensional case

$$\nu = \frac{c}{2a}\sqrt{n_x^2 + n_y^2 + n_z^2}$$

n_x, n_y, n_z are positive integers

Define $r^2 = n_x^2 + n_y^2 + n_z^2$

$$\nu = \frac{c}{2a} r$$

$N(\nu)\,d\nu$ = number of modes with frequency in the range between ν and $d\nu$

= number of lattice points enclosed by the shell r and $r+dr$ in the $\underset{\vec{n}}{(n_x, n_y, n_z)}$ space (with n_x, n_y, n_z positive)

Number of lattice point in the $\vec{n}$ space = volume in $\vec{n}$ space $= 4\pi r^2 dr$

Construct cuts at n_x, n_y, n_z at half-integers
It is obvious that there will be one lattice point in each unit volume cell.

$$N(\nu)\,d\nu = \frac{1}{8} 4\pi r^2 dr$$

$$= \frac{1}{8} 4\pi \cdot \left(\frac{2a\nu}{c}\right)^2 \frac{2a}{c} d\nu$$

$$= \frac{4\pi\nu^2}{c^3} V d\nu$$

For electromagnetic wave, for each mode, there are two possible states of polarization (vector field $\vec{E} \perp \vec{k}$)

$$\Rightarrow \quad N(\nu)\,d\nu = \frac{8\pi\nu^2}{c^3} V d\nu$$

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The law of equipartition of energy: classical kinetic theory

A system of gas molecules in thermal equilibrium at temperature T, the average kinetic energy of a molecule per degree of freedom is $\frac{1}{2}kT$

↳ Boltzmann constant

↓

Also applicable to any classical system containing, in equilibrium, a large number of entitle of the same kind.

↓

average kinetic energy of the sinusoildally standing wave is $\frac{1}{2}kT$

The average total energy of the sinusoilally standing wave is twice its average kinetic energy

↓

common property of physical systems, with one degree of freedom, that execute simple harmonic oscillation in time

$$\bar{E} = kT$$

Note: it is independent of ν.

Another derivation

$P(E)\,dE$ = probability of finding a given entity of a system with energy between E and $E+dE$

Many physical system, containing a large number of entities of the same kind in thermal equilibrium in temperature T, follows the Boltzmann distribution

$$P(E) = \frac{e^{-E/kT}}{\underbrace{kT}_{\text{normalization}}}$$

The set of simple harmonic oscillating standing waves in thermal equilibrium in a black-body cavity is also governed by the Boltzmann distribution

$$\bar{E} = \frac{\int_0^\infty E\,P(E)\,dE}{\int_0^\infty P(E)\,dE} = kT$$

Gustav Robert Kirchhoff 1824-1887

Max Planck 1858-1947

Max Planck (1858–1947, Germany). His work on the spectral distribution of radiation, which led to the quantum theory, was honored with the 1918 Nobel prize. In his later years, he wrote extensively on religious and philosophical topics.

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Difficulty with Rayleigh-Jean formula

$$U(T) = \int u(\nu, T)\, d\nu \to \infty \Rightarrow \text{ultraviolet catastrophe}$$

total radiation energy per unit volume

1900 Dec 14 Planck's paper → beginning of quantum theory

Thermodynamic consideration and interpolation

See T. Y. Wu, P. 32

⇒ Planck's formula

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{h\nu/kT} - 1}$$

h is an adjustable parameter with dimension of energy · time

Large ν limit

$$u(\nu, T) \longrightarrow \frac{8\pi h}{c^3} \nu^3 e^{-h\nu/kT} \qquad \text{Wien's formula}$$

Small ν limit

$$u(\nu, T) \longrightarrow \frac{8\pi\nu^2}{c^3} \frac{h\nu}{1 + \frac{h\nu}{kT} + \cdots - 1}$$

$$\rightarrow \frac{8\pi\nu^2}{c^3} kT \qquad \text{Rayleigh-Jean formula}$$

$$U(T) = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu$$

$$= \frac{8\pi h}{c^3} \left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1} \qquad x = \frac{h\nu}{kT}, \quad \nu = \frac{kT}{h} x, \quad d\nu = \frac{kT}{h} dx$$

$$= a T^4 \qquad \text{Stefan - Boltzmann law}$$

Claim. $\displaystyle\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$

Proof. $\displaystyle\int_0^\infty \frac{x^3 dx}{e^x - 1}$

$$= \int_0^\infty x^3 dx \sum_{n=1}^\infty e^{-nx}$$

$$= \sum_{n=1}^\infty \int_0^\infty x^3 dx\, e^{-nx} \qquad y = nx$$

$$= \sum_{n=1}^\infty \frac{1}{n^4} \int_0^\infty dy\, y^3 e^{-y}$$

$$= 6 \sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{15}$$

Derivation of Planck's formula

Postulate: each of these standing waves in the box cannot take on all possible energies, as classical physics implies but can only take on only discrete energies $E_n = nh\nu$ ($n = 1, 2, \cdots$)

$$p(E_n) = \frac{e^{-E_n/kT}}{\sum e^{-E_n/kT}}$$

$$\bar{E} = \sum E_n P(E_n) = \frac{\sum nh\nu\, e^{-nh\nu/kT}}{\sum e^{-nh\nu/kT}}$$

Let $x = \frac{h\nu}{kT}$

$$\bar{E} = \frac{xkT \sum n e^{-nx}}{\sum e^{-nx}} = \frac{-xkT \frac{d}{dx} \sum e^{-nx}}{\sum e^{-nx}}$$

Claim $\sum e^{-nx} = \frac{1}{1-e^{-x}}$

Proof $(1-y)^{-1} = 1 + y + y^2 + \cdots$
Take $y = e^{-x}$

$$\Rightarrow \bar{E} = \frac{-xkT \frac{d}{dx}\left(\frac{1}{1-e^{-x}}\right)}{\left(\frac{1}{1-e^{-x}}\right)} = \frac{xkT\, e^{-x}}{1-e^{-x}} = \frac{h\nu}{e^{h\nu/kT}-1}$$

$$\Rightarrow u(\nu, T)\, d\nu = \frac{8\pi\nu^2}{c^3} d\nu \frac{h\nu}{e^{h\nu/kT}-1}$$ Planck's formula

From experiments $\Rightarrow$ h = Planck's constant
$= 6.63 \cdot 10^{-27}$ erg·sec

Planck: for some unknown reasons the atoms in the walls of the cavity emitted radiation in "quanta" with energy $nh\nu$ ($n = 1, 2, 3, \cdots$)

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A dramatic example of an application of Planck's law on the current frontier of physics is in the tests of the predictions of the Big Bang theory of the formation and the present expansion of the universe

Current cosmological theory:

the universe originated in an extremely high temperature explosion

↓

infant universe was fill with radiation whose spectral distribution must be of black body radiation

↓

since that time, the universe has expanded to its present size and cooled to its present temperature T_{now}

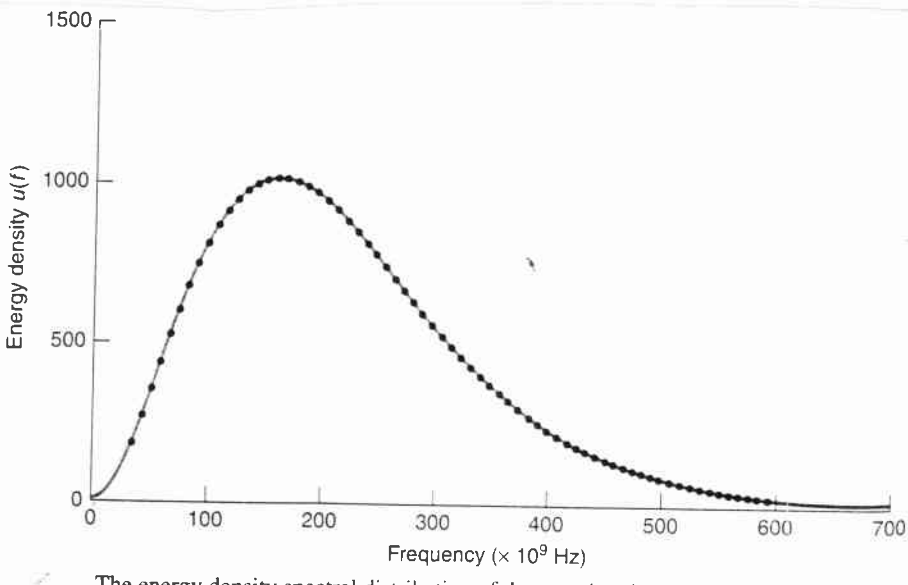
↓

the universe should still be filled with radiation whose spectral distribution should be that characteristic of black body at T_{max}

1965 Penzias and Wilson observed such radiation

Most recent data collected by Cosmic Blackground Explorer (COBE) ⇒ Plack's distribution at 2.735 K.

⇒ gives strong support for the Big Bang theory (See Chapter 18)



The energy density spectral distribution of the cosmic microwave background radiation. The solid line is Planck's law with $T = 2.735$ K. The measurements were made by the COBE satellite.

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