FROM FALLING BODIES TO RADIO WAVES

Classical Physicists and Their Discoveries

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Heinrich Hertz

From Wikipedia, the free encyclopedia

Heinrich Rudolf Hertz (22 February 1857 – 1 January 1894) was a German physicist who clarified and expanded James Clerk Maxwell's electromagnetic theory of light, which was first demonstrated by David Edward Hughes using non-rigorous trial and error procedures. Hertz is distinguished from Maxwell and Hughes because he was the first to conclusively prove the existence of electromagnetic waves by engineering instruments to transmit and receive radio pulses using experimental procedures that ruled out all other known wireless phenomena.[1] The scientific unit of frequency — cycles per second — was named the "hertz" in his honor.[2]

In 1883, Hertz took a post as a lecturer in theoretical physics at the University of Kiel.

In 1885, Hertz became a full professor at the University of Karlsruhe where he discovered electromagnetic waves.

The most dramatic prediction of Maxwell's theory of electromagnetism, published in 1865, was the existence of electromagnetic waves moving at the speed of light, and the conclusion that light itself was just such a wave. This challenged experimentalists to generate and detect electromagnetic radiation using some form of electrical apparatus.

The first successful radio transmission was made by David Edward Hughes in 1879, but it would not be conclusively proven to have been electromagnetic waves until the experiments of Heinrich Hertz in 1886. For the Hertz radio wave transmitter, he used a high voltage induction coil, a condenser (capacitor, Leyden jar) and a spark gap—whose poles on either side are formed by spheres of 2 cm radius—to cause a spark discharge between the spark gap's poles oscillating at a frequency determined by the values of the capacitor and the induction coil.

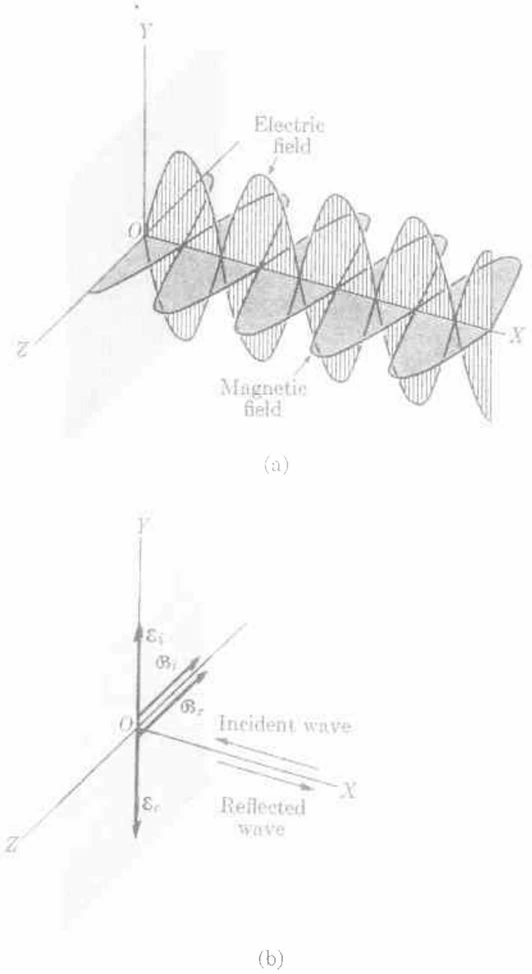
To prove there really was radiation emitted, it had to be detected. Hertz used a piece of copper wire, 1 mm thick, bent into a circle of a diameter of 7.5 cm, with a small brass sphere on one end, and the other end of the wire was pointed, with the point near the sphere. He bought a screw mechanism so that the point could be moved very close to the sphere in a controlled fashion. This "receiver" was designed so that current oscillating back and forth in the wire would have a natural period close to that of the "transmitter" described above. The presence of oscillating charge in the receiver would be signaled by sparks across the (tiny) gap between the point and the sphere (typically, this gap was hundredths of a millimeter).

In more advanced experiments, Hertz measured the velocity of electromagnetic radiation and found it to be the same as the light's velocity. He also showed that the nature of radio waves' reflection and refraction was the same as those of light and established beyond any doubt that light is a form of electromagnetic radiation obeying the Maxwell equations.

Hertz's experiments triggered broad interest in radio research that eventually produced commercially successful wireless telegraph, audio radio, and later television. In 1930 the International Electrotechnical Commission (IEC) honored Hertz by naming the unit of frequency—one cycle per second—the "hertz".[2]

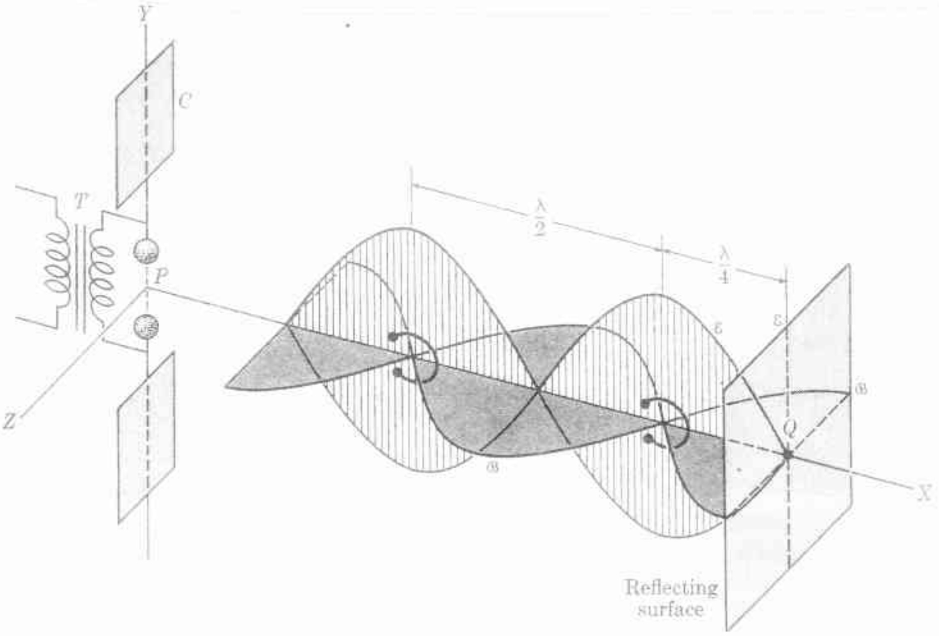
28.5 STANDING ELECTROMAGNETIC WAVES

Interference (and diffraction) phenomena are so characteristic of waves that their presence has always been accepted by physicists as conclusive proof that a process can be interpreted as a wave motion. For that reason, when in the seventeenth century Young, Grimaldi, and others observed interference (and diffraction) in their research on light, the wave theory of light became generally accepted. At that time electromagnetic waves were not known, and light was assumed to be an elastic wave in a subtle medium, called ether, that pervaded all matter. It was not until the end of the nineteenth century that Maxwell predicted the existence of electromagnetic waves, and Hertz, by means of interference experiments which gave rise to standing electromagnetic waves, experimentally verified the existence of electromagnetic waves in the radio-frequency range. Later their velocity was measured and found to be equal to that of light. The reflection, refraction, and polarization of electromagnetic waves was also found to be similar to those of light. The obvious conclusion was to identify light with electromagnetic waves of certain frequencies. At that time optics, to all intents and purposes, ceased to be an independent branch of physics and became simply a chapter of electromagnetic theory.



28.21 Standing electromagnetic waves produced by reflection from a conducting surface.

To understand the formation of standing electromagnetic waves, assume that the waves produced by an oscillating electric dipole are falling with perpendicular incidence on the plane surface of a perfect conductor (Fig. 28.21). Taking the X-axis as the direction of propagation and the Y- and Z-axes as being parallel to the electric and the magnetic fields, respectively, we have a wave that is plane polarized, with the electric field oscillating in the XY-plane. The electric field is then parallel to the surface of the conductor. But at the surface of a perfect conductor the electric field must be perpendicular to the conductor; that is, the electric field cannot have a tangential component. The only way to make this condition compatible with the orientation of the electric field in the incident wave is by requiring that the resultant electric field be zero at the surface of the conductor. This means



Hertz' experiment on interference of electromagnetic waves.

that the electric field of the reflected wave at the surface must be equal and opposite to that of the incident wave, thus giving

$$\mathcal{E} = \mathcal{E}_i + \mathcal{E}_r = 0$$

for $x = 0$. This condition is mathematically equivalent to the condition for the reflection of waves in a string with one end fixed, discussed in Section 28.4. Since the mathematics is the same, we may use Eq. (28.18) to write an expression for the resultant electric field,

$$\mathcal{E} = 2\mathcal{E}_0 \sin kx \sin \omega t.$$

The magnetic field oscillates in the XZ-plane. Using Eq. (24.8), we find that the magnetic field is expressed by

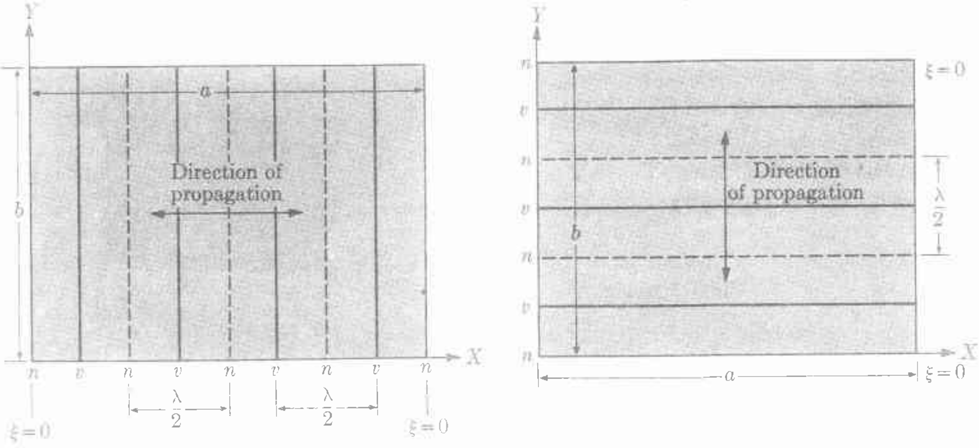
$$\mathcal{B} = 2\mathcal{B}_0 \cos kx \cos \omega t,$$

with $\mathcal{B}_0 = \mathcal{E}_0/c$. Therefore there is a phase difference of $\frac{1}{2}\lambda$ in the space variations and of $\frac{1}{2}P$ in the time variations of the two fields. From the mathematical expression for $\mathcal{B}$, note that the magnetic field has maximum amplitude at the surface. This can also be seen from the boundary condition at the surface: referring to Fig. 28.21(b), we see that if the electric field of the incident wave is along the $+Y$-axis, the magnetic field must be along the $-Z$-axis, according to the relative orientation of the two fields with respect to the direction of propagation of the incident waves, which is along the $-X$-axis. For a zero resultant electric field to exist at the surface, the electric field of the reflected wave must be along the $-Y$-axis, and since the reflected wave propagates along the X-axis, the magnetic field must be along the $-Z$-axis. Thus, although the electric fields interfere destructively at the surface, the magnetic fields interfere constructively there.

The amplitudes of the electric and magnetic fields of the resulting wave at a distance x from the surface are $2\mathcal{E}_0 \sin kx$ and $2\mathcal{B}_0 \cos kx$. They are indicated by the shaded lines in Fig. 28.21(a). At the points where

$$kx = n\pi \qquad \text{or} \qquad x = \tfrac{1}{2}n\lambda,$$

the electric field is zero and the magnetic field is



(a)

(b)

28.23 Standing waves on a rectangular membrane.

maximum. At the points where

$$kx = (n + \tfrac{1}{2})\pi \qquad \text{or} \qquad x = (2n + 1)\lambda/4,$$

the electric field has a maximum value but the magnetic field is zero.

Hertz' Experiment It is instructive to see how Heinrich Hertz, in 1888, with his primitive equipment, verified the theoretical predictions given above. Hertz' oscillator is shown on the left in Fig. 28.22. The transformer T charges the metallic plates C and C'. These plates discharge through the gap P, which becomes a dipole oscillator. Along the line PX, the direction of the electric field is parallel to the Y-axis and that of the magnetic field along the Z-axis. To observe the waves, Hertz used a short wire, bent in circular shape, but with a small gap. This device is called a *resonator*. The diameter of the resonator used in this kind of experiment must be very small compared with the wavelength of the waves. If the resonator is placed with its plane perpendicular to the magnetic field of the wave, the varying magnetic field induces an emf in the resonator, resulting in sparks at its gap. On the other hand, if the plane of the resonator is parallel to the magnetic field, no emf is induced and no sparks are observed at the gap.

To produce standing electromagnetic waves, Hertz placed a reflecting surface (made of a good conductor) at Q. In such a case, when the resonator is at a node of the magnetic field, no matter what its orientation, it will show no induced emf (or sparks). At an antinode of the magnetic field, however, the sparking is greatest when the resonator is oriented perpendicular to the magnetic field. By moving the resonator along the line PQ, Hertz found the position of the nodes and antinodes and the direction of the magnetic field. The results obtained by Hertz coincided with the theoretical analysis we have given. By measuring the distance between two successive nodes, Hertz could calculate the wavelength λ, and since he knew the frequency ν of the oscillator, he could calculate the velocity c of the electromagnetic waves by using the equation $c = \lambda\nu$. It was by this means that Hertz obtained the first experimental value for the velocity of propagation of electromagnetic waves.

Standing EM waves

Maxwell equation 1864

Hertz experiment (1888)
↓
figure

measure λ , $\lambda \nu = C$

from the node

from the emitter

Definitions

Radiators and Radiation

The energy emitted and absorbed by material objects in the form of electromagnetic waves is generally termed *electromagnetic radiation* or simply *radiation*.

Radiation is emitted at the expense of the energy of the radiating object; absorbed radiation adds to the energy of the absorbing object.

Any specific distribution of electromagnetic radiation, such as the colours of the rainbow, is termed a *spectrum*. If the distribution comprises a continuous region of frequencies or wavelengths, it is a *continuous spectrum*; if it comprises a series or group of discrete frequencies, it is a *line spectrum*.

The distribution of the electromagnetic radiation emitted by a body that acts as a source of radiation is its *emission spectrum* (Fig 2.1a). The rainbow is the sun's emission spectrum in the visible region.

The distribution of the electromagnetic radiation transmitted by an absorbing medium placed in the path of radiation that exhibits a continuous spectrum, is called an *absorption spectrum* (Fig 2.1b). When viewing a rainbow through a coloured filter, certain colours are seen to be missing; they are the ones absorbed by the filter. The spectrum of the transmitted light is the filter's absorption spectrum.

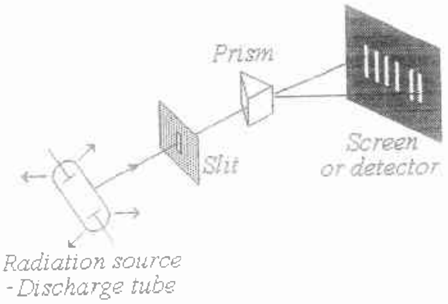


Fig 2.1a Viewing an emission spectrum in the visible range. The light emitted from the source is dispersed into its components by a prism or diffraction grating. These components (colours) illuminate the screen or detector and constitute the source's emission spectrum.

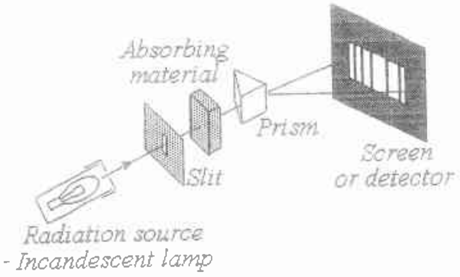


Fig 2.1b Viewing an absorption spectrum in the visible range. Light, such as that emitted by an incandescent lamp which exhibits a continuum of frequencies, is passed through a sample of the material whose absorption spectrum is being investigated. The frequencies absorbed by the sample appear as dark lines or bands on the continuous spectrum projected onto the screen.

spectrosopy.

Chapter

The Quantum Theory (1900-1925)

Experimental results required concepts totally incompatible with classical physics.

The new concepts are.
the particle properties of radiation
the wave properties of matter
the quantization of physical quantities.

1. Black-body Radiation - See Mattews, Dicke and Wittke

Body heat ⇒ it is seen to radiate

In equilibrium the light emitted over the whole spectrum of frequencies ν with a spectral distribution depend on ν and T

$\lambda \nu = c$ ⇒ we can either use λ or ν as variable
c ↳ velocity
λ ↓ wavelength

$E(\lambda, T)\,d\lambda$ = energy emitted per unit time in radiation with wavelength in the interval λ and $\lambda + d\lambda$ from a unit area of surface at absolute temperature T.

1859 Kirchhoff's law of radiation
$A(\lambda, T)$ = fraction of incident radiation λ that is absorbed by the body

$\dfrac{E(\lambda, T)}{A(\lambda, T)}$ ~~is same for all bodies~~

$A \equiv 1$ → blackbody
⇒ $E(\lambda, T)$ is a universal function independent of the detailed structure of the black body.

It is evident that the universal properties of the thermal radiation emitted by black bodies makes them of particular theoretical interest.

Best possible source of black-body

Radiation from a small hole in an enclosure heated to temperature T

Radiation incident upon the hole from the outside the cavity and is reflected back and forth by the wall of the cavity, eventually being absorbed by these walls.

If the hole is very small, compared with the area of the inner surface of the cavity, a negligible amount of the incident radiation will be reflected through the hole

⇒ all the radiation incident upon the hole is absorbed

⇒ $A = 1$ and the hole must have the properties of a black body.

Assume the walls of the cavity are uniformly heated to a temperature T

⇒ the wall will emit thermal radiation which will fill the cavity, the small fraction of this radiation incident from inside upon the hole will pass through the hole.

⇒ the hole will act as an emitter of thermal radiation

↓

$A = 1$

⇒ the radiation emitted by the hole must have a black-body spectrum

But the hole is merely sampling the thermal radiation present inside the cavity ⇒ radiation in the cavity must have a black-body spectrum characteristic of the temperature T

↓

problem is to understand the distribution of radiation inside a cavity whose walls are at a temperature T

Kirchhoff: radiation in the cavity must be isotropic and homogeneous

$u(\lambda, T)\, d\lambda$ = energy density inside the cavity, with wavelength between λ and $\lambda + d\lambda$, at temperature T.

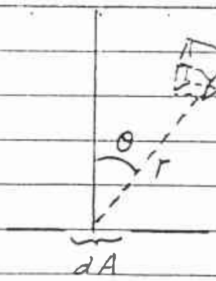
Relation between $E(\lambda, T)$ and $u(\lambda, T)$

$$u(\lambda, T) = \frac{4E(\lambda, T)}{c}$$

↓

theoretical interest

Proof.



Dimensional check

$$RHS = \frac{\text{energy / area} \cdot \text{time}}{\text{length / time}} = \frac{\text{energy}}{\text{area} \cdot \text{length}} \longrightarrow \text{energy density}$$

The shaded volume element $dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$

$$= r^2 dr \, d\Omega$$

The energy contained in the volume element dV

$$= u(\lambda, T) \, r^2 dr \sin\theta \, d\theta \, d\phi$$

The radiation is isotropic, the fraction passing through dA

$$= \frac{dA \cos\theta}{4\pi r^2}$$

The amount of energy emerging through dA from the volume element dV

$$= \frac{dA \cos\theta}{4\pi r^2} \cdot r^2 dr \sin\theta \, d\theta \, d\phi \; u(\lambda, T)$$

The amount of energy emerging per unit area from the volume element dV

$$= \frac{\cos\theta \sin\theta}{4\pi} \, d\theta \, d\phi \, dr \; u(\lambda, T)$$

In time interval Δt only r between 0 and $c\Delta t$ can contribute

$\Rightarrow$ The amount of energy emerging per unit area in time Δt

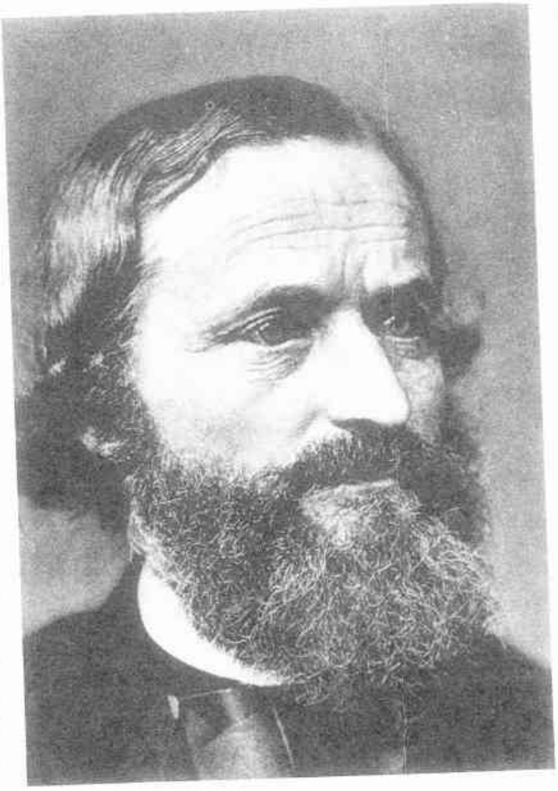
$$= u(\lambda, T) \int_0^{\pi/2} \int_0^{2\pi} \int_0^{c\Delta t} \frac{\cos\theta \sin\theta}{4\pi} \, d\theta \, d\phi \, dr$$

$$= u(\lambda, T) \frac{1}{4} c \Delta t$$

The amount of energy emerging per unit area per unit time

$$\equiv E(\lambda, T) = \frac{1}{4} c \, u(\lambda, T)$$

$$\Rightarrow \quad u(\lambda, T) = \frac{4E(\lambda, T)}{c}$$



Gustav Robert Kirchhoff (1824–1887) is remembered for his many contributions to physics. The distribution law of currents in circuits, the radiation law, spectral gens's principle are among them. A major exponent of classical theoretical physics, he taught primarily at Heidelberg and Berlin and wrote several influential books. (Courtesy of AIP Niels Bohr Library, W. F. Meggers Collection.)

Kirchhoff 1859

Thermal radiation
All warm bodies
radiates ⇒ emit EM waves
as a result of the
thermal motion of
the charged particle

We shall now turn to another puzzle confronting physicists at the turn of the century (1900): just how do heated bodies radiate? There was a general understanding of the mechanism involved—heat was known to cause the molecules and atoms of a solid to vibrate, and the molecules and atoms were themselves complicated patterns of electrical charges. (As usual, Newton was on the right track.) From the experiments of Hertz and others, Maxwell's predictions that oscillating charges emitted electromagnetic radiation had been confirmed, at least for simple antennas. It was known from Maxwell's equations that this radiation traveled at the speed of light and from this it was realized that light itself, and the closely related infrared heat radiation, were actually electromagnetic waves. The picture, then, was that when a body was heated, the consequent vibrations on a molecular and atomic scale inevitably induced charge oscillations. Assuming then that Maxwell's theory of electromagnetic radiation, which worked so well in the macroscopic world, was also valid at the molecular level, these oscillating charges would radiate, presumably giving off the heat and light observed.

How is Radiation Absorbed?

What is meant by the phrase "black body" radiation? The point is that the radiation from a heated body depends to some extent on the body being heated. To see this most easily, let's back up momentarily and consider how different materials *absorb* radiation. Some, like glass, seem to absorb light hardly at all—the light goes right through. For a shiny metallic surface, the light isn't absorbed either, it gets reflected. For a black material like soot, light and heat are almost completely absorbed, and the material gets warm. How can we understand these different behaviors in terms of light as an electromagnetic wave interacting with charges in the material, causing these charges to oscillate and absorb energy from the radiation? In the case of glass, evidently this doesn't happen, at least not much. Why not? A full understanding of why needs quantum mechanics, but the general idea is as follows: there are charges—electrons—in glass that are able to oscillate in response to an applied external oscillating electric field, *but* these charges are tightly bound to atoms, and can only oscillate at certain frequencies. (For quantum experts, these charge oscillations take place as an electron moves from one orbit to another. Of course, that was not understood in the 1890's, the time of the first precision work on black body radiation.) It happens that for ordinary glass *none of these frequencies corresponds to visible light*, so there is *no* resonance with a light wave, and hence little energy absorbed. That's why glass is perfect for windows! Duh. But glass *is* opaque at some frequencies *outside* the visible range (in general, both in the infrared and the ultraviolet). These are the frequencies at which the electrical charge distributions in the atoms or bonds can naturally oscillate.

How can we understand the *reflection* of light by a *metal* surface? A piece of metal has electrons free to move through the entire solid. This is what makes a metal a metal: it conducts both electricity and heat easily, both are actually carried by currents of these freely moving electrons. (Well, a little of the heat is carried by vibrations.) But metals are recognizable because they're shiny—why's that? Again, it's those free electrons: they're driven into large (relative to the atoms) oscillations by the electrical field of the incoming light wave, and this induced oscillating current radiates electromagnetically, just like a current in a transmitting antenna. This radiation *is* the reflected light. For a shiny metal surface, little of the incoming radiant energy is absorbed as heat, it's just reradiated, that is, reflected.

Now let's consider a substance that *absorbs* light: no transmission and no reflection. We come very close to perfect absorption with soot. Like a metal, it will conduct an electric current, but nowhere near as efficiently. There *are* unattached electrons, which can move through the whole solid, but they constantly bump into things—they have a short mean free path. When they bump, they cause vibration, like balls hitting bumpers in a pinball machine, so they give up kinetic energy into heat. Although the electrons in soot have a short mean free path compared to those in a good metal, they move very freely compared with electrons bound to atoms (as in glass), so they can accelerate and pick up energy from the electric field in the light wave. They are therefore very effective intermediaries in transferring energy from the light wave into heat.

Relating Absorption and Emission

Having seen how soot can absorb radiation and transfer the energy into heat, what about the reverse? Why does it radiate when heated? The pinball machine analogy is still good: imagine now a pinball machine where the barriers, etc., vibrate vigorously because they are being fed energy. The balls (the electrons) bouncing off them will be suddenly accelerated at each collision, and these accelerating charges emit electromagnetic waves. On the other hand, the electrons in a *metal* have very long mean free paths, the lattice vibrations affect them much less, so they are less effective in gathering and radiating away heat energy. It is evident from considerations like this that good absorbers of radiation are also good emitters.

In fact, we can be much more precise: **a body emits radiation at a given temperature and frequency *exactly* as well as it absorbs the same radiation**. This was proved by Kirchhoff: the essential point is that if we suppose a particular body can absorb better than it emits, then in a room full of objects all at the same temperature, it will absorb radiation from the other bodies better than it radiates energy back to them. This means it will get hotter, and the rest of the room will grow colder, contradicting the second law of thermodynamics. (We could use such a body to construct a heat engine extracting work as the room grows colder and colder!)

But a metal glows when it's heated up enough: why is that? As the temperature is raised, the lattice of atoms vibrates more and more, these vibrations scatter and accelerate the electrons. Even glass glows at high enough temperatures, as the electrons are loosened and vibrate.

Black-body Radiation

Common materials and objects do not absorb all the radiation incident upon them; they are not perfect absorbers of radiation. Nevertheless, we can imagine a perfect absorber, an ideal body which does absorb all the electromagnetic radiation that strikes it, whatever its wavelength or intensity. Because such a body would appear black in whatever light it is illuminated, it is called a *black-body*.

Since, by definition, a black-body is a perfect absorber, it must also be a perfect emitter, i.e., it must be able to emit radiation of every wavelength at any intensity. The heat radiation emitted by a black-body is called *black-body radiation*.

No substance is a perfect black-body, though soot, which absorbs some 95% of the visible and infra-red radiation incident upon it, closely approximates to one in this range. However, by making a small hole in the wall of a hollow object we can construct a device, which to all intents and purposes, absorbs all the radiation incident upon it. The device works somewhat like a fly-trap. Any radiation coming upon the hole from outside will pass through it, enter the cavity and be trapped inside (Fig 2.3). The hole is thus a perfect absorber of radiation; it is a black-body. By definition, it will also be a perfect emitter of radiation. Any radiation generated in the cavity that happens upon the hole will escape without hindrance, irrespective of its wavelength^2 or intensity.

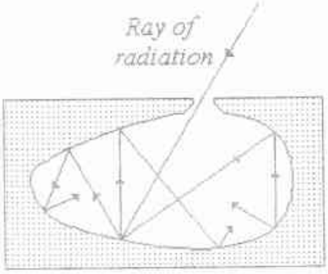


Fig 2.3 A hollow object with a small aperture in one of its walls. Any radiation that enters through the hole will be trapped inside.

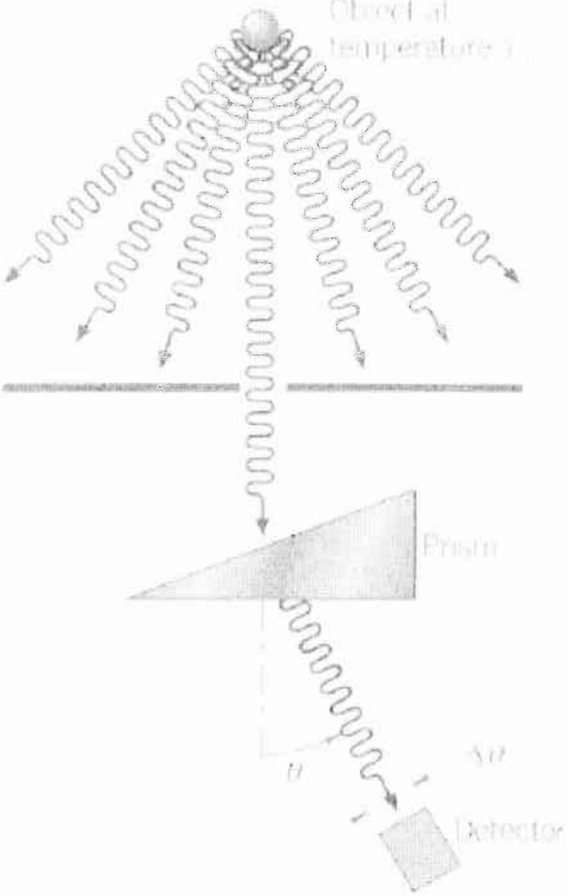
Suppose, that a hollow object with a small aperture in one of its walls is kept at a constant temperature. Every point on the inside surface of the cavity will be in thermal equilibrium with all the other points on the surface and heat radiation of the same quality and quantity will be emitted and absorbed by each point, irrespective of the material from which the inside surface of the cavity is made. The cavity will be filled by electromagnetic radiation that comprises all the wavelengths of the heat radiation characteristic of the object's temperature, each at its appropriate intensity. Any of this *cavity radiation* that happens upon the aperture from inside will escape through it unhindered. Viewed from outside, this representative sample of the cavity radiation will be the black-body radiation characteristic of the particular temperature.

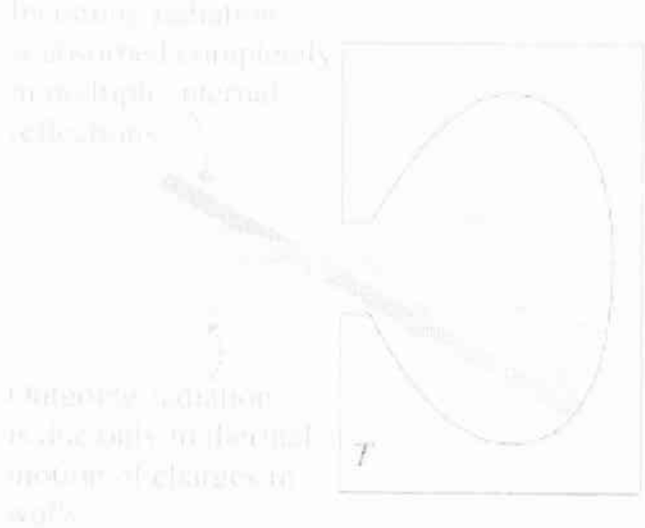
The "Black Body" Spectrum: a Hole in the Oven

Any body at any temperature above absolute zero will radiate to some extent, the intensity and frequency distribution of the radiation depending on the detailed structure of the body. To begin analyzing heat radiation, we need to be specific about the body doing the radiating: *the simplest possible case is an idealized body which is a perfect absorber, and therefore also (from the above argument) a perfect emitter. For obvious reasons, this is called a "**black body**".*

But we need to check our ideas experimentally: so how do we construct a perfect absorber? OK, nothing's perfect, but in 1859 Kirchhoff had a good idea: a small hole in the side of a large box is an excellent absorber, since any radiation that goes through the hole bounces around inside, a lot getting absorbed on each bounce, and has little chance of ever getting out again. So, we can do this *in reverse*: have an oven with a tiny hole in the side, and presumably the radiation coming out the hole is as good a representation of a perfect emitter as we're going to find. Kirchhoff challenged theorists and experimentalists to figure out and measure (respectively) the energy/frequency curve for this "cavity radiation", as he called it (in German, of course: hohlraumstrahlung, where hohlraum means hollow room or cavity, strahlung is radiation). *In fact, it was Kirchhoff's challenge in 1859 that led directly to quantum theory forty years later!*

Measurement of spectrum of thermal radiation





Experimental Results

The intensity of the heat radiation emitted from a blackbody is greater than that emitted by any other body at the same temperature.

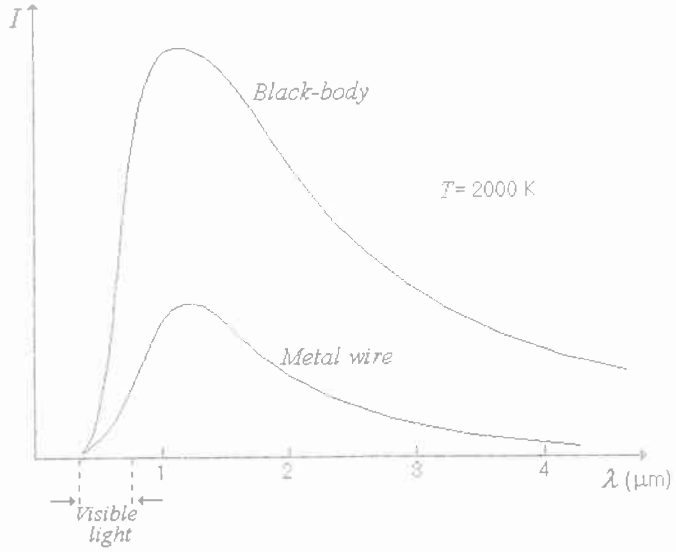


Fig 2.4 The intensity, *I*, of the heat radiation emitted by a black-body and the filament in an incandescent lamp at a temperature of 2000K as a function of the wavelength, *λ*, of the radiation.

The energy E radiated each second from each unit area of the surface of a blackbody $\propto T^4$

Stefan - Boltzmann law

The absolute temperature, T, of a black-body is inversely proportional to the wavelength *in vacuo*, $\lambda_{\max}$, of the radiation it emits with the greatest intensity (Fig 2.5):

$$T\lambda_{\max} = 0.29 \cdot 10^{-2}\,\mathrm{m} \cdot \mathrm{K} \qquad (2.2)$$

This rule is known as *Wien's*[5] *Law*.

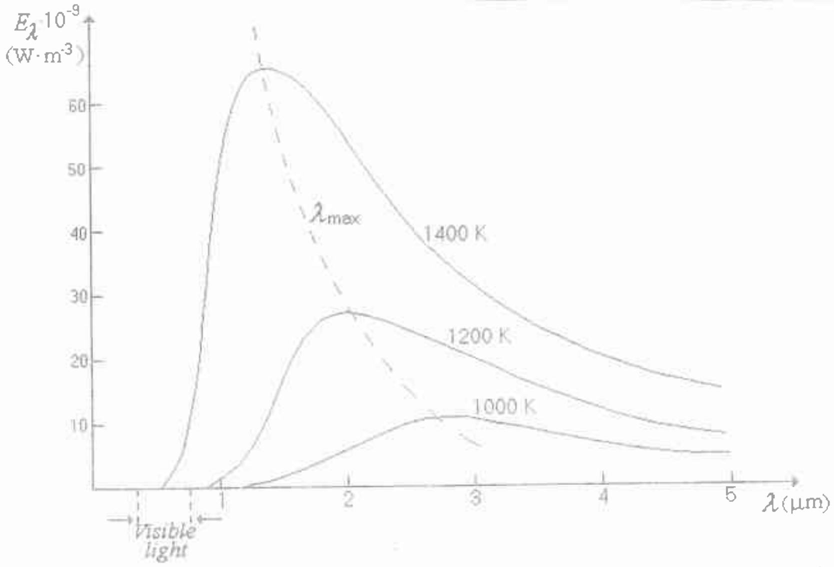


Fig 2.5 The distribution of the heat energy radiated by a black-body between the various wavelengths at a number of temperatures. E_λ is defined such that $E_\lambda \cdot \Delta\lambda$ is the energy radiated each second at the wavelengths between λ and $(\lambda + \Delta\lambda)$ from a unit area of the body's surface. The area under each curve gives the total energy radiated each second from each unit area at the particular temperature.

Wien's displace law

$$u(\lambda, T) = \lambda^{-5} \underset{\overset{\uparrow}{\text{function of a single variable } \lambda T}}{f(\lambda T)}$$

$$\Rightarrow \quad \frac{u(\lambda, T)}{T^5} \quad \text{is a universal function of } \lambda T.$$

$$u(\nu, T)\, d\nu = u(\lambda, T)\, d\lambda \qquad \text{change of variable}$$

$$\Rightarrow \quad u(\nu, T) = u(\lambda, T) \left| \frac{d\lambda}{d\nu} \right|$$

$$\lambda \nu = c \;, \quad \lambda = \frac{c}{\nu} \quad \Rightarrow \quad d\lambda = -\frac{c}{\nu^2} d\nu$$

$$\Rightarrow \quad u(\nu, T) = \frac{c}{\nu^2} u(\lambda, T)$$

$$= \frac{c}{\nu^2} \cdot \frac{1}{\lambda^5} f(\lambda T)$$

$$= \frac{c}{\nu^2} \cdot \frac{1}{(\frac{c}{\nu})^5} f(\lambda T)$$

$$= \nu^3 g\left(\frac{\nu}{T}\right)$$

Comments:

(i) Given the spectral distribution of blackbody radiation at one temperature, the distribution at any other temperature can be found with the help of the expression given above.

$u(\lambda, T_0)$ is known for all λ at T_0

Want to find $u(\lambda_1, T_1)$

$$u(\lambda_1, T_1) = \lambda_1^{-5} f(\lambda_1 T_1)$$

Find a λ_0 such that $\lambda_0 T_0 = \lambda_1 T_1 \Rightarrow \lambda_0 = \frac{\lambda_1 T_1}{T_0}$

$$u(\lambda_0, T_0) = \lambda_0^{-5} f(\lambda_0 T_0)$$

$$\frac{u(\lambda_1, T_1)}{u(\lambda_0, T_0)} = \left(\frac{\lambda_0}{\lambda_1}\right)^5$$

$$u(\lambda_1, T_1) = \underset{\overset{\downarrow}{\text{known}}}{u\left(\frac{\lambda_1 T_1}{T_0}, T_0\right)} \left(\frac{T_1}{T_0}\right)^5$$

(ii) $\lambda_{max}(T)$: the wavelength at which the energy density has its maximum value at temperature T

$$\Rightarrow \quad \lambda_{max} = \frac{b}{T} \leftarrow \text{universal constant}$$

$$u(\lambda, T) = \lambda^{-5} f(\lambda T)$$

$$\frac{\partial u}{\partial \lambda} = \lambda^{-5} f'(\lambda T) T - 5\lambda^{-6} f(\lambda T)$$

′ refers to derivative with respect to λT

$$= \lambda^{-6} [\underbrace{\lambda T f'(\lambda T) - 5 f(\lambda T)}_{\text{function of } \lambda T}]$$

Maximum occur at $\frac{\partial u}{\partial \lambda} = 0$

$\Rightarrow$ at $\lambda T = b =$ constant

$$\Rightarrow \quad \lambda_{max} = \frac{b}{T}$$

The problem is to specify $g(\frac{\nu}{T})$

Wien's model $\rightarrow$ $g(\frac{\nu}{T}) = c e^{-\beta\nu/T}$

c, β are adjustable parameters
works for high frequency

1900 Rayleigh - Jeans law

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3} kT$$

$$u(\nu, T) d\nu = [\underbrace{\frac{8\pi\nu^2}{c^3} V d\nu}_{\substack{\text{number of modes (i.e. degree of freedom)} \\ \text{for EM radiation with frequency in} \\ \text{the interval } \nu \text{ and } \nu + d\nu \text{ inside a} \\ \text{cavity of volume } V}} \overset{\text{average energy per mode}}{kT}] / V$$

average energy per mode ← from equipartition law

Thus, in classical physics, Rayleigh - Jean law is "inevitable"

$$u(\nu, T) = \nu^3 \frac{8\pi}{c^3} \frac{1}{\nu} kT$$

$$= \nu^3 g(\frac{\nu}{T})$$

$$g(\frac{\nu}{T}) = \frac{8\pi k}{c^3} \cdot (\frac{\nu}{T})^{-1}$$

The above equation works well for ν small (i.e. low frequency)

Number of modes with frequency between ν and $d\nu$

$= \frac{8\pi\nu^2}{c^3} d\nu \cdot V$ See Eisberg and Resnick P.8

Electromagatic wave (standing) in one dimension box ($x : 0 \to a$)

$$E(x,t) = E_0 \sin kx \sin \omega t$$

Boundary condition $E = 0$ at $x = 0$ and a

$ka = n\pi \leftrightarrow \frac{2\pi}{\lambda} = \frac{n\pi}{a}$ n = integers (positive)

$$\nu = \frac{c}{\lambda} = \frac{cn}{2a}$$

Generalize to three dimensional case

$$\nu = \frac{c}{2a}\sqrt{n_x^2 + n_y^2 + n_z^2}$$

n_x, n_y, n_z are positive integers

Define $r^2 = n_x^2 + n_y^2 + n_z^2$

$$\nu = \frac{c}{2a} r$$

$N(\nu)\, d\nu$ = number of modes with frequency in the range between ν and $d\nu$

= number of lattice points enclosed by the shell r and $r+dr$ in the $\underset{\vec{n}}{(n_x, n_y, n_z)}$ space (with n_x, n_y, n_z positive)

Number of lattice point in the $\vec{n}$ space = volume in $\vec{n}$ space

$= 4\pi r^2 dr$

Construct cuts at n_x, n_y, n_z at half-integers

It is obvious that there will be one lattice point in each unit volume cell.

$$N(\nu)\, d\nu = \frac{1}{8} 4\pi r^2 dr$$

$$= \frac{1}{8} 4\pi \cdot \left(\frac{2a\nu}{c}\right)^2 \frac{2a}{c} d\nu$$

$$= \frac{4\pi\nu^2}{c^3} V d\nu$$

For electromagnetic wave, for each mode, there are two possible states of polarization (vector field $\vec{E} \perp \vec{k}$)

$$\Rightarrow \quad N(\nu)\, d\nu = \frac{8\pi\nu^2}{c^3} V d\nu$$

分類：
編號：3. - 7
總號：

The law of equipartition of energy: classical kinetic theory

A system of gas molecules in thermal equilibrium at temperature T, the average kinetic energy of a molecule per degree of freedom is $\frac{1}{2}kT$

$\llcorner$ Boltzmann constant

↓

Also applicable to any classical system containing, in equilibrium, a large number of entitle of the same kind.

↓

average kinetic energy of the sinusoildally standing wave is $\frac{1}{2}kT$

The average total energy of the sinusoilally standing wave is twice its average kinetic energy

↓

common property of physical systems, with one degree of freedom, that execute simple harmonic oscillation in time

$\bar{E} = kT$

Note: it is independent of ν.

Difficulty with Rayleigh-Jean formula

$$U(T) = \int u(\nu, T)\, d\nu \to \infty \Rightarrow \text{ultraviolet catastrophe}$$

total radiation energy per unit volume

1900 Dec 14 Planck's paper → beginning of quantum theory

2. Planck's quantum theory (1900)

Thermodynamic consideration and interpolation

See T. Y. Wu, P. 32

⇒ Planck's formula

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{h\nu/kT} - 1}$$

h is an adjustable parameter with dimension of energy · time

Large ν limit

$$u(\nu, T) \longrightarrow \frac{8\pi h}{c^3} \nu^3 e^{-h\nu/kT} \qquad \text{Wien's formula}$$

Small ν limit

$$u(\nu, T) \longrightarrow \frac{8\pi\nu^2}{c^3} \frac{h\nu}{1 + \frac{h\nu}{kT} + \cdots - 1}$$

$$\to \frac{8\pi\nu^2}{c^3} kT \qquad \text{Rayleigh-Jean formula}$$

$$U(T) = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu$$

$$= \frac{8\pi h}{c^3} \left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1} \qquad x = \frac{h\nu}{kT}, \quad \nu = \frac{kT}{h} x, \quad d\nu = \frac{kT}{h} dx$$

$$= aT^4 \qquad \text{Stefan-Boltzmann law}$$

Claim. $\displaystyle\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$

Proof. $\displaystyle\int_0^\infty \frac{x^3 dx}{e^x - 1}$

$$= \int_0^\infty x^3 dx \sum_{n=1}^\infty e^{-nx}$$

$$= \sum_{n=1}^\infty \int_0^\infty x^3 dx\, e^{-nx} \qquad y = nx$$

$$= \sum_{n=1}^\infty \frac{1}{n^4} \int_0^\infty dy\, y^3 e^{-y}$$

$$= 6 \sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{15}$$

Derivation of Planck's formula

Postulate: each of these standing waves in the box cannot take on all possible energies, as classical physics implies but can only take on only discrete energies $E_n = nh\nu$ $(n = 1, 2, \cdots)$

$$p(E_n) = \frac{e^{-E_n/kT}}{\sum e^{-E_n/kT}}$$

derivation of the equation will be given in the appendix

$$\bar{E} = \sum E_n P(E_n)$$

$$= \frac{\sum nh\nu \, e^{-nh\nu/kT}}{\sum e^{-nh\nu/kT}}$$

Let $x = \frac{h\nu}{kT}$

$$\bar{E} = \frac{xkT \sum n e^{-nx}}{\sum e^{-nx}} = \frac{-xkT \frac{d}{dx} \sum e^{-nx}}{\sum e^{-nx}}$$

Claim $\sum e^{-nx} = \frac{1}{1-e^{-x}}$

Proof $(1-y)^{-1} = 1 + y + y^2 + \cdots$
Take $y = e^{-x}$

$$\Rightarrow \bar{E} = \frac{-xkT \frac{d}{dx}\left(\frac{1}{1-e^{-x}}\right)}{\left(\frac{1}{1-e^{-x}}\right)}$$

$$= \frac{xkT \, e^{-x}}{1-e^{-x}}$$

$$= \frac{h\nu}{e^{h\nu/kT}-1}$$

$$\Rightarrow u(\nu, T)\,d\nu = \frac{8\pi\nu^2}{c^3} d\nu \frac{h\nu}{e^{h\nu/kT}-1}$$ Planck's formula.

From experiments $\Rightarrow$ h = Planck's constant
$= 6.63 \cdot 10^{-27}$ erg·sec

Planck: for some unknown reasons the atoms in the walls of the cavity emitted radiation in "quanta" with energy $nh\nu$ $(n = 1, 2, 3, \cdots)$

Classical statistical physics is based on the following fundamental postulates:

(i) The particles of the system are identical but distinguishable

(ii) There is no restriction on the number of particles that may occupy a particular energy state

(iii) At thermal equilibrium, the distribution of particles among the accessible energy states is the most probable distribution consistent with prescribed constraints such as total energy and total number of particles

(iv) Every microstate of the system has equal a priori probability.

Boltzmann Factor

Wont to find $\{N_i\}$ that makes $W_{\{N_i\}}$ maximum under the restriction

$$\sum_{i=1}^{k} N_i = N \quad , \quad \sum_{i=1}^{k} N_i \epsilon_i = E$$

$$W_{\{N_i\}} = \frac{N!}{N_1 ! N_2 ! \cdots N_k !}$$

It is useful to take

$$\ln W = \ln N! - \sum_{i=1}^{k} \ln N_i !$$

Stirling approximation.

$$\ln N! \simeq N \ln N - N$$

$$\ln W = N \ln N - N - \left(\sum_{i=1}^{k} N_i \ln N_i - \sum_{i=1}^{k} N_i \right)$$

$$= N \ln N - \sum_{i=1}^{k} N_i \ln N_i \qquad \left(\sum_{i=1}^{k} N_i = N \right)$$

$$\delta \ln W = - \sum_{i=1}^{k} \left(\ln N_i \, \delta N_i + N_i \frac{\delta N_i}{N_i} \right) = 0$$

Constraint $\sum_{i=1}^{k} N_i = N \Rightarrow \sum_{i=1}^{k} \delta N_i = 0$

$$\sum_{i=1}^{k} N_i \epsilon_i = E \Rightarrow \sum_{i=1}^{k} \epsilon_i \, \delta N_i = 0$$

Introduce the Lagrange multiplier λ, μ

$$\sum_{i=1}^{k} (\ln N_i + \lambda + \mu \epsilon_i) \, \delta N_i = 0$$

With the presence of Lagrange multiplier λ, μ δN_i can be treated as independent

$$\Rightarrow \quad \ln N_i + \lambda + \mu \epsilon_i = 0$$

$$N_i = e^{-\lambda - \mu \epsilon_i}$$

$$\downarrow$$

$$f_{MB}(\epsilon_i) = e^{-\lambda - \mu \epsilon_i} = A e^{-\mu \epsilon_i} \qquad (\beta = \mu)$$

$$\downarrow$$

Maxwell - Boltzmann distribution

BINOMIAL AND MULTINOMIAL DISTRIBUTION

During the course of our discussion of the canonical ensemble, we shall encounter the problem of determining how many ways it is possible to divide N distinguishable systems into groups such that there are n_1 systems in the first group, n_2 systems in the second group, and so on, and such that $n_1 + n_2 + \cdots = N$, that is, all the systems are accounted for. This is actually one of the easiest problems in combinatorial analysis. To solve this, we first calculate the number of permutations of N distinguishable objects, that is, the number of possible different arrangements or ways to order N distinguishable objects. Let us choose one of the N objects and place it in the first position, one of the $N - 1$ remaining objects and place it in the second position, and so on, until all N objects are ordered. Clearly there are N choices for the first position, $N - 1$ choices for the second position, and so on, until finally there is only one object left for the Nth position. The total number of ways of doing this is then the product of all the choices,

$$N(N-1)(N-2)\cdots(2)(1) \equiv N! \qquad \text{(distinguishable objects)}$$

Next we calculate the number of ways of dividing N distinguishable objects into two groups, one group containing N_1 objects, say, and the other containing the remaining $N - N_1$. There are $N(N-1)\cdots(N-N_1+1)$ ways to form the first group, and $N_2! = (N - N_1)!$ ways to form the second group. The total number is, then, the product

$$N(N-1)\cdots(N-N_1+1) \times (N-N_1)! = \frac{N!}{(N-N_1)!} \times (N-N_1)! = N!$$

But this has overcounted the situation drastically, since the order in which we place N_1 members in the first group and N_2 in the second group is immaterial to the problem as stated. All $N_1!$ orders of the first group and $N_2!$ orders of the second group correspond to just one division of N objects into N_1 objects and N_2 objects. Therefore the desired result is

$$\frac{N!}{N_1!(N-N_1)!} = \frac{N!}{N_1!N_2!} \tag{1–75}$$

Since the combination of factorials in Eq. (1–75) occurs in the binomial expansion,

$$(x+y)^N = \sum_{N_1=0}^{N} \frac{N!x^{N-N_1}y^{N_1}}{N_1!(N-N_1)!} = \sum_{N_1N_2}{}^{*} \frac{N!x^{N_1}y^{N_2}}{N_1!N_2!} \tag{1–76}$$

$N!/N_1!(N-N_1)!$ is called a binomial coefficient. The asterisk on the second summation in Eq. (1–76) signifies the restriction $N_1 + N_2 = N$.

The generalization of Eq. (1–75) to the division of N into r groups, the first containing N_1, and so on, is easily seen to be

$$\frac{N!}{N_1!N_2!\cdots N_r!} = \frac{N!}{\prod_{j=1}^{r} N_j!} \tag{1–77}$$

where $N_1 + N_2 + \cdots + N_r = N$. This is known as a multinomial coefficient, since it occurs in the expansion

$$(x_1 + x_2 + \cdots + x_r)^N = \sum_{N_1=0}^{N}\sum_{N_2=0}^{N}\cdots\sum_{N_r=0}^{N}{}^{*} \frac{N!{x_1}^{N_1}\cdots {x_r}^{N_r}}{\prod_{j=1}^{r} N_j!} \tag{1–78}$$

where this time the asterisk signifies the restriction $N_1 + N_2 + \cdots + N_r = N$.

There are a number of other combinatorial formulas that are useful in statistical thermodynamics, but Eq. (1–77) is the most useful for our purposes. Combinatorial formulas can become rather demanding to derive. We refer to Appendix AVII of Mayer and Mayer* which contains a collection of formulas.

STIRLING'S APPROXIMATION

In statistical thermodynamics we often encounter factorials of very large numbers, such as Avogadro's number. The calculation and mathematical manipulation of factorials become awkward for large N. Therefore it is desirable to find an approximation for $N!$ for large N. Problems of this sort occur often in mathematics and are called asymptotic approximations, that is, an approximation to a function which improves as the argument of that function increases. Since $N!$ is actually a product, it is convenient to deal with $\ln N!$ because this is a sum. The asymptotic approximation to $\ln N!$ is called Stirling's approximation, which we now derive.

Since $N! = N(N-1)(N-2)\cdots(2)(1)$, $\ln N!$ is

$$\ln N! = \sum_{m=1}^{N} \ln m \qquad (1\text{–}73)$$

Figure 1–5 shows $\ln x$ plotted versus x. The sum of the areas under these rectangles up to N is $\ln N!$. Figure 1–5 also shows the continuous curve $\ln x$ plotted on the same graph. Thus $\ln x$ is seen to form an envelope to the rectangles, and this envelope becomes a steadily smoother approximation to the rectangles as x increases. We can approximate the area under these rectangles by the integral of $\ln x$. The area under $\ln x$ will poorly approximate the rectangles only in the beginning. If N is large enough (we are deriving an asymptotic expansion), this area will make a negligible contribution to the total area. We may write, then,

$$\ln N! = \sum_{m=1}^{N} \ln m \approx \int_1^N \ln x\, dx = N \ln N - N \qquad (N \text{ large}) \qquad (1\text{–}74)$$

which is Stirling's approximation to $\ln N!$. The lower limit could just as well have been taken as 0 in Eq. (1–74), since N is large. (Remember that $x \ln x \to 0$ as $x \to 0$.)

A more refined derivation of Stirling's approximation gives $\ln N! \approx N \ln N - N + \ln(2\pi N)^{1/2}$, but this additional term is seldom necessary. (See Problem 1–59.)

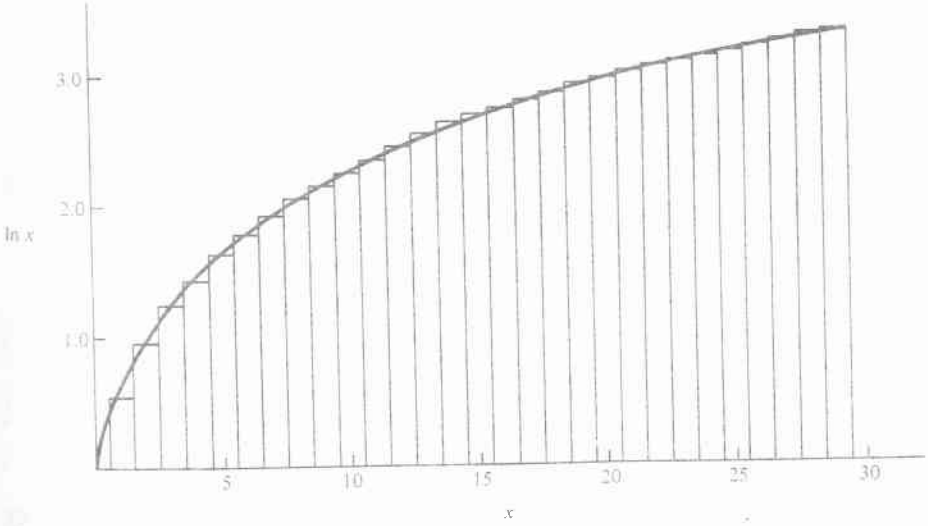


Figure 1–5. **A plot of $\ln x$ versus x, showing how the summation of $\ln m$ can be approximated by the integral of $\ln x$.**

METHOD OF LAGRANGE MULTIPLIERS

It will be necessary, later, to maximize Eq. (1–77) with the constraint $N_1 + N_2 + \cdots + N_r = \text{constant}$. This brings us to the mathematical problem of maximizing a function of several (or many) variables $f(x_1, x_2, \ldots, x_r)$ when the variables are connected by other equations, say $g_1(x_1, \ldots, x_r) = 0$, $g_2(x_1, \ldots, x_r) = 0$, and so on. This type of problem is readily handled by the method of Lagrange undetermined multipliers.

If it were not for the constraints, $g_j(x_1, x_2, \ldots, x_r) = 0$, the maximum of $f(x_1, \ldots, x_r)$ would be given by

$$\delta f = \sum_{j=1}^{r} \left(\frac{\partial f}{\partial x_j}\right)_0 \delta x_j = 0 \tag{1–79}$$

where the zero subscript indicates that this equation equals zero only when the r partial derivatives are evaluated at the maximum (or minimum) of f. Denote these values of x_j by x_j^0. If there were no constraints, each of the δx_j would be able to be varied independently and arbitrarily, and so we would conclude that $(\partial f/\partial x_j) = 0$ for every j, since δf must equal zero. This would give r equations from which the values of the $r\, x_j^0$ could be obtained.

On the other hand, if there is some other relation between the x's, such as $g(x_1, x_2, \ldots, x_r) = 0$, we have the additional equation

$$\delta g = \sum_{j=1}^{r} \left(\frac{\partial g}{\partial x_j}\right)_0 \delta x_j = 0 \tag{1–80}$$

This equation serves as a constraint that the δx_j must satisfy, thus making one of them depend upon the other $r - 1$. In the Lagrange method, one multiplies Eq. (1–80) by some parameter, say λ, and adds the result to Eq. (1–79) to get

$$\sum_{j=1}^{r} \left(\frac{\partial f}{\partial x_j} - \lambda \frac{\partial g}{\partial x_j}\right)_0 \delta x_j = 0 \tag{1–81}$$

The δx_j are still not independent, because of Eq. (1–80), and so they cannot be varied independently. Equation (1–80), however, can be treated as an equation giving one of the δx_j in terms of the other $r - 1$ independent ones. Pick any one of the $r\, \delta x_j$ as the dependent one. Let this be δx_μ.

The trick now is that we have not specified λ yet. We set it equal to $(\partial f/\partial x_\mu)_0/(\partial g/\partial x_\mu)_0$, making the coefficient of δx_μ in Eq. (1–81) vanish. The subscript zero here indicates that $(\partial f/\partial x_\mu)$ and $(\partial g/\partial x_\mu)$ are to be evaluated at values of the x_j such that f is at its maximum (or minimum) under the constraint of Eq. (1–80). Of course, we do not know these values of x_j yet, but we can nevertheless formally define λ in this manner. This leaves a sum of terms in Eq. (1–81) involving only the independent δx_j, which can be varied independently, yielding that

$$\left(\frac{\partial f}{\partial x_j}\right)_0 - \lambda \left(\frac{\partial g}{\partial x_j}\right)_0 = 0 \qquad j = 1, 2, \ldots, \mu - 1, \mu + 1, \ldots, r$$

---

* See Mayer and Mayer, *Statistical Mechanics* (New York: Wiley, 1940).

If we combine these $r - 1$ equations with our choice for λ, we have

$$\left(\frac{\partial f}{\partial x_j}\right)_0 - \lambda \left(\frac{\partial g}{\partial x_j}\right)_0 = 0 \tag{1–82}$$

for all j.

As we said above, the choice of λ here is certainly formal, since both $(\partial f/\partial x_\mu)_0$ and $(\partial g/\partial x_\mu)_0$ must be evaluated at these values of x_j which maximizes f, but these are known from Eq. (1–82) only in terms of λ. But this presents no difficulty, since in practice λ is determined by physical requirements. Examples of this will occur in the next two chapters.

Lagrange's method becomes no more difficult in the case in which there are several constraints. Let $g_1(x_1, \ldots, x_r)$, $g_2(x_1, \ldots, x_r)$, ... be a set of constraints. We introduce a Lagrange multiplier for each $g_i(x_1, \ldots, x_r)$ and proceed as above to get

$$\frac{\partial f}{\partial x_j} - \lambda_1 \frac{\partial g_1}{\partial x_j} - \lambda_2 \frac{\partial g_2}{\partial x_j} - \cdots = 0 \tag{1–83}$$

Specific Heat of Solid

Consider monatomic molecules

A mole of solid consists of N molecules
↳ Avogadro's number

Each molecule has three oscillating degree of freedom

$$U = 3\underbrace{Nk}_{R}T$$

R ← gas constant

U ↓ internal energy of a mole of solid

$\Rightarrow \quad C_V = 3R$ law of Dulong and Petit

At lower temperature, the molar heat capacities vary.
In fact $C_V \to T^3$ as $T \to 0$

1907 Einstein treat the solid as a set of harmonic oscillators, all have the same frequency ν_0, and calculate the average internal energy on the assumption that these oscillators can have only the discrete energies proposed by Planck, i.e., $nh\nu_0$

$$U = 3N \frac{h\nu_0}{e^{h\nu_0/kT}-1}$$

↓ average energy of oscillation per degree of freedom

Application of quantum concept to solid state physics

all oscillating with same ν

$$C_V = \frac{dU}{dT} = 3Nk\left(\frac{h\nu_0}{kT}\right)^2\left[e^{h\nu_0/kT}-1\right]^{-2}e^{h\nu_0/kT}.$$

reasonably agreement with experiments at reasonably low temperature

ν_0 is a parameter

Debye's theory (1912)

$$N(\nu)\,d\nu = \frac{4\pi V}{v^3}\nu^2\,d\nu$$

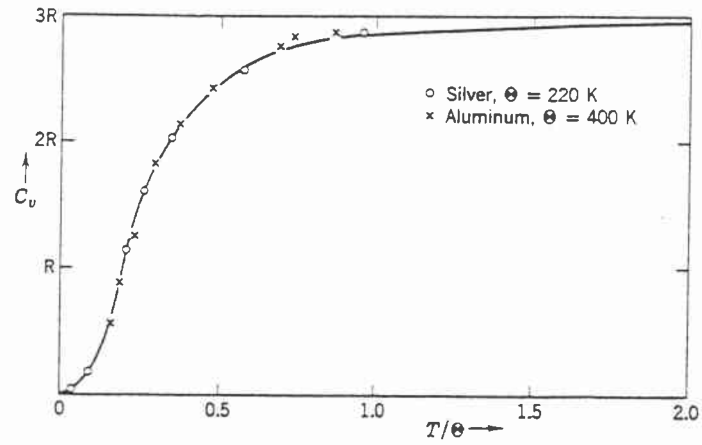
v ↙ velocity of sound

$$\int_0^{\nu_m} N(\nu)\,d\nu = 3N$$

$$\Rightarrow \quad \nu_m = v\left(\frac{9N}{4\pi V}\right)^{1/3}$$

$$\bar{E} = \frac{h\nu}{e^{h\nu/kT}-1}$$

$$\Rightarrow \quad U = \int_0^{\nu_m} \frac{4\pi V}{v^3}\nu^2\,d\nu\,\frac{h\nu}{e^{h\nu/kT}-1}.$$



Heat capacity at constant volume as a function of temperature. The solid curve is the Debye function (eq. 8-17). The curve was fitted to the data points for each metal in order to determine the Debye temperature Θ for the metal, and then the data were replotted as a function of T/Θ. (From F. Seitz, *Modern Theory of Solids*, McGraw-Hill, New York, 1940.)

Define $x = \frac{h\nu}{kT}$, $x_m = \frac{h\nu_m}{kT} \equiv \frac{\Theta}{T}$ ← Debye temperature

$$\Rightarrow \quad U = 3RT \frac{3}{x_m^3} \int_0^{x_m} \frac{x^3\,dx}{e^x - 1}$$

$$= 9R \frac{T^4}{\Theta^3} \int_0^{\Theta/T} \frac{x^3\,dx}{e^x-1} \qquad T \to 0 \quad \int_0^{\infty} \frac{x^3\,dx}{e^x-1} \to \text{constant}$$

$$\Rightarrow \quad C_V = \frac{dU}{dT} = 9R \left[4 \left(\frac{T}{\Theta}\right)^3 \int_0^{\Theta/T} \frac{x^3}{e^x-1}\,dx - \frac{\Theta}{T} \frac{1}{e^{\Theta/T}-1} \right]$$

This is Debye's theory of specific heat .

$$T >> \Theta \qquad U \cong 9R \frac{T^4}{\Theta^3} \int_0^{\Theta/T} x^2\,dx$$

$$= 9R \frac{T^4}{\Theta^3} \cdot \frac{\Theta^3}{3T^3} = 3RT$$

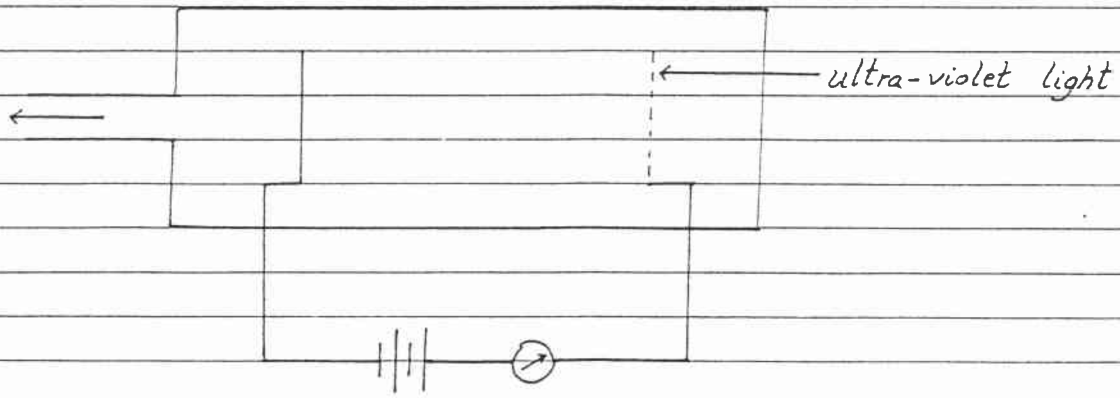
$$\Rightarrow \quad C_V = 3R \qquad \text{law of Dulong and Petit}$$

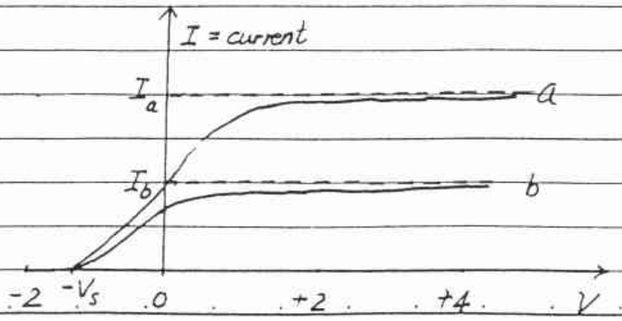
Furthermore $C_V \propto T^3$ as $T \to 0$

Photoelectric Effect

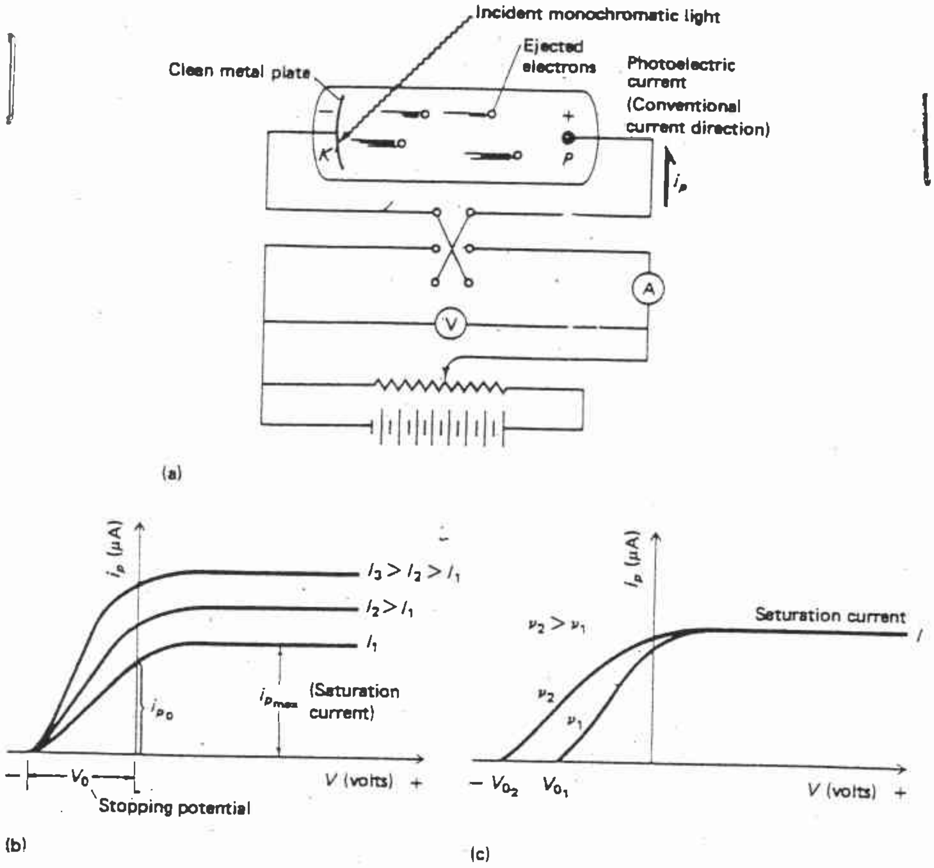
Definition : the release of electrons from matter under the influence of electromagnetic radiation

Discovered by Hertz in 1887

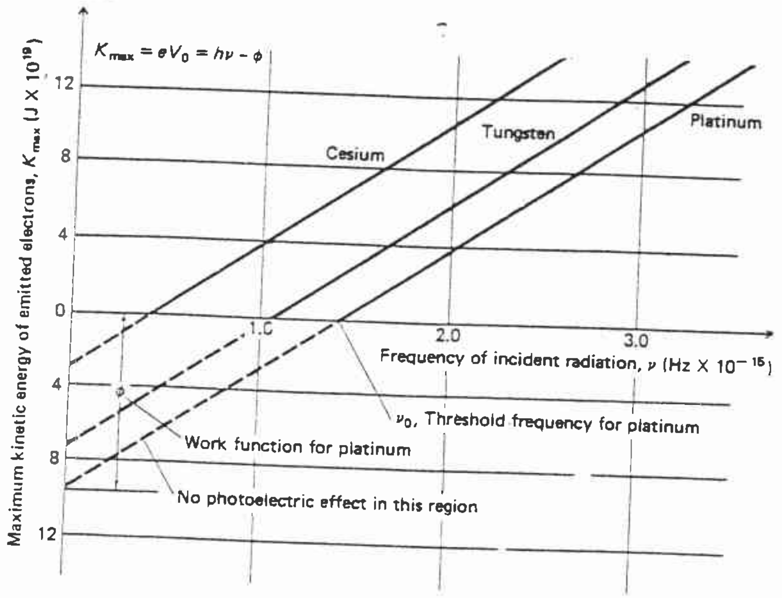




V = potential of the second electrode with respect to photocathode



(a) Schematic for photoelectric experiment. (b) Photoelectric current versus the accelerating potential V for incident monochromatic light of wavelength λ. (c) Photoelectric current versus accelerating potential to show frequency dependence.



The maximum kinetic energy of photoelectrons K_{max} $(= eV_0)$ versus the frequency of incident radiation.

Experimental results.

(i) The effect persists even the tube is evulated to a very low pressure ⇒ gaseous ions are not the carrier of the current

(ii) In curve b the intensity of incident light has been reduced to $\frac{1}{2}$ that of curve a
⇒ the magnitude of the current, when it exists, is proportional to the intensity of the light source

(iii) Some current still reaches the second electrode when $V<0$
⇒ photoelectrons are ejected from the photocathode with non-negligible kinetic energy
Well defined end point ⇒ well defined kinetic energy

$$(K.E.)_{max} = eV_s$$

Photoelectrons of maximum kinetic energy ↔ electrons emitted from the surfaces

Photoelectron with lower energy ↔ originated inside the surface and lost kinetic energy in the process of reaching the surface

$(K.E.)_{max}$ ↔ energy given to an electron in the photoelectric process

(iv) V_s in independent of light intensity ⇒ E_{max} is independent of the intensity of the incident light.

(v) Whether the electrons are emitted depend on the frequency of the light. In general, there will be a threshold that varies from metal to metal: only light with a frequency greater than a given threshold frequency will produce photoelectric effect.

Classical theory

Energy carried by an EM wave ∝ intensity of the light

↓

kinetic energy of the emitted electrons

↓

inconsistent with the experimental result.

Time delay problem

In classical theory of light, energy is uniformly distributed over the wave front

Assume the target is placed at 3 m from a weak light source whose power is 1 watt

Atomic radius $\sim 10^{-10}\,m$

Power fall on the target $\sim$ 1 watt $\frac{\pi(10^{-10}m)^2}{4\pi(3m)^2} \sim 28\times10^{-23}\,J/sec$

Time required to absorbed 1 eV of energy

$$\tau = \frac{1.6\cdot10^{-19}J}{28\cdot10^{-23}J/sec} \sim 572\ sec$$

↓

there should be a time lag of this order between the impinging of light on the surface and the ejection of photoelectrons

↓

No such delay (time) was observed

1905 Einstein's quantum theory of the photoelectric effect (Nobel prize, 1921)

Einstein. radiation consists of a collection of quanta (photons) with $E = h\nu$ (Einstein relation) which are absorbed individually in photoelectric process

$$K.E = h\nu - W$$

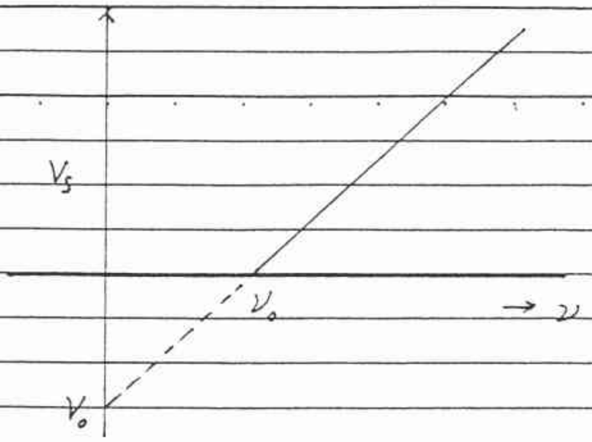
↳ work required to remove the electrons from the metal

$$K.E_{max} = h\nu - W_0$$

↳ minimum work required to remove the electron from the metal

$$K.E_{max} = eV_s$$

$$h\nu = eV_s + \underset{\overset{|||}{eV_0}}{W_0} \Rightarrow V_s = \frac{h}{e}\nu - V_0$$



V_s vs ν plot should be given as above

(i) Intercept → ν_0
↳ threshold frequency

(ii) Extrapolation of the line to $\nu = 0$ axis ⇒ V_0 → work function

(iii) Slope of the line ⇒ $\frac{h}{e}$ ⇒ h
↓
agree with the value
determined from
black-body radiation

These predictions were verified by Millikan in 1916 (Nobel prize, 1923)

A more intense source emits more photons and these photons in turn can liberate more photoelectrons ⇒ (ii)

Compton Effect (1923)

Monochromatic X-ray are scattered by a suitable scatter, the scattered radiation consists of two components
original wavelength λ_0
longer wavelength λ_1

$\lambda_1 - \lambda_0$ is a function of scattering angle only
independent of wavelength of the incident radiation,
and the scattering material.

Classical picture : electron will oscillate under the influence of the electric field of the incident wave
⇒ will radiate a wave of the same wavelength (frequency)

γ + "e^-" → γ + e^-

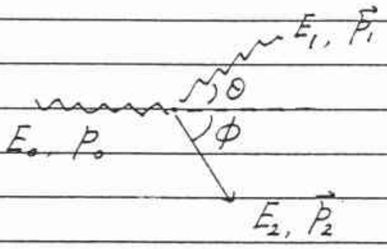
If the energy transfer to the electron is >> originâl binding energy ⇒ electrons can be treated approximately as "free electron"

Photon

$E = h\nu$

$p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda}$

↳ since photon has zero rest mass



Momentum conservation

$p_0 = |\vec{P}_1| \cos\theta + |\vec{P}_2| \cos\phi$

$|\vec{P}_1| \sin\theta = |\vec{P}_2| \sin\phi$

$$|\vec{P}_2|^2 = (p_0 - |\vec{P}_1|\cos\theta)^2 + |\vec{P}_1|^2 \sin^2\theta$$

$$= (P_0 - |\vec{P}_1|)^2 + 2P_0|\vec{P}_1|(1-\cos\theta) \qquad (1)$$

Energy conservation

$$E_0 + mc^2 = E_1 + E_2$$

$$E_2 = E_0 - E_1 + mc^2$$
$$= c(p_0 - |\vec{P}_1|) + mc^2$$

$$|\vec{P}_2|^2 = \frac{1}{c^2}\left[E_2^2 - m^2c^4\right]$$

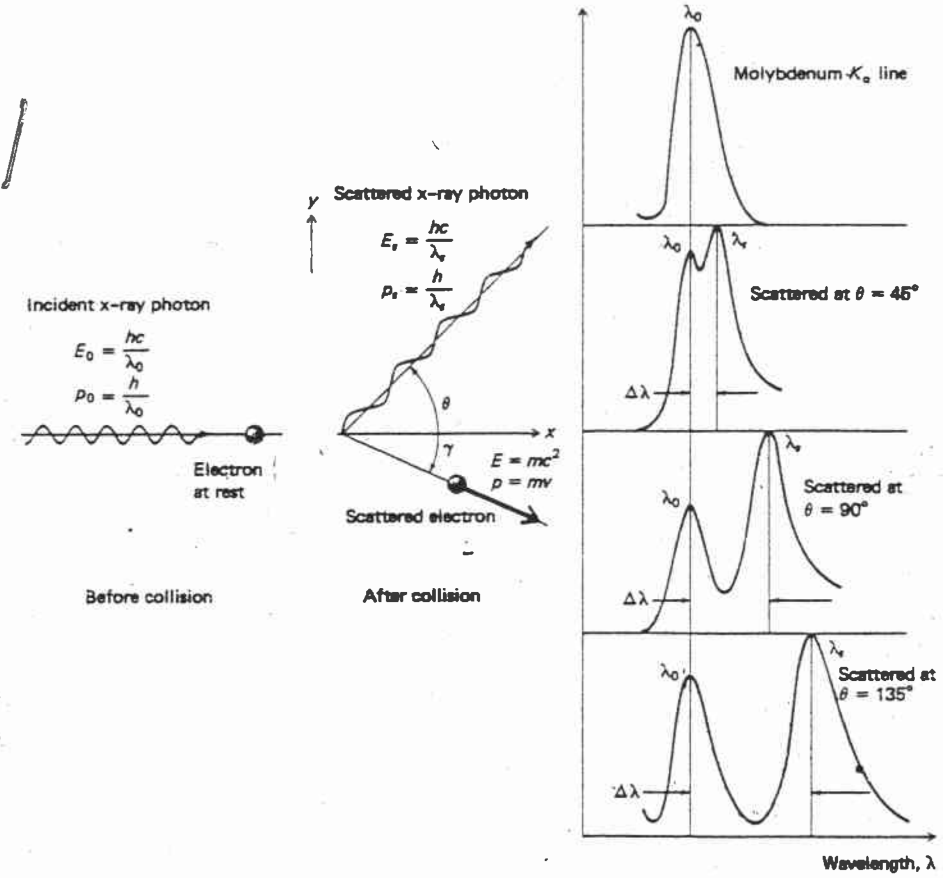
$$= \frac{1}{c^2}\left[c^2(p_0 - |\vec{P}_1|)^2 + 2mc^3(p_0 - |\vec{P}_1|) + m^2c^4 - m^2c^4\right]$$

$$= (p_0 - |\vec{P}_1|)^2 + 2mc(p_0 - |\vec{P}_1|) \qquad (2)$$

Combine (1), (2)

$$\Rightarrow \quad mc(p_0 - |\vec{P}_1|) = p_0|\vec{P}_1|(1-\cos\theta)$$

3-15a



Compton scattering of a photon from an electron at rest. The graphs at the right show the shift in the K_α radiation from molybdenum scattered from carbon.

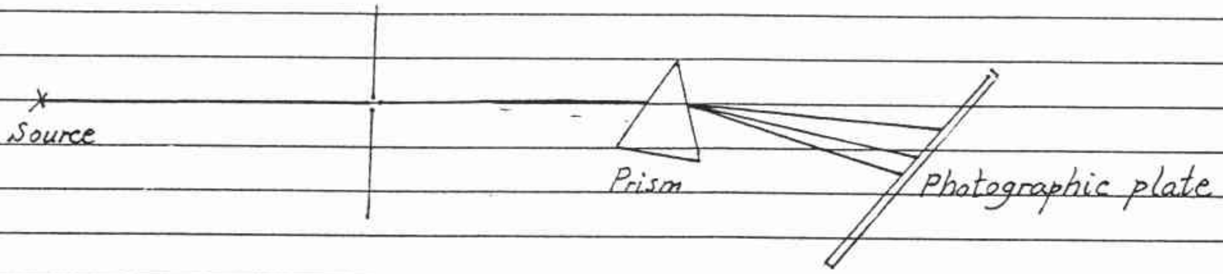
$$\frac{1}{|\vec{P}_1|} - \frac{1}{P_0} = \frac{1}{mc}(1-\cos\theta) \qquad \cdot h$$

$$\Rightarrow \quad \lambda_1 - \lambda_0 = \underset{\substack{\downarrow \\ 0.0243\ \text{Å}}}{\frac{h}{mc}}(1-\cos\theta) \quad \leftarrow \text{Compton formula}$$

↑ Compton wavelength of the electron

3. Bohr Model of the Atom

Atomic spectra



Source consists of electric discharge passing through a region containing a monatomic gas.

Collisions with electrons and with each other

⇒ some of the atoms become excited

From excited state ⟶ normal state

⇒ electromagnetic radiation

Radiation.

(i) collimated by slit

(ii) passes through a prism → break up into spectra

(iii) recorded on photographic plate

Results

(i) the electromagnetic radiations by free atoms are concentrated at discrete wavelengths

(ii) each discrete wavelength ↔ line

⇒ emission line spectra

Every element ↔ unique line spectra

Spectroscopy is a useful tool for analyzing the composition of unknown substances

Furthermore, wavelengths fall into definite set → spectral series

⇒ the wavelength in each series are specified by empirical formula

Hydrogen spectrum

1885 Balmer series

$$\lambda = 3646 \text{ Å} \quad \frac{n^2}{n^2-4}$$

1890 Rydberg

$$\frac{1}{\lambda} = \underset{\hookrightarrow}{R_H} \left(\frac{1}{2^2} - \frac{1}{n^2} \right) \qquad n = 3, 4, 5, \cdots \qquad \text{Balmer series}$$

Rydberg constant

$$\frac{1}{\lambda} = R_H \left(\frac{1}{1^2} - \frac{1}{n^2} \right) \qquad n = 2, 3, 4 \cdots \qquad \text{Lyman series}$$

$$\frac{1}{\lambda} = R_H \left(\frac{1}{m^2} - \frac{1}{n^2} \right) \qquad n > m$$

Bohr model (1913)

(i) Atoms exist in "stationary states" of definite energy, in which states they do not radiate and are stable.

(ii) Atoms emit or absorb radiation only when atom goes from one stationary state to another

$$\Delta E = h\nu$$

(iii) Correspondence principle : quantum theory should give the same results as classical theory in the limit of large system
[Similar to relativity they at $\frac{v}{c} \to 0 \Rightarrow$ Newtonian theory]

Hydrogen atom case

Correspondence principle ⇒ in the limit of large system, where the allowed energies approach a continuum the quantum radiation condition must yield the same result as classical calculation

If one pictures an electron orbiting about a nucleus
⇒ for very large orbital radii, such that the atom has macroscopic size, the frequency of the radiation emitted by the hydrogen should be the same as frequency of revolution of the electron

$$\frac{1}{\lambda_{nm}} = R_H \left(\frac{1}{m^2} - \frac{1}{n^2} \right)$$