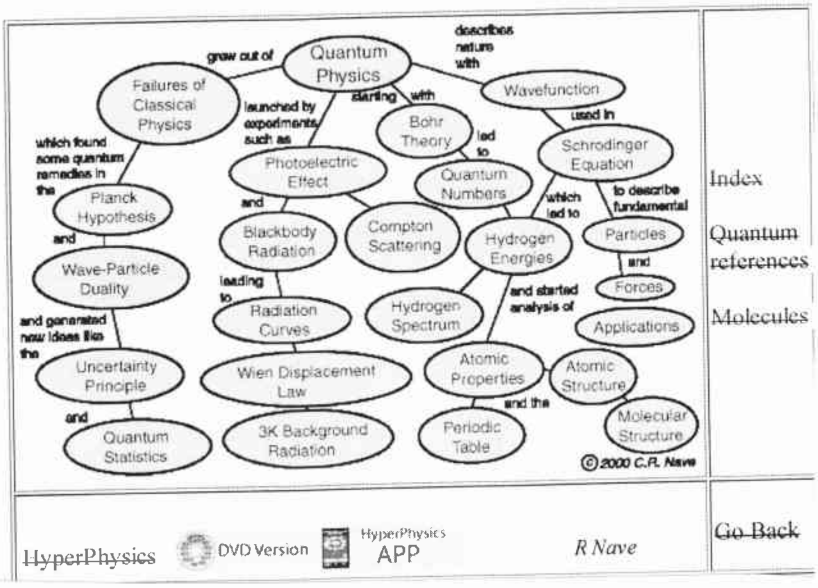
Quantum Mechanics

Summary



Quantum Statistics

I. Many Particle

Chapter 10

Many Particles

The multiparticle Schrodinger equation

$$H\,\psi(x_1, x_2, \cdots x_N, t) = i\hbar \frac{\partial}{\partial t} \psi(x_1, x_2, \cdots, x_N; t)$$

where

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \cdots + V(x_1, x_2, \cdots x_N)$$

$$\Rightarrow \left[\left(-\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} - \cdots\right) + V(x_1, x_2, \cdots x_N)\right]\psi(x_1, x_2, \cdots x_N; t)$$

$$= i\hbar \frac{\partial}{\partial t}\psi(x_1, x_2, \cdots x_N; t)$$

$\psi(x_1, x_2, \cdots x_N; t) \rightarrow$ multiparticle wave function.

$|\psi(x_1, x_2, \cdots x_N; t)|^2 dx_1\, dx_2 \cdots dx_N$

$\rightarrow$ probability of finding particle 1 in the range $(x_1, x_1 + dx_1)$
and particle 2 in the range $(x_2, x_2 + dx_2)$
and particle 3
⋮

For $V(x_1, x_2, \cdots x_N)$ independent of time, seperation of variable method may be used.

Ansatz $\quad \psi(x_1, x_2, \cdots x_N, t) = u(x_1, x_2, \cdots x_N)\, e^{-iEt/\hbar}$

Substitute into the time-dependent Schrodinger equation

$$\Rightarrow \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} - \cdots + V(x_1 \cdots x_N)\right) u(x_1, x_2, \cdots x_N)$$

$$= E\, u(x_1, x_2, \cdots x_N)$$

↓

time independent Schrodinger equation
energy eigenvalue equation

Remarks:

- This clearly is a generalization of one particle Schrodinger equation.
- The equation is written in one dimension form for simplification

$$x_i \longrightarrow \vec{r}_i$$

$$\frac{\partial^2}{\partial x_i^2} \longrightarrow \nabla_i^2$$

· Spin can be included also.

The problem can be further simplified for independent particles, i.e.,

$$V(x_1, x_2, \cdots x_N) = V_1(x_1) + V_2(x_2) + \cdots + V_N(x_N)$$

The time independent Schrodinger equation becomes

$$\left[\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} + V_1(x_1)\right) + \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_2(x_2)\right) + \cdots \right] u(x_1, x_2, \cdots x_N)$$

$$= E\, u(x_1, x_2, \cdots, x_N)$$

Ansatz: $\quad u(x_1, x_2, \cdots x_N) = u_1(x_1)\, u_2(x_2) \cdots u_N(x_N)$

· This ansatz comes from the fact that the probability of finding a particular particle somewhere is independent of the probabilities of finding particles anywhere else.

· This is essentially is to carry seperat of variable to the ultimate.

The problem is reduced to find $u_1(x_1), u_2(x_2), \cdots u(x_N)$

The results are

$$\left(-\frac{\hbar^2}{2m_1}\frac{d^2}{dx_1^2} + V_1(x_1)\right) u_1(x_1) = E_1 u_1(x_1)$$

$$\left(-\frac{\hbar^2}{2m_2}\frac{d^2}{dx_2^2} + V_2(x_2)\right) u_2(x_2) = E_2 u_2(x_2)$$

$$\vdots$$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx_N^2} + V_N(x_N)\right) u_N(x_N) = E_N u_N(x_N)$$

and

$$E = E_1 + E_2 + \cdots + E_N$$

· The problem is reduced to solve N one-body problem (which we have learned how to solve.

· Not all H can be written as $H = \sum_i H_i$, but we shall try to approximate H by $H \simeq \sum_i \overline{H}_i$

Identical Particles

If two particles are not identical and $\psi(1,2)$ represents a state of a high energy electron and a low energy proton then $\psi(2,1)$, in which the roles of the two particles are reversed, would represent a high energy proton and low energy electron, an entirely different situation.

But if $\psi(1,2)$ are the wave function of two identical particles, for instance, two electrons

The indistinguishability of particles requires the states represented by $\psi(1,2)$ and $\psi(2,1)$ (with the roles of 1 and 2 reversed) must be physically indistinguishable

$$\Downarrow$$

$$H(1,2) = H(2,1)$$

$$|\psi(2,1)|^2 = |\psi(1,2)|^2$$

$$\Rightarrow \quad \psi(2,1) = e^{i\delta}\,\psi(1,2)$$

Among these possibilities

$$\psi(2,1) = \psi(1,2) \quad \rightarrow \quad \text{symmetric under } 1 \leftrightarrow 2$$

$$\psi(2,1) = -\psi(1,2) \quad \rightarrow \quad \text{anti-symmetric under } 1 \leftrightarrow 2$$

Postulate :

For identical bosons (particles with integer spin) the multiparticle wave function must be symmetric under exchange (of any pair identical bosons)

$$\psi(1,2,\cdots i \cdots j \cdots) = +\psi(1,2,\cdots j,\cdots i,\cdots)$$

$$i \leftrightarrow j$$

i, j are identical bosons

For identical fermions (particles with half integral spins) the multiparticle wave function must be anti-symmetric under exchange of any pair of identical fermions.

$$\psi(1,2,\cdots i,\cdots j\cdots) = -\psi(1,2,\cdots j,\cdots i,\cdots)$$

$i \leftrightarrow j$ $\quad i, j$ are identical fermions

Identical Particles and Indistinguishability.

We say that two particles are identical if they have all the same intrinsic properties - same mass, same charge and same spin.

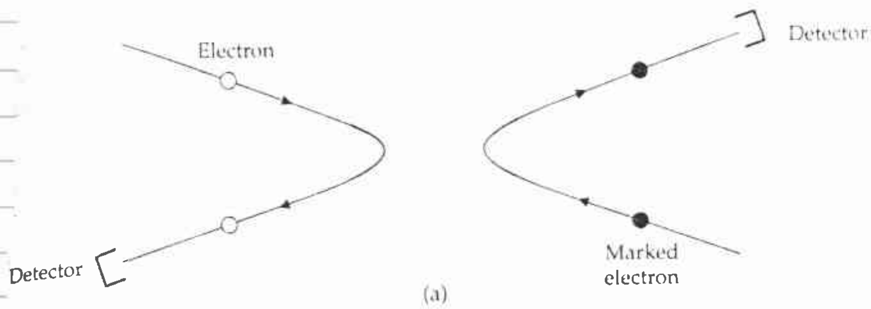
In classical mechanics, even identical particles are distinguishable in the sense that we could in principle keep track of which is which.

↓

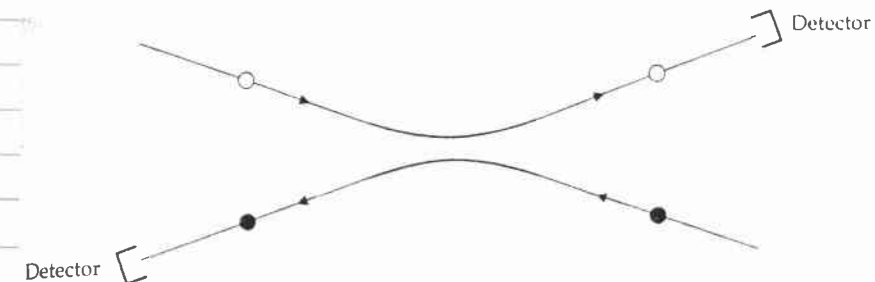
follow the trajectory

In quantum mechanics, two identical particles are indistinguishable.

When the wavefunction overlaps, there is absolutely no way of distinguish two identical particles.



(a)



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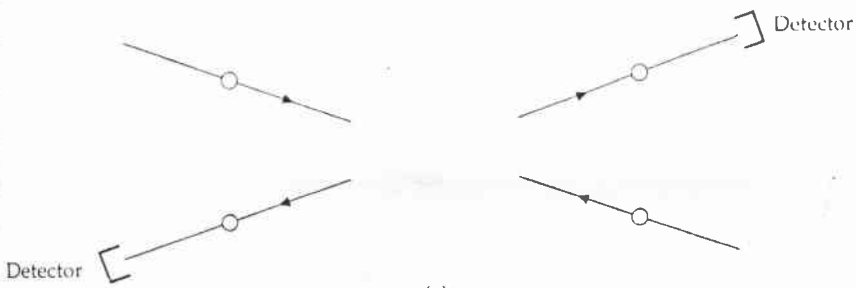

(c)

Consider two independent spin-zero bosons (with no interaction between them) in the same one dimensional potential infinite potential well.

$$H(1,2) = \frac{\vec{P}_1^2}{2\mu} + V(x_1) + \frac{\vec{P}_2^2}{2\mu} + V(x_2)$$

$$V(x) = \begin{cases} 0 & 0<x<L \\ \infty & \text{otherwise} \end{cases}$$

$$\left\{\left[-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x_1^2} + V(x_1)\right] + \left[-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x_2^2} + V(x_2)\right]\right\} u(x_1,x_2) = E\,u(x_1,x_2)$$

Ansatz $\quad u(x_1,x_2) = u(x_1)\,u(x_2)$

$$\Rightarrow \quad u(x_2)\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dx_1^2} + V(x_1)\right]u(x_1) + u(x_1)\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dx_2^2} + V(x_2)\right]u(x_2)$$

$$= E\,u(x_1)\,u(x_2)$$

$$\Rightarrow \quad \underbrace{\frac{1}{u(x_1)}\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dx_1^2} + V(x_1)\right]u(x_1)}_{\text{function of } x_1} = E - \underbrace{\frac{\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dx_2^2} + V(x_2)\right]u(x_2)}{u(x_2)}}_{\text{function of } x_2}$$

$$= E_1$$

$$\Rightarrow \left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dx_1^2} + V(x_1)\right]u(x_1) = E_1\,u(x_1)$$

the problem we have solved before.

$$u_n(x_1) = \sqrt{\frac{2}{L}}\,\sin\frac{n\pi x}{L} \qquad \text{with} \quad E_{1,n} = \frac{n^2\hbar^2\pi^2}{2\,L^2}$$

and

$$\left[-\frac{\hbar^2}{2}\frac{d^2}{dx^2} + V(x_2)\right]u(x_2) = \underset{\substack{\| \\ E-E_1}}{E_2}\,u(x_2)$$

$$u_m(x_2) = \sqrt{\frac{2}{L}}\,\sin\frac{m\pi x}{L} \qquad \text{with} \quad E_{2,m} = \frac{m^2\hbar^2\pi^2}{2\,L^2}$$

$u_m(x_1)\,u_n(x_2) = u_{m,n}(x_1,x_2)$ is the eigenfunction of the time-independent Schrodinger equation with eigenvalue $\quad E = E_1 + E_2$

Spin can take integral and half integral (in unit of $\hbar$) values

Particles with integral spin values are called bosons, while those with half-integral spin are fermions.

e, p, n are fermions
π, K, γ are bosons.
α

Electrons are fermions since their spin is $\frac{1}{2}\hbar$, it is possible for electrons to form a "complete pair" in some materials.

The composite can have integral spin and the pair then behaves as a boson.

This change over from fermion to boson-like behavior is manifested in electrical properties of materials.

For example, when electrons behave as fermions, materials have finite resistance.
Under certain conditions, when electrons couple together to form pairs (called Cooper pairs), the material becomes a superconductor and loses its resistance completely.

$$u_{m,n}(x_1, x_2) = \frac{2}{L} \sin\frac{n\pi x_1}{L} \sin\frac{m\pi x_2}{L}$$

with eigenvalue $E = \dfrac{(m^2+n^2)\hbar^2\pi^2}{2\mu L^2}$

This wave function is not symmetric under $\underset{1}{\underset{\downarrow}{x_1}} \leftrightarrow \underset{2}{\underset{\downarrow}{x_2}}$

Here x_1, x_2 are the only label in 1 and 2

For identical bosons.

To satisfy the requirement of symmetric under $1 \leftrightarrow 2$

$$u_{mn}(x_1, x_2) \Rightarrow \frac{1}{\sqrt{N_A}}\left[u_m(x_1)u_n(x_2) + u_n(x_1)u_m(x_2)\right]$$

with $E = \dfrac{(m^2+n^2)\hbar^2\pi^2}{2\mu L^2}$

Consider two zero spin identical bosons interacting with a potential $V(|\vec{r}_1 - \vec{r}_2|)$

The time-independent Schrodinger equation has the form

$$H_{12} = \frac{\vec{p}_1^{\,2}}{2\mu} + \frac{\vec{p}_2^{\,2}}{2\mu} + V(|\vec{r}_1 - \vec{r}_2|)$$

$$\Rightarrow \left[-\frac{\hbar^2}{2\mu}\nabla_1^2 - \frac{\hbar^2}{2\mu}\nabla_2^2 + V(|\vec{r}_1 - \vec{r}_2|)\right]u(\vec{r}_1, \vec{r}_2) = E\,u(\vec{r}_1, \vec{r}_2)$$

Introduce new variables

$\vec{R}$ and $\vec{r}$ (as we have done in Chapter 1)

$$\vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2), \quad \vec{r} = \vec{r}_1 - \vec{r}_2$$

(Remembering $m_1 = m_2$)

It can be shown that

$$H = H_1(\vec{R}) + H_2(\vec{r}) \quad \text{(See Appendix A)}$$

Note there is no cross term.

$H_1(\vec{R})$ is a free particle Hamiltonian

Hamiltonian is seperable.

$\Rightarrow \quad \psi(\vec{r}_1, \vec{r}_2) = \psi(\vec{R})\,\Phi(\vec{r})$

$\Phi(\vec{r})$ satisfies a Schrodinger equation with a central potential $V(r)$

The eigenfunction can be given by

$\Phi(\vec{r}) = R_{n\ell}(r)\, Y_{\ell m}(\theta, \phi)$ with eigenvalue $E_r \Rightarrow E^r_{n,\ell}$

Note: only for Coulomb potential $E^r_{n,\ell} \to E^r_n$

i.e., independent of ℓ

Now Bose statistics requires
$\psi(\vec{r}_1, \vec{r}_2)$ to be symmetric under $1 \leftrightarrow 2$

Under $1 \leftrightarrow 2$, $\vec{R} \to \vec{R}$, $\vec{r} \to -\vec{r}$

$\vec{r} \to -\vec{r}$ is spherical coordinate is

$r \to r$ $\qquad x = r\sin\theta\cos\phi$
$\theta \to \pi - \theta$ $\qquad y = r\sin\theta\sin\phi$
$\phi \to \pi + \phi$ $\qquad z = r\cos\theta$

$\Rightarrow \quad Y_{\ell m}(\theta, \phi) \to Y_{\ell m}(\pi - \theta, \pi + \phi) = (-1)^\ell\, Y_{\ell m}(\theta, \phi)$

$\psi(\vec{r}_1, \vec{r}_2) = (-1)^\ell\, \psi(\vec{r}_2, \vec{r}_1)$

In order the wave function to be symmetric
$\Rightarrow (-1)^\ell = 1 \Rightarrow \ell = 0, 2, 4, \ldots$

$\rho^0 \to \pi^0 \pi^0$

Spin of $\rho^0 = 1$

In the rest system of ρ^0, angular momentum conservation requires $\ell = 1$
(There is no spin to deal with)

However, bose statistics $\Rightarrow$ ℓ can be even only
$\Rightarrow \rho^0 \not\leftrightarrow \pi^0 \pi^0$
($\rho^0 \to \pi^+ \pi^-$ conspicuously)

The Total Spin of Two Electrons

$\chi_+(1) \rightarrow$ particle 1 has spin up. $S_{1z} = \frac{1}{2}\hbar$
$\chi_-(1) \rightarrow$ particle 1 has spin down. $S_{1z} = -\frac{1}{2}\hbar$
$\chi_+(2) \rightarrow$ particle 2 has spin up $S_{2z} = \frac{1}{2}\hbar$
$\chi_-(2) \rightarrow$ particle 2 has spin down. $S_{2z} = -\frac{1}{2}\hbar$

$\vec{S} = \vec{S}_1 + \vec{S}_2$

Two spin $\frac{1}{2}$ particles, such as two electrons

$\Rightarrow$ Total spin can be either 1 or 0

$S = 1$ $\quad S_z \quad \hbar, 0, -\hbar$
$\quad\quad\quad m_s \quad 1, 0, -1$

$S = 1, m_s = 1 \qquad \chi_+(1)\chi_+(2)$
$\quad\quad\;\; m_s = -1 \qquad \chi_-(1)\chi_-(2)$

Note: they are both symmetric under $1 \leftrightarrow 2$

$m_s = 0$ can be formed by linear combination of
$\chi_+(1)\chi_-(2)$; $\chi_+(2)\chi_-(1)$

$S = 1$, $m_s = 0$ should have the same property under interchange $1 \leftrightarrow 2$ $\Rightarrow \frac{1}{\sqrt{2}}[\chi_+(1)\chi_-(2) + \chi_+(2)\chi_-(1)]$

$$S = 1 \qquad \begin{array}{l} \chi_+(1)\chi_+(2) \\ \frac{1}{\sqrt{2}}(\chi_+(1)\chi_-(2) + \chi_+(2)\chi_-(1)) \\ \chi_-(1)\chi_-(2) \end{array}$$

$$S = 0 \qquad \frac{1}{\sqrt{2}}(\chi_+(1)\chi_-(2) - \chi_+(2)\chi_-(1))$$

Under $1 \leftrightarrow 2$
Spin wave function is symmetric for $S = 1$
anti-symmetric for $S = 0$

$S = 1$ triplet

$S = 0$ singlet

Two Electron System

The total wave function

$$\psi(1,2) = \psi(\vec{r}_1, \sigma_1 ; \vec{r}_2, \sigma_2)$$

$=$ spatial wave function $\cdot$ spin wave function

Sy for $\ell = 0, 2, 4, \cdots$ Sy for $S = 1$

A for $\ell = 1, 3, 5$ A for $S = 0$

$\psi(1,2)$ must be anti-symmetric under $1 \leftrightarrow 2$

For singlet $\ell = 0, 2, 4, \cdots$

triplet $\ell = 1, 3, 5, \cdots$

$$(-1)^{\ell} \quad (-1)^{S+1} = (-1)$$

$\Rightarrow$ $\ell + S$ must be even.

sum up the above result.

Now we shall go back to the general N-body problem

N identical particle

$H(1, 2, \cdots N)$
↓
Hamiltonian

$H(1, 2, \cdots i, \cdots j, \cdots N) = H(1, 2, \cdots j, \cdots i \cdots N)$
↓
N particles are identity

For the case of independent particles

$$H(1, 2, \cdots N) = \sum_{i=1}^{N} H(i)$$
↓
the independent Hamiltonian for the ith particle

$$H(i) = \frac{p_i^2}{2m} + V(\vec{r}_i)$$

Solve $H(i)\,\phi(i) = E\,\phi_i$
↓
essentially a one body problem

$\phi_\alpha(i) \rightarrow$ eigenfunction

α all the quantum numbers need to specified state
with corresponding Energy E_i

$\Rightarrow \psi_{\alpha_1 \alpha_2 \cdots \alpha_N}(1, 2 \cdots i \cdots j \cdots N) = \phi_{\alpha_1}(1)\,\phi_{\alpha_2}(2) \cdots \phi_{\alpha_i}(i)\ \phi_{\alpha_j}(j) \cdots \phi_{\alpha_N}(N)$

is an eigenstate of $H\psi = E\psi$
H: total Hamiltonian; E: total energy

$$E = E_{\alpha_1} + E_{\alpha_2} \cdots + E_{\alpha_N}$$

If we interchange $i \leftrightarrow j$

$$\psi_{\alpha_1 \alpha_2 \cdots \alpha_N}(1, 2, \cdots i, \cdots j, \cdots N) = \phi_{\alpha_1}(1) \cdots \phi_{\alpha_i}(j) \cdots \phi_{\alpha_j}(i) \cdots \phi_{\alpha_N}(N)$$

is also an eigenstate of $H\psi = E\psi$ with the same eigenvalue $E = E_{\alpha_1} + E_{\alpha_2} + \cdots + E_{\alpha_N}$

Bose-Einstein statistics requires the wavefunction to be symmetric under $i \leftrightarrow j$

$$\phi_{\alpha_1}(1)\phi_{\alpha_2}(2)\cdots\phi_{\alpha_i}(i)\cdots\phi_{\alpha_j}(j)\cdots\phi_{\alpha_N}(N)$$

+ all possible permutations

$\downarrow$

it is totally symmetric

Example (i) $\phi_n(1)\phi_m(2)$
$+ \phi_n(2)\phi_m(1)$ is clearly totally symmetric

$E = E_n + E_m$

(ii) $\phi_n(1)\phi_m(2)\phi_\ell(3)$

$$+\phi_n(1)\phi_m(3)\phi_\ell(2) + \phi_n(2)\phi_m(1)\phi_\ell(3) + \phi_n(2)\phi_m(3)\phi_\ell(1)$$
$$+\phi_n(3)\phi_m(1)\phi_\ell(3) + \phi_m(3)\phi_n(2)\phi_\ell(1)$$

clearly is a symmetric wave function.

Now Fermi-Dirac statistics requires the wavefunction to be anti-symmetric under $i \leftrightarrow j$

Fermi-Dirac statistics operates on identical fermions

We shall now add the spin m_s to our description

$\alpha_i \rightarrow \{\alpha_i, m_s\}$ extension

$m_s = \pm\frac{1}{2}$ corresponds to the spin up and spin down

$\phi_{\alpha_1}(1)\phi_{\alpha_2}(2)\cdots\phi_{\alpha_N}(N)$ is an eigenfunction of H

with eigenvalue $E = E_{\alpha_1} + E_{\alpha_2} + \cdots E_{\alpha_N}$

Fermi-Dirac statistics requires the wave function to be anti-symmetric under $i \leftrightarrow j$

This can be done using Slater determinant.

$$\psi^{A}_{\alpha_1 \cdots \alpha_N}(1,2,\cdots N) = \begin{vmatrix} \phi_{\alpha_1}(1) & \phi_{\alpha_1}(2) & \cdots\cdots & \phi_{\alpha_1}(N) \\ \phi_{\alpha_2}(1) & \phi_{\alpha_2}(2) & \cdots\cdots & \phi_{\alpha_2}(N) \\ \vdots & & & \vdots \end{vmatrix}$$

Obviously, it is anti-symmetric under $i \leftrightarrow j$

Example α_1 $n\uparrow$

α_2 $n\downarrow$

$$\phi^A = \begin{vmatrix} \phi_{n\uparrow}(1) & \phi_{n\uparrow}(2) \\ \phi_{n\downarrow}(1) & \phi_{n\downarrow}(2) \end{vmatrix}$$

$$= \phi_{n\uparrow}(1)\phi_{n\downarrow}(2) - \phi_{n\uparrow}(2)\phi_{n\downarrow}(1)$$

This is exactly the result we obtained before

Let us look at the three particle case discussed in the book

α_1 $u_{1\uparrow}$

α_2 $u_{1\downarrow}$

α_3 $u_{2\uparrow}$

$$\begin{vmatrix} u_{1\uparrow}(1) & u_{1\uparrow}(2) & u_{1\uparrow}(3) \\ u_{1\downarrow}(1) & u_{1\downarrow}(2) & u_{2\downarrow}(3) \\ u_{2\uparrow}(1) & u_{2\uparrow}(2) & u_{2\uparrow}(3) \end{vmatrix}$$

$$= u_{1\uparrow}(1) \begin{vmatrix} u_{1\downarrow}(2) & u_{1\downarrow}(3) \\ u_{2\uparrow}(2) & u_{2\uparrow}(3) \end{vmatrix}$$

$$- u_{1\uparrow}(2) \begin{vmatrix} u_{1\downarrow}(1) & u_{2\downarrow}(3) \\ u_{2\uparrow}(1) & u_{2\uparrow}(3) \end{vmatrix}$$

$$+ u_{1\uparrow}(3) \begin{vmatrix} u_{1\downarrow}(1) & u_{1\downarrow}(2) \\ u_{2\uparrow}(1) & u_{2\uparrow}(3) \end{vmatrix}$$

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$$= u_{1\uparrow}(1)\,u_{1\downarrow}(2)\,u_{2\uparrow}(3) - u_{1\uparrow}(1)\,u_{1\downarrow}(3)\,u_{2\uparrow}(2)$$

$$- u_{1\uparrow}(2)\,u_{1\downarrow}(1)\,u_{2\uparrow}(3) + u_{1\uparrow}(2)\,u_{1\downarrow}(3)\,u_{2\uparrow}(1)$$

$$+ u_{1\uparrow}(3)\,u_{1\downarrow}(1)\,u_{2\uparrow}(3) - u_{1\uparrow}(3)\,u_{1\downarrow}(2)\,u_{2\uparrow}(1)$$

This is exactly the result given in the textbook.

For $\alpha_i = \alpha_j$, then the anti-symmetric wave function does not exist.

$\Rightarrow$ No more than one electron can have a given set of quantum numbers (including the spin quantum number

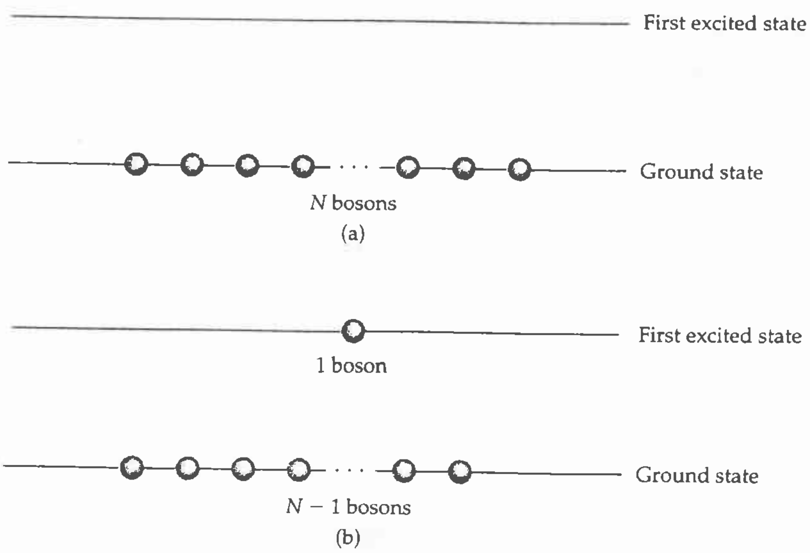
$\Rightarrow$ This is the Pauli exclusion principle.

The Fermi Energy

Ground state for N bosons in an infinite well.

No restriction of Pauli exclusion principle.

$E_g = N E_1$ ground state



Average energy per particle

$$\frac{E_g}{N} = E_1$$

First excited state

$N-1$ bosons in ground state

1 boson in the first excited state.

The energy of this excited system is

$$(N-1)E_1 + E_2$$

For fermions, the exclusion principle is operative

$N \rightarrow$ taken to be even, $\frac{N}{2}$ is an integer.

$$E_g = E_1 [2(1^2) + 2(2^2) + \cdots + 2(\tfrac{N}{2})^2] = 2E_1 \sum_{j=1}^{N/2} j^2$$

$[E_n = n^2 E_1]$

$$\sum_{j=1}^{n} j^2 = \frac{1}{6} n(n+1)(2n+1)$$

If u_n can be written as $f(n)-f(n-1)$, then

$$S_N = \sum_{n=1}^{N} u_n = f(N) - f(0)$$

difference method.

Want to find $f(n)-f(n-1) \longrightarrow n^2$

Answer: $f(n) = n(n+1)(2n+1)$

$$f(n-1) = (n-1)n(2n-1)$$

$$f(n)-f(n-1) = n[2n^2+3n+1-2n^2+3n-1]$$

$$= 6n^2$$

$$u_n = \frac{1}{6} N(N+1)(2N+1)$$

$$\sum_{j=1}^{N} j^2 = \frac{1}{6} N(N+1)(2N+1)$$

As N large the sum $\sim \frac{1}{3}\left(\frac{N}{2}\right)^3$

$$E_g \cong 2E_1 \frac{(N/2)^3}{3} \simeq \frac{N^3}{12} E_1$$

$$\frac{E_g}{N} \simeq \frac{N^2}{12} E_1$$

average energy per particle in the ground state.

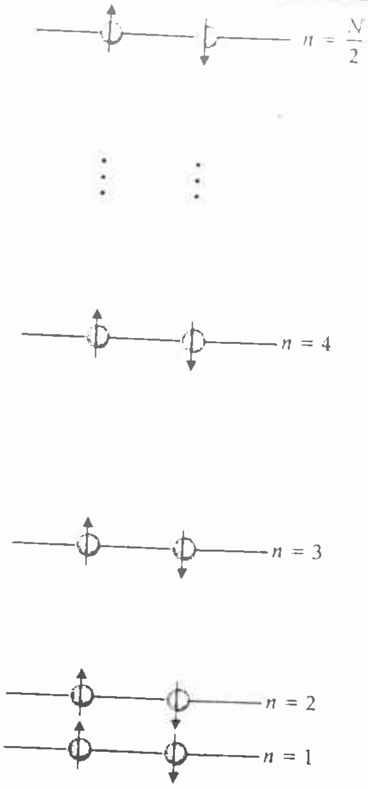
Please note the difference between the bosons and fermions.

The highest energy to be filled is known as the Fermi energy E_F

$$E_F = \frac{\pi^2\hbar^2\left(\frac{N}{2}\right)^2}{2mL^2} = \frac{\pi^2\hbar^2 N^2}{8mL^2}$$

In one dimensional case $n_e = \frac{N}{L}$

$$\Rightarrow E_F = \frac{\pi^2\hbar^2}{8m} n_e^2$$



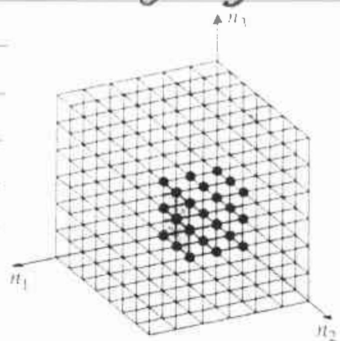
• **Figure 10–7** The lowest energy, or ground, state for N fermions has at most two particles in each single-particle level, one with spin up and one with spin down. All the levels up to $n = N/2$ are filled.

Now we go into the three dimensional box (cubic)

$$E_{\{n_1, n_2, n_3\}} = \frac{\pi^2\hbar^2}{2mL^2}(n_x^2+n_y^2+n_z^2) = E_1\cdot(n_1^2+n_2^2+n_3^2)$$

In the (n_1, n_2, n_3) space, there are

$\sim \frac{1}{8}\frac{4\pi R^3}{3}$ lattice points → integer n_1, n_2, n_3 with $n_1^2+n_2^2+n_3^2 \le R^2$



• **Figure 10–8** The energy states in a three-dimensional well are labeled by a set of three integers. These integers can be depicted as sitting on the intersection points of a three-dimensional cubic lattice.

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$$\frac{N}{2} = \frac{1}{8}\frac{4\pi R^3}{3}$$

$$R^2 = \frac{E_F}{E_1}$$

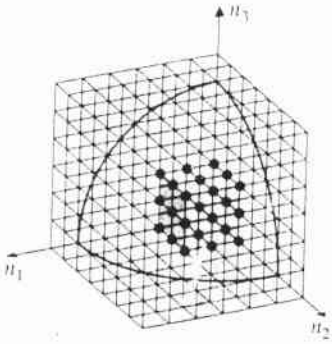
$$\Rightarrow \frac{N}{2} = \frac{4}{8}\frac{\pi}{3}\left(\frac{E_F}{E_1}\right)^{3/2}$$

$$= \frac{\pi}{6}\left(\frac{E_F}{E_1}\right)^{3/2} \quad \Rightarrow \quad E_F = E_1\left(\frac{3N}{\pi}\right)^{2/3}$$

$$\Rightarrow \quad E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{L^3}\right)^{2/3}$$

$$n_f = \frac{N}{L^3}$$

$$\Rightarrow \quad E_f = \frac{\hbar^2}{2m}(3\pi^2 n_f)^{2/3}$$



• **Figure 10–9** Because negative n-values for a three-dimensional well do not represent different states from those already labeled by the corresponding positive values, only positive values need be counted. This condition restricts us to one quadrant (one-eighth) of the full lattice. In other words, to find the points that correspond to the energy states and thus allow us to count the number of states with energy up to E, we restrict ourselves to points that lie within the octant that contains all positive coordinates of a sphere whose radius is proportional to E.

Total energy of the system

$$E_{tot} = \frac{\hbar^2\pi^2}{mL^2}\ \frac{1}{8}\int \vec{n}^2\, d^3\vec{n}$$

$\frac{1}{8}$ comes from the restriction to positive integers

$$= \frac{\hbar^2\pi^2}{8mL^2}\ 4\pi \int_0^R n^4\, dn$$

$n^2 dn\, d\Omega = d^3\vec{n}$

4π: angular integration.

$$= \frac{\hbar^2\pi^3}{10mL^2} R^5$$

[Remembering $\frac{N}{2} = \frac{1}{8}\frac{4\pi}{3}R^3$]

$$\Rightarrow \quad E_{tot} = \frac{\hbar^2\pi^3}{10mL^2}\left(\frac{3N}{\pi}\right)^{5/3}$$

Using $n = \frac{N}{L^3}$

n: number density.

$$\Rightarrow \quad E_{tot} = \frac{\hbar^2\pi^3}{10m}\left(\frac{3n}{\pi}\right)^{5/3} L^3$$

Remarks.

- The wave number k_F, defined by $E_f = \frac{\hbar^2 k_F^2}{2m}$

$$\Rightarrow \quad k_F = (3\pi^2 n)^{1/3}$$

$$k_F = \frac{2\pi}{\lambda_F}$$

$$\Rightarrow \quad \lambda_F = 2.03\, n^{-\frac{1}{3}}$$

$n^{-\frac{1}{3}} \sim$ interparticle spacing d

$$d \sim \frac{\lambda_F}{2}$$

$\Rightarrow$ Since the exclusion principle forbids two electrons with identical quantum numbers to be on top of each other

$\Rightarrow$ ~~the closest that two electrons can get to each other~~

is roughly half de Brogile wavelength $\leftrightarrow$ Fermi energy

If we keep a fixed number of electrons, then

$$E_{tot} = \frac{\hbar^2\pi^2}{10m}\left(\frac{3N}{\pi}\right)^{5/3} V^{-2/3}$$

If N is large, it can be shown that the result is independent of the shape of the volume.

Degeneracy pressure

If the electron gas is compressed, the electrons are pushed closer to each other $\Rightarrow$ decrease the de Broglie wavelength $\Leftrightarrow$ increases the kinetic energy
$\Rightarrow$ the compression is resisted, and the pressure resisting the compression is called the degeracy pressure

$$P_{deg} = -\frac{\partial E_{tot}}{\partial V} = \frac{\hbar^2\pi^3}{15m}\left(\frac{3n}{\pi}\right)^{5/3}$$

The bulk modulus B of a material is defined by

$$B = V\frac{\partial p}{\partial V}$$

if $p = P_{deg}$, then $B = 5P_{deg}/3 = \frac{\hbar^2\pi^2}{9m}\left(\frac{3n}{\pi}\right)^{5/3}$

The use of a degenerate electron model for a metal gives the correct order of magnitude for the bulk modulus.

Example: for copper

$$n_e = 8.47\times10^{28}\ \text{electron}/m^3$$

$$\downarrow$$

using above equation

$$\Rightarrow B = 6.4\cdot10^{10}\ N/m^2$$

Experimental result $\Rightarrow 14\times10^{10}\ N/m^2$

• **Figure 10–4** Paul A. M. Dirac was one of the pioneers of quantum mechanics. He developed the quantum theory of radiation and did ground-breaking work in the creation of relativistic quantum mechanics. He predicted the existence of anti-matter. The French physicist Leon Brillouin, himself an important contributor, is in the background.

• **Figure 10–3** The Italian-born American physicist Enrico Fermi (1901–1954) was both a brilliant theorist and an equally brilliant experimental physicist. Among his many accomplishments was his leadership in the construction of the first nuclear reactor. He is shown here on a hike with Niels Bohr in 1931.

• **Figure 10–2** Wolfgang Pauli was born in Vienna in 1900 and died in Zurich in 1958. He was one of the most brilliant theoretical physicists of this century.

Satyendranath Bose

(1894–1974, Indian)

Bose was born and educated in Calcutta, India. In a paper written in 1924 he derived the Planck formula for blackbody radiation by treating the photons as what we would now call bosons. This paper drew the attention of Einstein and secured an invitation for Bose to visit Europe, where he met Einstein, de Broglie, Born, and others. Einstein extended Bose's ideas, and the rules that govern bosons are now called Bose–Einstein statistics. We will see some of the dramatic consequences of these ideas in Chapter 13.

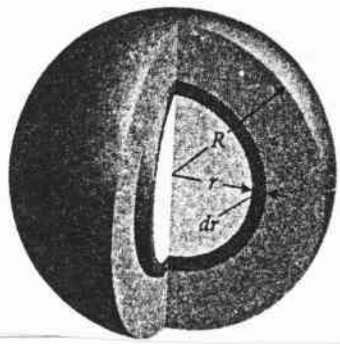
Astrophysical Applications

Stellar evolution (H. Bethe)

Stars start out as hydrogen gas, which "burns" by undergoing a succession of nuclear reactions

At some point the burning stops

$\Rightarrow$ then the only effects we have to worry about are
- gravitational force $\rightarrow$ work to compress the stellar matter
- degeneracy pressure of the electrons in the star, which works to hold the electrons, and with them all the stellar matter apart



• Figure 10–10 To find the gravitational potential energy of a uniform sphere of matter, we add up the separate contributions of thin spherical shells.

Gravitational pressure

$$dU = -\frac{G\,m\,dm}{r} = -G\,\frac{\frac{4\pi\rho r^3}{3}\cdot 4\pi\rho r^2 dr}{r}$$

$$= -\frac{16\pi^2 G\rho^2}{3} r^4 dr$$

$$U_g = -\frac{16\pi^2 G\rho^2}{3}\int_0^R r^4\, dr$$

$$= -\frac{16\pi^2 G\rho^2}{15} R^5$$

$$\rho = \frac{N m_p}{V}$$

m_p = mass of proton
N = number of nucleons

$$V = \frac{4\pi R^3}{3}$$

$$\Rightarrow \quad U_g = -\frac{3}{5}\left(\frac{4\pi}{3}\right)^{1/3} G (N m_p)^2 V^{-1/3}$$

$$\Rightarrow \quad P_g = -\frac{\partial U_g}{\partial V} = \frac{1}{5}\left(\frac{4\pi}{3}\right)^{1/3} G (N m_p)^2 V^{-4/3}$$

$$P_{deg} = \frac{6}{5}\left(\frac{\pi^4}{3}\right)^{1/3}\frac{\hbar^2}{2m_e}\left(\frac{N}{2}\right)^{5/3} V^{-5/3}$$

Equate $P_g = P_{deg}$.

$$\Rightarrow \quad R = \left(\frac{81\pi^2}{128}\right)^{1/3}\frac{\hbar^2}{G m_p^2 m_e} N^{-\frac{1}{3}} \cong (1.15\cdot 10^{23}\,\text{km})\, N^{-\frac{1}{3}}$$

For a star with the mass of the Sun.

$M_{sum} \sim 2 \times 10^{30}$ Kg $\Rightarrow$ $N \simeq 1.2 \cdot 10^{57}$

After nuclear burning $\longrightarrow$ white dwarfs
↓
gravitational pressure balanced by degeneracy pressure

[Note E_{tot} (degeneracy energy) $\sim \frac{1}{m}$
⇓
electrons plays the dominant role.

For more massive (than the Sun) star
$N \rightarrow$ larger
↓
large the electrons' average energy
↓
electrons becomes relativistic
↓
pressures cannot balance
↓
the gravitational pressure wins out.
the star continues to collapse
↓
forcing all the particles closer together
$e^- + p \rightarrow n + \nu$
↳ escape

$\Rightarrow$ neutron star
↓
degeneracy pressure due to neutron balance the gravitational pressure
$R \sim 10$ km.

For even more massive star, gravitational pressure dominate

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the degeneracy pressure even for neutrons

↓

gravitational collape continues

↓

black hole

[Exchange force will be discussed in Chapter 11]

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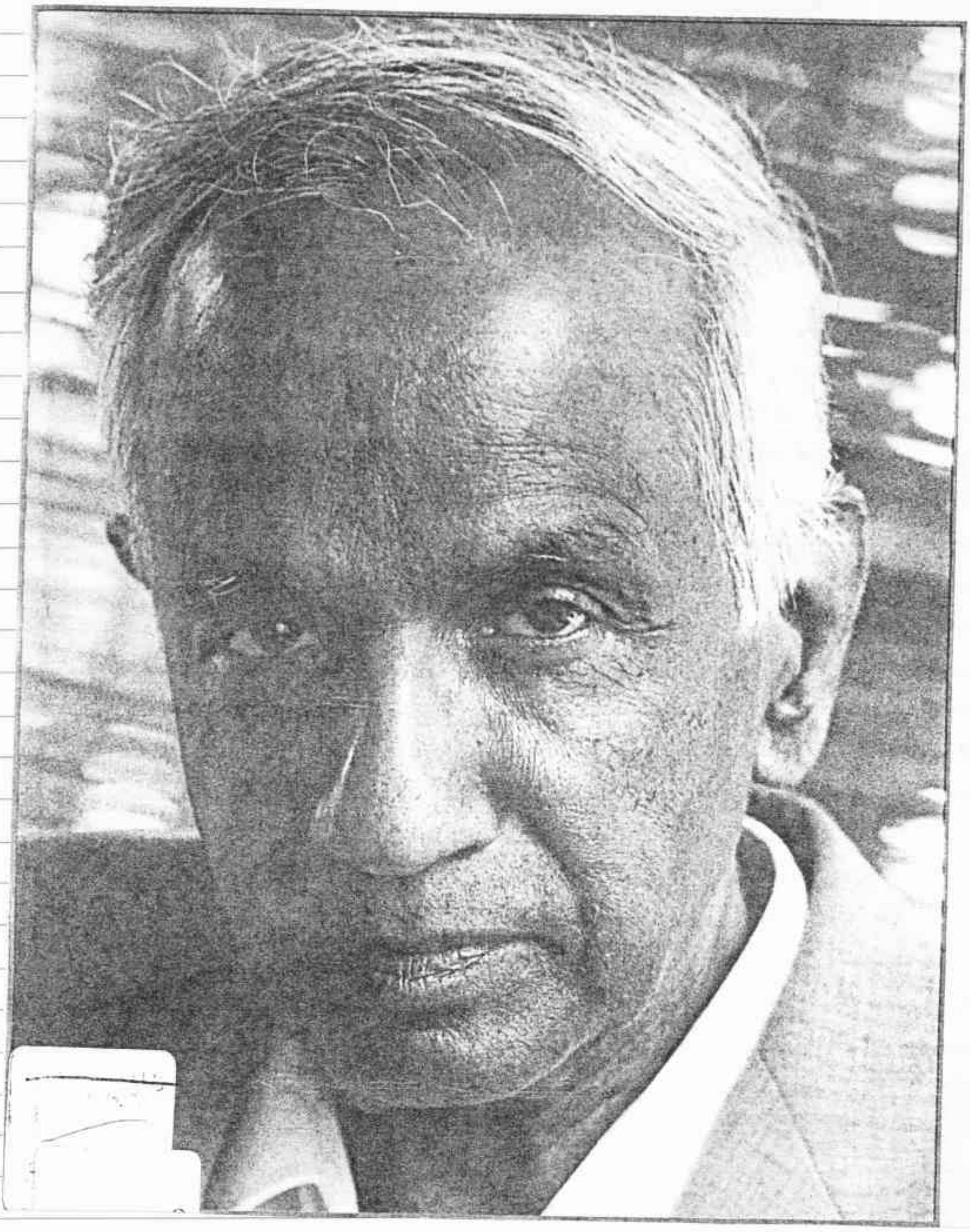
Hans Bethe

(Born 1906, German-American)

After postdoctoral work with Rutherford in Cambridge and Fermi in Rome, Bethe taught in Germany for a few years before coming to the United States in 1935. Among many contributions to atomic and nuclear physics, he is best known for finding the two nuclear cycles by which most stars get their energy. For this discovery, he won the 1967 Nobel Prize in physics.

CHANDRA

A Biography of S. Chandrasekhar



1. Multipart Schrodinger Equation

One dimensional problem

2. Independent Particle Model

Hartree method

3. Identical Particles

Indistinguishability

Example : (a) particle in a box

(b) two idential boson

such as $\pi^0 \pi^0$

$\Rightarrow$ three dimension

(c) spin wave function for two electrons

4 N - identical Independent Particle Wave Function

Totally symmetric wave function

Totally anti-symmetric wave function.

Slater determinant

Pauli exclusion relation.

5. Ground State Energy

one dimensional box

For bosons $E_g = NE_1$

For fermions $E_g \cong \frac{N^2}{12}$

Fermi energy $E_F = \frac{\pi^2 \hbar^2 N^2}{8 m L^2}$

three dimensional box

$E_f = \frac{\hbar^2}{2m} (3\pi^2 n_f)^{3/2}$

n_f: density of the Fermi gas

Total energy of the system E_{tot}

$$E_{tot} = \frac{\hbar^2\pi^2}{10m}\left(\frac{3N}{\pi}\right)^{5/3} V^{-2/3}$$

6. Degeneracy Pressure

Application. Bulk modulus of copper

7. Astrophysical Applications