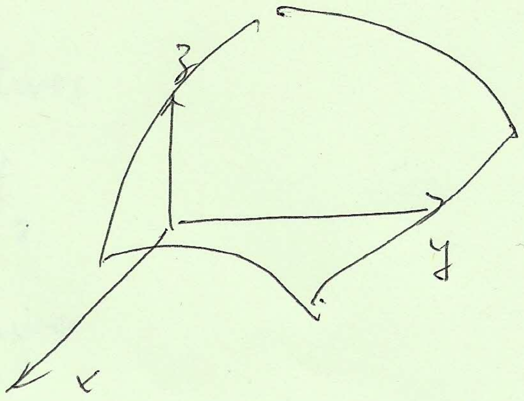
一. Partial Derivative, Tangent plane, Gradient

1. Function of two variables

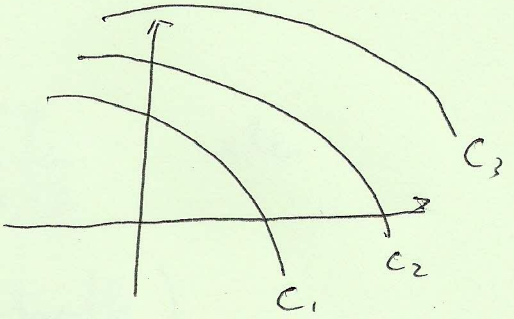
$$z = f(x, y)$$

Graph : surface



Level curves :

$$f(x, y) = c$$



2. Fuction of three variables

$$u = f(x, y, z)$$

Level surfaces

$$f(x, y, z) = c$$

3. partial derivatives

$$z = f(x, y)$$

1st order partial derivatives

$$\frac{\partial f}{\partial x} = f_x \quad , \quad \frac{\partial f}{\partial y} = f_y$$

2nd order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} \quad , \quad \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\frac{\partial^2 f}{\partial y \partial x} = f_{xy} = f_{yx} = \frac{\partial^2 f}{\partial x \partial y} \qquad \text{etc.}$$

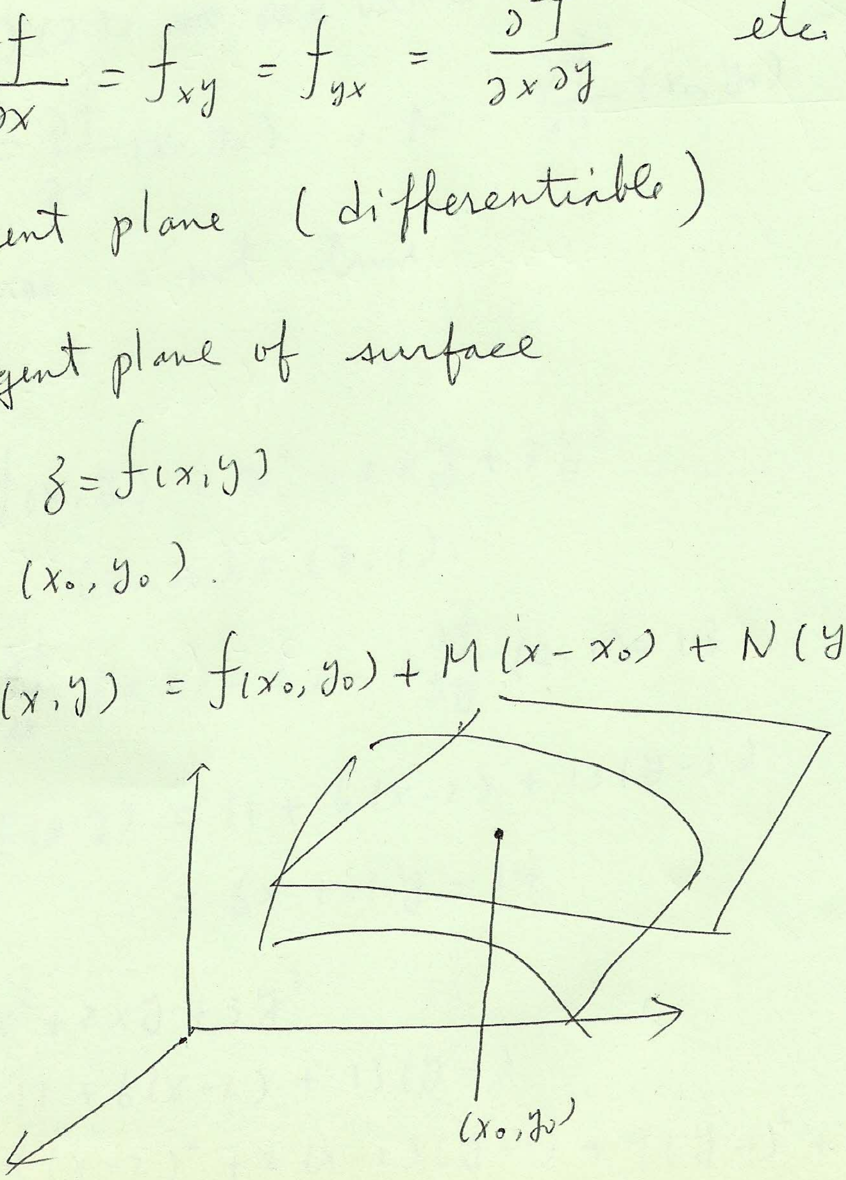
4. Tangent plane (differentiable)

Tangent plane of surface

$$z = f(x, y)$$

at (x_0, y_0).

$$T(x, y) = f(x_0, y_0) + M(x - x_0) + N(y - y_0)$$



with property

$$f(x,y) = f(x_0, y_0) + M(x-x_0) + N(y-y_0)$$
$$+ \varepsilon_1(x, x_0, y, y_0)\ (x-x_0) \qquad (1)$$
$$+ \varepsilon_2(x_1, x_0, y_1, y_0)\ (y-y_0)$$

where

$$\lim_{(x,y)\to(x_0,y_0)} \varepsilon_1 = \lim_{(x,y)\to(x_0,y_0)} \varepsilon_2 = 0 \qquad (2)$$

If (1) (2) ~~is~~ are valid, then

$$M = \frac{\partial f}{\partial x}(x_0, y_0), \quad N = \frac{\partial f}{\partial y}(x_0, y_0)$$

converse is not true.

5. Ex.

$$f(x,y) = x^2 + 2xy + 3y^3$$
$$(x_0, y_0) = (2, 1)$$
$$\frac{\partial f}{\partial x}(2,1) = 6, \quad \frac{\partial f}{\partial y}(2,1) = 13$$
$$T(x,y) = 11 + 6(x-2) + 13(y-3)$$
$$= 6x + 13y - 14$$

$$x^2 + 2xy + 3y^3$$
$$= 11 + 6(x-2) + 13(y-1)$$
$$+ (x-2)^2 + 2(x-2)(y-1) + 9(y-1)^2 + 3(y-1)^3$$

$$\varepsilon_1 = (x-2) + 2(y-1)$$
$$\varepsilon_2 = 9(y-1) + 3(y-1)^2$$

Ex.
$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$$

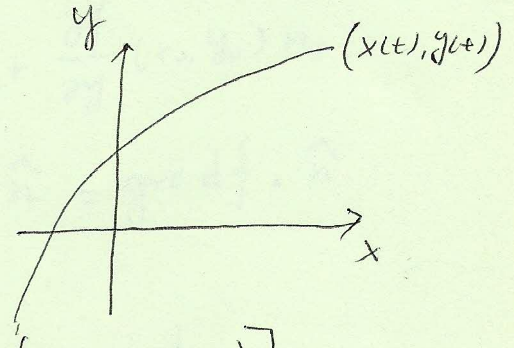
$$\frac{xy}{x^2+y^2} = 0 + 0 + 0 + \varepsilon_1 x + \varepsilon_2 y$$

Let $x = y \to 0$

$$\frac{1}{2} = 0 \qquad \text{contradiction}$$

6. Chain rule

$$f(x(t), y(t))$$

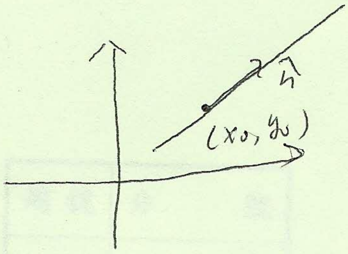


$$\frac{d}{dt} f(x(t), y(t))$$
$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big[f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t)) \Big]$$
$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big[f(x(t) + x'(t)\Delta t + \varepsilon_1 \Delta t,\ y(t) + y'(t)\Delta t,\ \varepsilon_2 \Delta t) - f(x(t), y(t)) \Big]$$
$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big[f(x(t), y(t)) + \frac{\partial f}{\partial x}\cdot(x'(t)\Delta t + \varepsilon_1 \Delta t) + \frac{\partial f}{\partial y}\cdot(y'(t)\Delta t + \varepsilon_2 \Delta t) + \varepsilon_3 \cdot (x'(t)\Delta t + \varepsilon_1 \Delta t) + \varepsilon_4 \cdot (y'(t)\Delta t + \varepsilon_2 \Delta t) - f(x(t), y(t)) \Big]$$

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \quad \text{total differential}$$

7. Directional derivative, Gradient



$z = f(x,y)$, (x_0, y_0)

$\hat{n} = (n_1, n_2)$

$$\begin{cases} x(t) = x_0 + n_1 t \\ y(t) = y_0 + n_2 t \end{cases}$$

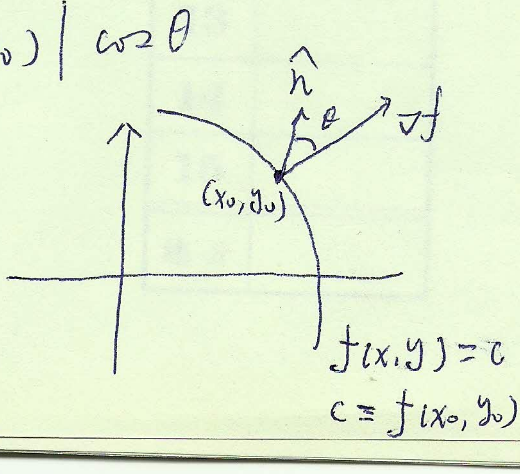
Directional derivative of f at (x_0, y_0) along

$$\begin{aligned} D_{\hat{n}} f(x_0, y_0) &\equiv \frac{d}{dt} f(x(t)\ y(t)) \\ &= \frac{\partial f}{\partial x}(x_0, y_0)\, n_1 + \frac{\partial f}{\partial y}(x_0, y_0)\, n_2 \\ &\equiv \nabla f(x_0, y_0) \cdot \hat{n} = \text{grad} f \cdot \hat{n} \end{aligned}$$

Gradient (vector)

$$\nabla f(x,y) \equiv i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} = \text{grad} f$$

$$\therefore\ D_{\hat{n}} f(x_0, y_0) = |\nabla f(x_0, y_0)| \cos\theta$$



$|\nabla f(x_0, y_0)|$: maximum chang of rate of f at (x_0, y_0)

$\nabla f(x_0, y_0)$ perpendicular to the level curve

3. 3-dim

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

Del operator

$$\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

vector field

$$\vec{F}(x, y, z) = F_x\,\hat{i} + F_y\,\hat{j} + F_z\,\hat{k}$$

curl

$$\text{curl}\,\vec{F} \equiv \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\text{div.}\vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\text{curl grad } f = \nabla \times \nabla f = 0$$

$$\text{div curl } \vec{F} = \nabla \cdot \nabla \times \vec{F} = 0$$

三. line integral

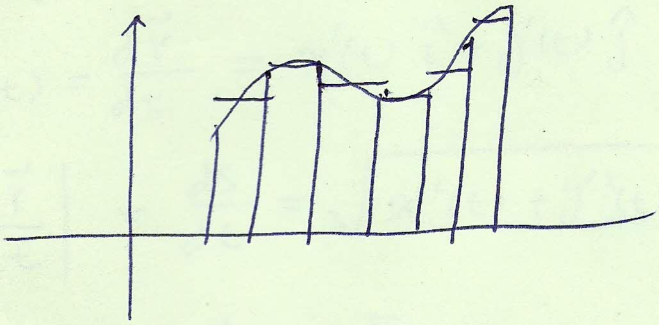
1. Definite integral

$f(x)$ defined on $[a, b]$

$$a = x_0 < x_1 < x_2 < \cdots < x_{i-1} < x_i < \cdots < x_n = b$$

$$\xi_i \in [x_{i-1}, x_i] \qquad \Delta x_i \equiv x_i - x_{i-1}$$

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} f(\xi_i)\,\Delta x_i$$



2. Fundamental Theorem of calculus

$$\int_a^b \Phi'(x)\,dx = \Phi(b) - \Phi(a)$$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

3. curve

$$C: \vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k} \qquad a \le t \le b$$

curve length

$$s(t) = \int_a^t \sqrt{x'(\tau)^2 + y'(\tau)^2 + z'(\tau)^2}\; d\tau$$

Tangent vector

$$\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = x'(t)\,\hat{i} + y'(t)\,\hat{j} + z'(t)\,\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \frac{ds}{dt} = \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}$$

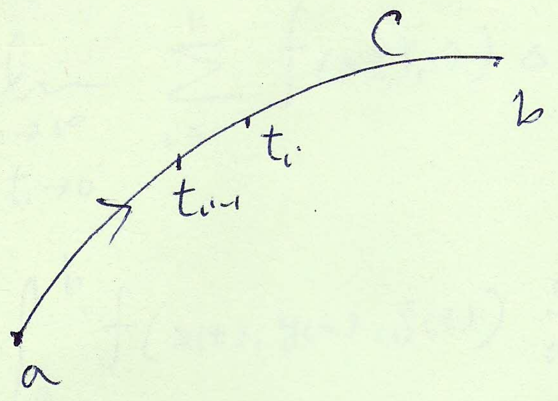
unit tangent vector

$$\vec{T} = \frac{d\vec{r}}{dt}\Big/\left|\frac{d\vec{r}}{dt}\right| = \frac{d\vec{r}}{dt}\Big/\frac{ds}{dt} = \frac{d\vec{r}}{ds}$$

$$|\vec{T}| = 1$$

4. line integral

$$a = t_0 < t_1 < \cdots < t_{i-1} < t_i < \cdots < t_n = b$$



$$\xi_i \in [t_{i-1}, t_i]$$

Define line integral

$$\int_C f(x, y, z)\, ds$$

$$= \lim_{\substack{n\to\infty \\ \Delta t_i \to 0}} \sum_{i=1}^{n} f(\vec{r}(\xi_i))\, \Delta s_i$$

$$= \lim_{\substack{n\to\infty \\ \Delta t_i \to 0}} \sum_{i=1}^{n} f(\vec{r}(\xi_i))\, \frac{\Delta s_i}{\Delta t_i}\, \Delta t_i$$

$$= \int_a^b f(x(t), y(t), z(t))\, \frac{ds}{dt}\, dt$$

'In general path dependent

Define another line integral

$$\int_c f(x,y,z)\, dx$$

$$= \lim_{\substack{n \to \infty \\ \Delta t_i \to 0}} \sum_{i=1}^{n} f(r(\xi_i))\, \Delta x_i \qquad \Delta x_i = x(t_i) - x(t_{i-1})$$

$$= \int_a^b f(x(t), y(t), z(t))\, \frac{dx}{dt}\, dt$$

$$\int_c f(x,y,z)\, dy \quad , \quad \int_c f(x,y,z)\, dz$$

5.

For a vector field

$$\vec{F}(x,y,z) = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

Work

$$W = \lim_{\substack{n \to \infty \\ \Delta t_i \to 0}} \sum_{i=1}^{n} \vec{F} \cdot \vec{T}(\xi_i)\, \Delta s_i$$

$$= \int_c \left(F_x \frac{dx}{ds} + F_y \frac{dy}{ds} + F_z \frac{dz}{ds} \right) ds$$

$$= \int_c \left(F_x\, dx + F_y\, dy + F_z\, dz \right)$$

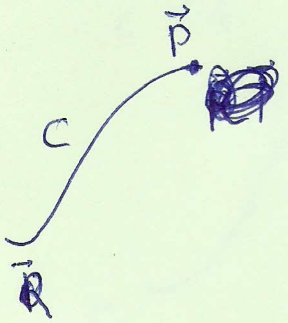
$$= \int_C \vec{F} \cdot d\vec{r}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

6. Fundamental theorem of line integral

$C : \vec{r} = \vec{r}(t) \qquad a \le t \le b \qquad \vec{r}(a) = \vec{Q}$

$\vec{r}(b) = \vec{P}$



The following four statements are equivalent

(a) $\displaystyle {}_C\!\int_{\vec{Q}}^{\vec{P}} \vec{F} \cdot d\vec{r}$ independent of path C.

(b) $\vec{F} = -\nabla(\Phi + c)$

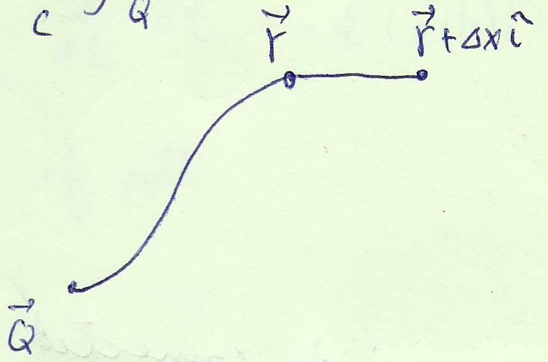
(c) $\nabla \times \vec{F} = 0$

(d) $\oint \vec{F} \cdot d\vec{r} = 0$

Pf. (a) $\Rightarrow$ (b)

If $\displaystyle {}_C\!\int_{\vec{Q}}^{\vec{P}} \vec{F}\cdot d\vec{r}$ is independent of C.

Define $$ {}_C\!\int_{\vec{Q}}^{\vec{r}} \vec{F}\cdot d\vec{r} = -\Phi(\vec{r})$$



$$\frac{\partial \Phi}{\partial x} = \lim_{\Delta x\to 0} \frac{\Phi(x+\Delta x, y, z) - \Phi(x,y,z)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{-1}{\Delta x}\int_{\vec{r}}^{\vec{r}+\Delta x\hat{i}} F_x\, dx = -F_x$$

$$\therefore \quad \vec{F} = -\nabla\Phi$$

(b) $\Rightarrow$ (a)

If $\vec{F} = -\nabla \Phi$

$$\int_{C\,\vec{Q}}^{\vec{P}} \vec{F}\cdot d\vec{r} = -\int_{C\,\vec{Q}}^{\vec{P}} \frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy + \frac{\partial \Phi}{\partial z}dz$$

$$= -\int_a^b \frac{d\Phi}{dt}\,dt = -\Phi(\vec{r}(b)) + \Phi(\vec{r}(a))$$

indep. of path

(c) $\Rightarrow$ (b)

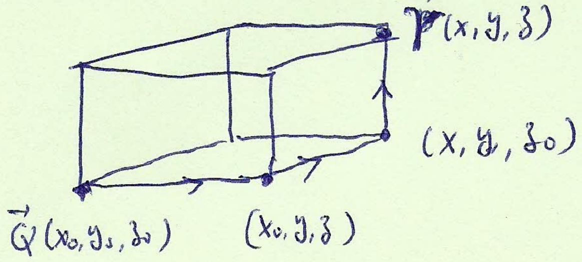
Assume $\nabla \times \vec{F} = 0$

i.e. $\dfrac{\partial F_y}{\partial z} = \dfrac{\partial F_z}{\partial y}$, $\dfrac{\partial F_x}{\partial y} = \dfrac{\partial F_y}{\partial x}$, $\dfrac{\partial F_x}{\partial z} = \dfrac{\partial F_z}{\partial x}$

Define

$$\Phi(x,y,z) = -\int_C \vec{F}\cdot d\vec{r}$$

where C is



i.e. $\Phi(x, y, z)$

$$= -\int_{y_0}^{y} F_y(x_0, y', z_0)\, dy'$$

$$- \int_{x_0}^{x} F_x(x', y, z_0)\, dx'$$

$$- \int_{z_0}^{z} F_z(x, y, z')\, dz'$$

check $\dfrac{\partial \Phi}{\partial x} = -F_x$,

$$\frac{\partial \Phi}{\partial y} = -F_y$$

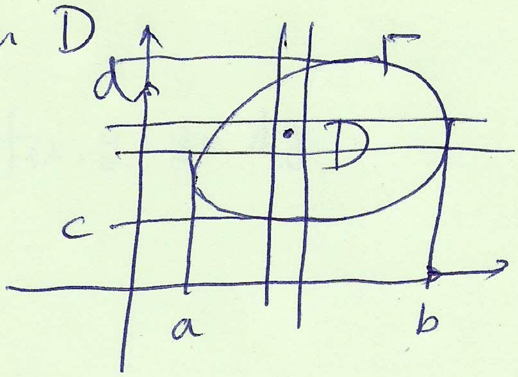
$$\frac{\partial \Phi}{\partial z} = -F_z$$

!

三. Double integral , Green's theorem

1.

$z = f(x,y)$ define on D



$a = x_0 < x_1 < \cdots < x_n = b$

$c = y_0 < y_1 < \cdots < y_m = d$

$(\xi_i, \eta_j) \in \Delta x_i \times \Delta y_j$

Define

$$\iint_D f = \iint_D f(x,y)\,dx\,dy$$

$$= \lim_{\substack{n,m\to\infty \\ \Delta x_i \to 0 \\ \Delta y_j \to 0}} \sum_{i,j=1}^{n,m} f(\xi_i, \eta_j)\,\Delta x_i \Delta y_j$$

2. Volume

3. iterated integral

$\Gamma :\ y = g(x) \cup y = h(x)$

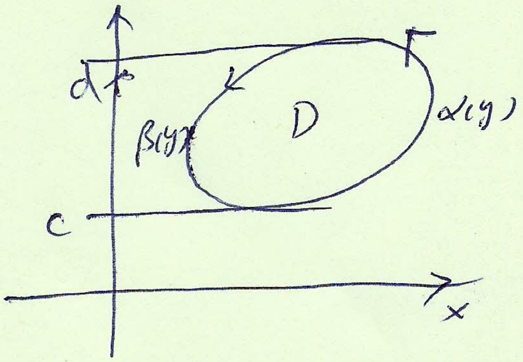
$x = \alpha(y) \cup x = \beta(y)$

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \int_{h(x)}^{g(x)} f(x,y)\,dy\,dx$$

$$= \int_c^d \int_{\beta(y)}^{\alpha(y)} f(x,y)\,dx\,dy.$$

4. Green's theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint_\Gamma P\,dx + Q\,dy$$



Pf

$$\iint_D \frac{\partial Q}{\partial x}\,dx\,dy$$

$$= \int_c^d \int_{\beta(y)}^{\alpha(y)} \frac{\partial Q}{\partial x}\,dx\,dy$$

$$= \int_c^d \left[Q(\alpha(y), y) - Q(\beta(y), y) \right] dy$$

$$= \oint_\Gamma Q\,dy$$

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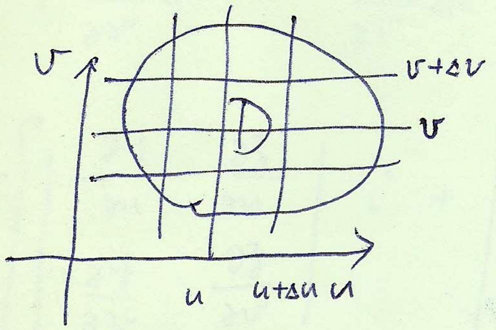
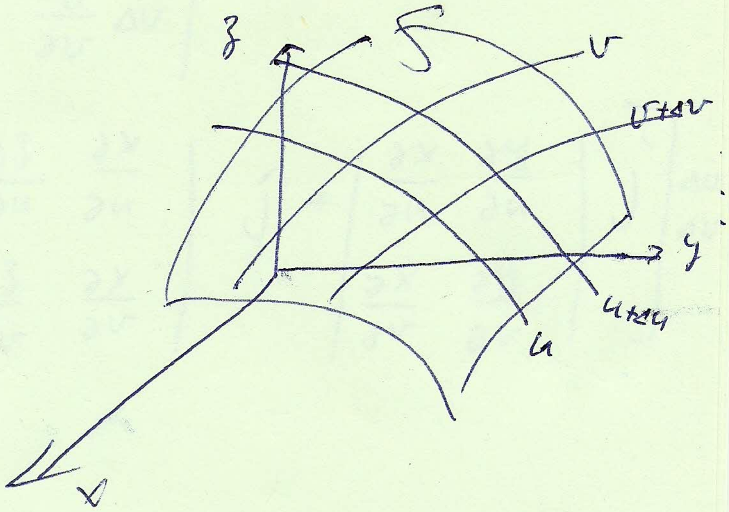
18 Surface integral

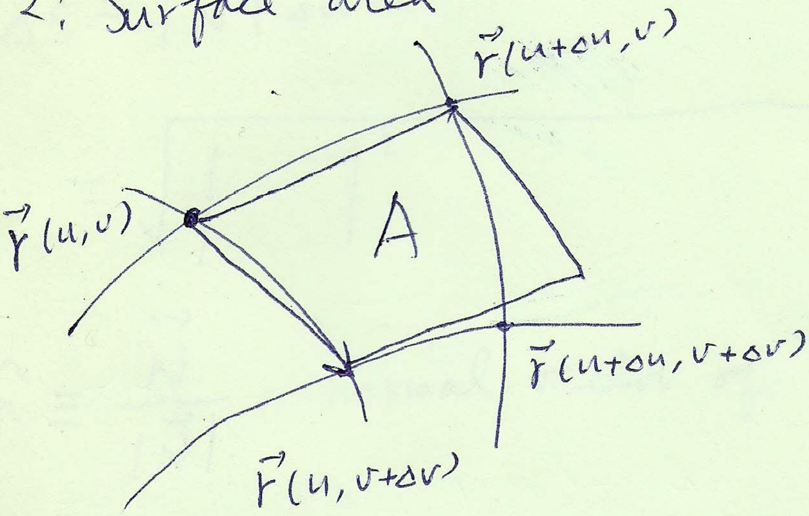
1. Surface

$$\vec{r} = \vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$





2. Surface area



Area of para

$$\left(\vec{r}(u+\Delta u, v) - \vec{r}(u,v)\right) \times \left(\vec{r}(u, v+\Delta v) - \vec{r}(u,v)\right)$$

$$= \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\, \Delta u \Delta v$$

$$= \begin{vmatrix} i & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u & \frac{\partial z}{\partial u}\Delta u \\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v & \frac{\partial z}{\partial v}\Delta v \end{vmatrix}$$

$$= \left[\begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \hat{i} + \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix} \hat{j} + \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \hat{j} \right] \Delta u \Delta v$$

$$\equiv \vec{N}(u,v)\, \Delta u \, \Delta v$$

$$\Delta S = |\vec{N}| \, \Delta u \, \Delta v$$

$$= \sqrt{\left| \quad \right|^2 + \left| \quad \right|^2 + \left| \quad \right|^2}$$

$\hat{n} \equiv \frac{\vec{N}}{|\vec{N}|}$ normal vector of surface

3. Surface integral

① $$\iint_S f(x,y,z)\, dS$$

$$= \lim_{\substack{\Delta S_i \to 0 \\ n \to \infty}} \sum_{i=1}^{n} f(\vec{r}(u_i, v_i))\, \Delta S_i$$

$$= \lim_{\substack{\Delta S_i \to 0 \\ n \to \infty}} \sum_{i=1}^{n} f(\vec{r}(u_i, v_i))\, |\vec{N}(u_i, v_i)|\, \Delta u_i \Delta v_i$$

$$= \iint_D f(\vec{r}(u,v)) \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\, dv$$

② $$\iint_S f(\vec{r})\, dx\, dy$$

$$\equiv \lim_{\substack{n \to \infty \\ \Delta S_i \to 0}} \sum_{i=1}^{n} f(\vec{r}(u_i, v_i))\, |\vec{N}|\, \Delta u \Delta v \, \frac{\frac{\partial(x,y)}{\partial(u,v)}}{|\vec{N}|}$$

$$= \iint_A f(\vec{r}(u,v)) \frac{\partial(x,y)}{\partial(u,v)}\, du\, dv$$

$$\iint_S f(\vec{r})\, dy\, dz \,, \quad \iint_S f(\vec{r})\, dz\, dx$$

4. Flux.

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

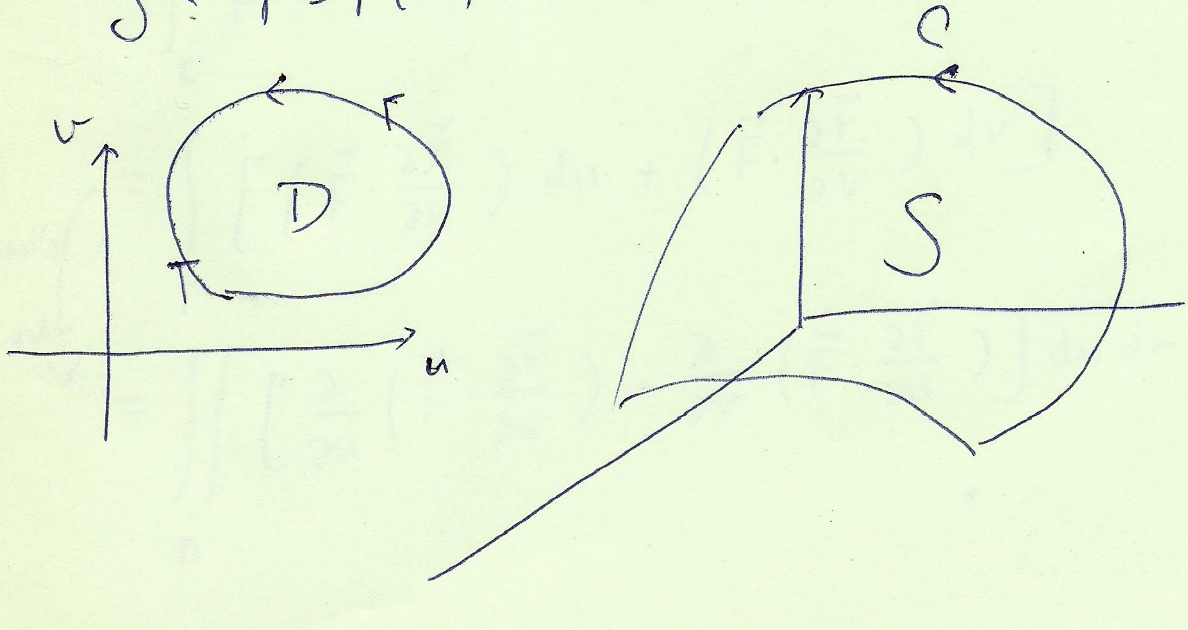
$$\text{Flux} \equiv \iint_S \vec{F} \cdot \hat{n}\, dS \equiv \iint_S \vec{F} \cdot d\vec{S}$$

$$= \iint_S \vec{F} \cdot \frac{\vec{N}}{|\vec{N}|} |\vec{N}|\, du\, dv$$

$$= \iint_S F_x\, dy\, dz + F_y\, dz\, dx + F_z\, dx\, dy$$

5. Stokes's theorem

$S: \vec{r} = \vec{r}(u,v)$ on $(u,v) \in D$



$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$

Pf.

i) $$d\vec{r} = \frac{\partial \vec{r}}{\partial u}\,du + \frac{\partial \vec{r}}{\partial v}\,dv$$

$$\frac{\partial}{\partial u} = \frac{\partial \vec{r}}{\partial u}\cdot\nabla$$

$$\frac{\partial}{\partial v} = \frac{\partial \vec{r}}{\partial v}\cdot\nabla$$

$$\left(\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right)\times\nabla = \frac{\partial \vec{r}}{\partial v}\left(\frac{\partial \vec{r}}{\partial u}\cdot\nabla\right) - \frac{\partial \vec{r}}{\partial u}\left(\frac{\partial \vec{r}}{\partial v}\cdot\nabla\right)$$

ii) $$\int_C \vec{F}\cdot d\vec{r}$$

$$= \int_\Gamma \left[\left(\vec{F}\cdot\frac{\partial \vec{r}}{\partial u}\right)du + \left(\vec{F}\cdot\frac{\partial \vec{r}}{\partial v}\right)dv\right]$$

Green's theorem

$$= \iint_D \left[\frac{\partial}{\partial u}\left(\vec{F}\cdot\frac{\partial \vec{r}}{\partial v}\right) - \frac{\partial}{\partial v}\left(\vec{F}\cdot\frac{\partial \vec{r}}{\partial u}\right)\right]du\,dv$$

$$= \iint_D \left[\frac{\partial \vec F}{\partial u}\cdot\frac{\partial \vec r}{\partial v} + \cancel{\vec F \frac{\partial^2 \vec r}{\partial u\,\partial v}} - \cancel{\vec F \frac{\partial^2 \vec r}{\partial v\,\partial u}} - \frac{\partial \vec F}{\partial v}\cdot\frac{\partial \vec r}{\partial u}\right] du\,dv$$

$$= \iint_D \left(\frac{\partial \vec r}{\partial v}\cdot\frac{\partial}{\partial u} - \frac{\partial \vec r}{\partial u}\frac{\partial}{\partial v} \right)\cdot \vec F \, du\,dv$$

$$= \iint_D \left[\frac{\partial \vec r}{\partial v}\left(\frac{\partial \vec r}{\partial u}\cdot\nabla\right) - \frac{\partial \vec r}{\partial u}\left(\frac{\partial \vec r}{\partial v}\cdot\nabla\right)\right]\cdot \vec F \, du\,dv$$

$$= \iint_D \left(\frac{\partial \vec r}{\partial u}\times\frac{\partial \vec r}{\partial v} \right)\times \nabla\cdot \vec F \, du\,dv$$

$$= \iint_D \left(\frac{\partial \vec r}{\partial u}\times\frac{\partial \vec r}{\partial v} \right)\cdot \nabla\times \vec F \, du\,dv$$

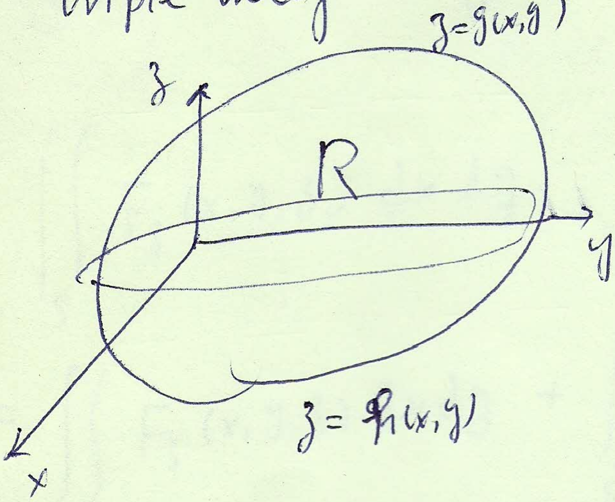
$$= \iint_D \operatorname{curl}\vec F\cdot\left(\frac{\partial \vec r}{\partial u}\times\frac{\partial \vec r}{\partial v} \right) du\,dv$$

$$= \iint_S \operatorname{curl}\vec F\cdot d\vec S$$

6. $$\operatorname{curl}\vec F\cdot\hat n = \lim_{\Delta S\to 0} \oint_C \vec F\cdot d\vec r$$

II. triple integral, Gauss theorem

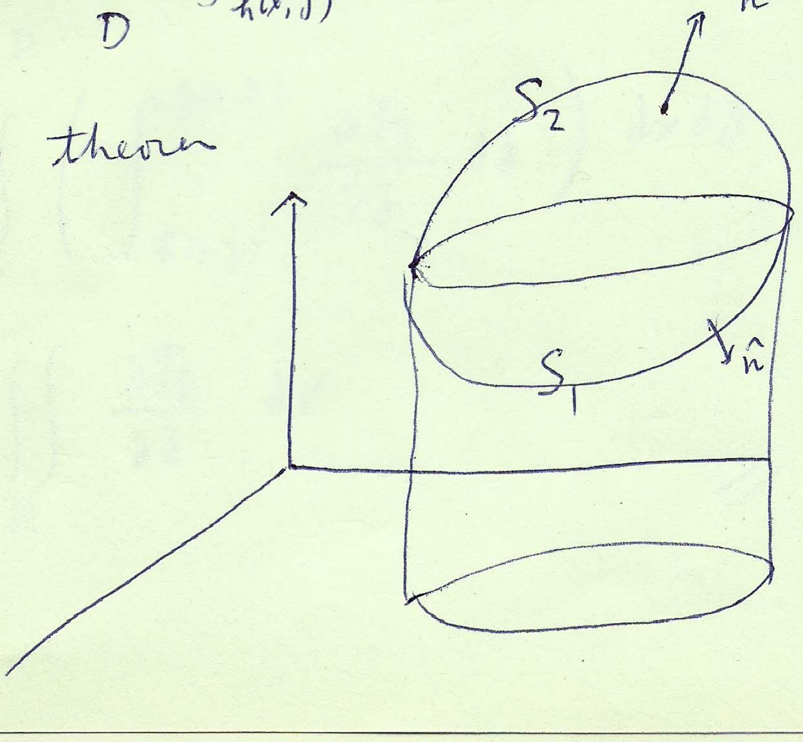
1. triple integral



$$\iiint_R f(x,y,z)\,dV$$

$$= \iint_D \int_{h(x,y)}^{g(x,y)} f(x,y,z)\,dz\,dy\,dx$$

2. Gauss theorem



$$\oiint_S \vec{F}\cdot\hat{n}\, dS \equiv \iiint_V \nabla\cdot\vec{F}\, dv$$

Pf.

$$\iint_S F_z(x,y,z)\, dx\, dy$$

$$= \iint_{S_1} F_z(x,y,z)\, dx\, dy + \iint_{S_2} F_z(x,y,z)\, dx\, dy$$

$$= \iint_D F_z(x,y,g(x,y))\, dx\, dy$$

$$- \iint_D F_z(x,y,h(x,y))\, dx\, dy$$

$$= \iint_D \left(\int_{h(x,y)}^{g(x,y)} \frac{\partial F_z}{\partial z}\, dz \right) dx\, dy$$

$$\equiv \iiint_R \frac{\partial F_z}{\partial z}\, dv.$$

//

3. $$\operatorname{div} \vec{F} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_S \vec{F} \cdot d\vec{S}$$

4. $$\vec{F} = \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|^3} = -\nabla \frac{1}{|\vec{r} - \vec{r}_0|}$$

$$\nabla \cdot \vec{F} = 0 \qquad \vec{r} \neq \vec{r}_0$$

$$\nabla^2 \frac{1}{|\vec{r} - \vec{r}_0|} = 0 \qquad \vec{r} \neq \vec{r}_0$$