

Lab 19

RC and RLC circuits

A. Purpose

To study the charge relaxation in the RC circuit and the damped charge oscillation and the resonance in the RLC circuit.

B. Introduction

An RC circuit is a circuit with a resistor and a capacitor. Fig. 1 shows a simple RC circuit with a DC source ϵ , a resistor R , a capacitor C , and a two-position switch. The circuit allows the capacitor to be charged or discharged, depending on the position of the switch.

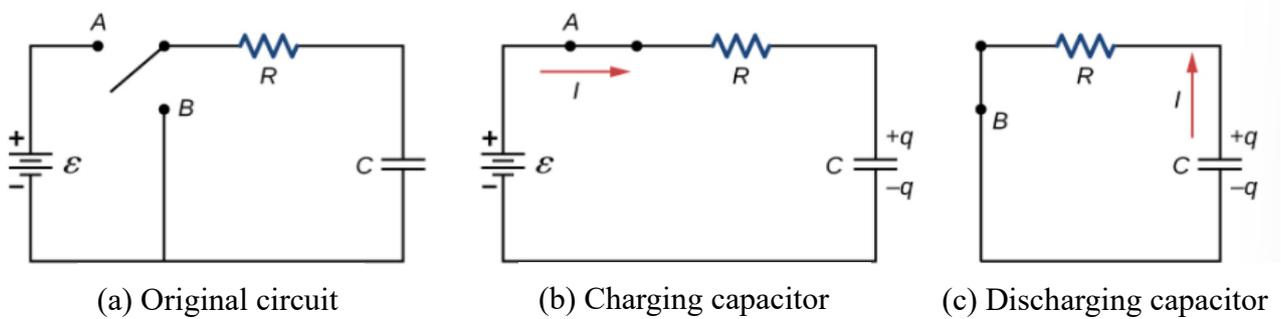


Fig. 1. RC circuits with a two-pole switch used to charge and discharge a capacitor.

If an inductor L is connected in series to the circuit, as Fig. 2(a) shows, this circuit is called the RLC circuit. Given the capacitor is initially fully charged, the charge oscillation begins when the switch is closed, as Fig. 2(b) shows. As you can see from the figure, the oscillating pattern is the same as a damped mass-spring system. This is the general dynamics for any linearly-damped oscillator.

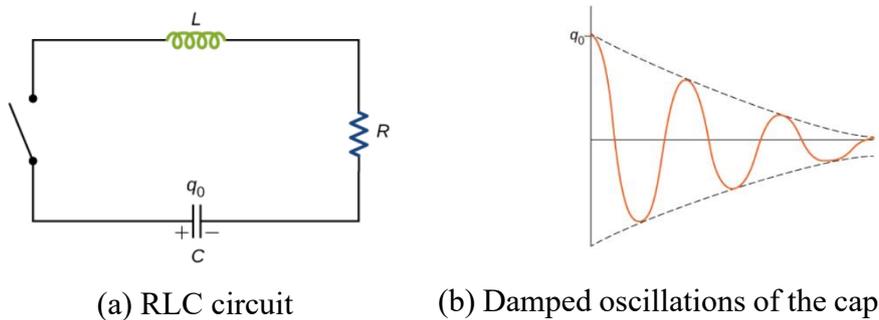


Fig. 2. RC circuits with a two-pole switch

This experiment first focuses on the charging and discharging of RC circuits to understand the behavior of RC circuits and then turns to a simple RLC circuit to learn the dynamics of a linearly-damped oscillator. Besides, you will learn how to simulate the dynamical systems by iterations (Euler’s method).

C. Theory

1. RC circuit

While charging a capacitor (Fig.1(b)), with Kirchhoff's loop rule, one has

$$\varepsilon - V_R - V_C = 0 \Rightarrow \varepsilon - R \frac{dq}{dt} - \frac{q}{C} = 0 \tag{1}$$

This differential equation can be integrated to find the function of time for the charging.

Therefore, the charge on the charging capacitor as a function of time is

$$q(t) = C\varepsilon(1 - e^{-t/RC}) \equiv q_{\max}(1 - e^{-t/\tau}) \tag{2}$$

where for $t \rightarrow \infty$, the charge approaches the maximum charge $q_{\max} = C\varepsilon$, and since the unit of RC is the second, unit of time, this quantity is known as the relaxation time constant $\tau = RC$.

As for discharging the capacitor (Fig.1(c)), with Kirchhoff's loop rule, one has

$$-V_R - V_C = 0 \Rightarrow R \frac{dq}{dt} = -\frac{q}{C} \tag{3}$$

Therefore, the discharging as a function of time is

$$q(t) = q_{\max} e^{-t/\tau} \tag{4}$$

Moreover, the time for $q = q_{\max}$ to $q = q_{\max}/2$ is called the half-life period $T_{1/2}$.

$$T_{1/2} = \tau \ln 2 \tag{5}$$

The charging function and discharging function are as shown in Fig. 3.

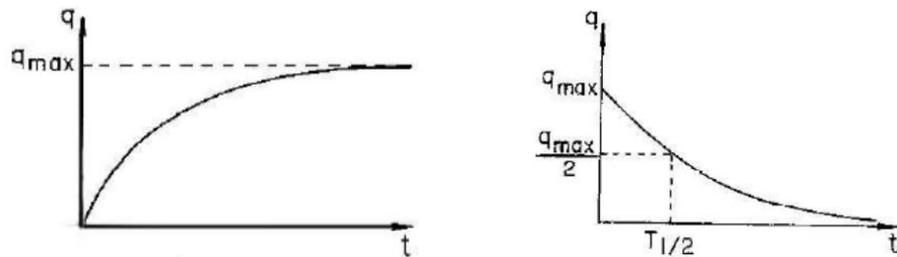


Fig. 3. Charging and discharging function of RC circuit

This lab uses a square wave from a function generator as the voltage source. As you can imagine, the first half period of the square provides the condition of charging while the second half is about the discharging.

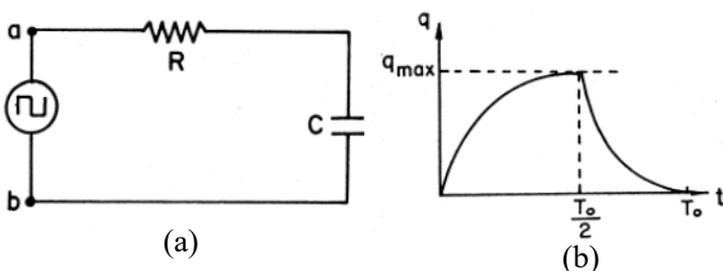


Fig. 4. (a) RC circuit with a square wave as the voltage source (b) charging and discharging behavior for the period T_0 of the square wave larger than 2τ .

2. RLC circuit

Consider a simple RLC circuit shown in Fig. 2. Again, with Kirchhoff’s loop rule, one has

$$L \frac{dI}{dt} + IR + \frac{q}{C} = 0 \Rightarrow L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C}q = 0 \tag{6}$$

Now recall the equation of motion for a damped mass-spring system.

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0 \tag{7}$$

As we discussed in Lab 9, the solution of this differential equation depends on whether the natural frequency $\omega_0 (= \sqrt{k/m})$ of the undamped mass-spring system is greater than, equal to, or less than the damping coefficient $\beta (= b/2m)$. Therefore, the result can be underdamped ($\omega_0 > \beta$), critically damped ($\omega_0 = \beta$), or overdamped ($\omega_0 < \beta$). Analogously, the solution $Q(t)$ to the RLC differential equation has the same feature. By replacing m by L , b by R , k by $1/C$, and x by q , the charge vs. time $q(t)$ for the underdamped condition is

$$q(t) = [q_0 e^{-\beta t}] \cos(\omega_1 t - \delta) \tag{8}$$

As you can see from eq(8), the amplitude is exponentially decaying with the oscillating frequency¹ ω_1 . That is, if we connect all the peaks (valleys) of the charge curve, the envelope will be formed and be decaying exponentially as shown in Fig. 5.

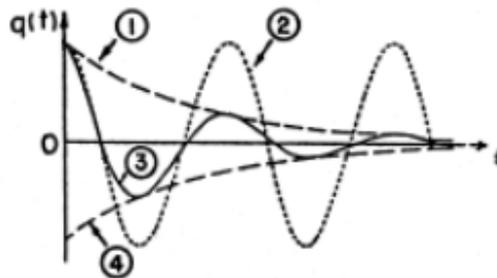


Fig. 5. Damped oscillation of charge in the RLC circuit with $\delta = 0$. ($\beta^2 < \omega_0^2$)

- ① Envelope of peaks ② $\beta = 0$ (without damping)
- ③ $\beta = 0.2\omega_0$ (underdamped) ④ Envelope of valleys

If we focus on the envelope, the time for $q = q_0$ to $q = q_0/2$ is, as we do in the RC circuit, called the half-life period $T_{1/2}$. Again, the half-life period is related to the time constant τ of the circuit.

$$T_{1/2} = \ln 2 / \beta = \tau \ln 2 \tag{9}$$

During the experiment, $T_{1/2}$ is measured to calculate the damping coefficient β .

¹ Although we call it the oscillating frequency, the oscillator (charge) does not go back to the initial state after a period.

If the damping coefficient is increased by modulating the total resistance, capacitance, of inductance of the circuit, the RLC circuit will become critically-damped when $\beta^2 = \omega_0^2$ (or $\omega_1 = 0$), and overdamped when $\beta^2 > \omega_0^2$ (or ω_1 becomes imaginary). For these two cases, charges will not oscillate but gradually decrease to zero as Fig. 6 shows.

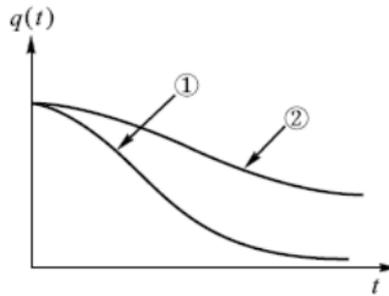


Fig. 6. Damped oscillation of charge in the RLC circuit.
 ① Critical damping ($\beta^2 = \omega_0^2$) ② Overdamping ($\beta^2 > \omega_0^2$)

If the RLC circuit is connected to a DC source ε_0 , one has

$$L \frac{dI}{dt} + \frac{q}{C} + IR = \varepsilon_0 \tag{10}$$

The charge oscillation can be solved to be

$$q(t) = \frac{\varepsilon_0/L}{\omega_0^2} [1 - Ae^{-\beta t} \cos(\omega_1 t - \delta)] \xrightarrow{q(0)=\delta=0} \frac{a}{\omega_0^2} [1 - e^{-\beta t} \cos(\omega_1 t)] \xrightarrow{t \rightarrow \infty} \frac{a}{\omega_0^2} \tag{11}$$

Fig. 7 shows the charge oscillation in RLC circuit of the DC source with different β . In the experiment, the square wave from the function generator provides the charging (first-half period) and discharging (second-half period) condition of the capacitor as Fig. 8 shows.

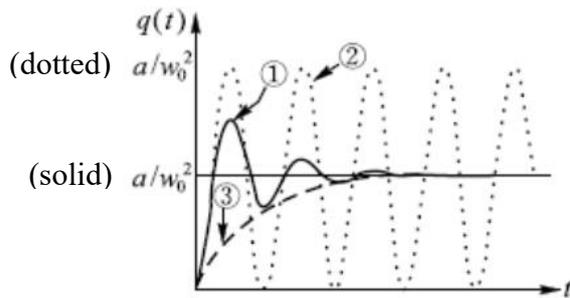


Fig. 7. Charge oscillation of RLC circuit with the DC power source
 ① $\beta = 0.2\omega_0$ ② $\beta = 0$ ③ $\beta = \omega_0$

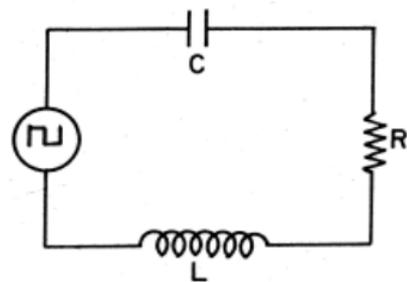


Fig. 8. Circuit diagram of RLC circuit with square wave source

If the RLC circuit is connected to an AC source $\varepsilon_0 \sin \omega t$, one has

$$L \frac{dI}{dt} + \frac{q}{C} + IR = \varepsilon_0 \sin \omega t \tag{12}$$

The solution to this differential equation has two parts: the transient part and the steady part.

Here we only focus on the steady state of the charge, and the solution is

$$q(t) = \frac{\varepsilon_0/L}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \sin(\omega t - \delta) = q_s \sin(\omega t - \delta) \quad (13)$$

$$\Rightarrow I(t) = q_s \omega \cos(\omega t - \delta) = \frac{\varepsilon_0}{\sqrt{R^2 + (1/\omega C - \omega L)^2}} \sin(\omega t - \phi) \quad (14)$$

where

$$\delta = \tan^{-1} \frac{2\omega\beta}{\omega_0^2 - \omega^2} = \tan^{-1} \frac{R}{(1/\omega C) - \omega L} \quad \text{and} \quad \phi = \delta + \frac{\pi}{2} = \tan^{-1} \frac{\omega L - 1/\omega C}{R} \quad (15)$$

The charge oscillates with the same frequency of the driving force as you can see from eq(13). Moreover, the amplitude of the oscillation and the phase difference between the charge and the voltage is also related to the driving frequency as indicated by eq (15), where ϕ is the phase difference between the current and the voltage. Fig. 9 shows the results of charge oscillation in the series RLC circuit. As indicated by Fig. 9(b), there exists a peak in amplitude, and the frequency for the maximum amplitude is called the resonant frequency ω_R .

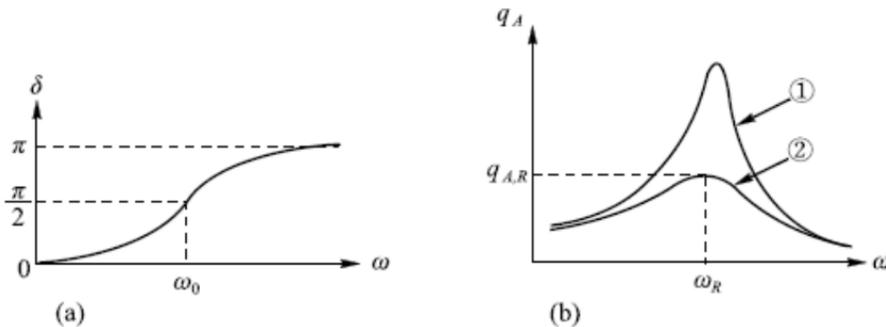


Fig. 9. Relationship between the driving frequency and (a) the phase and (b) amplitude, where $\beta_1 < \beta_2$

For the alternating current (AC) circuit, it's more convenient to define the reactances for resistor, inductor, and capacitor. Consider a resistor in an AC circuit $\varepsilon = \varepsilon_0 \sin \omega t$ with current $i = i_0 \sin \omega t$ as Fig. 10 (a) shows. The instantaneous potential difference across the resistor is

$$v_R = i_R R = i_0 R \sin \omega t \equiv V_R \sin \omega t \quad (16)$$

The current and the potential difference are *in phase* as Fig. 11(a) shows. Replace the resistor by a inductor as Fig. 10(b) shows. The potential difference across the inductor is

$$v_L = L \frac{di}{dt} = L i_0 \omega \cos \omega t \equiv i_0 X_L \sin\left(\omega t + \frac{\pi}{2}\right) = V_L \sin\left(\omega t + \frac{\pi}{2}\right) \quad (17)$$

where X_L is called the reactance of an inductor. For this case, the potential difference leads the current by 90° as shown in Fig. 11(b). Finally, for the capacitor case as Fig. 10(c) shows, the potential difference across the capacitor is

$$v_C = \frac{q}{C} = -\frac{i_0}{\omega C} \cos \omega t \equiv i_0 X_C \sin\left(\omega t - \frac{\pi}{2}\right) = V_C \sin\left(\omega t - \frac{\pi}{2}\right) \quad (18)$$

where X_C is called the reactance of a capacitor. For this case, the potential difference lags the current by 90° as shown in Fig. 11(c).

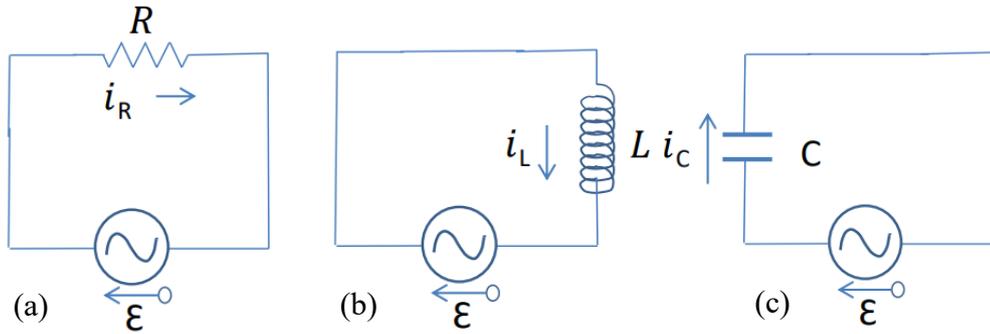


Fig. 10. Three simple AC circuits. (a) resistive load (b) inductive load (c) capacitive load

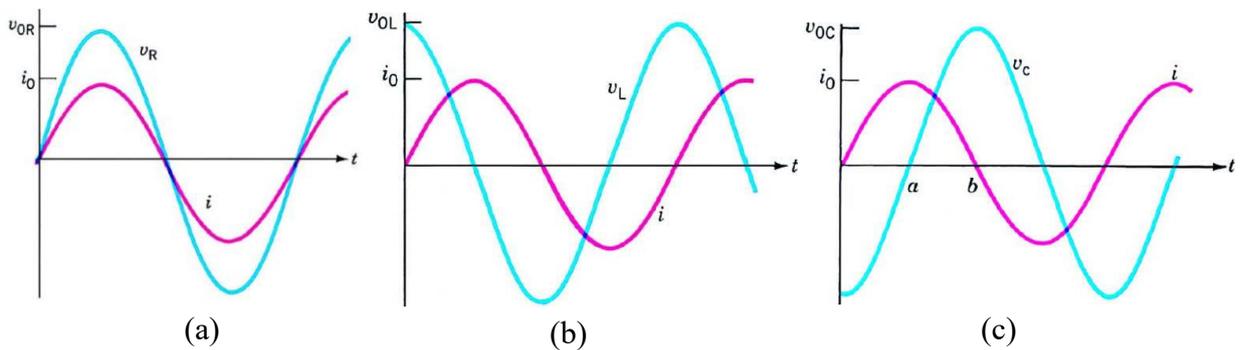


Fig. 11. Relationship between the current and (a) resistor (b) inductor (c) capacitor

Let's go back to the series RLC circuit. If we put the voltage differences across R, L, and C in a phasor diagram as phasors as Fig. 12 shows, then since the phasors V_R, V_L, V_C are *in phase*, 90° *ahead of*, and 90° *behind* the phasor of the current, respectively, the relation between the voltage and the current is

$$V_0 = i_0 \sqrt{R^2 + (X_L - X_C)^2} \equiv i_0 Z \quad (19)$$

where Z is the impedance of the circuit.

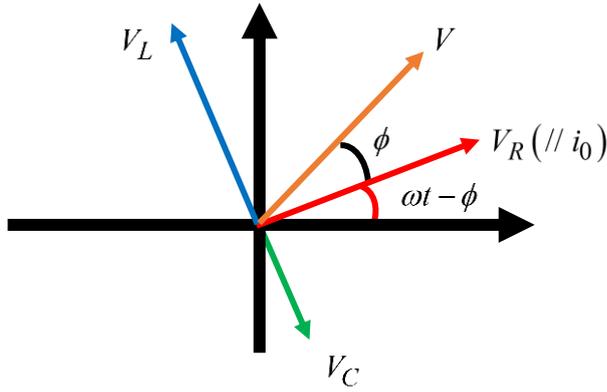


Fig. 12. Phasor diagram of the series RLC circuit. V_R, V_L, V_C are in phase with, 90° ahead of, and 90° behind the phasor of the current, respectively.

One should try to relate the phasor diagram to eq(14), which is obtained by directly solving the differential equation.

D. Procedures

1. Pre-lab assignments (hand in before the experiment)
 - (1) Make a flowchart of this experiment and answer the questions below.
 - (2) Prove eq (14) by the phasor diagram given in Fig. 12.
 - (3) Prove that the resonant frequency ω_R of the series RLC circuit with AC source is

$$\omega_R = \sqrt{\omega_0^2 - 2\beta^2} \tag{20}$$

Also prove that when the damping is small ($R \ll \sqrt{L/C}$), the amplitude A_{\max} is

$$A_{\max} = \frac{\varepsilon_0}{\omega_0 R} \tag{21}$$

which indicates the relationship between the resistance and the max amplitude.

- (4) Following (3), in physics, the quality factor (Q-factor)² is a dimensionless parameter defined as

$$Q \equiv \frac{\omega_R}{2\beta} \tag{22}$$

The bandwidth (BW) $\Delta\omega$ of a resonant system is defined as the total number of cycles below and above the resonant frequency ω_R for which *the amplitude is equal to or greater than $A_{\max}/\sqrt{2}$* . The two frequencies ω_{\pm} in the curve that are at $A_{\max}/\sqrt{2}$ are called half-power frequencies or **cutoff frequencies**. With eq (13) and eq (21), prove that for $\beta^2 \ll \omega_0^2$, the Q-factor and the BW have the relation

$$Q \approx \frac{\omega_R}{\Delta\omega} = \frac{\omega_R}{\omega_+ - \omega_-} \tag{23}$$

- (5) Consider a spring-mass system with the spring constant $k = 1$ N/m and the mass $m = 1$ kg. Given the initial condition of the mass is $x_0 = 1$ m and $v_0 = 0$ m/s. After the release, the mass is under a damping force $f = -\beta v$ (N).

² High Q factor corresponds to the less energy loss of an oscillator or a resonator.

- (i) Simulate the system with $\beta = 0.5, 1.0, 1.5, 2.0, 2.5$ kg/s. Put the results in one graph with a legend that specifies the curves.
- (ii) Following (i), if the system is under a driving force $F = \sin t$ (N), do the simulation again. Observe the result with different time scales. You will see the transient state and the steady state of the mass.

2. In-lab activities

For each part, take pictures as the proof of your work. Also note that the output impedance of the function generator is 50Ω , and the inductors of 10 mH and 22 mH have resistances about 30Ω and 50Ω , respectively.

(1) Charging and discharging behavior of the capacitor (an RC circuit)

- (i) Connect the $10 \text{ k}\Omega$ resistor, the $0.1 \mu\text{F}$ capacitor, the function generator, and the scope as Fig. 13 shows. Set the output voltage source to be a square wave with $\text{Ampl} = 5 \text{ V}_{PP}$, and the period of the square wave to be $T_0 = 10RC$.
- (ii) Use BNC T adapter so that the output of the function generator can be the source of the circuit and simultaneously be the signal to CH1 of the scope. Use CH2 of the scope to observe the voltage signal across the capacitor.
- (iii) Press "MATH" to subtract the two signals in CH1 and CH2. Observe the signal of the square function from the function generator and the signals across both the capacitor and the resistor at the same time.
- (iv) Measure the half-life periods for the circuits and use the results to determine the relaxation time constants.
- (v) Compare the results with the product of the given R and C .

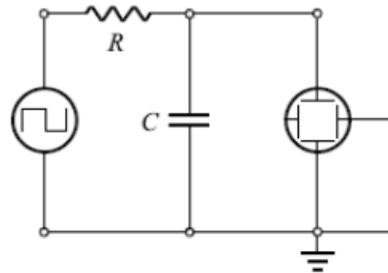


Fig. 13. Electrical circuit of the observation of charging and discharging the RC circuit

- (6) Change the period of the square wave to $T_0 = RC$. Observe the difference.
- (2) **(Optional)** Frequency response of an RC circuit (an RC low passfilter)
- (i) With the same setup in the previous experiment (1), measure the peak-to-peak difference V_{diff} of the voltage signal across the capacitor under the conditions that $T_0/RC = 0.1, 0.2, 0.3, \dots, 0.9, 1.0, 2.0, \dots, 9.0, 10.0$.
 - (ii) Draw a graph of the gain versus the period (in log scale) with the gain defined as

$$\text{Gain (in dB)} = -20 \cdot \log(V_{diff} / \varepsilon_0).$$

This graph is known as the gain plot of the Bode diagram, usually used to show the pass-band of a filter, indicating the measure of how much the initial input signal is allowed to pass through the circuit. Conventionally, the $-3dB$ point is defined as the cutoff, corresponding to the output amplitude of $A_{max}/\sqrt{2}$.

- (iii) Change to the sine wave input, do the steps (i) and (ii) again, and compare the results. For those who want to explore the result theoretically, see Q4 below.
 - (iv) How about the signals across the resistor? (It'll become a high passfilter.)
- (3) Damped oscillation of the series RLC circuit
- (i) Connect a 22 mH inductor, a $0.001 \mu F$ capacitor, and a 100Ω resistor in series as shown in Fig. 14. Use the oscilloscope to observe the voltage difference across the capacitor with the period of output square wave from the function generator being $100/\beta$. Measure $T_{1/2}$ and ω_1 by the cursors for the underdamped case and compare the results with the theoretical value.
 - (ii) Replace the resistor by a $30 k\Omega$ variable resistor. Slowly increase the resistance until you find the critical damping. Measure the total resistance of the circuit and compare with the theoretical value. **(What is the theoretical value?)**

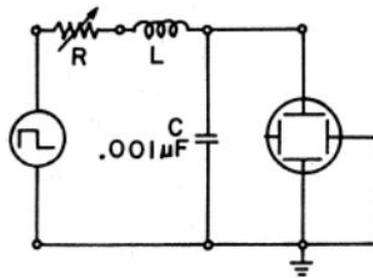


Fig. 14. Electrical circuit of the observation of charge oscillation in RLC circuit

- (4) RLC resonance (voltage difference across the capacitor versus the frequency)
- Following exp 3.(i), change the output of the function generator to the sine wave of $V_{PP} = 1 V$. Replace the inductor and the capacitor by the 10 mH and $0.001 \mu F$ one.
- (i) Set the driving frequency to be 100, 300, 1k, 3k, 10k, 30k, 100k, and 300 kHz. Record the amplitudes and the phase differences.
 - (ii) Find the resonant frequency that makes the maximum amplitude. Record the amplitude A_{max} and the phase difference δ .
 - (iii) Centered at the resonant frequency, increase and decrease the driving frequency by 500 Hz for at least 20 times. Record the amplitude and the phase difference for each trial. Also, find and record the amplitude as well as the phase difference for the amplitude being $0.2A_{max}$, $0.4A_{max}$, $0.6A_{max}$, $0.8A_{max}$
 - (iv) Find the two frequencies that make the amplitude $A_{max}/\sqrt{2}$. Record the amplitude and the phase difference. Calculate the quality factor of this RLC

circuit. Compare the result with the theory. (This can be known as the bandpass filter, compared with an RC passfilter.)

- (v) Collect the data from all the trials above. Draw the semi-log graphs of the amplitude versus the driving frequency and phase difference versus the driving frequency. The driving frequency should be in the log scale.

(5) Matlab Simulation

Simulate the electrical circuits in each experiment.

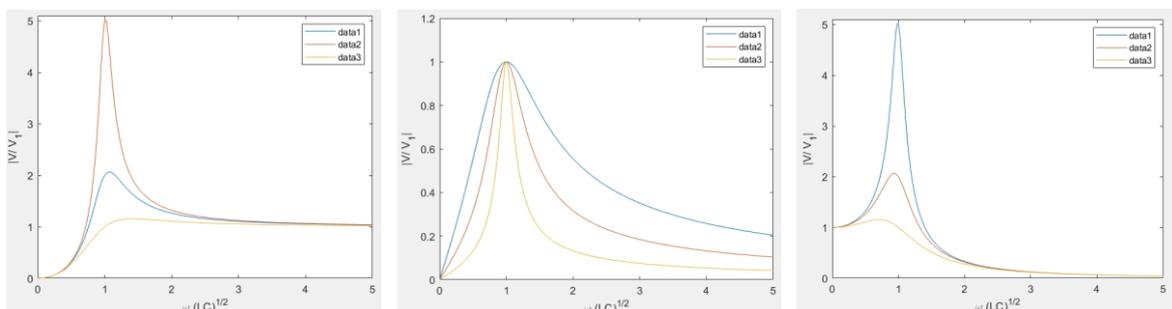
(Optional) Compare the simulation graphs of amp vs. freq and phase diff vs. freq with experimental results by putting the data in the Matlab simulation graph.

3. Post-lab report

- (1) Recopy and organize your data from the in-lab tables in a neat and more readable form.
- (2) Analyze the data you obtained in the lab and answer the given questions

E. Questions

1. Suppose the output function in Fig. 4 is a square wave whose amplitude is V_0 in the first half period and 0 in the second half. Find the theoretical expression of the potential drop across the resistor. Graph the potential drop versus time via Matlab.
2. For the series RLC circuit with AC source, the potential difference across the capacitor is said to lag the current by a phase. Show it by your experimental results and explain it by a physical picture without mathematics.
3. In the experiment, we study the frequency response of the voltage difference across the capacitor. Similarly, we can see the frequency response of the voltage difference across the resistor or the inductor. Suppose the input voltage is a sinusoid with phasor V_1 .
 - (1) Find the units of resistance, inductance, and capacitance in SI units. Prove that $R\sqrt{C/L}$ and $\omega\sqrt{LC}$ are dimensionless.
 - (2) Suppose you are given three different apparatus with $R\sqrt{C/L} = 0.2, 0.5, 1$. The simulated frequency response across the capacitor, the inductor, or the resistor are shown below.
 - (i) Which graph is for capacitor? for inductor? and for resistor? Explain.
 - (ii) Find the corresponding $R\sqrt{C/L}$ value of each data in each graph. Explain.



4. **(Optional)** Suppose the output function in Fig. 3 (a) is a square wave whose amplitude is

ε_0 in the first half period and 0 in the second half. If the period T_0 of the output function is under the condition $T_0 \ll RC$, before the steady state, the charging and discharging are in the transient state. Given $Q_{P,N}$ being the N-th peak value of the charging and $Q_{V,N}$ being the N-th valley value of the discharging,

(1) Derive that

$$\begin{pmatrix} Q_{P,N} \\ Q_{V,N} \end{pmatrix} = C\varepsilon_0 \frac{1 - e^{-NT_0/RC}}{1 + e^{-T_0/2RC}} \begin{pmatrix} 1 \\ e^{-T_0/2RC} \end{pmatrix}.$$

Also, from the result, prove that the average voltage of the steady state is always $C\varepsilon_0/2$.

(2) Simulate the RC circuit with this transient effect.

(3) Compare the theoretical gain plot of the Bode diagram with the measurement.

F. References

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Appendix: Matlab codes --- Euler Method

Consider a spring-block system with the equation of motion

$$m \frac{d^2x}{dt^2} = -kx$$

Since it is a 2nd order differential equation, to simulate this by the iterations, we first rewrite it into two 1st order ordinary differential equations

$$\begin{cases} \frac{dx}{dt} = v \\ \frac{dv}{dt} = -\frac{k}{m}x \end{cases} \Rightarrow \begin{cases} x(t + \Delta t) = x(t) + \dot{x}(t)\Delta t + \frac{1}{2}\ddot{x}(t)\Delta t^2 + \frac{1}{6}\dddot{x}(t)\Delta t^3 + \dots \\ v(t + \Delta t) = v(t) + \dot{v}(t)\Delta t + \frac{1}{2}\ddot{v}(t)\Delta t^2 + \frac{1}{6}\dddot{v}(t)\Delta t^3 + \dots \end{cases}$$

With the given initial conditions, the system can evolve using Euler method (keep only linear term)

$$\begin{array}{ccc} t = \Delta t & t = 2\Delta t & t = 3\Delta t \\ x_1 = x_0 + v_0\Delta t & x_2 = x_1 + v_1\Delta t & \dots \\ v_1 = v_0 + \left(-\frac{k}{m}x_0\right)\Delta t & v_2 = v_1 + \left(-\frac{k}{m}x_1\right)\Delta t & \dots \end{array}$$

From this, we can use for loop in Matlab to simulate this simple harmonic oscillation.

```
clear
clf % clear all the figures

dt = 0.0001; % Timestep
N = round(50/dt); % Number of iterations

% System parameters
k = 1; m = 1;

% Initial condition
t = 0; x = 1; % Maximum of position (m)
v = 0; % Velocity (m/s)

%arrays for t,x,v
t_arr = []; x_arr = []; v_arr = [];

% euler method (iteration)
for i = 1:N
    t = t + dt;
    x_new = x + v*dt;
    v_new = v + (-k/m*x)*dt;
    x = x_new; v = v_new;
    t_arr(end+1) = t;
    x_arr(end+1) = x;
    v_arr(end+1) = v;
end

%The followings are for the graph.
figure(1)
plot(t_arr, x_arr, '-.r', 'DisplayName', 'block system without damping')
hold on
figure(2)
plot(x_arr, v_arr, '-.r','DisplayName', 'block system without damping')
hold on

figure(1);
title('Simple Harmonic Oscillation');
xlabel('Time(s)'); ylabel('Position(m)');
figure(2);
title('Phase diagram');
xlabel('Position(m)'); ylabel('Velocity(m/s)');
hold off
figure(1), legend;
figure(2), legend;
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