Inflationary cosmology

References:

**[1]Kazuharu Bamba and Sergei D. Odintsov,
JCAP 04 (2008) 024, e-print arXiv：0801.0954 [astro-ph]**

**[2]Kazuharu Bamba, Shin'ichi Nojiri and Sergei D. Odintsov,
e-print arXiv：0803.3384 [hep-th]**

Particle Seminar at NTHU on 15th May, 2008

Presenter : **Kazuharu Bamba** (***National Tsing Hua University***)

Collaborators : **Shin'ichi Nojiri** (***Nagoya University***)

Sergei D. Odintsov (***ICREA and IEEC-CSIC***)

< Contents >

- Inflation and late-time cosmic acceleration in non-minimal Maxwell theory in general relativity and modified gravity
- Generation of large-scale magnetic fields

I. Introduction

- It is observationally confirmed not only that inflation occurred in the early universe, but that the current expansion of the universe is accelerating.

[Spergel et al., Astrophys. J. Suppl. 148, 175 (2003)]

< Scenarios to explain the late-time cosmic acceleration >

(1) General relativistic approaches: Dark energy

(2) Modified gravity approaches: Dark gravity

→ Modifications to the Einstein-Hilbert action:
Addition of an arbitrary function of the scalar curvature to it.

⇒ ***F*(*R*)-gravity**

R : Ricci scalar
$F(R)$: Arbitrary function of R

[Nojiri and Odintsov, Int. J. Geom. Meth. Mod. Phys. 4, 115 (2007)]

- Hu and Sawicki have proposed a very realistic modified gravitational theory that evade solar-system tests.

[Hu and Sawicki, Phys. Rev. D 76, 064004 (2007)]

→ Although this theory is successful in explaining the late-time acceleration of the universe, the possibility of the realization of inflation has not been discussed.

< Proposals in which both inflation and the late-time acceleration can be realized >

(1) Modified gravities **[Nojiri and Odintsov, Phys. Lett. B 657, 238 (2007)]**

(2) Coupling between the scalar curvature and matter Lagrangian

[Nojiri and Odintsov, Phys. Lett. B 599, 137 (2004)]

→ It is known that the coupling between the scalar curvature and the Lagrangian of the electromagnetic field arises in curved spacetime due to one-loop vacuum-polarization effects in Quantum Electrodynamics (QED).

[Drummond and Hathrell, Phys. Rev. D 22, 343 (1980)]

- Such a non-minimal gravitational coupling of the electromagnetic field breaks the conformal invariance of the electromagnetic field.

⇨ Electromagnetic quantum fluctuations can be generated at the inflationary stage even in the Friedmann-Robertson-Walker (FRW) spacetime, which is conformally flat.

[Turner and Widrow, Phys. Rev. D 37, 2743 (1988)]

→ They can appear as large-scale magnetic fields at the present time because their scale is made longer than Hubble horizon due to inflation.

< Cosmic magnetic fields >

(1) Galactic magnetic fields

$$B_{\mathrm{gal}} \sim \mu \mathrm{G}$$

[Sofue et al., Annu. Rev. Astron. Astrophys. 24, 459 (1986)]

(2) Magnetic fields in clusters of galaxies

[Clarke et al., Astrophys. J. 547, L111 (2001)]

$$B_{\mathrm{ICM}} : 0.1 - 10 \mu\mathrm{G}, \quad L : 10\,\mathrm{kpc} - 1\mathrm{Mpc}$$

< Inflationary cosmology >

- **In the early universe, the scale of the universe grew exponentially in time when the potential energy of a scalar field, called an "inflaton", dominated.**

⟶ **Inflation accounts for the observed degree of homogeneity, isotropy, and flatness of the present universe.**

⟶ **Inflation naturally produces effects on very large scales, larger than Hubble horizon, starting from microphysical processes operating on a causally connected volume.**

⇩

The most natural origin of large-scale magnetic fields:

Electromagnetic quantum fluctuations generated at the inflationary stage

→ We consider inflation and the late-time acceleration of the universe in non-minimal electromagnetism, in which the electromagnetic field couples to a function of the scalar curvature.

⇩

(1) We show that power-law inflation can be realized due to the non-minimal gravitational coupling of the electromagnetic field.

(2) We show that large-scale magnetic fields can be generated due to the breaking of the conformal invariance of the electromagnetic field through its non-minimal gravitational coupling.

(3) We demonstrate that both inflation and the late-time acceleration of the universe can be realized in a modified Maxwell-*F*(*R*) gravity.

(4) We also consider classically equivalent form of non-minimal Maxwell-*F*(*R*) gravity.

II. Inflation in general relativity

< II A. Model >

< Action >

$g = \det(g_{\mu\nu})$

$$S_{\rm GR} = \int d^4x \sqrt{-g} \left[\mathcal{L}_{\rm EH} + \mathcal{L}_{\rm EM} \right]$$

$$\mathcal{L}_{\rm EH} = \frac{1}{2\kappa^2} R$$

$$\mathcal{L}_{\rm EM} = -\frac{1}{4} I(R) F_{\mu\nu} F^{\mu\nu}$$

← **Breaking of the conformal invariance**

$$I(R) = 1 + f(R)$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$: Electromagnetic field-strength tensor

A_μ : $U(1)$ gauge field

$f(R)$: Arbitrary function of R

$\kappa^2 \equiv 8\pi / {M_{\rm Pl}}^2$, $M_{\rm Pl}$: Planck mass

$\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$: Covariant d'Alembertian

∇_μ : Covariant derivative operator

$R_{\mu\nu}$: Ricci curvature tensor

$T^{\rm (EM)}_{\mu\nu}$: Energy-momentum tensor of the electromagnetic field

< Gravitational field equation >

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T^{\rm (EM)}_{\mu\nu}$$

$$T^{\rm (EM)}_{\mu\nu} = I(R) \left(g^{\alpha\beta} F_{\mu\beta} F_{\nu\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) + \frac{1}{2} \left\{ f'(R) F_{\alpha\beta} F^{\alpha\beta} R_{\mu\nu} + g_{\mu\nu} \Box \left[f'(R) F_{\alpha\beta} F^{\alpha\beta} \right] - \nabla_\mu \nabla_\nu \left[f'(R) F_{\alpha\beta} F^{\alpha\beta} \right] \right\}$$

< Electromagnetic field equation >

$$-\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}I(R)F^{\mu\nu}\right)=0$$

< Spatially flat FRW space-time >

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=-dt^2+a^2(t)d\boldsymbol{x}^2 \;=a^2(\eta)(-d\eta^2+d\boldsymbol{x}^2)$$

$a(t)$: Scale factor, η : Conformal time

$$\longrightarrow \quad g_{\mu\nu} \;=\; \mathrm{diag}\left(-1,a^2(t),a^2(t),a^2(t)\right)$$

$$R_{00}=-3\left(\dot{H}+H^2\right),\; R_{0i}=0,\; R_{ij}=\left(\dot{H}+3H^2\right)g_{ij},\; R=6\left(\dot{H}+2H^2\right)$$

$H=\dot{a}/a$: Hubble parameter, $\dot{}=\partial/\partial t$

< Equations of motion of *U*(1) gauge field $A_\mu(t,\boldsymbol{x})$ >

Coulomb gauge: $\partial^j A_j(t,\boldsymbol{x})=0$ and the case of $A_0(t,\boldsymbol{x})=0$

$$\longrightarrow \quad \ddot{A}_i(t,\boldsymbol{x})+\left(H+\frac{\dot{I}}{I}\right)\dot{A}_i(t,\boldsymbol{x})-\frac{1}{a^2}\overset{(3)}{\Delta}A_i(t,\boldsymbol{x})=0$$

$\overset{(3)}{\Delta}=\partial^i\partial_i$: Flat three dimensional Laplacian

< II B. Evolution of large-scale electric and magnetic fields >

< Quantization of $A_\mu(t, \boldsymbol{x})$ >

- **Canonical momenta:** $\pi_0 = 0\,, \qquad \pi_i = I a(t) \dot{A}_i(t, \boldsymbol{x})$

- **Canonical commutation relation:**

$$\left[A_i(t, \boldsymbol{x}), \pi_j(t, \boldsymbol{y}) \right] = i \int \frac{d^3 k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right)$$

< Expression for $A_i(t, \boldsymbol{x})$ >

$\boldsymbol{k}$: Comoving wave number, $\quad k = |\boldsymbol{k}|$

$$A_i(t, \boldsymbol{x}) = \int \frac{d^3 k}{(2\pi)^{3/2}} \sum_{\sigma=1,2} \left[\hat{b}(\boldsymbol{k}, \sigma) \epsilon_i(\boldsymbol{k}, \sigma) A(t, k) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{b}^\dagger(\boldsymbol{k}, \sigma) \epsilon_i^*(\boldsymbol{k}, \sigma) A^*(t, k) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \right]$$

$$\left[\hat{b}(\boldsymbol{k}, \sigma), \hat{b}^\dagger\left(\tilde{\boldsymbol{k}}, \tilde{\sigma}\right) \right] = \delta_{\sigma,\tilde{\sigma}} \delta^3\left(\boldsymbol{k} - \tilde{\boldsymbol{k}}\right), \ \left[\hat{b}(\boldsymbol{k}, \sigma), \hat{b}\left(\tilde{\boldsymbol{k}}, \tilde{\sigma}\right) \right] = \left[\hat{b}^\dagger(\boldsymbol{k}, \sigma), \hat{b}^\dagger\left(\tilde{\boldsymbol{k}}, \tilde{\sigma}\right) \right] = 0$$

$\epsilon_i(\boldsymbol{k}, \sigma) \ (\sigma = 1, 2)$: Polarization vector

$\hat{b}(\boldsymbol{k}, \sigma)$: Annihilation operator

$\hat{b}^\dagger(\boldsymbol{k}, \sigma)$: Creation operator

< Equation for the mode function $A(k,t)$ >

$$\ddot{A}(k,t) + \left(H + \frac{\dot{I}}{I}\right)\dot{A}(k,t) + \frac{k^2}{a^2}A(k,t) = 0$$

▪ **Normalization condition:** $A(k,t)\dot{A}^*(k,t) - \dot{A}(k,t)A^*(k,t) = \dfrac{i}{Ia}$

$\longrightarrow$ $t \longrightarrow \eta$

$$\frac{\partial^2 A(k,\eta)}{\partial \eta^2} + \frac{1}{I(\eta)}\frac{dI(\eta)}{d\eta}\frac{\partial A(k,\eta)}{\partial \eta} + k^2 A(k,\eta) = 0$$

- Although it is impossible to obtain the exact solution of the above equation for the case when I is given by a general function of η, we can obtain an approximate solution with sufficient accuracy by using the WKB approximation on subhorizon scales ($k|\eta| \gg 1$) and the long-wavelength approximation on superhorizon scales ($k|\eta| \ll 1$), and matching these solutions at the horizon crossing ($-k\eta = 1$).

[KB and Sasaki, JCAP <u>0702</u>, 030 (2007)]

< Solution for $A(k,\eta)$ >

1. WKB subhorizon solution (Subhorozon scale: $k|\eta| \gg 1$)

$$A_{\rm in}(k,\eta) = \frac{1}{\sqrt{2k}} I^{-1/2} e^{-ik\eta}$$

← We have assumed that the vacuum in the short-wavelength limit is the standard Minkowski vacuum.

2. Solution on superhorizon scales (Superhorozon scale: $k|\eta| \ll 1$)

- **Long-wavelength expansion:** $A_{\rm out} = A_0(\eta) + k^2 A_1(\eta) + O(k^4)$

→ By matching this solution with the WKB subhorizon solution at the horizon crossing, we find

η_k : Conformal time at the horizon-crossing

$\eta_{\rm f}$: Conformal time at the end of inflation

< Lowest order solution for $A_{\rm out}$ >

$$A_{\rm out}(k,\eta) = C(k) + D(k) \int_{\eta}^{\eta_{\rm f}} \frac{1}{I(\tilde{\eta})} d\tilde{\eta}$$

(Decaying mode)

$$C(k) = \frac{1}{\sqrt{2k}} I^{-1/2}(\eta) \left[1 - \left(\frac{1}{2} \frac{dI(\eta)}{d\eta} + ikI(\eta) \right) \int_{\eta}^{\eta_{\rm f}} \frac{1}{I(\tilde{\tilde{\eta}})} d\tilde{\tilde{\eta}} \right] e^{-ik\eta} \Bigg|_{\eta=\eta_k}$$

$$D(k) = \frac{1}{\sqrt{2k}} I^{-1/2}(\eta) \left(\frac{1}{2} \frac{dI(\eta)}{d\eta} + ikI(\eta) \right) e^{-ik\eta} \Bigg|_{\eta=\eta_k}$$

$\Longrightarrow$ (We have neglected the decaying mode.)

$$|A(k,\eta)|^2 = |C(k)|^2 = \frac{1}{2kI(\eta_k)} \left| 1 - \left[\frac{1}{2} \frac{1}{kI(\eta_k)} \frac{dI(\eta_k)}{d\eta} + i \right] e^{-ik\eta_k} k \int_{\eta_k}^{\eta_f} \frac{I(\eta_k)}{I(\tilde{\tilde{\eta}})} d\tilde{\tilde{\eta}} \right|^2$$

< Proper electric and magnetic fields >

$$E_i^{\text{proper}}(t,\boldsymbol{x}) = a^{-1} E_i(t,\boldsymbol{x}) = -a^{-1} \dot{A}_i(t,\boldsymbol{x})$$

$$B_i^{\text{proper}}(t,\boldsymbol{x}) = a^{-1} B_i(t,\boldsymbol{x}) = a^{-2} \epsilon_{ijk} \partial_j A_k(t,\boldsymbol{x})$$

$E_i(t,\boldsymbol{x})$: Comoving electric field

$B_i(t,\boldsymbol{x})$: Comoving magnetic field

ϵ_{ijk} : Totally antisymmetric tensor

$\Longrightarrow$ $$|E^{\text{proper}}(k,\eta)|^2 = 2\frac{1}{a^4}\left|\frac{\partial A(k,\eta)}{\partial \eta}\right|^2 = 2\frac{1}{a^4}\frac{|D(k)|^2}{|I(\eta)|^2}$$

$\longrightarrow$ $$|E^{\text{proper}}(L,\eta)|^2 = \frac{4\pi k^3}{(2\pi)^3} |E^{\text{proper}}(k,\eta)|^2$$

$$|B^{\text{proper}}(k,\eta)|^2 = 2\frac{k^2}{a^4}|A(k,\eta)|^2 = 2\frac{k^2}{a^4}|C(k)|^2$$

$\longrightarrow$ $$|B^{\text{proper}}(L,\eta)|^2 = \frac{4\pi k^3}{(2\pi)^3} |B^{\text{proper}}(k,\eta)|^2$$

< Energy density of the electric and magnetic fields >

$$\rho_E(L,\eta) = \frac{1}{2}|E^{\text{proper}}(L,\eta)|^2 I(\eta) = \frac{|D(k)|^2}{2\pi^2 k}\frac{k^4}{a^4}\frac{1}{I(\eta)} \quad \underline{\propto 1/I(\eta)}$$

$$\rho_B(L,\eta) = \frac{1}{2}|B^{\text{proper}}(L,\eta)|^2 I(\eta) = \frac{k|C(k)|^2}{2\pi^2}\frac{k^4}{a^4} I(\eta) \quad \underline{\propto I(\eta)}$$

$L = 2\pi/k$: Comoving scale

⟶ **If I increases in time, magnetic fields becomes dominant.**

[KB, JCAP 0710, 015 (2007)]

< Example >

We consider the case of a specific form for the function I .

$$I(\eta) = I_s \left(\frac{\eta}{\eta_s}\right)^{-\alpha}$$

η_S : Some fiducial time during inflation

α : Constant, I_S : Value of I at $\eta = \eta_S$

⟶ $k|C(k)|^2 = \mathcal{C}/\left[2I(\eta_k)\right]$

$\mathcal{C}$: Constant of order unity

⇨ $$\rho_B(L,\eta) = \frac{\mathcal{C}}{(2\pi)^2}\left(\frac{k}{a}\right)^4 \frac{I(\eta)}{I(\eta_k)}$$

< II C. Power-law inflation >

- $(\mu, \nu) = (0,0)$ component of the gravitational field equation:

$$H^2 = \frac{\kappa^2}{3}\left\{ I(R)\left(g^{\alpha\beta}F_{0\beta}F_{0\alpha} - \frac{1}{4}g_{00}F_{\alpha\beta}F^{\alpha\beta} \right) \right.$$
$$+ \frac{3}{2}\left[-f'(R)\left(\dot{H} + H^2\right) + 6f''(R)H\left(\ddot{H} + 4H\dot{H}\right) \right] F_{\alpha\beta}F^{\alpha\beta}$$
$$\left. + \frac{3}{2}f'(R)H\left(F_{\alpha\beta}F^{\alpha\beta}\right)^{\bullet} - \frac{1}{2}f'(R)\frac{1}{a^2}\overset{(3)}{\Delta}\left(F_{\alpha\beta}F^{\alpha\beta}\right) \right\}$$

- Trace part of $(\mu, \nu) = (i, j)$ component: $\left(F_{\alpha\beta}F^{\alpha\beta}\right)^{\bullet} = \partial\left(F_{\alpha\beta}F^{\alpha\beta}\right)/\partial t$

$$2\dot{H} + 3H^2 = \frac{\kappa^2}{2}\left\{ \frac{1}{6}I(R)F_{\alpha\beta}F^{\alpha\beta} + \left[-f'(R)\left(\dot{H} + 3H^2\right) \right.\right.$$
$$\left. + 6f''(R)\left(\dddot{H} + 7H\ddot{H} + 4\dot{H}^2 + 12H^2\dot{H}\right) + 36f'''(R)\left(\ddot{H} + 4H\dot{H}\right)^2 \right] F_{\alpha\beta}F^{\alpha\beta}$$
$$+ 3\left[f'(R)H + 4f''(R)\left(\ddot{H} + 4H\dot{H}\right) \right]\left(F_{\alpha\beta}F^{\alpha\beta}\right)^{\bullet} + f'(R)\left(F_{\alpha\beta}F^{\alpha\beta}\right)^{\bullet\bullet}$$
$$\left. - \frac{2}{3}f'(R)\frac{1}{a^2}\overset{(3)}{\Delta}\left(F_{\alpha\beta}F^{\alpha\beta}\right) \right\}$$

$$g^{\alpha\beta} F_{0\beta} F_{0\alpha} - \frac{1}{4} g_{00} F_{\alpha\beta} F^{\alpha\beta} = \frac{1}{2} \left(|E_i^{\text{proper}}(t, \boldsymbol{x})|^2 + |B_i^{\text{proper}}(t, \boldsymbol{x})|^2 \right)$$

$$F_{\alpha\beta} F^{\alpha\beta} = 2 \left(|B_i^{\text{proper}}(t, \boldsymbol{x})|^2 - |E_i^{\text{proper}}(t, \boldsymbol{x})|^2 \right)$$

$$\longrightarrow \left(F_{\alpha\beta} F^{\alpha\beta} \right)^{\bullet} = 8 \Bigg\{ -H |B^{\text{proper}}(L, \eta)|^2 + \left[H + 3 \frac{f'(R)}{1 + f(R)} \left(\ddot{H} + 4 H \dot{H} \right) \right] |E^{\text{proper}}(L, \eta)|^2 \Bigg\}$$

- We consider the case in which magnetic fields are mainly generated rather than electric fields because we are interested in the generation of large-scale magnetic fields.

  ↑

  This situation is realized if I increases rapidly in time during inflation.

  ($\longrightarrow$ We neglect terms in electric fields.) **[KB, JCAP 0710, 015 (2007)]**

- We consider the case in which $\overset{(3)}{\Delta} \left(F_{\alpha\beta} F^{\alpha\beta} \right)$ is very small because it corresponds to the second order spatial derivative of the quadratic quantity of electromagnetic quantum fluctuations, so that it can be neglected.

- $(\mu, \nu) = (0, 0)$ component of the gravitational field equation:

$$H^2 = \kappa^2 \left[\frac{1}{6} I(R) - f'(R) \left(\dot{H} + 5H^2 \right) + 6 f''(R) H \left(\ddot{H} + 4H\dot{H} \right) \right] \frac{k|C(k)|^2}{\pi^2} \frac{k^4}{a^4}$$

- Trace part of $(\mu, \nu) = (i, j)$ component:

$$2\dot{H} + 3H^2 = \kappa^2 \left[\frac{1}{6} I(R) + f'(R) \left(-5\dot{H} + H^2 \right) + 6 f''(R) \left(\dddot{H} - H\ddot{H} + 4\dot{H}^2 - 20H^2\dot{H} \right) + 36 f'''(R) \left(\ddot{H} + 4H\dot{H} \right)^2 \right] \frac{k|C(k)|^2}{\pi^2} \frac{k^4}{a^4}$$

→ Eliminating $I(R)$ from these equations, we obtain

$$\dot{H} + H^2 = \kappa^2 \left[f'(R) \left(-2\dot{H} + 3H^2 \right) + 3 f''(R) \left(\dddot{H} - 2H\ddot{H} + 4\dot{H}^2 - 24H^2\dot{H} \right) + 18 f'''(R) \left(\ddot{H} + 4H\dot{H} \right)^2 \right] \frac{k|C(k)|^2}{\pi^2} \frac{k^4}{a^4}$$

▪ We consider the case in which $f(R)$ is given by the following form:

$$f(R) = f_{\rm HS}(R) \equiv \frac{c_1 \left(R/m^2\right)^n}{c_2 \left(R/m^2\right)^n + 1}$$

$c_1,\ c_2$: Dimensionless constants

n : Positive constant

m : Mass scale

(1) $\displaystyle \lim_{R\to\infty} f_{\rm HS}(R) = \frac{c_1}{c_2} = \text{const}\,,$ (2) $\displaystyle \lim_{R\to 0} f_{\rm HS}(R) = 0$

[Hu and Sawicki, Phys. Rev. D 76, 064004 (2007)]

→ At the inflationary stage, because $R/m^2 \gg 1$, we are able to use the following approximate relations:

$$f_{\rm HS}(R) \approx \frac{c_1}{c_2}\left[1 - \frac{1}{c_2}\left(\frac{R}{m^2}\right)^{-n}\right], \quad f'_{\rm HS}(R) \approx \frac{n c_1}{{c_2}^2}\frac{1}{m^2}\left(\frac{R}{m^2}\right)^{-(n+1)}$$

→ We consider the case in which the scale factor is given by

$$a(t) = \bar{a}\left(t/\bar{t}\right)^p$$

p : Constant, $\bar{t}$: Some fiducial time during inflation

$\bar{a}$: Value of $a(t)$ at $t = \bar{t}$

→ Substituting these equations into the gravitational field equation, we find

$$p = \frac{n+1}{2}$$

$$\frac{\bar{a}}{\bar{t}^p} = \left\{ \frac{1}{3^{n+1}\pi^2} \frac{1}{(n-1)\left[n(n+1)\right]^n} \frac{(-c_1)}{{c_2}^2} k|C(k)|^2 k^4 \kappa^2 m^{2n} \right\}^{1/4}$$

→ **If $n \gg 1$, p becomes much larger than unity, so that power-law inflation can be realized.**

⇨ **The electromagnetic field with a non-minimal gravitational coupling can be a source of inflation.**

III. Inflation and late-time cosmic acceleration in modified gravity

< III A. Inflation >

< Action >

$$S_{\rm MG} = \int d^4x \sqrt{-g} \left[\mathcal{L}_{\rm MG} + \underline{\mathcal{L}_{\rm EM}} \right]$$

$$\mathcal{L}_{\rm EM} = -\frac{1}{4} I(R) F_{\mu\nu} F^{\mu\nu}$$

$$I(R) = 1 + f(R)$$

$$\mathcal{L}_{\rm MG} = \frac{1}{2\kappa^2} \left[R + F(R) \right]$$

$F(R)$: Arbitrary function of R

< Gravitational field equation >

$$\left[1 + F'(R)\right] R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left[R + F(R)\right] + g_{\mu\nu} \Box F'(R) - \nabla_\mu \nabla_\nu F'(R) = \kappa^2 T^{(\rm EM)}_{\mu\nu}$$

[Nojiri and Odintsov, Phys. Lett. B 657, 238 (2007)]

- $(\mu, \nu) = (0, 0)$ component of the gravitational field equation:

$$H^2 + \underline{\frac{1}{6} F(R) - F'(R) \left(\dot{H} + H^2 \right)} = \frac{\kappa^2}{3} \left\{ I(R) \left(g^{\alpha\beta} F_{0\beta} F_{0\alpha} - \frac{1}{4} g_{00} F_{\alpha\beta} F^{\alpha\beta} \right) \right.$$
$$+ \frac{3}{2} \left[-f'(R) \left(\dot{H} + H^2 \right) + 6 f''(R) H \left(\ddot{H} + 4 H \dot{H} \right) \right] F_{\alpha\beta} F^{\alpha\beta}$$
$$\left. + \frac{3}{2} f'(R) H \left(F_{\alpha\beta} F^{\alpha\beta} \right)^{\bullet} - \frac{1}{2} f'(R) \frac{1}{a^2} \overset{(3)}{\Delta} \left(F_{\alpha\beta} F^{\alpha\beta} \right) \right\}$$

- Trace part of $(\mu, \nu) = (i, j)$ component:

$$2\dot{H} + 3H^2 + \frac{1}{2}F(R) - F'(R)\left(\dot{H} + 3H^2\right) + 6F''(R)\left[\dddot{H} + 4\left(\dot{H}^2 + H\ddot{H}\right)\right] + 36F'''(R)\left(\ddot{H} + 4H\dot{H}\right)^2$$

$$= \frac{\kappa^2}{2}\left\{\frac{1}{6}I(R)F_{\alpha\beta}F^{\alpha\beta} + \left[-f'(R)\left(\dot{H} + 3H^2\right) + 6f''(R)\left(\dddot{H} + 7H\ddot{H} + 4\dot{H}^2 + 12H^2\dot{H}\right) + 36f'''(R)\left(\ddot{H} + 4H\dot{H}\right)^2\right]F_{\alpha\beta}F^{\alpha\beta} + 3\left[f'(R)H + 4f''(R)\left(\ddot{H} + 4H\dot{H}\right)\right]\left(F_{\alpha\beta}F^{\alpha\beta}\right)^{\bullet} + f'(R)\left(F_{\alpha\beta}F^{\alpha\beta}\right)^{\bullet\bullet} - \frac{2}{3}f'(R)\frac{1}{a^2}\overset{(3)}{\Delta}\left(F_{\alpha\beta}F^{\alpha\beta}\right)\right\}$$

- We consider the case in which terms in electric fields and $\overset{(3)}{\Delta}\left(F_{\alpha\beta}F^{\alpha\beta}\right)$ are negligible.

→ Eliminating $I(R)$ from these equations, we find

$$\dot{H} + H^2 + \left\{\frac{1}{6}F(R) - F'(R)H^2 + 3F''(R)\left[\dddot{H} + 4\left(\dot{H}^2 + H\ddot{H}\right)\right] + 18F'''(R)\left(\ddot{H} + 4H\dot{H}\right)^2\right\}$$

$$= \kappa^2\left[f'(R)\left(-2\dot{H} + 3H^2\right) + 3f''(R)\left(\dddot{H} - 2H\ddot{H} + 4\dot{H}^2 - 24H^2\dot{H}\right)\right.$$

$$\left. + 18f'''(R)\left(\ddot{H} + 4H\dot{H}\right)^2\right]\frac{k|C(k)|^2}{\pi^2}\frac{k^4}{a^4}$$

- We consider the case in which $F(R)$ is given by the following form:

$$F(R) = -M^2\frac{\left[(R/M^2) - (R_0/M^2)\right]^{2l+1} + (R_0/M^2)^{2l+1}}{c_3 + c_4\left\{\left[(R/M^2) - (R_0/M^2)\right]^{2l+1} + (R_0/M^2)^{2l+1}\right\}}$$

$$\lim_{R\to\infty} F(R) = -\frac{M^2}{c_4} = \text{const}$$

$$\lim_{R\to 0} F(R) = 0$$

$c_3,\ c_4$: Dimensionless constants, l : Positive constant, M : Mass scale

[Nojiri and Odintsov, Phys. Lett. B 657, 238 (2007)]

- At the very early stage: $\lim_{R\to\infty} F(R) = -M^2 \frac{1}{c_4} = -2\Lambda_{\rm i}$

$\Lambda_{\rm i} \left(\gg {H_0}^2\right)$: Effective cosmological constant in the very early universe

$$H_0 \ = \ 100h \, {\rm km} \, {\rm s}^{-1} \, {\rm Mpc}^{-1} \ = \ 2.1h \times 10^{-42}{\rm GeV} \ \approx \ 1.5 \times 10^{-33}{\rm eV}$$

: Current value of the Hubble constant

$h = 0.70$

[Freedman et al., Astrophys. J. <u>553</u>, 47 (2001)]

- At the present time: $F(R_0) = -M^2 \frac{\left(R_0/M^2\right)^{2l+1}}{c_3 + c_4 \left(R_0/M^2\right)^{2l+1}} = -2R_0$

$R_0 \left(\approx {H_0}^2\right)$: Current curvature

$$\Longrightarrow \quad c_3 = \frac{1}{2}\left(\frac{R_0}{M^2}\right)^{2l}\left(1 - \frac{R_0}{\Lambda_{\rm i}}\right) \approx \frac{1}{2}\left(\frac{R_0}{M^2}\right)^{2l}, \quad c_4 = \frac{1}{2}\frac{M^2}{\Lambda_{\rm i}}$$

(We have used $\left(R_0/\Lambda_{\rm i}\right) \ll 1$.)

- We consider the case in which $f(R)$ is given by the following form:

$$f(R) = f_{\rm NO}(R) \equiv \frac{\left[(R/M^2) - (R_0/M^2)\right]^{2q+1} + (R_0/M^2)^{2q+1}}{c_5 + c_6 \left\{\left[(R/M^2) - (R_0/M^2)\right]^{2q+1} + (R_0/M^2)^{2q+1}\right\}}$$

$$\lim_{R\to\infty} f_{\rm NO}(R) = \frac{1}{c_6} = \text{const}\,, \qquad \lim_{R\to 0} f_{\rm NO}(R) = 0$$

$c_5,\ c_6$: Dimensionless constants, q : Positive constant

[Nojiri and Odintsov, Phys. Lett. B 657, 238 (2007)]

[Nojiri, Odintsov and Tretyakov, arXiv:0710.5232 [hep-th]]

→ At the inflationary stage, because $R/M^2 \gg 1$ and $R/M^2 \gg R_0/M^2$ we are able to use the following approximate relations:

$$F(R) \approx -M^2 \frac{1}{c_4} \left[1 - \frac{c_3}{c_4} \left(\frac{R}{M^2}\right)^{-(2l+1)}\right]$$

$$f_{\rm NO}(R) \approx \frac{1}{c_6} \left[1 - \frac{c_5}{c_6} \left(\frac{R}{M^2}\right)^{-(2q+1)}\right]$$

- At the very early stage, because $R \to \infty$, we obtain

$$\dot{H} + H^2 = \frac{\Lambda_{\rm i}}{3} \longrightarrow a(t) \propto \exp\left(\sqrt{\frac{\Lambda_{\rm i}}{3}} t\right)$$

$\Longrightarrow$ **Exponential inflation can be realized.**

< Equation for the scale factor >

$$\dot{H} + H^2 + \left\{\frac{1}{6}F(R) - F'(R)H^2 + 3F''(R)\left[\dddot{H} + 4\left(\dot{H}^2 + H\ddot{H}\right)\right] + 18F'''(R)\left(\ddot{H} + 4H\dot{H}\right)^2\right\}$$

$$= \kappa^2\left[f'(R)\left(-2\dot{H} + 3H^2\right) + 3f''(R)\left(\dddot{H} - 2H\ddot{H} + 4\dot{H}^2 - 24H^2\dot{H}\right) + 18f'''(R)\left(\ddot{H} + 4H\dot{H}\right)^2\right]\frac{k|C(k)|^2}{\pi^2}\frac{k^4}{a^4}$$

Terms in $F(R)$ as well as the right-hand side can be a source of inflation.

If the contribution of terms in $f(R)$ to inflation is dominant, power-law inflation can be realized.

$$a(t) \propto t^{\tilde{p}}$$

$$\longrightarrow \tilde{p} = q + 1$$

If $q \gg 1$, $\tilde{p} \gg 1$.

< III B. Late-time cosmic acceleration >

- At the early stage of the universe, at which the curvature is very large, inflation can be realized due to the terms in $F(R)$ and/or those in $f(R)$.

⇨ As curvature becomes small, the contribution of these terms to inflation becomes small, and then inflation ends.

⇨ After inflation, radiation becomes dominant, and subsequently matter becomes dominant.

⇨ When the energy density of matter becomes small and the value of curvature becomes R_0 , there appears the small effective cosmological constant at the present time.

→ Hence, the current cosmic acceleration can be realized.

- In the limit $R \to R_0$, because $R/M^2 - R_0/M^2 \ll 1$, we are able to use the following approximate relations:

$$F(R) \approx -M^2 \frac{c_3}{\left[c_3 + c_4 \left(R_0/M^2\right)^{2l+1}\right]^2}$$

$$\times \left\{ \left(\frac{R}{M^2} - \frac{R_0}{M^2} \right)^{2l+1} + \left[\frac{c_3 + c_4 \left(R_0/M^2\right)^{2l+1}}{c_3} \right] \left(\frac{R_0}{M^2} \right)^{2l+1} \right\}$$

$$f_{\mathrm{NO}}(R) \approx \frac{c_5}{\left[c_5 + c_6 \left(R_0/M^2\right)^{2q+1}\right]^2}$$

$$\times \left\{ \left(\frac{R}{M^2} - \frac{R_0}{M^2} \right)^{2q+1} + \left[\frac{c_5 + c_6 \left(R_0/M^2\right)^{2q+1}}{c_5} \right] \left(\frac{R_0}{M^2} \right)^{2q+1} \right\}$$

(If $q > l$, $f_{\mathrm{NO}}(R)$ becomes constant more rapidly than $F(R)$ in the limit $R \to R_0$.)

- In the limit $R \to R_0$, we obtain

$$\dot{H} + H^2 = \frac{R_0}{3} \longrightarrow a(t) \propto \exp\left(\sqrt{\frac{R_0}{3}} t\right)$$

$$\longrightarrow \frac{\ddot{a}(t)}{a(t)} = \frac{R_0}{3} > 0$$

⇒ **Late-time acceleration can be realized.**

< Important feature of the present model >

Equation for the scale factor

$$\dot{H} + H^2 + \left\{ \frac{1}{6} F(R) - F'(R) H^2 + 3F''(R) \left[\dddot{H} + 4 \left(\dot{H}^2 + H \ddot{H} \right) \right] + 18 F'''(R) \left(\ddot{H} + 4H\dot{H} \right)^2 \right\}$$

$$= \kappa^2 \left[f'(R) \left(-2\dot{H} + 3H^2 \right) + 3 f''(R) \left(\dddot{H} - 2H\ddot{H} + 4\dot{H}^2 - 24 H^2 \dot{H} \right) + 18 f'''(R) \left(\ddot{H} + 4H\dot{H} \right)^2 \right] \frac{k|C(k)|^2}{\pi^2} \frac{k^4}{a^4}$$

Non-minimal electromagnetic coupling

Modified gravity

IV. Classically equivalent form of non-minimal Maxwell-*F*(*R*) gravity

- $S_{\rm MG}$ can be rewritten by using auxiliary fields, ζ and ξ, as follows:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} [\zeta + F(\zeta)] + I(\zeta) \underline{\mathcal{L}_{\rm M}} + \xi (R - \zeta) \right\}$$

$$\mathcal{L}_{\rm M} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

(Using $\zeta = R$, which is derived by taking variation of S with respect to ξ, this form is reduced to $S_{\rm MG}$.)

- Taking variation of S with respect to ζ, we find

$$\xi = \frac{1}{2\kappa^2} [1 + F'(\zeta)] + I'(\zeta) \mathcal{L}_{\rm M}$$

→ Substituting this equation into S and eliminating ξ from S, we find

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} [1 + \underline{F'(\zeta)] R} + [I(\zeta) + I'(\zeta)(R - \zeta)] \mathcal{L}_{\rm M} + \frac{1}{2\kappa^2} [F(\zeta) - F'(\zeta)\zeta] \right\}$$

We make the following conformal transformation of the above form:

$$g_{\mu\nu} \to \hat{g}_{\mu\nu} = e^{\varphi} g_{\mu\nu} \,, \qquad e^{\varphi} = 1 + F'(\zeta)$$

φ : Scalar field

(The hat denotes quantities in a new conformal frame in which the term in the coupling between $F'(\zeta)$ and R in the first term on the right-hand side of the above form of S disappears.)

$\zeta(\varphi)$ is obtained by solving this equation with respect to ζ as $\zeta = \zeta(\varphi)$.

< Form of S in the new conformal frame >

$$S_{\mathrm{N}} = \int d^4x \sqrt{-\hat{g}} \left[\frac{1}{2\kappa^2} \left(\hat{R} - \frac{3}{2} \hat{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \right) \right.$$

$$+ \left(e^{-2\varphi} \left\{ I\left[\zeta(\varphi)\right] - I'\left[\zeta(\varphi)\right] \zeta(\varphi) \right\} + e^{-\varphi} I'\left[\zeta(\varphi)\right] \left(\hat{R} + 3\hat{\Box}\varphi - \frac{3}{2} \hat{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \right) \right) \hat{\mathcal{L}}_{\mathrm{M}}$$

$$\left. + \frac{1}{2\kappa^2} e^{-2\varphi} \left\{ F\left[\zeta(\varphi)\right] - \left(e^{\varphi} - 1\right) \zeta(\varphi) \right\} \right]$$

$$\hat{\Box}\varphi = \frac{1}{\sqrt{-\hat{g}}} \partial_\mu \left(\sqrt{-\hat{g}} \hat{g}^{\mu\nu} \partial_\nu \varphi \right)$$

This form is close to that of the electromagnetic field with the coupling to the dilaton, namely, the Lagrangian of non-minimal Maxwell-*F*(*R*) gravity is qualitatively similar to Lagrangian describing dilaton electromagnetism.

< Model of dilaton electromagnetism >

$$S = \int d^4x \sqrt{-g} [\mathcal{L}_{\text{inflaton}} + \mathcal{L}_{\text{dilaton}} + \mathcal{L}_{\text{EM}}]$$

(KB and Yokoyama, Phys. Rev. D 69, 043507 [2004])

$$\mathcal{L}_{\text{inflaton}} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - U[\phi]$$

$$\mathcal{L}_{\text{dilaton}} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - V[\Phi]$$

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4} f(\Phi) F_{\mu\nu} F^{\mu\nu}$$

ϕ : Inflaton field

Φ : Dilaton field

λ, $\tilde{\lambda}$: Dimensionless constants

$\bar{V}$: Constant

$$f(\Phi) = e^{-\lambda\kappa\Phi}$$: Dilaton coupling

$$V[\Phi] = \bar{V} \exp(-\tilde{\lambda}\kappa\Phi)$$: Dilaton potential

$$f(\Phi) = f[\Phi(t)] = f[\Phi(a(t))] \equiv \bar{f} a^{\beta-1}$$

$\bar{f}$: Constant

β : parameter

< Current energy density of large-scale magnetic fields >

$$\rho_B(L,t_0) \propto H_{\rm inf}^{\ 4}\left(\frac{a_{\rm R}}{a_0}\right)^4\left(\frac{k}{a_{\rm R}H_{\rm inf}}\right)^{\underline{-|\beta|+5}}$$

$H_{\rm inf}$: Hubble constant at the inflationary stage

$a_{\rm R}$: Value of $a(t)$ at the end of inflation

a_0 : Value of $a(t)$ at the present time

$\beta \approx 5.0$ **: Scale-invariant spectrum**

→ Magnetic fields on 1 Mpc with as large as 10^{-10}G can be generated.

↑

This strength is enough to account for the magnetic fields observed in galaxies and clusters of galaxies through only adiabatic compression without requiring any dynamo amplification.

V. SUMMARY

We have considered inflation and the late-time acceleration in the expansion of the universe in non-minimal electromagnetism, in which the electromagnetic field couples to a function of the scalar curvature .

We have shown the following points:

(1) **Power-law inflation can be realized due to the non-minimal gravitational coupling of the electromagnetic field.**

(2) **Large-scale magnetic fields can be generated due to the breaking of the conformal invariance of the electromagnetic field through its non-minimal gravitational coupling.**

(3) **Both inflation and the late-time acceleration of the universe can be realized in a modified Maxwell-$F(R)$ gravity.**

(4) **The Lagrangian of non-minimal Maxwell-$F(R)$ gravity is qualitatively similar to Lagrangian describing dilaton electromagnetism.**

→ **Results in non-minimal Maxwell-$F(R)$ gravity can be generalized to a non-minimal YM-$F(R)$ gravity and a non-minimal vector-$F(R)$ gravity (Ref. [2]).**

< Observational deviation of a non-minimal electromagnetic theory from the ordinary Maxwell theory >

- In the case of exponential inflation, the scalar curvature is proportional to the square of the Hubble parameter.
- It is known that the root-mean-square (rms) amplitude of curvature perturbations is also proportional to the square of the Hubble parameter.
- In a non-minimal electromagnetic theory, because magnetic fields couple to the scalar curvature, there can exist the cross correlations between magnetic fields and curvature perturbations through the Hubble parameter.

If

(1) the primordial large-scale magnetic fields are detected by future experiments such as PLANCK, SPIDERS (post-PLANCK) and Inflation Probe (CMBPol mission) in the Beyond Einstein program of NASA on the anisotropy of the cosmic microwave background (CMB) radiation,

(2) there exist the cross correlations between the primordial large-scale magnetic fields and curvature perturbations,

→ it is observationally suggested that at the inflationary stage there should exist a non-minimal gravitational coupling of the electromagnetic field.

(1) Non-minimal Yang-Mills-*F*(*R*) gravity

$$S_{\rm GR} = \int d^4x \sqrt{-g} \left[\mathcal{L}_{\rm EH} + \mathcal{L}_{\rm YM} \right]$$

$$\mathcal{L}_{\rm YM} = -\frac{1}{4} I(R) F^a_{\mu\nu} F^{a\mu\nu} \left[1 + b\tilde{g}^2 \ln \left| \frac{-(1/2)\, F^a_{\mu\nu} F^{a\mu\nu}}{\mu^4} \right| \right]$$

$$I(R) = 1 + f(R) \,, \quad b = \frac{1}{4} \frac{1}{8\pi^2} \frac{11}{3} N$$: Asymptotic freedom constant

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$$: Field strength tensor

A^a_μ : $SU(N)$ field strength, f^{abc} : Structure constants

μ : Mass scale of the renormalization point

$$\tilde{g}^2(X) = \frac{\tilde{g}^2}{1 + b\tilde{g}^2 \ln |X/\mu^4|} \,, \quad X \equiv -\frac{1}{2} F^a_{\mu\nu} F^{a\mu\nu}$$

$\tilde{g}$: Value of the running coupling constant when $X = \mu^4$

(2) Non-minimal Vector-*F*(*R*) gravity

$$\bar{S}_{\mathrm{MG}} = \int d^4x \sqrt{-g} \left[\mathcal{L}_{\mathrm{MG}} + \mathcal{L}_{\mathrm{V}} \right]$$

$$\mathcal{L}_{\mathrm{V}} = I(R) \left\{ -\frac{1}{4} F^{a}_{\mu\nu} F^{a\mu\nu} - V[A^{a2}] \right\}$$

$$V[A^{a2}] = \bar{V} \left(\frac{A^{a2}}{\bar{m}^2} \right)^{\bar{n}}, \quad \bar{n}(>1)$$: Positive integer

$\bar{V}$: Constant

$\bar{m}$: Mass scale

⇩

Results in non-minimal Maxwell-*F*(*R*) gravity can be generalized to a non-minimal YM-*F*(*R*) gravity and a non-minimal vector-*F*(*R*) gravity.

< Asymptotic freedom versus non-minimal coupling >

→ We propose the origin of the non-minimal gravitational coupling function based on renormalization-group considerations.

- The effective renormalization-group improved Lagrangian for the $SU(2)$ gauge theory in matter sector has been found for a de Sitter background as follows:

[Elizalde, Odintsov and Romeo, Phys. Rev. D 54, 4152 (1996)]

$$\mathcal{L}_{SU(2)} = -\frac{1}{4}\frac{\tilde{g}^2}{\tilde{g}^2(\tilde{t})} G^a_{\mu\nu} G^{a\mu\nu}$$

$$\tilde{g}^2(\tilde{t}) = \frac{\tilde{g}^2}{1 + 11\tilde{g}^2\tilde{t}/\left(12\pi^2\right)}$$

$$\tilde{t} = \frac{1}{2}\ln\frac{R/4 + \tilde{g}\tilde{H}}{\mu^2}$$

$$G^a_{\mu\nu} G^{a\mu\nu}/2 = \tilde{H}^2$$

$$1 + f(R)$$

$G^a_{\mu\nu}$: $SU(2)$ field strength

$\tilde{g}(\tilde{t})$: Running $SU(2)$ gauge coupling constant

$\tilde{g}$: Value of $\tilde{g}(\tilde{t})$ in the case $\tilde{t} = 0$

$\tilde{t}$: Renormalization-group parameter

μ : Mass parameter

$\tilde{H}$: Magnetic field in the $SU(2)$ gauge theory

The running gauge coupling constant typically shows asymptotically free behavior: it goes to zero at very high energy.

→ We try to relate the asymptotic freedom in a non-Abelian gauge theory with non-minimal Maxwell-modified gravity.

If $f(R) = f_{\mathrm{HS}}(R) \equiv \dfrac{c_1 \left(R/m^2\right)^n}{c_2 \left(R/m^2\right)^n + 1}$, we find

$$\frac{c_1 \left(R/m^2\right)^n}{c_2 \left(R/m^2\right)^n + 1} = \frac{11\tilde{g}^2}{12\pi^2}\tilde{t}$$

→ (1) If $R/m^2 \gg 1$, $\tilde{t} \approx \left[12\pi^2/\left(11\tilde{g}^2\right)\right]\left(c_1/c_2\right)$.

(2) In the limit $R \to 0$, $\tilde{t} \to 0$.

⇒ **Asymptotic freedom induces the appearance of the non-minimal gravitational gauge coupling in (non-) Abelian gauge theories at high energy.**

Appendix A

< I B. Scenarios for the origin of cosmic magnetic fields >

1. Astrophysical processes

→ A kind of Plasma instability

(1) Biermann battery mechanism

(Biermann and Schlüter, Phys. Rev. 82, 863 [1951])

(2) Weibel instability **(Weibel, Phys. Rev. Lett. 2, 83 [1959])**

2. Cosmological processes

(1) First-order cosmological electroweak phase transition (EWPT) **(Baym, Bödeker, and McLerran, Phys. Rev. D 53, 662 [1996])**

(2) Quark-hadron phase transition (QCDPT)

(Quashnock, Loeb, and Spergel, Astrophys. J. 344, L49 [1989])

(3) Generation of the magnetic fields from primordial density perturbations before the epoch of recombination **(Matarrese *et al*., Phys. Rev. D 71, 043502 [2005])**
(Ichiki *et al*., Science 311, 827 [2006])

< I E. Breaking mechanisms >

1. Coupling of a scalar field to electromagnetic fields

(Ratra, Astrophys. J. 391, L1 [1992])

(KB and Yokoyama, Phys. Rev. D 69, 043507 [2004])

$$\mathcal{L} = -\frac{1}{4} f(\Phi) F_{\mu\nu} F^{\mu\nu}$$

$$f(\Phi) = e^{-\lambda\kappa\Phi}$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$: Electromagnetic field-strength tensor

A_μ : $U(1)$ gauge field

Φ : Dilaton field, $\kappa = \sqrt{8\pi G}$, G : Newton's constant

λ : Dimensionless constant

2. Non-minimal gravitational coupling to electromagnetic fields

(Turner & Widrow, Phys. Rev. D 37, 2743 [1988])

$$(R/m^2) F_{\mu\nu} F^{\mu\nu}$$

$\longleftarrow$ Such a term is known to arise in curved spacetime due to one-loop vacuum-polarization effects.

(Drummond & Hathrell, Phys. Rev. D 22, 343 [1980])

R : Ricci scalar

m : Mass scale

3. The conformal anomaly in the trace of the energy-momentum tensor induced by quantum corrections to Maxwell electrodynamics

(Dolgov, Phys. Rev. D 48, 2499 [1993])

Appendix B

< Equation for the mode function $A(k,t)$ >

$$\ddot{A}(k,t) + \left(H + \frac{\dot{I}}{I} \right) \dot{A}(k,t) + \frac{k^2}{a^2} A(k,t) = 0$$

- **The normalization condition** $A(k,t)\dot{A}^*(k,t) - \dot{A}(k,t)A^*(k,t) = \frac{i}{Ia}$

$\longrightarrow$ $t \longrightarrow \eta$

$$A''(k,\eta) + \frac{I'}{I} A'(k,\eta) + k^2 A(k,\eta) = 0$$

The prime denotes differentiation with respect to the conformal time η .

- In the exact de Sitter background, we have $a = 1/(-H\eta)$ where H is the de Sitter Hubble parameter.

$\longrightarrow$ The horizon-crossing, which is defined by $H = k/a$, is given by $-k\eta = 1$.

$\Rightarrow$

- Subhorozon scale: $k|\eta| \gg 1$
- Superhorozon scale: $k|\eta| \ll 1$

< III B. Solution for $A(k,\eta)$ >

1. WKB subhorizon solution

(Subhorozon scale: $k|\eta| \gg 1$)

$$A_{\rm in}(k,\eta) = \frac{1}{\sqrt{2k}} I^{-1/2} e^{-ik\eta}$$

← We have assumed that the vacuum in the short-wavelength limit is the standard Minkowski vacuum.

2. Solution on superhorizon scales

(Superhorozon scale: $k|\eta| \ll 1$)

- **Long-wavelength expansion:** $A_{\rm out} = A_0(\eta) + k^2 A_1(\eta) + O(k^4)$

⟶ Let the two independent solutions for $A_{\rm out}$ be u and v .

- **The boundary condition:** $u \to 1$ and $v \to 0$ as $\eta \to \eta_{\rm R}$

$$A_0'' + \frac{I'}{I} A_0' = 0 \longrightarrow u_0 = 1, \quad v_0 = \int_\eta^{\eta_{\rm R}} \frac{1}{I(\tilde{\eta})} d\tilde{\eta}$$

$$A_1'' + \frac{I'}{I} A_1' + A_0 = 0 \longrightarrow u = u_0 + k^2 u_0 \int_\eta^{\eta_{\rm R}} d\eta' I(\eta') \int_{\eta'}^{\eta} \frac{d\eta''}{I(\eta'')}$$

$$v = v_0 + k^2 \int_\eta^{\eta_{\rm R}} d\eta' v_0(\eta') I(\eta') \int_{\eta'}^{\eta} \frac{d\eta''}{I(\eta'')}$$

$\eta_{\rm R}$: Conformal time at the time of reheating after inflation

$$\Rightarrow \quad A_{\rm out} = Cu + \underline{Dv}$$

(Decaying mode)

C, D : Constants

< Junction conditions for $A_{\rm in}$ and $A_{\rm out}$ >

$$A_{\rm in}(\eta_k) = A_{\rm out}(\eta_k), \quad A'_{\rm in}(\eta_k) = A'_{\rm out}(\eta_k)$$

η_k : Horizon crossing ($\eta_k \approx -1/k$)

< Lowest order solution for $A_{\rm out}$ >

$$A_{\rm out} = C(k) + D(k) \int_{\eta}^{\eta_{\rm R}} \frac{1}{I(\tilde{\eta})} d\tilde{\eta}$$

$$C(k) = \frac{1}{\sqrt{2k}} I^{-1/2} \left[1 - \left(\frac{1}{2} I' + ikI \right) \int_{\eta}^{\eta_{\rm R}} \frac{1}{I(\eta')} d\eta' \right] e^{-ik\eta} \Big|_{\eta=\eta_k}$$

$$D(k) = \frac{1}{\sqrt{2k}} I^{-1/2} \left(\frac{1}{2} I' + ikI \right) e^{-ik\eta} \Big|_{\eta=\eta_k}$$

< Evolution of electric and magnetic fields after inflation >

- The conductivity of the universe in the inflationary stage is negligibly small, because there are few charged particles at that time. Hence the electric fields can exist during inflation.

→ After reheating following inflation, however, a number of charged particles are produced, so that the conductivity of the universe immediately becomes much larger than the Hubble parameter at that time.

→ Hence the electric fields accelerate charged particles and finally dissipate, and only the magnetic fields can survive up to the present time.

Appendix C

< Determination of C and D accurate to $O(k^2)$ >

$$\left.\begin{pmatrix} A_{\rm in} \\ A'_{\rm in} \end{pmatrix}\right|_{\eta=\eta_k} = \mathcal{M}\begin{pmatrix} C \\ D \end{pmatrix}, \quad \mathcal{M} \equiv \left.\begin{pmatrix} 1-k^2 I_1 & I_2\left(1-k^2 I_1\right) \\ k^2 I_3/I & \left[-1+k^2\left(I_1+I_2 I_3\right)\right]/I \end{pmatrix}\right|_{\eta=\eta_k}$$

$$I_1(\eta) \equiv \int_{\eta}^{\eta_{\rm R}} \left[\frac{\int_{\tilde{\eta}}^{\eta_{\rm R}} I\left(\tilde{\tilde{\eta}}\right) d\tilde{\tilde{\eta}}}{I\left(\tilde{\eta}\right)}\right] d\tilde{\eta}, \qquad I_2(\eta) \equiv \int_{\eta}^{\eta_{\rm R}} \frac{1}{I\left(\tilde{\eta}\right)} d\tilde{\eta}$$

$$I_3(\eta) \equiv \int_{\eta}^{\eta_{\rm R}} I\left(\tilde{\eta}\right) d\tilde{\eta}$$

$$\Longrightarrow \quad \begin{pmatrix} C \\ D \end{pmatrix} = \mathcal{M}^{-1} \left.\begin{pmatrix} A_{\rm in} \\ A'_{\rm in} \end{pmatrix}\right|_{\eta=\eta_k}$$

$$\mathcal{M}^{-1} = \frac{1}{\left(1-k^2 I_1\right)^2} \left.\begin{pmatrix} 1-k^2\left(I_1+I_2 I_3\right) & I I_2\left(1-k^2 I_1\right) \\ k^2 I_3 & -I\left(1-k^2 I_1\right) \end{pmatrix}\right|_{\eta=\eta_k}$$

< Adiabatic compression >

- $\sigma >> H \Rightarrow$ $B \propto a^{-2}$, $\rho \propto a^{-3}$ $\Rightarrow$ $B \propto \rho^{2/3}$

- $\dfrac{\rho_{\rm gal}}{\bar{\rho}} \sim 10^5 - 10^6 \longrightarrow B_{\rm gal} \approx 10^3 - 10^4\, B_{\rm prim}$

- $\dfrac{\rho_{\rm cg}}{\bar{\rho}} \sim 10^2 - 10^3 \longrightarrow B_{\rm cg} \approx 10^1 - 10^2\, B_{\rm prim}$

$\bar{\rho}$: Average cosmic energy density

$$B_{\rm gal} \approx 10^{-6}\,{\rm G}$$
$$B_{\rm cg} \approx 10^{-7}\,{\rm G}$$

$\Rightarrow$

$$B_{\rm prim} \approx 10^{-10} - 10^{-9}\,{\rm G}$$