Lectures on exact renormalization group-3

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1.Non-perturbative (Wilsonian) renormalization group equation

- Large coupling region
- the perturbatively nonrenormalizable interaction terms

  (higher dim. operators, beyond 4-dim.)
- Non-perturbative phenomena (nonrenormalizable theorem)

Gauge invariance and renormalization group

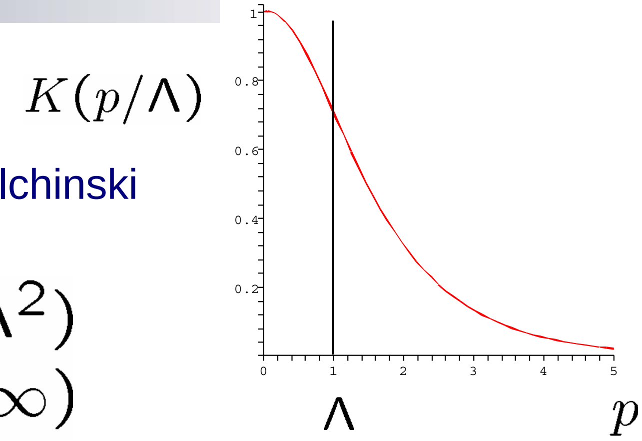
Cutoff vs Gauge invariance

Gauge transformation: $\delta A_\mu(p) = -ig \int_k A_\mu(p-k)c(k)$

Mix UV fields and IR fields

- Wegner-Houghton equation (sharp cutoff)
- Polchinski equation (smooth cutoff)
- Exact evolution equation (for 1PI effective action)

Polchinski equation



Nucl. Phys. B231 (1984) 269, Polchinski

Smooth Cutoff fn.

$$K(p/\Lambda) \quad \rightarrow \quad \begin{cases} 1 & (p^2 < \Lambda^2) \\ 0 & (p^2 \rightarrow \infty) \end{cases}$$

$$Z(J) = \int d\phi \exp \int_p -\frac{1}{2}\phi(p)\phi(-p)(p^2+m^2)K^{-1}(p^2/\Lambda_0^2)$$

$$+L_{int} + J(p)\phi(-p)$$

Introduce a new lower cutoff scale Λ.

Integrate out UV fields.

$$Z(J,\Lambda) = \int d\phi \exp \int_p -\frac{1}{2}\phi(p)\phi(-p)(p^2+m^2)K^{-1}(p^2/\Lambda^2)$$

$$+L(\phi,\Lambda) + J(p)\phi(-p)$$

general interaction Lagrangian

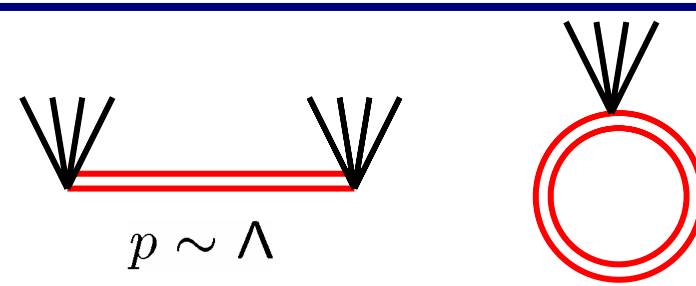
Assumption: Z does not depend on the cutoff scale Λ .

$$\Lambda \frac{d}{d\Lambda} Z = 0$$

Polchinski eq.

$$\frac{\partial}{\partial t} L = \int d^4p (2\pi)^4 \frac{1}{2(p^2+m^2)} \partial_t K(p) \left[\frac{\partial L}{\partial \phi(-p)} \frac{\partial L}{\partial \phi(p)} - \frac{\partial \partial L}{\partial \phi(-p) \partial \phi(p)} \right]$$

Infinite sum of following diagrams.



- Exact equation
- Infinite number of interaction terms
  (Including some perturbatively nonrenormalizable terms)

Gauge invariance and renormalization group

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Identity for the BRS invariance

- Master equation for BRS symmetry
- modified Ward-Takahashi identity (Ellwanger `94
  Sonoda `07)

2.Review of IIS (QED)

Igarashi, Itoh and Sonoda （IIS） (arXiv:0704.2349)

Outline of the IIS’s paper

They got the modified Ward-Takahashi identity which is the W-T id. for the effective action.

J. Phys. A40 (2007) 9675, Sonoda

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From the identity, they define the modified BRS transformation.

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However, the modified BRS transformation breaks nilpotency.

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They introduce the anti-field as a source of the modified BRS transformation.

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They found the Master action order by order in the anti-fields.

(fixed the anti-field dependence)

modified Ward –Takahashi id. in QED

J. Phys. A40 (2007) 9675, Sonoda

$$S[\phi] = \frac{1}{2}\phi \cdot D \cdot \phi + \mathcal{S}_I$$

$$\begin{aligned}
\frac{1}{2}\phi^A \cdot D_{AB} \cdot \phi^B &= \int_k \frac{1}{2} A_\mu(-k)(k^2\delta_{\mu\nu} - k_\mu k_\nu)A_\nu(k) + \bar{c}(-k)ik^2 c(k) \\
&\quad - B(-k)\left(ik_\mu A_\mu(k) + \frac{\alpha}{2}B(k)\right) + \bar{\psi}(-p)(p^\mu\gamma_\mu + m)\psi(p) \\
\mathcal{S}_I &= -e\int_{p,k} \bar{\psi}(-p-k)\gamma_\mu A_\mu(k)\psi(p)
\end{aligned}$$

The action is invariant under usual BRS tr.

$$\begin{aligned}
\delta A_\mu(k) &= -ik_\mu c(k),\ \delta\bar{c} = iB(k),\ \delta c(k) = \delta B(k) = 0, \\
\delta\psi(p) &= -ie\int_k \psi(p-k)c(k),\ \delta\bar{\psi}(-p) = ie\int_k \bar{\psi}(-p-k)c(k)
\end{aligned}$$

The partition functional for the QFT

$$Z_\phi[J] = \int D\phi \exp(-\mathcal{S}[\phi] + J \cdot \phi)$$

$$\mathcal{S}[\phi] = \frac{1}{2}\phi \cdot D \cdot \phi + \mathcal{S}_I$$

to decompose IR and UV fields $\phi = \Phi + \tilde{\phi}$

Insert the following gaussian integral $\theta = (1-K)\Phi - K\tilde{\phi}$

$$\int D\theta \exp - \left\{ \frac{1}{2}(\theta - J(1-K)D^{-1}) \cdot \frac{D}{K(1-K)} \cdot (\theta - (-)^J D^{-1}(1-K)J) \right\} = const.$$

- IR field Φ $K(p)D^{-1}(p)$
- UV field $\tilde{\phi}$ $(1-K(p))D^{-1}(p)$

Relation between the original partition functional and the one for IR theory

$$Z_\phi[J] = N_J Z_\Phi[J]$$

$$\begin{aligned} Z_\Phi[J] &= \int D\Phi \,\mathrm{exp}(-S[\Phi] + J\cdot K^{-1}\Phi) \\ S[\Phi] &\equiv \frac{1}{2}\Phi\cdot K^{-1}D\cdot\Phi + \underline{S_I[\Phi]} \end{aligned}$$

$$e^{-S_I[\Phi]} \equiv \int \mathcal{D}\tilde{\phi}\,\mathrm{exp}-(\frac{1}{2}\tilde{\phi}\cdot(1-K)^{-1}D\cdot\tilde{\phi} + \mathcal{S}_I[\Phi+\tilde{\phi}])$$

Standard Ward-Takahashi identity

under the BRS tr. $\delta\phi^A = R^A[\phi]\lambda$

Ward-Takahashi operator

$$\langle 0|[Q_B, \exp(-S + J\cdot\phi)]|0\rangle = 0$$

$$Z_\phi[J] = \int D\phi \exp(-S + J\cdot\phi)$$

$$\langle\Sigma\rangle = Z_\phi^{-1} J\cdot R[\partial_J] Z_\phi[J] = 0$$

Ward-Takahashi id. for the IR theory (Modified Ward-Takahashi id.)

$$
\begin{aligned}
Z_\phi^{-1}(J\cdot R)Z_\phi[J] &= Z_\Phi^{-1}[J]N_J^{-1}(J\cdot R)N_J Z_\Phi[J] \\
&= \langle \Sigma_\Phi \rangle = 0
\end{aligned}
$$

- the normalization term N_J nonlinear effects for WT id.
- the cutoff function $K(p)$ scaling effects

$$
\begin{aligned}
\Sigma_\Phi &= \int_k \left\{ \frac{\partial S}{\partial A_\mu(k)}(-ik_\mu)c(k) + \frac{\partial^r S}{\partial \bar{c}(k)} iB(k) \right\} \\
&\quad -ie\int_{p,k}\left[\frac{\partial^r S}{\partial \psi(p)}\frac{K(p)}{K(p-k)}\psi(p^k) - \frac{K(p)}{K(p+k)}\bar{\psi}(-p-k)\frac{\partial^l S}{\partial \bar{\psi}(-p)}\right]c(k) \\
&\quad -ie\int_{p,k}\left[\frac{\partial^l S}{\partial \bar{\psi}(-p+k)}\frac{\partial^r S}{\partial \psi(p)} - \frac{\partial^l \partial^r S}{\partial \bar{\psi}(-p+k)\partial \psi(p)}\right]U(-p,p-k)c(k)
\end{aligned}
$$

Define the BRS tr. for the IR fields as follow:

$$\begin{aligned}
\delta A_\mu(k) &= -ik_\mu c(k),\ \delta\bar{c} = iB(k),\ \delta c(k) = \delta B(k) = 0, \\
\delta\psi(p) &= ie\int_k c(k)\left[\frac{K(p)}{K(p-k)}\psi(p-k) - U(-p,p-k)\frac{\partial^l S}{\partial\bar{\psi}(-p+k)}\right], \\
\delta\bar{\psi}(-p) &= ie\int_k \frac{K(p)}{K(p+k)}\bar{\psi}(-p-k)c(k)
\end{aligned}$$

- the BRS tr. depends on the action itself.
- It is not nilpotent. $\delta\delta\psi \neq 0$

Then, the modified WT id. can be written as

$$\Sigma_\Phi = \frac{\partial^r S}{\partial\Phi^A}\delta\Phi^A + ie\frac{\partial^l\partial^r S}{\partial\bar{\psi}\partial\psi}cU = 0$$

usual WT id. Jacobian

3. BV formalism and Master equation

Batalin-Vilkoviski（BV） formalism：

Local (global) symmetry

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Action（S）satisfies the quantum Master equation

Introduce the anti-field $\hat{S} \equiv S[\phi] + \phi^* \delta_Q \phi$

classical Master equation: $(\hat{S}, \hat{S}) = 0 \Rightarrow \frac{\partial^r S}{\partial \Phi^A} \delta \Phi^A = 0$ Ward-Takahashi id.

quantum Master equation： $\Sigma[\Phi, \Phi^*] \equiv \frac{1}{2}(S_M, S_M) - \Delta S_M = 0$

This action is the Master action.

$$(X, Y) \equiv \frac{\partial^r X}{\partial \Phi^A} \frac{\partial^l Y}{\partial \Phi^*_A} - \frac{\partial^r X}{\partial \Phi^*_A} \frac{\partial^l Y}{\partial \Phi^A}, \Delta \equiv (-)^{\epsilon_A + 1} \frac{\partial^r}{\partial \Phi^A} \frac{\partial^r}{\partial \Phi^*_A}$$

quantum BRS transformation in anti-field formalism

- Classical (usual) BRS transformation

$$\delta_Q X = (X, \widehat{S})$$

- quantum BRS transformation

$$\delta_Q X \equiv (X, S_M) - \Delta X$$

- Master action is invariant under the quantum BRS transf.
- Nilpotency $\delta_Q^2 X = (X, \Sigma[\Phi, \Phi^*]) = 0$
- The BRS tr. depends on the Master action

Polchinski eq for the Master action:

- Master eq.
- Polchinski eq.

quantum Master equation:

$$\Sigma[\Phi, \Phi^*] \equiv \frac{1}{2}(S_M, S_M) - \Delta S_M = 0$$

The scale dependence of the Master action is

$$\begin{aligned} \partial_t \Sigma &= (\partial_t S_M, S_M) - \Delta \partial_t S_M \\ &= \delta_Q \partial_t S_M = 0 \end{aligned}$$

quantum BRS invariant

The problem is to solve the Master equation and to construct the explicit form of the Master action.

- for chiral lattice theory in interacting fermions Nucl.Phys.B640:95-118,2002. (Igarashi, So and Ukita)

Ginsparg-Wilson relation＝Quantum Master equation

- for gauge theory (QED)

Igarashi, Itoh and Sonoda （IIS） (arXiv:0704.2349)

the power expansion of the anti-fields

$$
\begin{aligned}
S_M[\Phi.\Phi^*] \;&=\; S[\Phi] + \Phi^*_A \delta\Phi^A \\
&\quad + \Phi^*_A \Phi^*_B C^{AB}[\Phi] + \cdots
\end{aligned}
$$

IIS's Master action

Extend the Wilson action to the Master action order by order of the anti-fields.

$$S_M[\Phi.\Phi^*] \;=\; S[\Phi] + \Phi^*_A \delta\Phi^A + \Phi^*_A \Phi^*_B C^{AB}[\Phi] + \cdots$$

the anti-field is the source for modified BRS tr.

The solution of the quantum Master equation in Abelian gauge theory

$$\begin{aligned} S_M[\Phi,\,\Phi^*] \;&=\; \frac{1}{2}\Phi'\cdot K^{-1}D\cdot\Phi' + S_I'[\Phi'] \\ &+ \int_k (A^*_\mu(-k)(-ik^\mu C(k)) + \bar{C}^*(-k)iB(k)) \\ &+ ie\int_{p,k}\left(K(p)\Psi^*(-p)C(k)\frac{\Psi(p-k)}{K(p-k)} + \frac{\bar{\Psi}(p-k)}{K(p-k)}K(p)\bar{\Psi}^*(-p)C(k)\right) \end{aligned}$$

$$\begin{aligned} \Phi'^A \;&=\; \{A_\mu,\; B,\; C,\; \bar{C},\; \Psi,\; \bar{\Psi}'\}\,, \\ \bar{\Psi}'(-p) \;&=\; \bar{\Psi}(-p) - ie\int_k \Psi^*(-p-k)C(k)U(-p-k,\,p) \end{aligned}$$

Remarks on IIS Master action

- only the anti-fermion field is shifted
- only linear dependence of the anti-field

4.Our method

T.Higashi,E.I and T.Kugo :Prog. Theo. Phys. 118 (2007) 1

$$\mathcal{Z}_{\phi}[J,\ \phi^*] = \int \mathcal{D}\phi\ \exp\left(-\mathcal{S}[\phi] + J\cdot\phi - \phi^*\cdot F(\phi)\right)$$

$$\delta_Q \phi^A = F^A(\phi)$$

The anti-field is the source of usual BRS tr.

The action is the Yang-Mills action.

To decompose the IR and UV fields, we insert the gaussian integral.

$$\mathcal{Z}_\phi[J,\,\phi^*] \;=\; N_J \int \mathcal{D}\Phi\mathcal{D}\tilde{\phi}\;\mathrm{exp}-(\frac{1}{2}\Phi\cdot K^{-1}D\cdot\Phi+\frac{1}{2}\tilde{\phi}\cdot(1-K)^{-1}D\cdot\tilde{\phi} \\ +\mathcal{S}_I[\Phi+\tilde{\phi}]+\phi^*\cdot F(\Phi+\tilde{\phi})-J\cdot K^{-1}\Phi)$$

$$Z_\Phi[K^{-1}J,\,\Phi^*]=\int\mathcal{D}\Phi\;\mathrm{exp}\left(-S[\Phi,\,\Phi^*]+K^{-1}J\cdot\Phi\right)$$

$$S[\Phi,\,\Phi^*]\equiv\frac{1}{2}\Phi\cdot K^{-1}D\cdot\Phi+S_I[\Phi,\,\Phi^*]$$

$$e^{-S_I[\Phi,\,\Phi^*]}\equiv\int\mathcal{D}\tilde{\phi}\;\mathrm{exp}-(\frac{1}{2}\tilde{\phi}\cdot(1-K)^{-1}D\cdot\tilde{\phi}+\mathcal{S}_I[\Phi+\tilde{\phi}]+K\Phi^*\cdot\,F(\Phi+\tilde{\phi}))$$

Now $S[\Phi,\,\Phi^*]$ is the Wilsonian action which includes the anti-fields.

Ward-Takahashi identity

The action and the anti-field term are the BRS invariant, then the external source term is remained.

$$J\cdot K^{-1}\frac{\delta^l}{\delta\Phi^*}Z_\Phi[K^{-1}J,\ \Phi^*] = \langle J\cdot K^{-1}\frac{\delta^l S}{\delta\Phi^*}\rangle_{K^{-1}J,\Phi^*} = 0$$

Act the total derivative on the identity.

$$0 \;=\; \int \mathcal{D}\Phi\,\frac{\delta^r}{\delta\Phi^A}\left(\frac{\delta^l S}{\delta\Phi^*_A}e^{(-S[\Phi,\,\Phi^*]+K^{-1}J\cdot\Phi)}\right)$$

$$\langle\frac{\delta^r\delta^l S}{\delta\Phi^A\delta\Phi^*_A} - \frac{\delta^r S}{\delta\Phi^A}\frac{\delta^l S}{\delta\Phi^*_A}\rangle_{K^{-1}J,\ \Phi^*} = 0$$

The Wilsonian action satisfies the Master equation.

Construction of the Master action (QED)

$$\mathcal{Z}_\phi[J,\,\phi^*] \;=\; N_J \int \mathcal{D}\Phi \mathcal{D}\tilde{\phi}\, \exp -(\frac{1}{2}\Phi \cdot K^{-1} D \cdot \Phi + \frac{1}{2}\tilde{\phi}\cdot(1-K)^{-1} D \cdot \tilde{\phi} + \mathcal{S}_I[\Phi+\tilde{\phi}] + \phi^* \cdot F(\Phi+\tilde{\phi}) - J\cdot K^{-1}\Phi)$$

The linear term of UV fields can be absorbed into the kinetic terms by shifting the integration variables:

$$\Phi \to \Phi' = \Phi - (f(\Phi)\cdot\Phi^*)$$

$$S[\Phi,\,\Phi^*] \;=\; \frac{1}{2}\Phi' \cdot K^{-1} D \cdot \Phi' + S_I'[\Phi'] + (\text{linear terms in } \Phi^*) + (\text{quadratic terms in } \Phi^*)$$

$$\begin{cases} A'_\mu(k) = A_\mu(k) + \dfrac{k_\mu}{k^2}(1-K(k))K(k)\bar{C}^*(k), \\ \Psi'(p) = \Psi(p) - ie\dfrac{1-K(p)}{p^\mu\gamma_\mu + m}\displaystyle\int_k K(p-k)\bar{\Psi}^*(p-k)C(k), \\ \bar{\Psi}'(-p) = \bar{\Psi}(-p) - ie\displaystyle\int_k K(p+k)\Psi^*(-p-k)C(k)\dfrac{1-K(p)}{p^\mu\gamma_\mu + m} \end{cases}$$

(linear terms in Φ^*)

$$= K\Phi^* \cdot F(\Phi') + \Phi' \cdot K^{-1}D \cdot (f(\Phi)\cdot\Phi^*)$$

$$= \int_k (K(k)A^*_\mu(-k)(-ik^\mu C(k)) + \bar{C}^*(-k)iB(k))$$

$$+ ie\int_{p,k}\left(K(p)\Psi^*(-p)C(k)\frac{\Psi'(p-k)}{K(p-k)} + \frac{\bar{\Psi}'(p-k)}{K(p-k)}K(p)\bar{\Psi}^*(-p)C(k)\right)$$

(quadratic terms in Φ^*)

$$= K\Phi^* \cdot F'(\Phi')(f(\Phi)\cdot\Phi^*) + \frac{1}{2}(f(\Phi)\cdot\Phi^*)\cdot\left[-\frac{1}{1-K} + \frac{1}{K}\right] D\cdot(f(\Phi)\cdot\Phi^*)$$

Remarks on our Master action

- the gauge field and fermion field are also shifted.
- there are quadratic term of the anti-field.

Relation between IIS's and our Master action

Master action is defined by the solution of the Master equation.

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there is a freedom of doing the canonical transformation in the field and anti-field space.

We found the following functional gives the canonical transformation.

$$
\begin{aligned}
&W[\Phi,\ \Phi^*_{IIS}] \\
&= \int_k \left[A^{*\mu}_{IIS}(-k)(A_\mu(k) + \frac{k_\mu}{k^2} K(k)(1-K(k))\bar{C}^*_{IIS}(k)) + \bar{C}^*_{IIS}(-k)\bar{C}(k) \right] \\
&\quad + \int_p \Big[\Psi^*_{IIS}(-p)(\Psi(p) - ie(1-K(p)) \int_k (p^\mu \gamma_\mu + m)^{-1} K(p-k) \bar{\Psi}^*_{IIS}(p-k) C(k)) \\
&\qquad\qquad + \bar{\Psi}(p) \bar{\Psi}^*_{IIS}(-p) \Big]
\end{aligned}
$$

5. Summary

- We introduce the anti-field as the source term for the usual BRS transformation.
- We show the Wilsonian effective action satisfies the Master eq.
- In the case of abelian gauge theory, we can solve the Master equation.
- We show that our Master action equals to IIS action via the canonical transformation.
- The BRS invariant RG flows exist.

Discussion

- Approximation method (truncate the interaction terms)
- To find the explicit form of the quantum BRS invariant operators.
- To solve the Master eq. for the non-abelian gauge theory

  Because of the non-trivial ghost interaction terms, the quadratic terms of UV field cannot be eliminated.

  The Master action cannot be represented by the shift of the fields.
- We can investigate ････

  4-dim. QCD nonperturbatively,

  the renormalizability of the beyond 4-dimensional gauge theories,

  and so on.