Near BPS Wilson loop in AdS/CFT Correspondence

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Outline

AdS/CFT correspondence and Wilson loop

- The AdS/CFT correspondence states the equivalence of string theory on $AdS_5 \times S^5$ to the $\mathcal{N} = 4$ supersymmetric Yang-Mills.
- According to this correspondence, there exists a map between gauge invariant operators in the field theory and states in the string theory. e.g. half BPS local operators where the dual string states are D-branes in the bulk.
- The Wilson loop operator is another important class of gauge invariant observable which is non-local. It measures the dynamical effects of external quark source and acts as an order parameter for confinement.

- BPS Wilson loop operators in the fundamental representation is dual to a fundamental string worldsheet (minimal surface) ending on the AdS_5 boundary. (Maldacena; Rey, Yee)

$$\langle W \rangle = e^{-\mathrm{Area}}$$

- Later it was found that Wilson loop in the symmetric or antisymmetric representation has supergravity dual in terms of D3-brane or D5-brane with worldvolume RR flux. (Drukker, Fiol; Yamaguchi; Okuyama, Semenoff)
- Most generally, Wilson loop in a general representations (described by a Young tableaus) has a SUGRA dual which is a bound state of a certain array of D3 branes or D5-branes. (Gomis, Passerini)

Wilson loop in $\mathcal{N} = 4$ SYM theory

- A general Wilson loop operator in the $\mathcal{N} = 4$ SYM theory:

$$W_R[C] = \frac{1}{N}\mathrm{Tr}_R\, P \exp\left(\oint_C d\tau (iA_\mu \dot{x}^\mu + \varphi_i \dot{y}^i)\right).$$

  R is the representation of the gauge group $G = U(N)$.

- The loop C is parametrized by the variables:

$$x^\mu(\tau) : \text{actual loop in four dimensions}$$
$$y^i(\tau)) : \text{six arbitrary functions}$$

  $(y^i(\tau))$ can be thought of as determining the shape of the loop in the tranverse 6 dimensions (hidden from 4d point of view).

- $(x^\mu(\tau), y^i(\tau))$ determine the coupling to the gauge fields and the scalar fields. It has to satisify the constraint

$$\dot{x}^2 = \dot{y}^2$$

(Drucker, Gross, Ooguri)

The constraint can be understood in a number of ways:

- Dynamical effect of external quark. Breaking $U(N+1) \to U(N) \times U(1)$. The gauge boson in the adjoint decomposes into a gauge boson and a quark in fundamental representation. The Wilson loop can be derived from a certain correlation function.
- Perturbation theory. The vev of the Wilson loop is divergent unless the constraint is satisfied.
- Requirement of local susy. The scalar field and the gauge boson are in the same $\mathcal{N} = 4$ multiplet. SUSY transformation relates them to each other. The Wilson loop satisfying the constraint is locally half BPS.
- Using minimal surface in $AdS_5 \times S^5$.

i. Perturbation theory

- In general the Wilson loop is divergent unless the constraint is satisfied. To see this, in the leading order of $g_{YM}^2 N$,

$$\langle W \rangle = 1 - g_{YM}^2 N \oint ds \oint ds' \quad \Big[\quad \dot{x}^{\mu}(s)\dot{x}^{\nu}(s')G_{\mu\nu}(x(s) - x(s')) \\ -\dot{y}^{i}(s)\dot{y}^{j}(s')G_{ij}(y(s) - y(s'))\Big],$$

where $G_{\mu\nu}(x) = \frac{\delta_{\mu\nu}}{x^2+\epsilon^2}$, $G_{ij}(x) = \frac{\delta_{ij}}{x^2+\epsilon^2}$ are gauge and scalar prop.

- Contribution from A_μ contains

$$-\frac{\lambda}{8\pi^2}\oint ds \oint_{-\frac{\epsilon}{|\dot{x}|}}^{\frac{\epsilon}{|\dot{x}|}} ds' \dot{x}^{\mu}(s)\dot{x}^{\nu}(s')\frac{\delta_{\mu\nu}}{\epsilon^2} = -\frac{\lambda}{(2\pi)^2\epsilon}\oint ds|\dot{x}| = -\lambda\frac{L}{(2\pi)^2\epsilon}$$

- Similarly, contribution from scalars contains

$$\frac{\lambda}{8\pi^2}\oint ds \oint_{-\frac{\epsilon}{|\dot{x}|}}^{\frac{\epsilon}{|\dot{x}|}} ds' \dot{y}^{\mu}(s)\dot{y}^{\nu}(s')\frac{\delta_{ij}}{\epsilon^2} = \frac{\lambda}{(2\pi)^2\epsilon}\oint ds|\dot{x}|\frac{\dot{y}^2}{\dot{x}^2}$$

- Combining these terms together,

$$\langle W \rangle = 1 - \frac{\lambda}{(2\pi)^2\epsilon}\oint ds|\dot{x}|(1 - \frac{\dot{y}^2}{\dot{x}^2}) + \text{finite}$$

ii. Supersymmetry

The Wilson loop satisfying the constraint is half BPS locally.

- The supersymmetry transformations in $\mathcal{N} = 4$ is given by

$$\delta A_\mu = i\bar{\epsilon}\gamma_\mu\lambda, \quad \delta\phi_I = i\bar{\epsilon}\gamma_I\lambda$$

- Preservation of half of the Poincare supersymmetries yields

$$P\epsilon = (\gamma_\mu \dot{x}^\mu + \gamma_I \dot{y}^I)\epsilon = 0$$

  Therefore there are invariant spinors at each point of the loop iff $\dot{x}^2 = \dot{y}^2$.

- It is natural to associate the UV finiteness of the Wilson loop as being due to the existence of local SUSY. However we will see that this is not true.

iii. Minimal surface in $AdS_5 \times S^5$

- AdS/CFT correspondence states that

$$\langle W[C]\rangle = \int_{\partial X = C} \mathcal{D}X \exp(-\sqrt{\lambda} S[X])$$

- In the large N limit,

$$\langle W[C]\rangle \approx \exp(-\sqrt{\lambda} S)$$

  where S is the Legendre transformation of the minimal surface in AdS bounded by the loop.

- Boundary condition of the string is specified by:

$$X^\mu(\sigma_1, 0) = x^\mu(\sigma_1)$$
$$J_1^\alpha \partial_\alpha Y^i(\sigma_1, 0) = \dot{y}^i(\sigma_1)$$

  where $J_\alpha^\beta = \frac{1}{\sqrt{g}} g_{\alpha\gamma}\epsilon^{\gamma\beta}$ is the complex structure on the worldsheet and $g_{\alpha\beta}$ is the induced metric.

- The loop constraint is most easily derived using the Hamilton-Jacobi form of the equation of equation (more later):

$$G^{IJ}\frac{\delta A}{\delta X^I}\frac{\delta A}{\delta X^J} = G_{IJ}\partial_1 X^I \partial_1 X^J.$$

  One can show that the minimal surface can terminates at the boundary only if the consraint $\dot{x}^2 = \dot{y}^2$ is satisfied.

- Moreover we need to use the Legendre transform of the area functional.

$$\tilde{A} = A - \oint d\sigma_1 P_i Y^i$$

  This new action processes the same EOM and still solved by minimal surface.
  Reason: the Nambu-Goto action is a functional of X^μ and Y^i and is appropriate for full Dirichlet BC. Now with our mixed BC, we need to perform the Legendre transformation.

- However this "regulated" action has different values. In particular, it gives finite area for smooth loop when the constraint is satisfied.

Our motivation

- Recently some new minimal surface, as well as D3 and D5 brane solution have been constructed in the Lunin-Maldacena background. These solutions have been proposed as dual to Wilson loops in the corresponding $\mathcal{N} = 1$ SYM theory.
  However the form of the field theory operators that are in dual with the supergravity configurations has not been identified.
- Supersymmetry is usually a guidance. However since $\mathcal{N} = 1$ SUSY does not mix scalars and gauge bosons, hence there does not seem that locally BPS Wilson loop operator could exist.
- Q. What is the appropiate form of the Wilson loop operator?
- Q. What happens in a more generic less/non supersymmetric setting?

Our goal is to construct the appropriate Wilson loop operator and to study its AdS/CFT correspondence.

β-deformed SYM

- The β-deformation of the $\mathcal{N}=4$ SYM is obtained by replacing the superpotential as

$$\begin{aligned} W &= \mathrm{tr}[\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2] \\ \rightarrow & \quad \mathrm{tr}[e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2], \end{aligned}$$

- The resulting theory preserves $\mathcal{N}=1$ superconformal symmetry and has a global $U(1)\times U(1)$ symmetry

$$\begin{aligned} U(1)_1 : &\qquad (\Phi_1,\Phi_2,\Phi_3) \rightarrow (\Phi_1, e^{i\delta_1}\Phi_2, e^{-i\delta_1}\Phi_3) \\ U(1)_2 : &\qquad (\Phi_1,\Phi_2,\Phi_3) \rightarrow (e^{-i\delta_2}\Phi_1, e^{i\delta_2}\Phi_2, \Phi_3). \end{aligned}$$

  The $U(1)_R$ symmetry acts as

$$U(1)_R : \quad (\Phi_1,\Phi_2,\Phi_3) \rightarrow e^{i\delta}(\Phi_1,\Phi_2,\Phi_3)$$

- All together, the $\mathcal{N}=1$ β-deformed SYM theory is invariant under a $U(1)^3$ symmetry. It's action on the scalar components is

$$\Phi_k \rightarrow e^{i\delta_k}\,\Phi_k, \quad \text{for arbitrary constants } \delta_k, \quad (k=1,2,3).$$

Lunin-Maldacena supergravity background

- Lunin-Maldacena proposed the following SUGRA dual:

$$
\begin{aligned}
ds^2 &= R^2\left[ds^2_{AdS_5} + \sum_i \left(d\mu_i^2 + G\mu_i^2 d\phi_i^2\right) + \hat{\gamma}^2 G\mu_1^2\mu_2^2\mu_3^2\Big(\sum_i d\phi_i\Big)^2\right], \\
e^{2\phi} &= g_s G\,, \\
B &= R^2\hat{\gamma}\; G\,\left(\mu_1^2\mu_2^2 d\phi_1\wedge d\phi_2 + \mu_2^2\mu_3^2 d\phi_2\wedge d\phi_3 + \mu_3^2\mu_1^2 d\phi_3\wedge d\phi_1\right),
\end{aligned}
$$

  $C_2, C_4 \neq 0$

  where $R^4 = 4\pi g_s N$ and $G^{-1} = 1 + \hat{\gamma}^2(\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_3^2\mu_1^2)$.
- The SUGRA description is valid in the limit of $R \gg 1, \quad R\beta \ll 1$ with

$$R^2\beta := \hat{\gamma} \text{ fixed}$$

- The background has the $U(1)^3$ symmetry

$$\phi_k \to e^{i\delta_k}\phi_k, \quad \text{for arbitrary constant } \delta_k, \quad (k = 1, 2, 3).$$

  This is in correspondence with the $U(1)^3$ symmetry of the β-deformed SYM theory.

- For the studies of minimal surface dual to the Wilson loop, it is more convenient to write the metric in the Cartesian form. Introduce Cartesian coords:

$$\begin{aligned} Y^1 &= Y\theta^1 = Y\mu_1 \cos\phi_1, \qquad & Y^4 &= Y\theta^4 = Y\mu_1 \sin\phi_1, \\ Y^2 &= Y\theta^2 = Y\mu_2 \cos\phi_2, \qquad & Y^5 &= Y\theta^5 = Y\mu_2 \sin\phi_2, \\ Y^3 &= Y\theta^3 = Y\mu_3 \cos\phi_3, \qquad & Y^6 &= Y\theta^6 = Y\mu_3 \sin\phi_3. \end{aligned}$$

$$ds^2 = \frac{R^2}{Y^2}\left(\sum_{\mu=0}^{3} dX^\mu dX^\mu + dY^2 + Y^2 d\tilde{\Omega}_5^2\right) = R^2\left(\frac{1}{Y^2}\sum_{\mu=0}^{3} dX^\mu dX^\mu + \sum_{i=1}^{6} G_{ij} dY^i d$$

- The metric satisfies a remarkable identity

$$Y^i G_{ij} Y^j = 1,$$

which leads to

$$\theta^i g_{ij} \theta^j = 1, \quad \text{where } G_{ij} = g_{ij}/Y^2.$$

- Another interesting property of the deformed metric is that

$$\theta^i (\partial_\alpha g_{ij}) \theta^j = 0, \quad \text{where } \partial_\alpha \text{ is an arbitrary derivative.}$$

- The B-field also satisfies an interesting identity:

$$B_{ik} \partial_\sigma Y^k Y^i = 0.$$

Form of the Wilson loop operator

- We propose to consider Wilson loop operator of the same form with the same loop constraint as in the $\mathcal{N} = 4$ theory.
- In perturbation theory, one can show that the Wilson loop operator has a expectation value that is free from UV divergence up to order $(g^2N)^2$ when the constraint is satisfied.

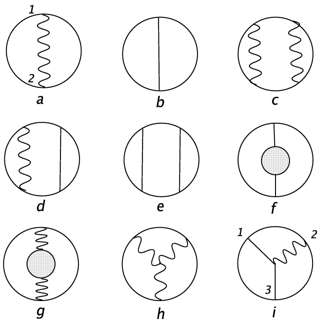

Figure: Feynman diagrams of leading and next-to-leading orders

- The finiteness is not true for a generic non-BPS Wilson loop.
- We call this operator a *near BPS Wilson loop operator*.
- An analogous example is the BMN operator in the $\mathcal{N} = 4$ SYM theory. The BMN operator is not a BPS operator, but it has a finite anomalous dimensions in a particular double scaling limit.

Next let us look at SUGRA side for justification of the constraint.

Hamilton-Jacobi equation

- The HJ eqn is basically the EOM written in Hamiltonian form.
- Consider the action for the string

$$S = \int d^2\sigma \left(\sqrt{\det g} - iB_{IJ}\partial_1 X^I \partial_2 X^J \right)$$

where $g_{\alpha\beta} := G_{IJ}\partial_\alpha X^I \partial_\beta X^J$, $\alpha, \beta = 1, 2$. The conjugate momentum is

$$P_I = \frac{\delta S}{\delta(\partial_2 X^I)} = \frac{1}{\sqrt{g}} G_{IJ}(g_{11}\partial_2 X^J - g_{12}\partial_1 X^J) + iB_{IJ}\partial_1 X^J := \mathcal{P}_I + iB_{IJ}\partial_1 X^J,$$

- The Hamiltonian is

$$H = \frac{\sqrt{g}}{g_{11}}(G^{IJ}\mathcal{P}_I\mathcal{P}_J - g_{11}).$$

And we obtain the HJ equation $H = 0$,

$$G^{IJ}\mathcal{P}_I\mathcal{P}_J = G_{IJ}\partial_1 X^I \partial_1 X^J.$$

Deformed BC

- For the LM background, the Hamilton-Jacobi equation takes the form

$$g_{ij} {J_1}^{\alpha} {J_1}^{\beta} \partial_\alpha Y^i \partial_\beta Y^j + {J_1}^{\alpha} {J_1}^{\beta} \partial_\alpha X^\mu \partial_\beta X^\mu = g_{ij} \partial_1 Y^i \partial_1 Y^j + (\partial_1 X^\mu)^2,$$

- Due to the presence of the B-field, the general mixed boundary condition takes the form

$$J_1^\alpha \partial_\alpha Y^k(\sigma_1, 0) + i{B^k}_l \partial_1 Y^l(\sigma_1, 0) = {\Lambda^k}_l \dot{y}^l(\sigma_1)$$

for some invertible matrix ${\Lambda^k}_l$.

- For a minimal surface to terminate at the boundary of AdS_5, we have the Dirichlet conditions $Y^i(\sigma_1, 0) = 0$, which means $\partial_1 Y^i(\sigma_1, 0) = 0$. And so

$$J_1^\alpha \partial_\alpha Y^k(\sigma_1, 0) = {\Lambda^k}_l \dot{y}^l(\sigma_1).$$

Also we have

$$X^\mu(\sigma_1, 0) = x^\mu(\sigma_1).$$

- Inserting the boundary conditions in the HJ equation we find

$$\dot{x}^2 - {\Lambda^k}_m {\Lambda^l}_n g_{kl} \dot{y}^m \dot{y}^n = ({J_1}^{\alpha} \partial_\alpha X^\mu)^2.$$

- As usual the term $(J_1{}^{\alpha}\partial_{\alpha}X^{\mu})^2$ has to be zero near a smooth boundary, otherwise it costs infinite area. Therefore, we arrived at the constraint

$$\dot{x}^2 = g_{kl}\Lambda^k{}_m\Lambda^l{}_n\,\dot{y}^m\dot{y}^n.$$

- In particular, the constraint derived from supergravity agrees with the constraint derived from field theory considerations if

$$g_{kl}\Lambda^k{}_m\Lambda^l{}_n = \delta_{mn}.$$

This means that the *boundary condition matrix* $\Lambda^k{}_m$ is the vielbein of the deformed metric g_{kl}.

- Moreover, one can identify the UV divergences in the Wilson loop and find that they cancel provided the identities for the metric and the B-field are satisfied:

$$\theta^i g_{ij} \theta^j = 1,$$

$$\theta^i (\partial_\alpha g_{ij}) \theta^j = 0,$$

  and

$$B_{ik} \partial_\sigma Y^k Y^i = 0.$$

- The fact that the UV divergence cancels and a well-defined Wilson loop is obtained for both small and large λ leads us to the conjecture that the Wilson loop is well-defined and has finite vev in the $\mathcal{N} = 1$ β-deformed SYM theory.

BPS Wilson loop and Matrix model

- For $\mathcal{N} = 4$ theory, it is possible to construct globally supersymmetric Wilson. These are circular Wilson loop with a path of radius R_0 in space

  $$x^1 = R_0 \cos \tau, \quad x^2 = R_0 \sin \tau,$$

  and the coupling to the three scalars $\varphi_1, \varphi_2, \varphi_5$ is parametrized by

  $$\theta^1 = \cos \theta_0, \quad \theta^2 = \sin \theta_0 \cos \tau, \quad \theta^5 = \sin \theta_0 \sin \tau,$$

  with an arbitrary fixed θ_0. The operator is $1/2$ BPS ($\theta_0 = 0$) or $1/4$ BPS ($\theta \neq 0$).
- The circular loop is related to the straight line by a conformal transformation, one can therefore relate the circular Wilson loop to the expectation value 1 of the Wilson straight line.
- However due to a conformal anomaly, the gluon propagator is modified by a singular total derivative which gives non-zero contribution only when both ends of the propagator are located at the point which is conformally mapped to the infinity.

- It was conjectured that diagrams with internal vertexes cancel precisely. This is supported by a direct calculation at order $g^4 N^2$.
- Assuming this is true, the sum of all the non-interacting diagrams can be written as a Hermitian matrix model

$$\langle W_R \rangle = \Big\langle \frac{1}{N} \mathrm{Tr}_R \big[e^M \big] \Big\rangle = \frac{1}{Z} \int \mathcal{D}M \frac{1}{N} \mathrm{Tr}_R \big[e^M \big] \exp \big(- \frac{2N}{\lambda} \mathrm{Tr} M^2 \big).$$

This is exact to all order in λ and $1/N$. (Drukker, Gross)

- Explicit evaluation of the integral and hence the Wilson loop expectation value has been performed. For example for fundamental reps.,

$$\langle W_{\mathrm{circular}} \rangle = \frac{2}{\sqrt{\lambda'}} I_1(\sqrt{\lambda'}) + \frac{\lambda'}{48 N^2} I_2(\sqrt{\lambda'}) + \cdots$$

where $\lambda' = \lambda \cos^2 \theta_0$.

- In the large N and large λ limit, this gives

$$\langle W_{\mathrm{circular}} \rangle \sim e^{\sqrt{\lambda} \cos \theta_0}$$

This agrees with the (regulated) area of the dual minimal surface constructed in SUGRA.

SUGRA computation for $\beta \neq 0$

- For the β-deformed case, the circular Wilson loop operator is not BPS.
- The Euclidean AdS_5 metric is

$$ds^2 = du^2 + \cosh^2 u(d\rho^2 + \sinh^2 \rho d\psi^2) + \sinh^2 u(d\chi^2 + \sin^2 \chi d\phi^2). \quad (2)$$

  and we parametrize the deformed sphere's μ_i coord. via

$$\mu_1 = \cos\theta, \quad \mu_2 = \sin\theta\cos\alpha, \quad \mu_3 = \sin\theta\sin\alpha$$

- The action reads

$$\begin{aligned} S &= \frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau\Big[\rho'^2 + \dot{\rho}^2 + \sinh^2\rho(\psi'^2 + \dot{\psi}^2) + \theta'^2 + \dot{\theta}^2 + G\cos^2\theta(\phi_1'^2 + \dot{\phi}_1^2) \\ &\qquad + G\sin^2\theta(\phi_2'^2 + \dot{\phi}_2^2) - 2i\hat{\gamma}G\sin^2\theta\cos^2\theta(\dot{\phi}_1\phi_2' - \phi_1'\dot{\phi}_2)\Big]. \end{aligned}$$

- Consider an ansatz

$$\rho = \rho(\sigma), \quad \psi = \tau, \quad \theta = \theta(\sigma), \quad \phi_1 = \phi_1(\sigma), \quad \phi_2 = \tau.$$

  Solving the equation of motion and Virasoro constraints, we obtain

$$\phi_1' = \hat{\gamma}\sin^2\theta, \quad \rho'^2 = \sinh^2\rho, \quad \theta'^2 = \sin^2\theta.$$

- Substituting back to the action, we find that the G-depedence disappears

$$S_{\rm bulk} = \frac{\sqrt{\lambda}}{2\pi}\int d\sigma d\tau(\sinh^2\rho + \sin^2\theta) = \sqrt{\lambda}(\coth\rho_{\rm max} \mp \cos\theta_0).$$

- Hence the vev is the same as in the undeformed case

$$\langle W\rangle \sim \exp\big(\pm\sqrt{\lambda}\cos\theta_0\big),$$

- Remark: in addition to this supergravity solution which involves 3 angles in S^5, one can also construct a solution which involves only the two angles

$$\theta = \theta(\sigma), \quad \alpha = \tau.$$

  This solution is exactly the same as the undeformed one! Moreover it gives rises to the same expectation value for the dual Wilson loop.
- This is nontrivial since we do not have anymore the $SO(6)$ symmetry!
- Although the $SO(6)$ symmetry is broken by the β-deformation, it seems that the ability to construct $SO(6)$ invariant like constraint is reasonable for this coincidence.

Discussions

- In $\mathcal{N} = 4$ SYM, Wilson loop satisfing the loop constraint is locally BPS and has finite vev. It was previously thought that the existence of local susy is responsible for the finiteness. We show that this is not true.
- For β-deformed SYM, we provided a definition of a Wilson loop operator which has finite vev. The finiteness of the Wilson loop relies on some remarkable properties satisfied by the metric and the B-field of the Lunin-Maldacena background.

- More generally, as long as these identities holds, the SUGRA minimal surface will have finite regulated area, suggesting that the Wilson loop operator has finite vev.
  - For example, one can show that these identities holds also for the multi-parameters β-deformations. These background is non-supersymmetric! On the other hand, those $\mathcal{N} = 1$ SCFT with Sasaki-Einstein dual does not satisify these identities.
- Perhaps it is integrability which guarantee the finiteness of the Wilson loop. Will be interesting to establish this link.