國立清華大學

Electromagnetism

Introduction to Electrodynamics 4th David J. Griffiths

Chap.7

Prof. Tsun Hsu Chang

TA: Hung Chun Hsu, Yi Wen Lin, and Tien Fu Yang

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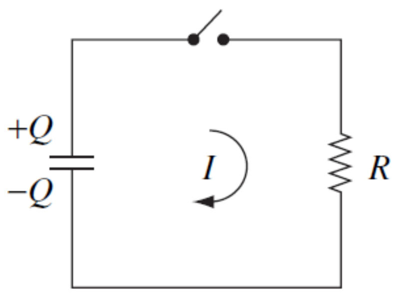
Exercise List

2, 6, 8, 11

Problem 7.2 A capacitor C has been charged up to potential V_0; at time $t = 0$, it is connected to a resistor R, and begins to discharge (Fig. 7.5a).



(a)

(a) Determine the charge on the capacitor as a function of time, $Q(t)$. What is the current through the resistor, $I(t)$?

$$V = IR = \frac{Q}{C},\ \frac{dQ}{dt} = -I = -\frac{V}{R} = -\frac{Q}{RC}$$

$$\Rightarrow Q(t) = Q(0)e^{-\frac{t}{RC}} = CV_0 e^{-\frac{t}{RC}} \Rightarrow I(t) = -\frac{dQ(t)}{dt} = \frac{V_0}{R}e^{-\frac{t}{RC}}$$

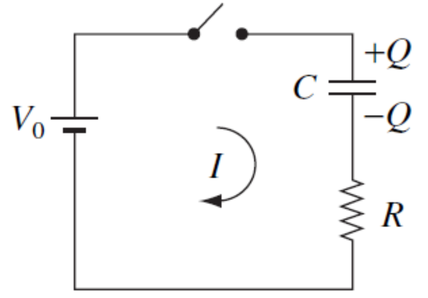
(b) What was the original energy stored in the capacitor (Eq. 2.55)? By integrating Eq. 7.7, confirm that the heat delivered to the resistor is equal to the energy lost by the capacitor.

$$(\text{Eq. } 2.55)\, W = \frac{1}{2}CV^2 = \frac{1}{2}CV_0^2$$

$$(\text{Eq. } 7.7)\, P = I^2R \Rightarrow \int_0^\infty P dt = \int_0^\infty I^2 R dt = \frac{V_0^2}{R}\int_0^\infty e^{-\frac{2t}{RC}} dt = \frac{V_0^2}{R}\left(-\frac{RC}{2}\right)(-1) = \frac{1}{2}CV_0^2$$

Problem 7.2

Now imagine *charging up* the capacitor, by connecting it (and the resistor) to a battery of voltage V_0, at time $t = 0$ (Fig. 7.5b).



(b)

(c) Again, determine $Q(t)$ and $I(t)$.

$$V_0 = \frac{Q}{C} + IR,\ \frac{dQ}{dt} = I = \frac{1}{R}\left(V_0 - \frac{Q}{C}\right) = \frac{CV_0 - Q}{RC} \Rightarrow \frac{dQ}{CV_0 - Q} = \frac{dt}{RC}$$

$$\Rightarrow -\ln\left(CV_0 - Q\right) = \frac{t}{RC} + \underline{const.} \Rightarrow Q(t) = CV_0 - \underline{A}e^{-\frac{t}{RC}} = CV_0\left(1 - e^{-\frac{t}{RC}}\right) \Rightarrow I(t) = \frac{dQ(t)}{dt} = \frac{V_0}{R}e^{-\frac{t}{RC}}$$

(d) Find the total energy output of the battery ($\int V_0 I\, dt$). Determine the heat delivered to the resistor. What is the final energy stored in the capacitor? What fraction of the work done by the battery shows up as energy in the capacitor? [Notice that the answer is independent of R!]

$$\text{(Battery)}\ P = IV \Rightarrow \int_0^\infty IVdt = \frac{V_0^2}{R}\int_0^\infty e^{-\frac{t}{RC}}dt = \frac{V_0^2}{R}(-RC)(-1) = CV_0^2$$

Heat $= \frac{1}{2}CV_0^2$, since the current I keep the same.

Therefore the fianl energy stored in C is $CV_0^2 - \frac{1}{2}CV_0^2 = \frac{1}{2}CV_0^2$

Problem 7.6 A rectangular loop of wire is situated so that one end (height h) is between the plates of a parallel-plate capacitor (Fig. 7.9), oriented parallel to the field $\mathbf{E}$. The other end is way outside, where the field is essentially zero. What is the emf in this loop? If the total resistance is R, what current flows? Explain. [*Warning:* This is a trick question, so be careful; if you have invented a perpetual motion machine, there's probably something wrong with it.]

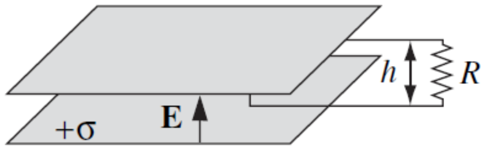


FIGURE 7.9

$$V = \begin{cases} \dfrac{\sigma}{\varepsilon_0} h, & \text{ideally} \\ 0, & \text{considering fringing field} \end{cases}$$

Problem 7.8 A square loop of wire (side a) lies on a table, a distance s from a very long straight wire, which carries a current I, as shown in Fig. 7.18.

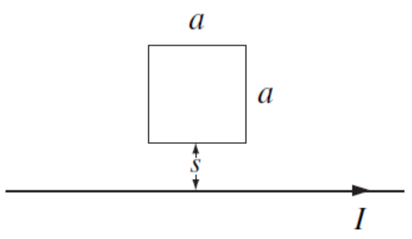


FIGURE 7.18

(a) Find the flux of **B** through the loop.

$$\text{Flux} = \int \mathbf{B}\cdot d\mathbf{a} = \int_0^a \int_s^{s+a} \frac{\mu_0 I}{2\pi s'} ds' dl' = \frac{\mu_0 I a}{2\pi} \ln\frac{s+a}{s}$$

(b) If someone now pulls the loop directly away from the wire, at speed v, what emf is generated? In what direction (clockwise or counterclockwise) does the current flow?

$$\frac{ds}{dt} = v$$

$$V = -\frac{d(\text{Flux})}{dt} = \frac{\mu_0 I a}{2\pi}\frac{d}{dt}\left(\ln\frac{s+a}{s}\right) = \frac{\mu_0 I a}{2\pi}\frac{v/s - v(s+a)/s^2}{s+a/s} = \frac{\mu_0 I a}{2\pi}\frac{sv - sv + av}{s^2 + sa} = \frac{\mu_0 I a^2 v}{2\pi s(s+a)}$$

(c) What if the loop is pulled to the *right* at speed v?

$$V = 0$$

Problem 7.11 A square loop is cut out of a thick sheet of aluminum. It is then placed so that the top portion is in a uniform magnetic field **B**, and is allowed to fall under gravity (Fig. 7.20). (In the diagram, shading indicates the field region; **B** points into the page.) If the magnetic field is 1 T (a pretty standard laboratory field), find the terminal velocity of the loop (in m/s). Find the velocity of the loop as a function of time. How long does it take (in seconds) to reach, say, 90% of the terminal velocity? What would happen if you cut a tiny slit in the ring, breaking the circuit? [*Note:* The dimensions of the loop cancel out; determine the actual *numbers*, in the units indicated.]

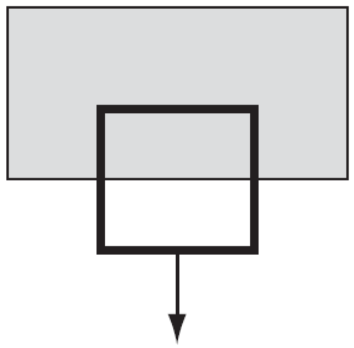


FIGURE 7.20

$$mg - F_B = ma, \quad |\mathbf{F_B}| = \int |\mathbf{I}\times\mathbf{B}|\,dl = IBL \quad V = \frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{a} = BLv = IR$$

$$\Rightarrow mg - IBL = mg - \frac{BLv_t}{R}BL = 0 \Rightarrow \boxed{v_t = \frac{mgR}{B^2L^2}}$$

$$\Rightarrow mg - \frac{(BL)^2 v}{R} = m\frac{dv}{dt} \Rightarrow \frac{dv}{dt} = g - \frac{(BL)^2 v}{mR} \Rightarrow v(t) = \frac{g}{(BL)^2/mR}\left(1 - e^{\frac{-(BL)^2 t}{mR}}\right) = \boxed{v_t\left(1 - e^{\frac{-B^2L^2 t}{mR}}\right)}$$

$$t_{90\%} \Rightarrow \frac{v(t)}{v_t} = 0.9 = 1 - e^{\frac{-B^2L^2 t_{90\%}}{mR}} \Rightarrow t_{90\%} = \frac{-mR}{B^2L^2}\ln(0.1) = \boxed{\frac{mR}{B^2L^2}\ln(10)}$$

If we cut the loop: free falling