九十七學年第一學期 PHYS2310 電磁學 期中考試題(共兩頁)

[Griffiths Ch.1-3] **補考** 2008/12/23 10:10am–12:00am, 教師：張存績

記得寫上學號，班別及姓名等。請依題號順序每頁答一題。

◇ Useful formulas

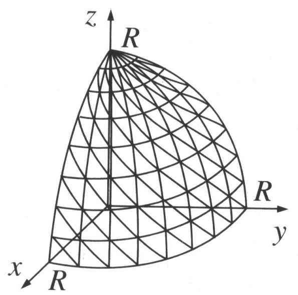
$$\nabla V = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\varphi}} \quad \text{and} \quad \nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}v_\phi$$

1. (8%,12%) $\mathbf{v} = r^2\cos\theta\hat{\mathbf{r}} + r^2\cos\phi\hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\hat{\boldsymbol{\varphi}}$

   (a) Compute $\nabla\cdot\mathbf{v}$.

   (b) Check the divergence theorem using the volume shown in the figure (one octant of the sphere of radius R).

   [Hint: Make sure you include the entire surface.]

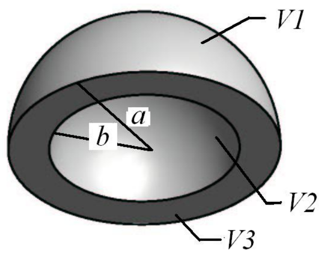


2. (10%, 10%) Suppose the potential at the surface of a hollow hemisphere is specified, as shown in the figure, where $V_1(a,\theta)=0$, $V_2(b,\theta)=V_0(2\cos\theta-5\cos\theta\sin^2\theta)$, $V_3(r,\pi/2)=0$. V_0 is a constant.

   (a) Show the general solution in the region $b \le r \le a$ and determine the potential in the region $b \le r \le a$, using the boundary conditions.

   (b) When $V_2(b,\theta)=V_0\sin\theta$ and $V_1(a,\theta)=V_3(r,\pi/2)=0$, how do you solve this problem? Please explain as detailed as possible.

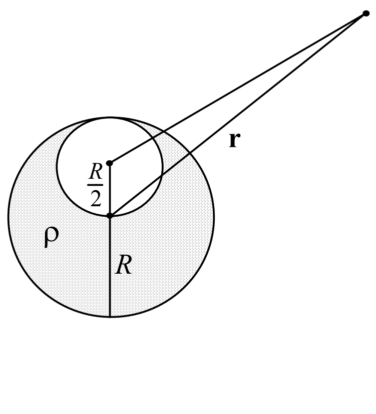
   [Hint: $P_0(x)=1$, $P_1(x)=x$, $P_2(x)=(3x^2-1)/2$, and $P_3(x)=(5x^3-3x)/2$.]



3. (7%, 7%, 6%) The potential of some configuration is given by the expression $V(\mathbf{r}) = Ae^{-\lambda r}/r$, where A and λ are constants.
   (a) Find the energy density (energy per unit volume).
   (b) Find the charge density $\rho(\mathbf{r})$.
   (c) Find the total charge Q (do it two different ways) and verify the divergence theorem.

4. (7%,7%,6%) A uniform line charge λ is placed on an infinite straight wire, a distance d above a grounded conducting plane.
   (a) Find the potential V in the region above the plane.
   (b) Find the surface charge density σ induced on the conducting plane.
   (c) Find the force on the wire per unit length.
   [Hint: Use the method of images.]

5. (8%, 6%, 6%) Consider a hollowed charged sphere with radius R and uniform charge density ρ as shown in the figure. The inner radius of the spherical cavity is $R/2$.
   (a) If the observer is very far from the charged sphere, find the multiple expansion of the potential V in power of $1/r$
   (b) Find the dipole moment $\mathbf{p}$.
   (c) Find the electric field $\mathbf{E}$ up to the dipole term.
   [Note: Specify a vector with both magnitude and direction.]



1.

(a)

$$\mathbf{v} = r^2\cos\theta\hat{\mathbf{r}} + r^2\cos\phi\hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\hat{\boldsymbol{\phi}}$$

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}v_\phi \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta r^2\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi) \\ &= 4r\cos\theta + r\frac{\cos\theta}{\sin\theta}\cos\phi - r\frac{\cos\theta}{\sin\theta}\cos\phi \\ &= 4r\cos\theta\end{aligned}$$

(b)

The divergence theorem $\int_V \nabla\cdot\mathbf{v}d\tau = \oint_S \mathbf{v}\cdot d\mathbf{a}$

$$\begin{aligned}\int_V \nabla\cdot\mathbf{v}d\tau &= \int_0^R\int_0^{\pi/2}\int_0^{\pi/2}(4r\cos\theta)r^2\sin\theta drd\theta d\phi = \frac{\pi}{2}\int_0^R\int_0^{\pi/2}(4r^3\cos\theta\sin\theta)drd\theta \\ &= \frac{\pi}{16}\int_0^R\int_0^{\pi/2}(4\sin 2\theta)dr^4 d\theta = \frac{\pi}{4}R^4\int_0^{\pi/2}\sin 2\theta d2\theta = \frac{\pi}{4}R^4\end{aligned}$$

$\oint_S \mathbf{v}\cdot d\mathbf{a} = xy\text{-plane} + yz\text{-plane} + zx\text{-plane} + \text{curved surface}$

xy-plane: $d\mathbf{a} = -rdrd\theta\hat{\boldsymbol{\phi}},\ \phi = 0,\ \mathbf{v}\cdot d\mathbf{a} = (r^2\cos\theta\sin\phi)rdrd\theta = 0,$

yz-plane: $d\mathbf{a} = rdrd\theta\hat{\boldsymbol{\phi}},\ \phi = \pi/2,\ \mathbf{v}\cdot d\mathbf{a} = -(r^2\cos\theta\sin\phi)rdrd\theta = -r^3\cos\theta drd\theta,\ \int_0^R\int_0^{\pi/2} -(r^3\cos\theta)drd\theta = -\frac{1}{4}R^4$

zx-plane: $d\mathbf{a} = rdrd\phi\hat{\boldsymbol{\theta}},\ \theta = \pi/2,\ \mathbf{v}\cdot d\mathbf{a} = (r^2\cos\phi)rdrd\phi = r^3\cos\phi drd\phi,\ \int_0^R\int_0^{\pi/2}(r^3\cos\phi)drd\phi = \frac{1}{4}R^4$

curved surface: $d\mathbf{a} = R^2\sin\theta d\theta d\phi\hat{\mathbf{r}},\ r = R,\ \mathbf{v}\cdot d\mathbf{a} = (R^2\cos\theta)R^2\sin\theta d\theta d\phi = \frac{R^4}{2}\sin 2\theta d\theta d\phi,$

$$\int_0^{\pi/2}\int_0^{\pi/2}(\frac{R^4}{2}\sin 2\theta)d\theta d\phi = \frac{\pi}{4}R^4$$

$$\oint_S \mathbf{v}\cdot d\mathbf{a} = 0 - \frac{1}{4}R^4 + \frac{1}{4}R^4 + \frac{\pi}{4}R^4 = \frac{\pi}{4}R^4 = \int_V \nabla\cdot\mathbf{v}d\tau$$

2.

(a)

Boundary condition $\begin{cases}\text{(i) } V_1(a,\theta) = 0 \\ \text{(ii) } V_2(b,\theta) = V_0(2\cos\theta - 5\cos\theta\sin^2\theta) = V_0(5\cos^3\theta - 3\cos\theta) = 2V_0P_3 \\ \text{(ii) } V_3(r,\theta = \pi/2) = 0\end{cases}$

General solution $V(r,\theta) = \sum_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-(\ell+1)})P_\ell(\cos\theta)$

B.C. (i) $\rightarrow V(a,\theta)=\sum_{\ell=0}^{\infty}(A_\ell a^\ell + B_\ell a^{-(\ell+1)})P_\ell(\cos\theta)=0 \quad \Rightarrow B_\ell = -A_\ell a^{2\ell+1}$

B.C. (ii) $\rightarrow V(b,\theta)=\sum_{\ell=0}^{\infty}(A_\ell b^\ell + B_\ell b^{-(\ell+1)})P_\ell(\cos\theta)=2V_0P_3(\cos\theta)$

Comparing the coefficiency $\Rightarrow A_3b^3 + B_3b^{-4} = 2V_0,\ A_\ell = B_\ell = 0$ for $\ell = 0,1,2,4,5,...$

B.C. (iii) $\rightarrow V(r,\theta=\frac{\pi}{2})=(A_3r^3+B_3r^{-4})P_3(0)=0$

$\Rightarrow A_\ell = B_\ell = 0$ except $\ell = 3$,

$$A_3 = \frac{2V_0b^4}{b^7-a^7} \text{ and } B_3 = -\frac{2V_0b^4a^7}{b^7-a^7}$$

$$\therefore V(r,\theta)=\left(\frac{2V_0}{b^7-a^7}b^4r^3 - \frac{2V_0}{b^7-a^7}b^4a^7r^{-4}\right)\left(\frac{5\cos^3\theta-3\cos\theta}{2}\right)$$

(b)

Boundary condition $\begin{cases}\text{(i) } V_1(a,\theta)=0\\ \text{(ii) } V_2(b,\theta)=V_0\sin\theta\\ \text{(ii) } V_3(r,\theta=\pi/2)=0\end{cases}$

General solution $V(r,\theta)=\sum_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-(\ell+1)})P_\ell(\cos\theta)$

B.C. (i) $\sum_{\ell=0}^{\infty}(A_\ell a^\ell + B_\ell a^{-(\ell+1)})P_\ell(\cos\theta)=0 \quad \Rightarrow \quad B_\ell = -A_\ell a^{2\ell+1}$

B.C. (iii) $\sum_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-(\ell+1)})P_\ell(0)=0 \quad \Rightarrow$ ℓ=1,3,5,... only odd terms survive

B.C. (ii) $\sum_{\ell=0}^{\infty}A_\ell(b^\ell - \frac{a^{2\ell+1}}{b^{\ell+1}})P_\ell(\cos\theta)=V_0\sin\theta$

$$\int_{-1}^{1}P_\ell(x)P_{\ell'}(x)dx=\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta d\theta = \begin{cases}0 & \text{if } \ell' \neq \ell\\ \frac{2}{2\ell+1}, & \text{if } \ell' = \ell\end{cases}$$

$$\sum_{\ell=0}^{\infty}A_\ell(b^\ell - \frac{a^{2\ell+1}}{b^{\ell+1}})\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta d\theta = \int_0^\pi V_0\sin\theta P_{\ell'}(\cos\theta)\sin\theta d\theta$$

$$A_\ell = (\frac{b^{\ell+1}}{b^{2\ell+1}-a^{2\ell+1}})\frac{2\ell+1}{2}\int_0^\pi V_0\sin\theta P_\ell(\cos\theta)\sin\theta d\theta$$

But $A_\ell = 0$ for ℓ=1,3,5,... It does not make sense. Why?

Add an artifical boundary condition $V_2(b,\theta)=\begin{cases}V_0\sin\theta & \text{for } 0\le\theta\le\frac{\pi}{2}\pi\\ -V_0\sin\theta & \text{for } \frac{\pi}{2}\le\theta\le\pi\end{cases}$

or $V_2(b,\theta)=\begin{cases}V_0\sin\theta & \text{for } 0\le\theta\le\frac{\pi}{2}\pi\\ 0 & \text{for } \frac{\pi}{2}\le\theta\le\pi\end{cases}$

3.

(a)

$$\mathbf{E} = -\nabla V = -A\frac{\partial}{\partial r}(\frac{e^{-\lambda r}}{r})\hat{\mathbf{r}} = -A\left\{\frac{-\lambda r e^{-\lambda r} - e^{-\lambda r}}{r^2}\right\}\hat{\mathbf{r}} = A\frac{(\lambda r+1)e^{-\lambda r}}{r^2}\hat{\mathbf{r}}$$

$$\text{Energy density} = \frac{\varepsilon_0}{2}E^2 = \frac{\varepsilon_0}{2}A^2\frac{(\lambda r+1)^2 e^{-2\lambda r}}{r^4}$$

(b)

$$\rho = \varepsilon_0 \nabla\cdot\mathbf{E} = \varepsilon_0 A(\nabla\cdot\frac{(\lambda r+1)e^{-\lambda r}}{r^2}\hat{\mathbf{r}}) = \varepsilon_0 A(\lambda r+1)e^{-\lambda r}(\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2}) + \varepsilon_0 A\frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla((\lambda r+1)e^{-\lambda r})$$

$$(\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2}) = 4\pi\delta^3(\mathbf{r}) \quad \text{and} \quad (\lambda r+1)e^{-\lambda r}\delta^3(\mathbf{r}) = \delta^3(\mathbf{r})$$

$$\frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla((\lambda r+1)e^{-\lambda r}) = \frac{\hat{\mathbf{r}}}{r^2}\cdot\hat{\mathbf{r}}\frac{\partial}{\partial r}((\lambda r+1)e^{-\lambda r}) = \frac{1}{r^2}\frac{\partial}{\partial r}((\lambda r+1)e^{-\lambda r}) = -\frac{\lambda^2}{r}e^{-\lambda r}$$

$$\rho = \varepsilon_0 A\left[4\pi\delta^3(\mathbf{r}) - \frac{\lambda^2}{r}e^{-\lambda r}\right]$$

(c)

$$Q = \int_v \rho d\tau = \int_v \varepsilon_0 A[4\pi\delta^3(r) - \frac{\lambda^2}{r}e^{-\lambda r}]d\tau = 4\pi\varepsilon_0 A[1 + \int_{r=0}^{\infty}\frac{\lambda^2}{r}e^{-\lambda r}r^2 dr]$$

$$\int_{r=0}^{\infty}\frac{\lambda^2}{r}e^{-\lambda r}r^2 dr = \int_{r=0}^{\infty}e^{-\lambda r}\lambda^2 r dr = -\int_{r=0}^{\infty}\lambda r de^{-\lambda r} = -\int_{x=0}^{\infty}x de^{-x} = -1 \quad \Rightarrow \quad Q = \int_v \rho d\tau = 0$$

Use Gauss's law, the charge enclosed in a sphere of radius R

$$Q_R = \oint_S \varepsilon_0 \mathbf{E}\cdot d\mathbf{a} = 4\pi\varepsilon_0 A(\lambda R+1)e^{-\lambda R} \quad \Rightarrow \quad \text{The total charge } Q_{R\to\infty} = 4\pi\varepsilon_0 A(\lambda R+1)e^{-\lambda R}\Big|_{R=\infty} = 0$$

4.

(a)

Assume the image line charge of $-\lambda$ is placed at a distance d below the plane.

Using the Gauss's law, the electric field outside a line charge λ is $\mathbf{E} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}\hat{\mathbf{r}}$.

So $V = -\int_{r_0}^{r}\mathbf{E}\cdot d\mathbf{l} = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r_0}{r} = V(r) - V_{ref}(r_0)$

$$V = V_+ + V_- = \frac{\lambda}{2\pi\varepsilon_0}\left(\ln\frac{r_0}{\sqrt{(x-d)^2+y^2}} - \ln\frac{r_0}{\sqrt{(x+d)^2+y^2}}\right) = \frac{\lambda}{4\pi\varepsilon_0}\left(\ln\frac{(x+d)^2+y^2}{(x-d)^2+y^2}\right)$$

(b)

$$\sigma = \varepsilon_0 \mathbf{E}\cdot\hat{\mathbf{n}} = \varepsilon_0 E_x = -\frac{\partial}{\partial x}\frac{\lambda}{4\pi}\left(\ln\frac{(x+d)^2+y^2}{(x-d)^2+y^2}\right)\Bigg|_{x=0} = -\frac{\lambda}{4\pi}\left(\frac{2(x+d)}{(x+d)^2+y^2} - \frac{2(x-d)}{(x-d)^2+y^2}\right)\Bigg|_{x=0}$$

$$= -\frac{\lambda}{4\pi}\frac{4d}{d^2+y^2} = -\frac{\lambda}{\pi}\frac{d}{d^2+y^2}$$

Simple check: $\lambda' = \int_{-\infty}^{\infty}\sigma dy = \int_{-\infty}^{\infty} -\frac{\lambda}{\pi}\frac{d}{d^2+y^2}dy$

Let $y = d\tan\theta$, $dy = d\sec^2\theta d\theta$

$$\lambda' = -\frac{\lambda}{\pi}\int_{-\pi/2}^{\pi/2}\frac{d^2\sec^2\theta}{d^2\sec^2\theta}d\theta = -\lambda$$

(c)

$$dF = Edq = E\lambda d\ell$$

$$\frac{dF}{d\ell} = E\lambda = \frac{\lambda}{2\pi\varepsilon_0(2d)}\lambda = \frac{\lambda^2}{4\pi\varepsilon_0 d}$$

5.

(a)

Consider this problem as two charge spheres, one with charge density ρ the other with opposite charge density $-\rho$.

$$V_{big} = \frac{1}{4\pi\varepsilon_0 r}(\rho\frac{4\pi}{3}R^3) \quad \text{and} \quad V_{small} = \frac{1}{4\pi\varepsilon_0\left|\mathbf{r}-\frac{1}{2}\mathbf{R}\right|}(-\rho\frac{4\pi}{3}(\frac{R}{2})^3)$$

$$\frac{1}{\left|\mathbf{r}-\frac{1}{2}\mathbf{R}\right|} = \frac{1}{r}(1+(\frac{\frac{1}{2}R}{r})\cos\theta+\ldots)$$

Using the principle of superposition, we find,

$$V = \frac{1}{4\pi\varepsilon_0 r}(\rho\frac{4\pi}{3}R^3) - \frac{1}{4\pi\varepsilon_0 r}(\rho\frac{4\pi}{3}(\frac{R}{2})^3)(1+(\frac{\frac{1}{2}R}{r})\cos\theta+\ldots)$$

$$= \frac{1}{4\pi\varepsilon_0 r}\frac{7}{8}(\rho\frac{4\pi}{3}R^3) - \frac{1}{4\pi\varepsilon_0 r}(\rho\frac{4\pi}{3}(\frac{R}{2})^3)(\frac{R}{2r})\cos\theta+\ldots, \quad \text{let } Q = \rho\frac{4\pi}{3}R^3$$

$$= \frac{1}{4\pi\varepsilon_0 r}\frac{7Q}{8} - \frac{1}{4\pi\varepsilon_0 r^2}(\frac{Q}{8}\frac{R}{2})\cos\theta+\ldots$$

(b)

$$Q = \rho\frac{4\pi}{3}R^3$$

$$V = \frac{1}{4\pi\varepsilon_0 r}\frac{7Q}{8} - \frac{1}{4\pi\varepsilon_0 r^2}(\frac{Q}{8}\frac{R}{2})\cos\theta+\ldots$$

The first term is the monopole term and the second term is the dipole term.

So the dipole moment $\mathbf{p} = -\frac{QR}{16}\hat{\mathbf{z}}$.

(c)

$$V = \frac{1}{4\pi\varepsilon_0 r}\frac{7Q}{8} - \frac{1}{4\pi\varepsilon_0 r^2}\left(\frac{Q}{8}\frac{R}{2}\right)\cos\theta + \ldots$$

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} - \frac{1}{r\sin\theta}\cancel{\frac{\partial V}{\partial \phi}}\hat{\boldsymbol{\varphi}}$$

$$= \left(\frac{1}{4\pi\varepsilon_0 r^2}\frac{7Q}{8} - \frac{2p}{4\pi\varepsilon_0 r^3}\cos\theta\right)\hat{\mathbf{r}} - \frac{p}{4\pi\varepsilon_0 r^3}\sin\theta\hat{\boldsymbol{\theta}}$$