Chapter 2 Electrostatics

2.1 The Electric Field: 2.1.1 Introduction

What is the force on the test charge Q due to a source charge q?

We shall consider the special case of the **electrostatics** in which all the source charges are stationary.

The **principle of superposition** states that the interaction between any two charges is completely unaffected by the presence of others.

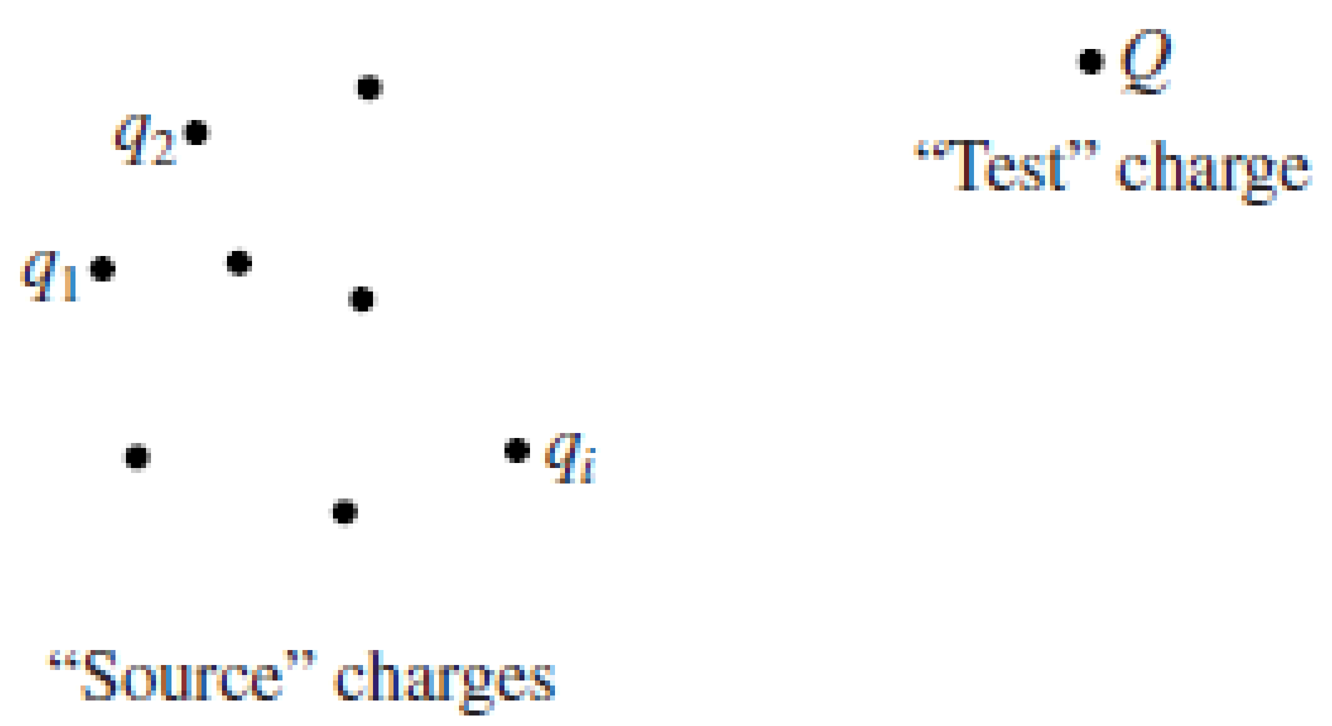

FIGURE 2.1

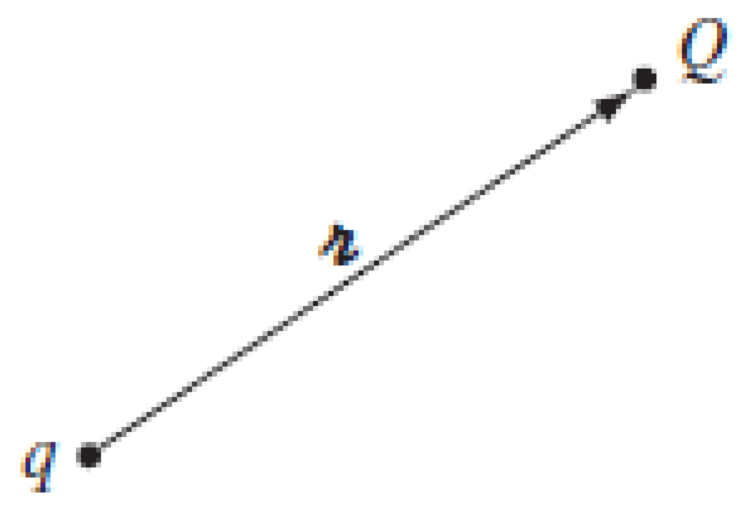

FIGURE 2.2

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2.1.2 Coulomb's Law

Coulomb's law quantitatively describe the interaction of charges.

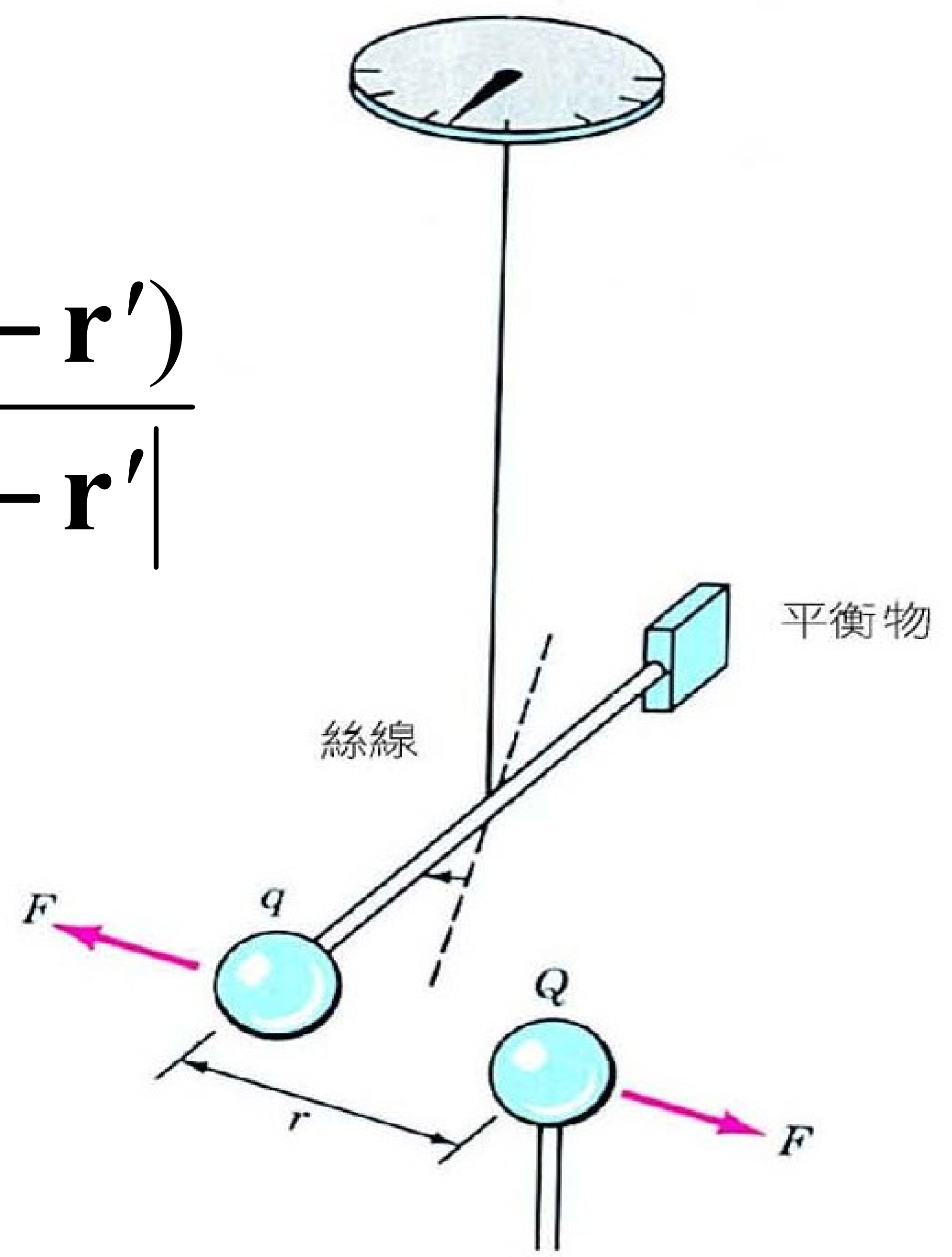
Coulomb determined the force law for electrostatic charges directly by experiment.

$$\mathbf{F} = \frac{kqQ}{\mathscr{r}^2}\hat{\mathscr{r}} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{|\mathbf{r}-\mathbf{r}'|^2}\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\text{Where } k = 9\times10^9\ \frac{\mathrm{N\cdot m^2}}{\mathrm{C^2}}$$

$$\text{and } \varepsilon_0 = 8.85\times10^{-12}\ \frac{\mathrm{C^2}}{\mathrm{N\cdot m^2}}$$

epsilon naught



Action at a distance

Coulomb's law, like Newton's law of gravitation, involves the concept of action at a distance.

It simply states how the particles interact but provides no explanation of the mechanism by which the force is transmitted from one point to the other.

Even Newton himself is not comfortable with this aspect of his theory.

What is the concept of action at a distance? This leads to the gravitational, electric, and magnetic fields.

2.1.3 The Electric Field

How does one particle sense the presence of the other?

The electric charge creates an electric field in the space around it. A second charged particle does not interact directly with the first; rather, it responds to whatever field it encounters. In this sense, the field acts as an *intermediary* between the particles. KK[ˌɪntəˋmidɪˌɛrɪ]

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots = \frac{Q}{4\pi\varepsilon_0}\left(\frac{q_1}{\mathscr{r}_1^{\,2}}\hat{\mathscr{r}}_1 + \frac{q_2}{\mathscr{r}_2^{\,2}}\hat{\mathscr{r}}_2 + \cdots\right) = Q\mathbf{E}$$

$$\text{where } \mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{\mathscr{r}_1^{\,2}}\hat{\mathscr{r}}_1 + \frac{q_2}{\mathscr{r}_2^{\,2}}\hat{\mathscr{r}}_2 + \cdots\right) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^{\,2}}\hat{\mathscr{r}}_i$$

The electric field strength is defined as the force per unit charge placed at that point.

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Example

On a clear day there is an electric field of approximately 100 N/C directed vertically down at the earth's surface. Compare the electrical and gravitational forces on an electron.

$$1\ \mathrm{N/C} = 1\ \mathrm{V/m} \Rightarrow$$

$$1\ \mathrm{N\cdot m} = 1\ \mathrm{C\cdot V} = 1\ \mathrm{J}$$

Solution:

The magnitude of the electrical force is

$F_e = eE = 1.6\times10^{-19}\times100 = 1.6\times10^{-17}$ N. (upward)

The magnitude of the gravitational force is

$F_g = mg = 9.11\times10^{-31}\times9.8 = 8.9\times10^{-30}$ N. (downward)

2.1.4 Continuous Charge Distributions

In order to find the electric field due to a continuous distribution of charge, one must divide the charge distribution into infinitesimal elements of charge dq which may be considered to be point charges.

$$d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\mathscr{r}^2}\hat{\mathscr{r}} \qquad \Rightarrow \qquad \mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{\mathscr{r}^2}\hat{\mathscr{r}}$$

Thus the electric field of a line charge is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_P \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\, dl';$$

for a surface charge,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\, da';$$

and for a volume charge,

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\, d\tau'.}$$

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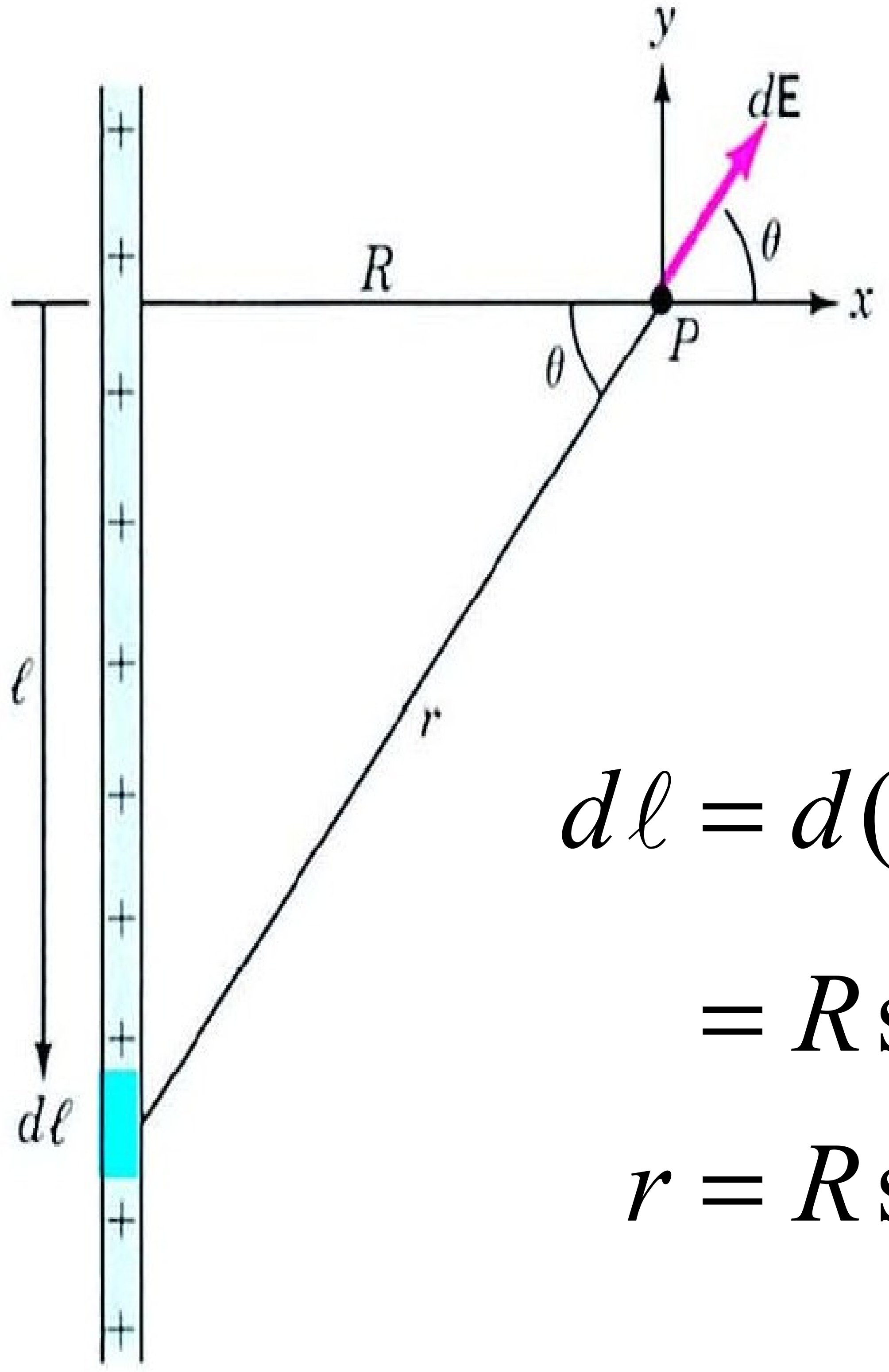
Example 2.1

What is the field strength at a distance R from an infinite line of charge with linear charge density λ C/m.

Solution:

Since the charge carrier is infinitely long, the electric field in y-direction completely cancels out. Thus the resultant field is along the x-axis.



$$d\ell = d(R\tan\theta)$$

$$= R\sec^2\theta d\theta$$

$$r = R\sec\theta$$

$$dE_x = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\cos\theta d\ell}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R\sec^2\theta\cos\theta\ d\theta}{(R\sec\theta)^2}$$

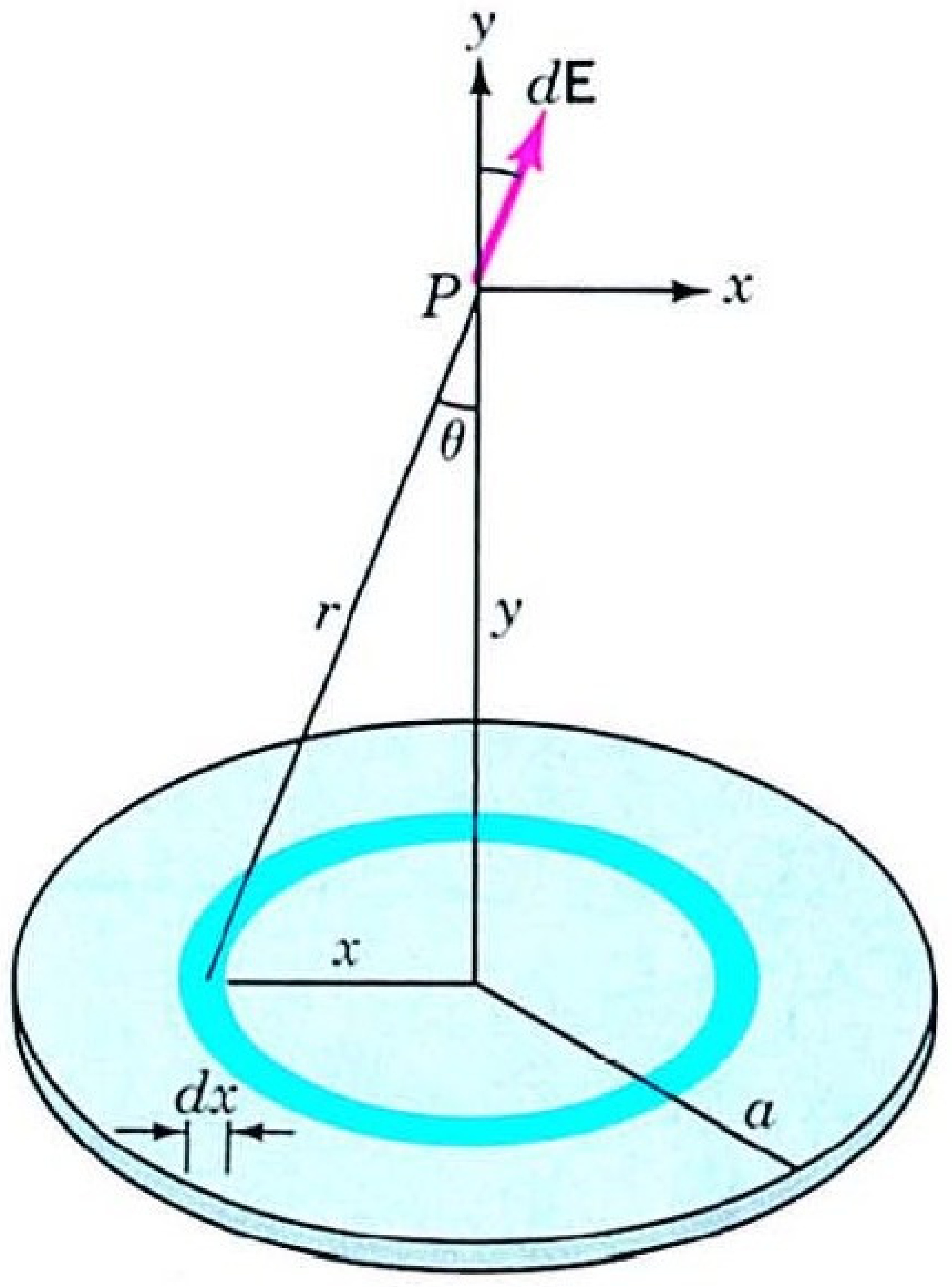
$$= \frac{1}{4\pi\varepsilon_0}\frac{\lambda\cos\theta d\theta}{R}$$

$$E_x = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{R}\int_{-\pi/2}^{\pi/2}\cos\theta d\theta = \frac{\lambda}{2\pi\varepsilon_0 R}$$

Example

Non-conducting disk of radius a has a uniform surface charge density σ C/m². What is the field strength at a distance y from the center *along the central axis*.

Solution:



The y-component of the field is

$$dE_y = dE\cos\theta = \frac{kdq}{r^2}\frac{y}{r}$$

where $r^2 = \mathrm{x}^2 + \mathrm{y}^2$ and $dq = \sigma(2\pi x dx)$

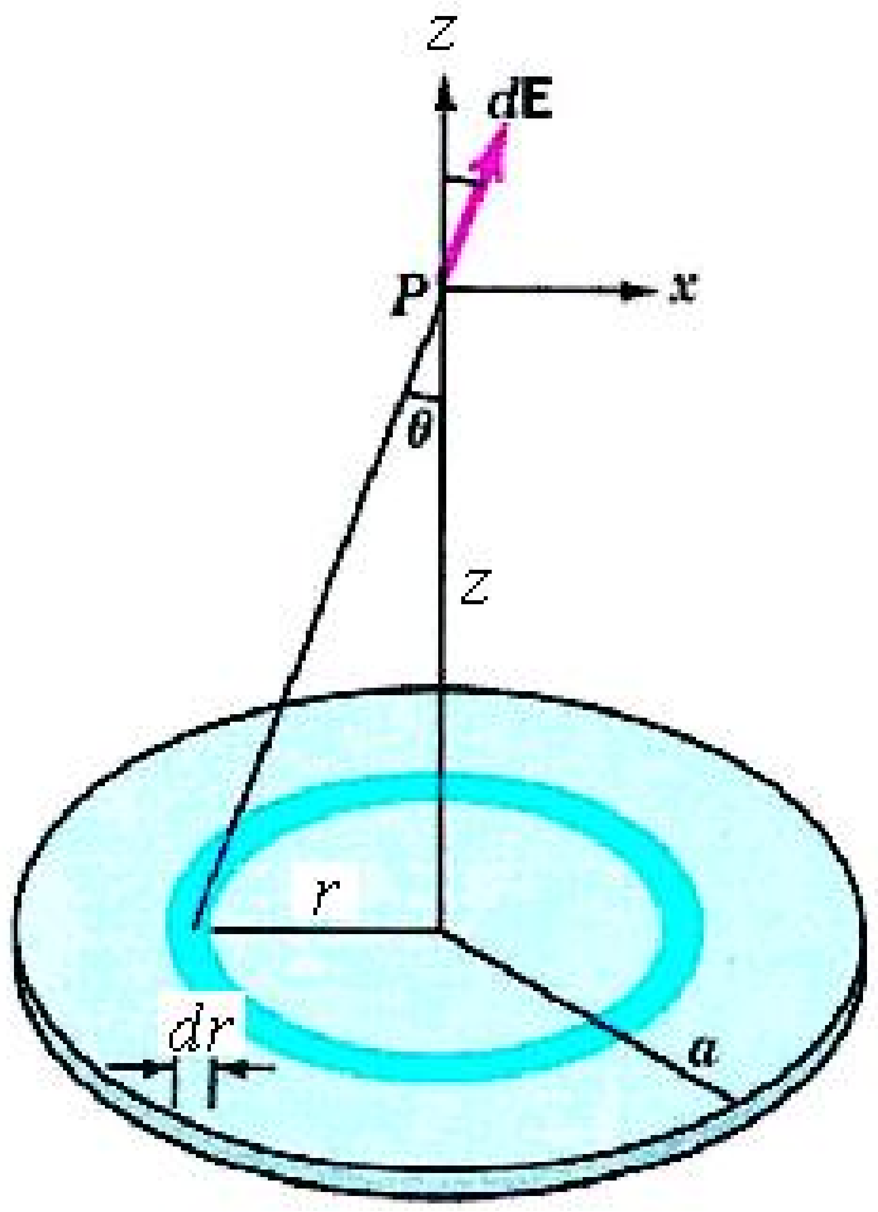
$$E_y = \pi k\sigma y\int_0^a \frac{2xdx}{(x^2+y^2)^{3/2}}$$

$$= \pi k\sigma y\int_0^a \frac{dx^2}{(x^2+y^2)^{3/2}}$$

$$= 2\pi k\sigma y\left(\frac{1}{y} - \frac{1}{\sqrt{y^2+a^2}}\right) = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{y}{\sqrt{y^2+a^2}}\right)$$

Example: Use cylindrical coordinates

Non-conducting disk of radius a has a uniform surface charge density σ C/m². What is the field strength at a distance z from the center *along the central axis*.



Solution: The z-component of the field is

Observer $P=(0,0,z)$ and sources $(x',y',0)$

$$\vec{\mathcal{r}}=(-x',-y',z),\ \mathcal{r}^2=(x'^2+y'^2+z^2)$$

$$\begin{cases} x'=r\cos\phi \\ y'=r\sin\phi \end{cases} \Rightarrow \mathcal{r}^2=(x'^2+y'^2+z^2)=r^2+z^2$$

$$dE_z=dE\cos\theta=\frac{kdq}{(r^2+z^2)}\frac{z}{\sqrt{(r^2+z^2)}},\ \text{where } dq=\sigma(2\pi r dr)$$

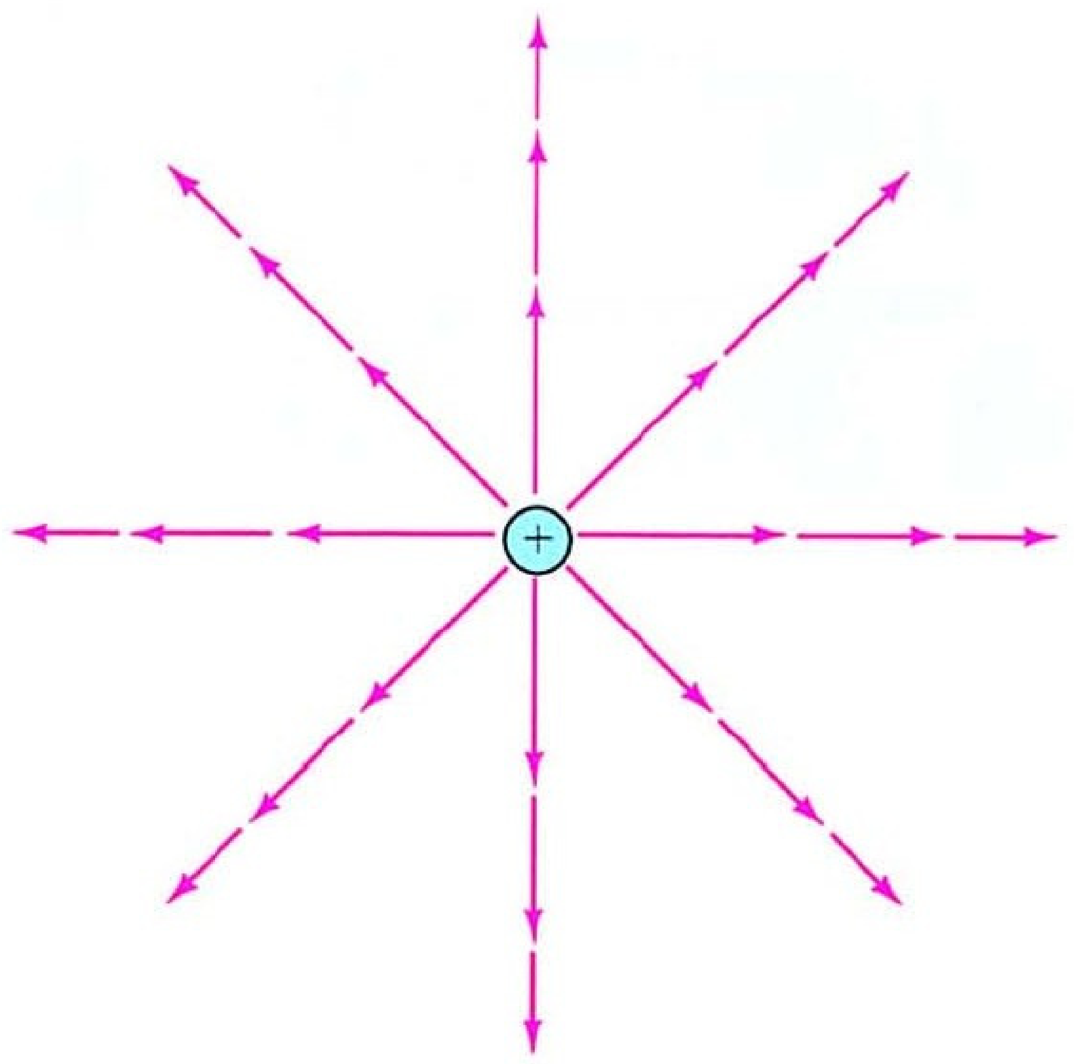
$$E_z=\pi k\sigma z\int_0^a\frac{2rdr}{(r^2+z^2)^{3/2}}=\pi k\sigma z\int_0^a\frac{dr^2}{(r^2+z^2)^{3/2}}=2\pi k\sigma z\left(\frac{1}{\sqrt{z^2}}-\frac{1}{\sqrt{z^2+a^2}}\right)$$

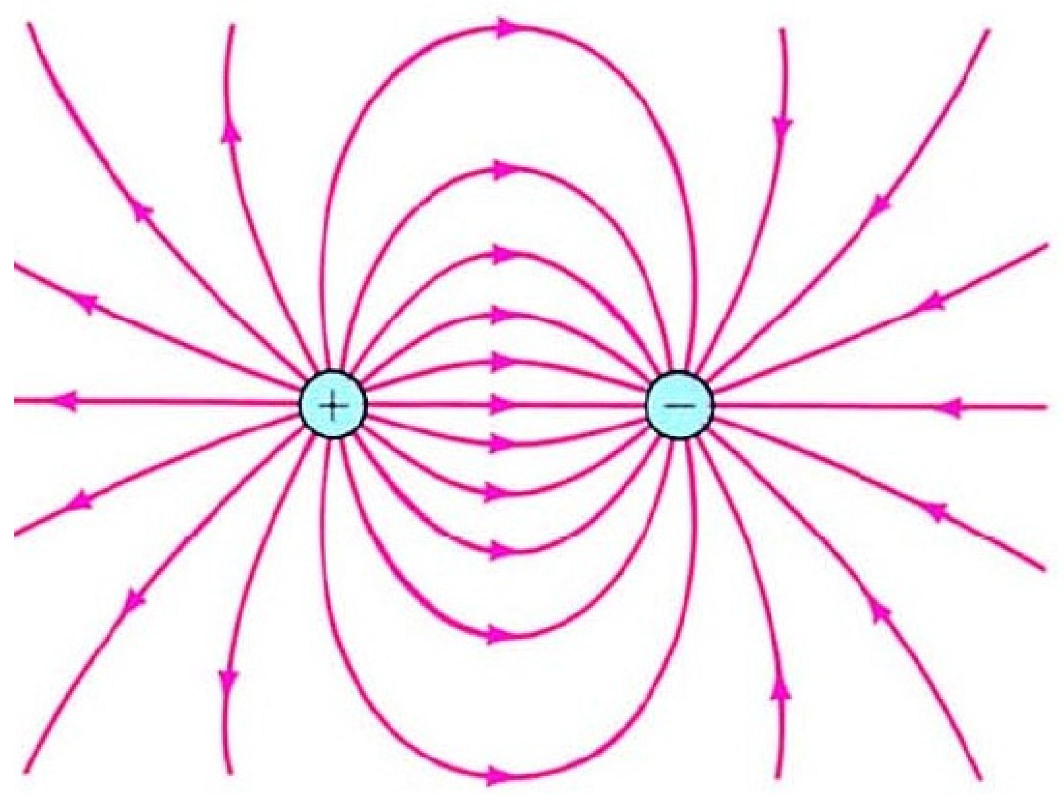
$$=\frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{z^2+a^2}}\right),\ \text{for } z\ge 0$$

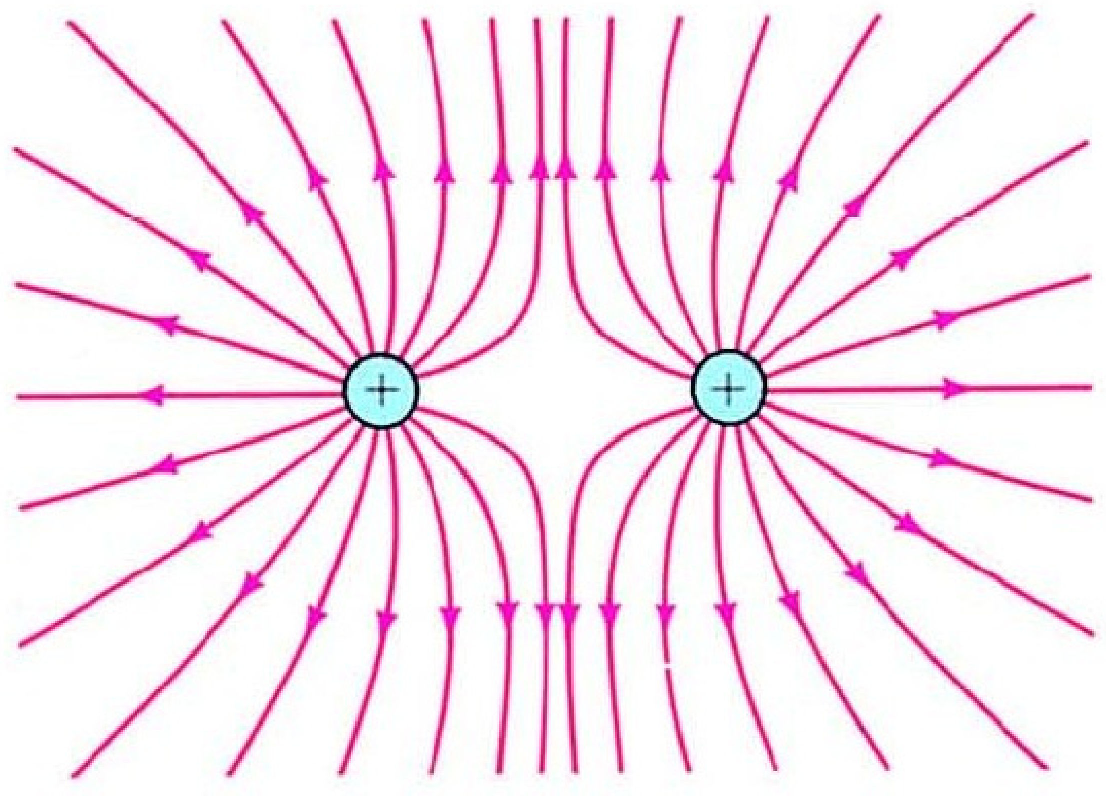
2.2 Divergence and Curl of Electrostatics Fields
2.2.1 Field Lines, Flux, and Gauss's Law

How do we express the magnitude and vector properties of the field strength?

The field strength at any point could be represented by an arrow drawn to scale. However, when several charges are present, the use of arrows of varying length and orientations becomes confusing. Instead we represent the electric field by continuous field lines or lines of force.







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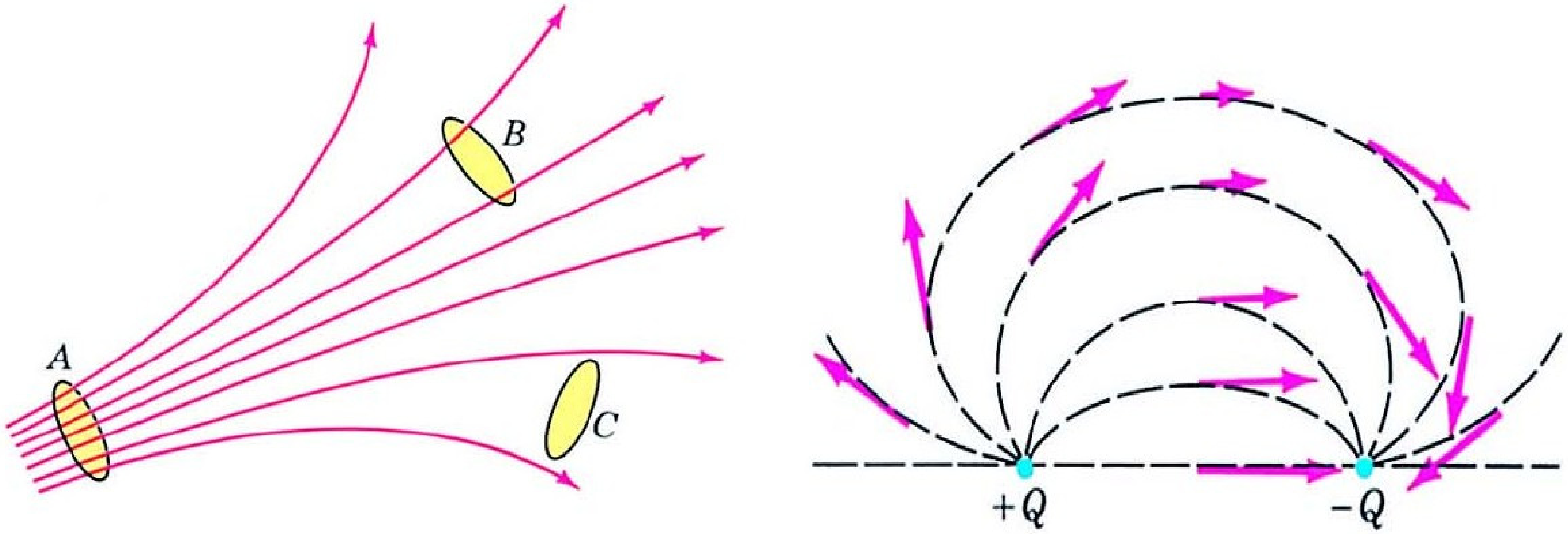
Field Lines

How do we determine the field strength from the field lines?

The lines are crowed together when the field is strong and spread apart where the field is weak. The field strength is proportional to the density of the lines.



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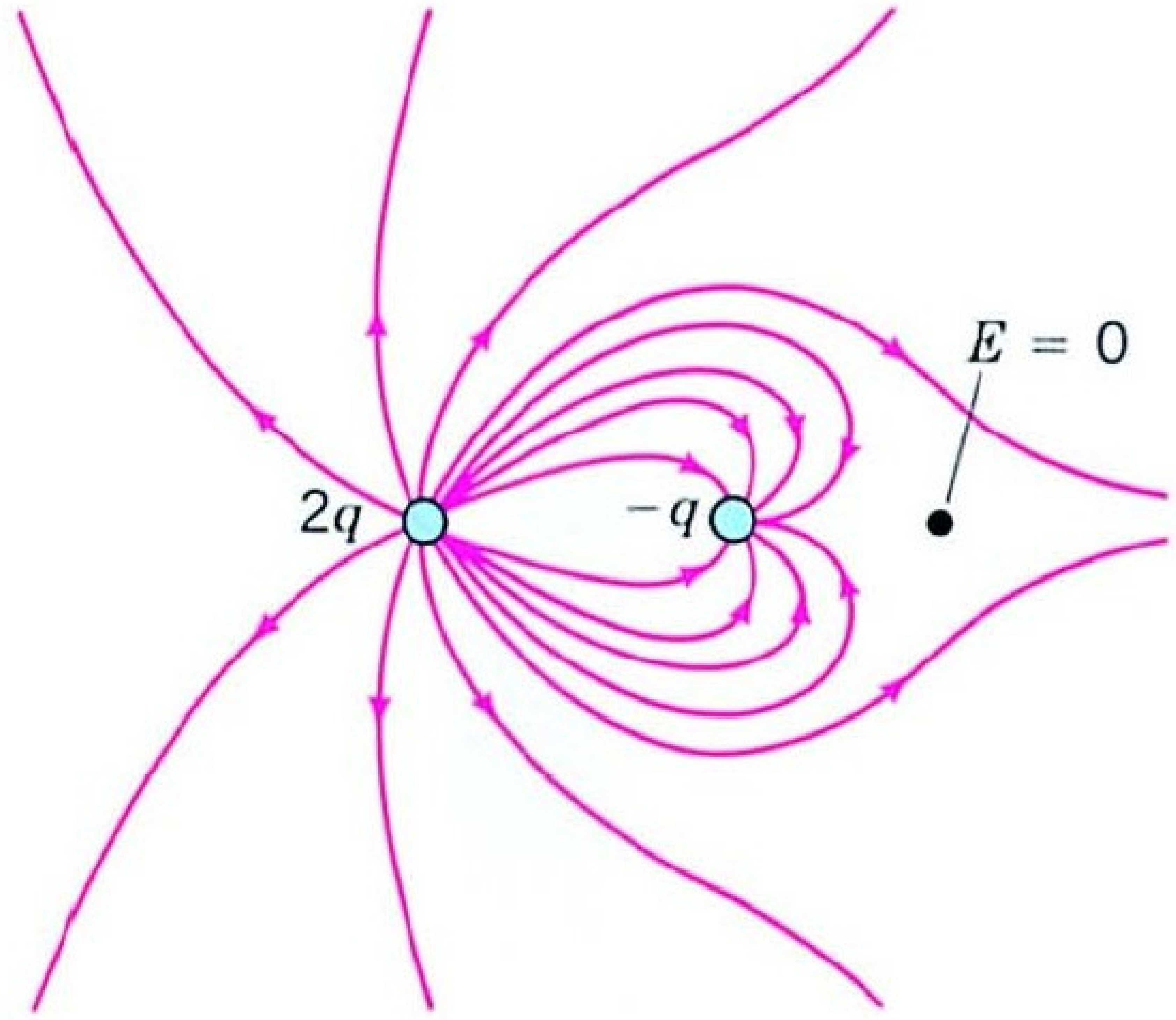
Example

Sketch the field lines for two point charges $2q$ and $-q$.

Solution:



(a)Symmetry

(b)Near field

(c)Far field

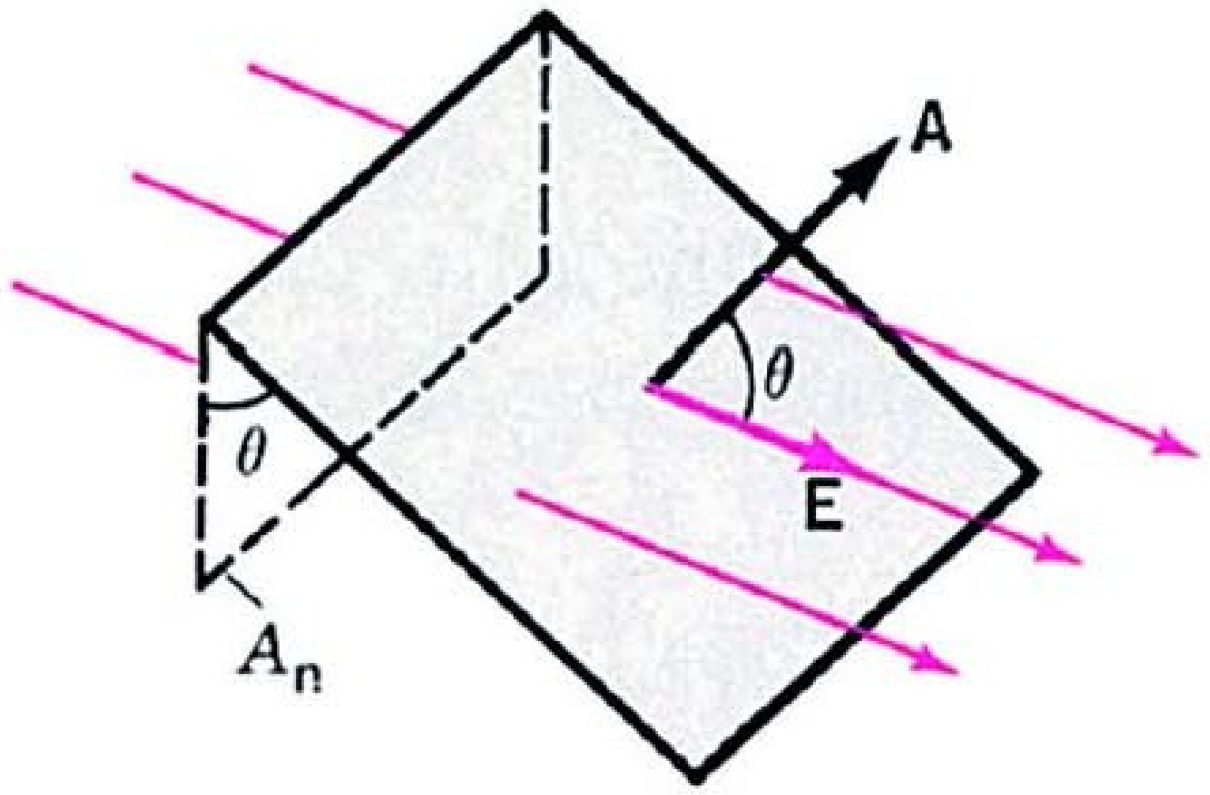
(d)Null point

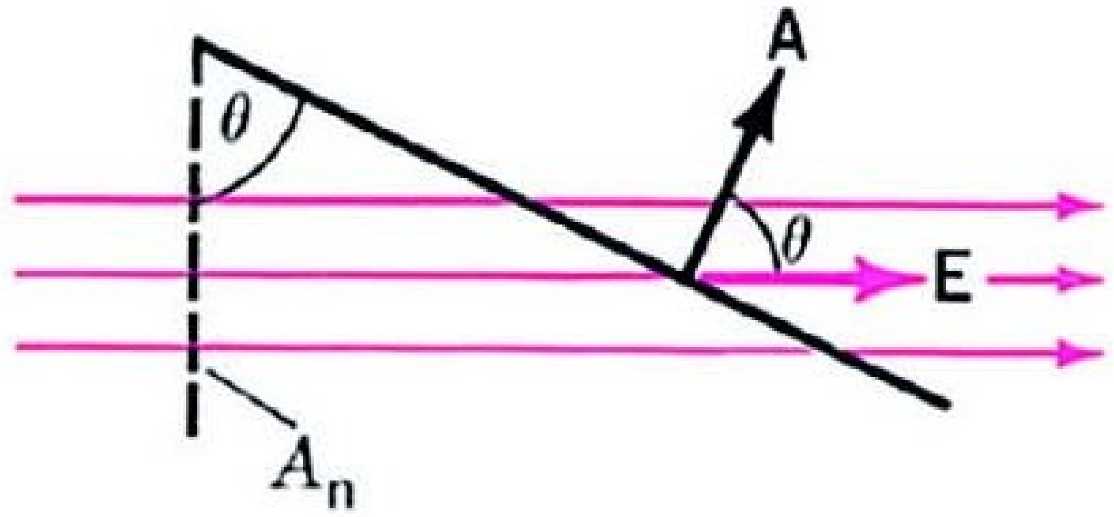
(e)Number of lines

Flux

The electric flux Φ_E through this surface is defined as

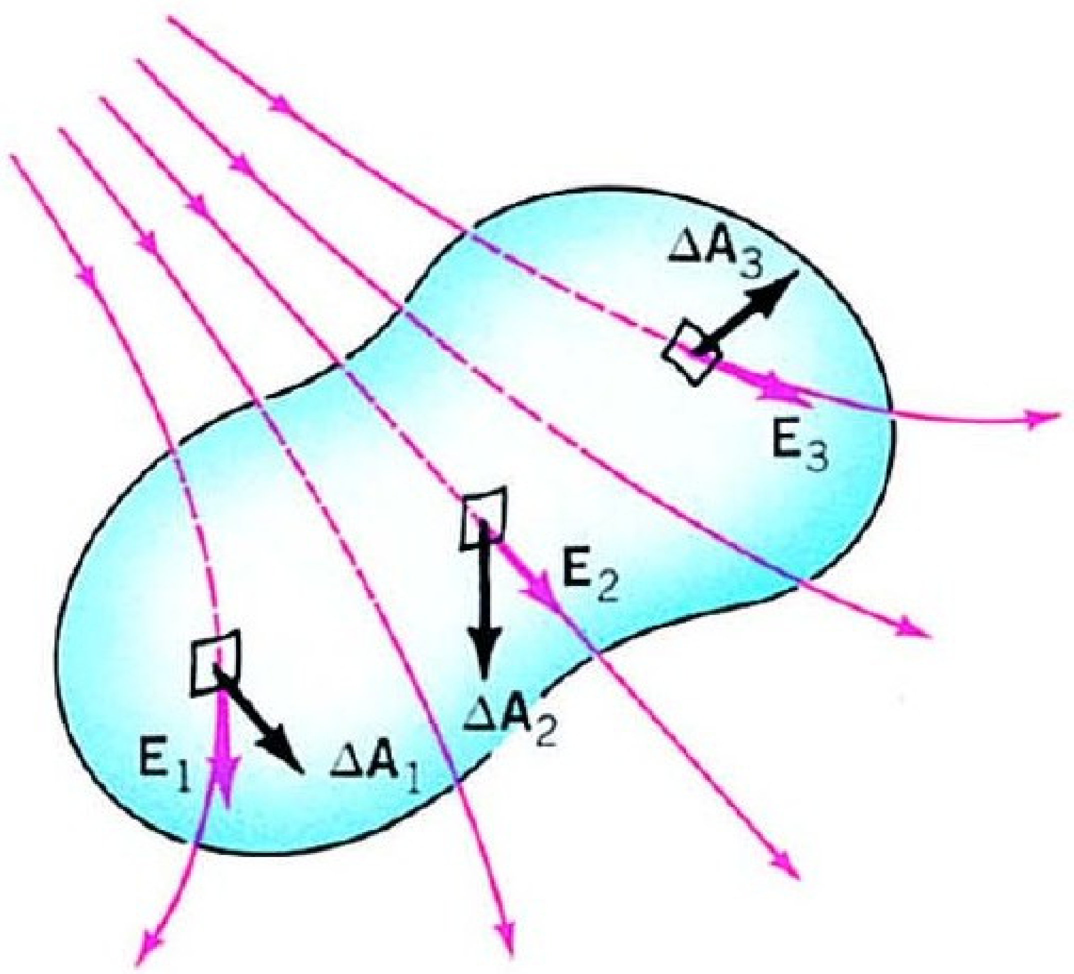
$$\Phi_E = EA\cos\theta$$
$$= \mathbf{E}\cdot\mathbf{A}$$





For a nonuniform electric field

$$\Phi_E = \int \mathbf{E}\cdot\hat{\mathbf{n}}\,da$$



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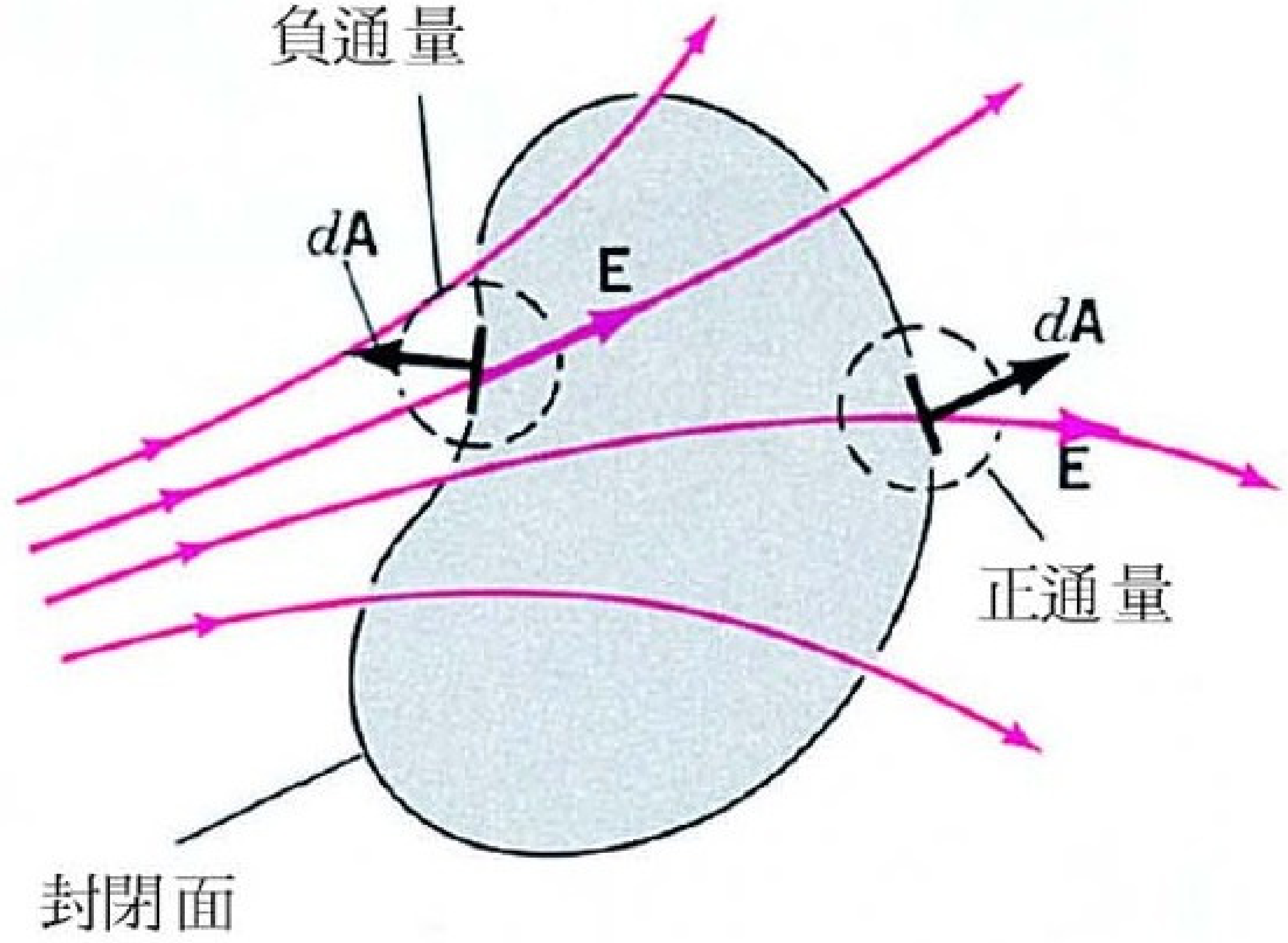
Flux

Flux leaving a closed surface is positive, whereas flux entering a closed surface is negative.

The net flux through the surface is zero if the number of lines that enter the surface is equal to the number that leave.



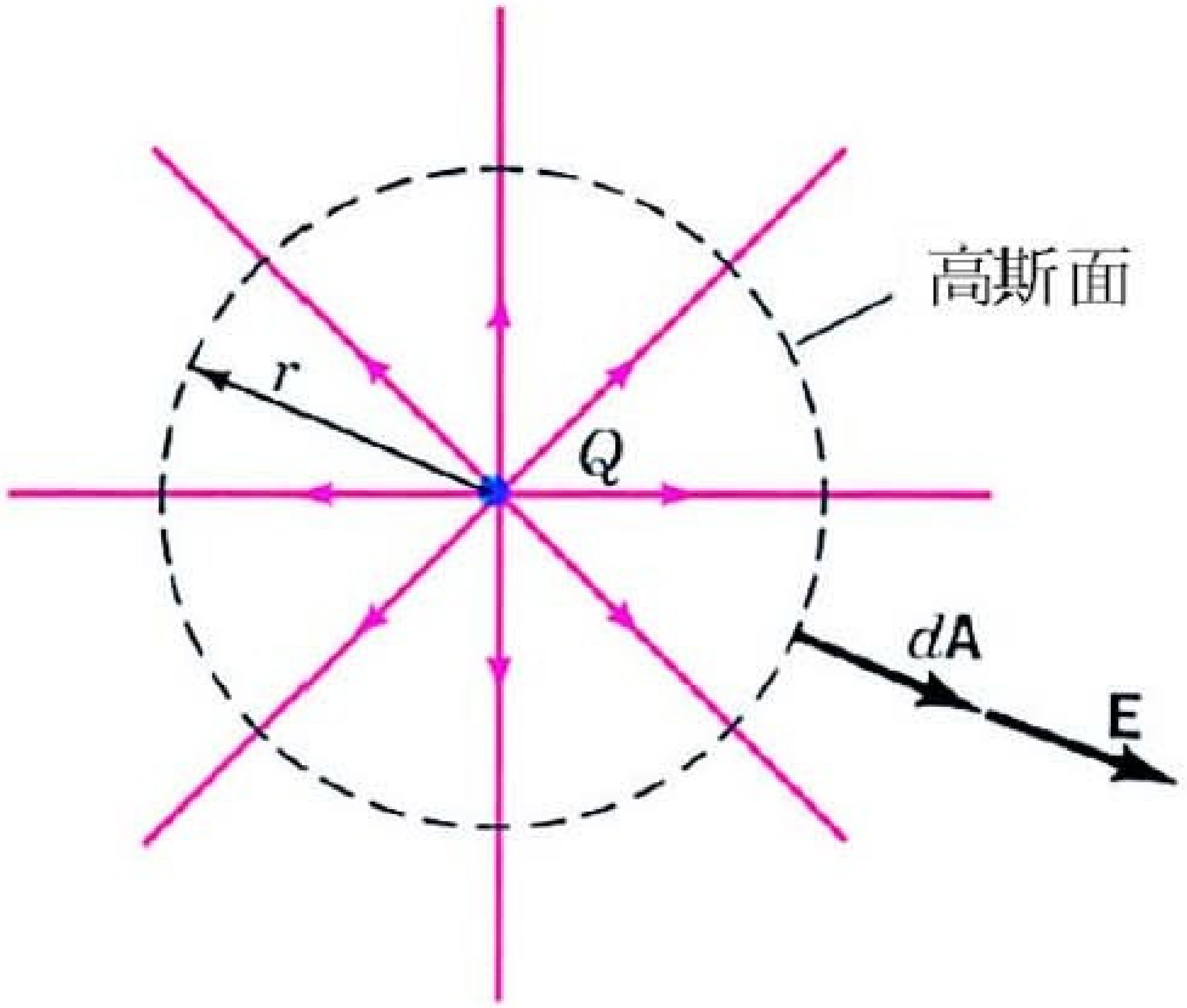
Gauss's Law

How much is the flux for a spherical Gaussian surface around a point charge?

The total flux through this closed Gaussian surface is

$$\Phi_E = \oint \mathbf{E} \cdot \hat{\mathbf{n}} da$$

$$= \frac{Q}{4\pi\varepsilon_0 r^2} \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0}$$



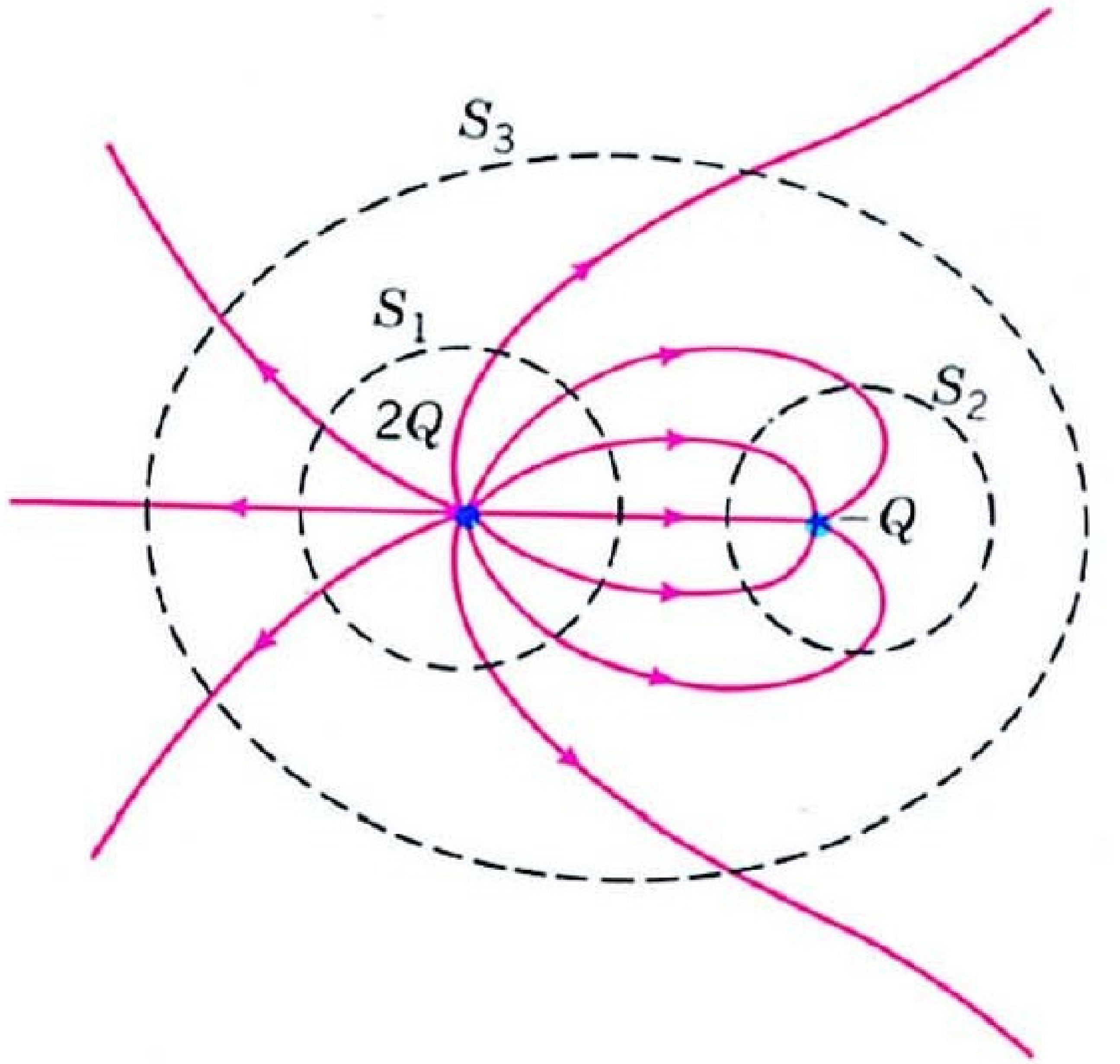
The net flux through a closed surface equals $1/\varepsilon_0$ times the net charge enclosed by the surface.

Can we prove the above statement for arbitrary closed shape?

Gauss's Law (II)

•The circle on the integral sign indicates that the Gaussian surface must be enclosed.

•The flux through a surface is determined by the ***net*** charge enclosed.



How do we apply Gauss's law?

1. Use symmetry.
2. Properly choose a Gaussian surface (E//A or E⊥A).

Turn Gauss's Law
from integral equation into differential form

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \oint_S \mathbf{E} \cdot \hat{\mathbf{n}} da = \frac{Q_{enc}}{\varepsilon_0}$$

By applying the divergence theorem

$$\oint_S \mathbf{E} \cdot \hat{\mathbf{n}} da = \int_v (\nabla \cdot \mathbf{E}) d\tau \quad \text{and} \quad \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_v \rho \, d\tau$$

So $\displaystyle \int_v (\nabla \cdot \mathbf{E}) d\tau = \frac{1}{\varepsilon_0} \int_v \rho d\tau$

Since this holds for any volume, the integrands must be equal

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho$$ $\longleftarrow$ **Gauss's Law in differential form.**

2.2.2 The Divergence of Electric Field & 2.2.3 Application of Gauss's Law

The electric field can be expressed in the following form

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_{\text{all space}} \frac{dq}{\mathscr{r}^2}\hat{\mathscr{r}} = \frac{1}{4\pi\varepsilon_0}\int_{\text{all space}} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho(\mathbf{r}')d\tau'$$

Divergence of the electric field is

Why doesn't the divergence operator apply on the $\mathbf{r}'$ coordinate?

$$\nabla\cdot\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_{\text{all space}} \left(\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\rho(\mathbf{r}')d\tau'$$

Since $\left(\nabla\cdot\dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\vec{\mathscr{r}})$,

$$\nabla\cdot\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_{\text{all space}} 4\pi\delta^3(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')d\tau' = \frac{1}{\varepsilon_0}\rho(\mathbf{r})$$

Example 2.2

A *non-conducting* charged sphere of radius R has a total charge Q uniformly distributed throughout its volume. Find the field (a) inside, and (b) outside the sphere.

Solution:

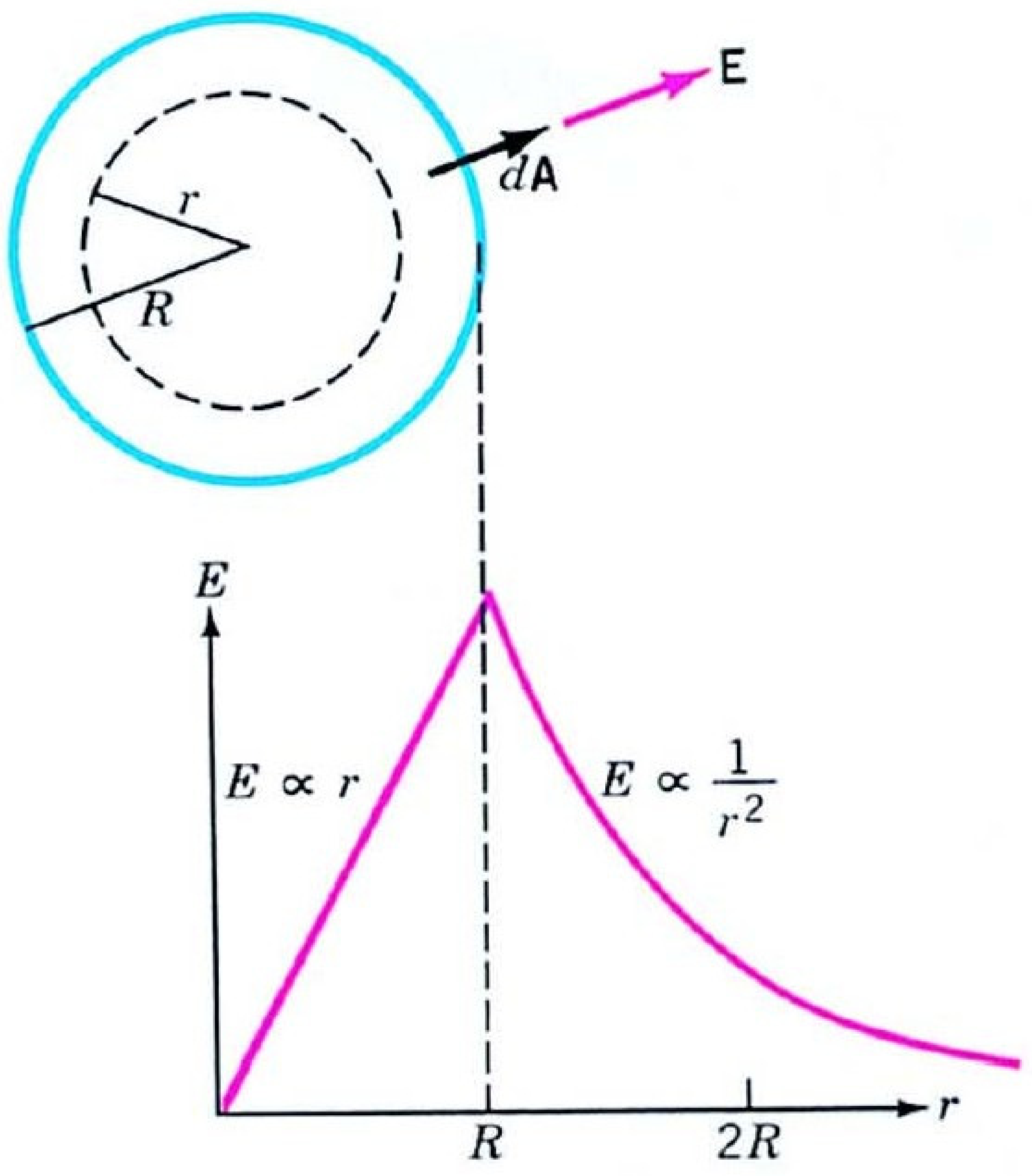
(a) inside

$$\mathbf{E} = \frac{\Phi_{enc}}{4\pi r^2}\hat{\mathbf{r}} = \left(\frac{Q\frac{4}{3}\pi r^3}{\varepsilon_0 \frac{4}{3}\pi R^3}\right)\frac{1}{4\pi r^2}\hat{\mathbf{r}}$$

$$= \frac{Q}{4\pi\varepsilon_0 R^3} r\hat{\mathbf{r}}$$

(b) outside

$$\mathbf{E} = \frac{\Phi}{4\pi r^2}\hat{\mathbf{r}} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$



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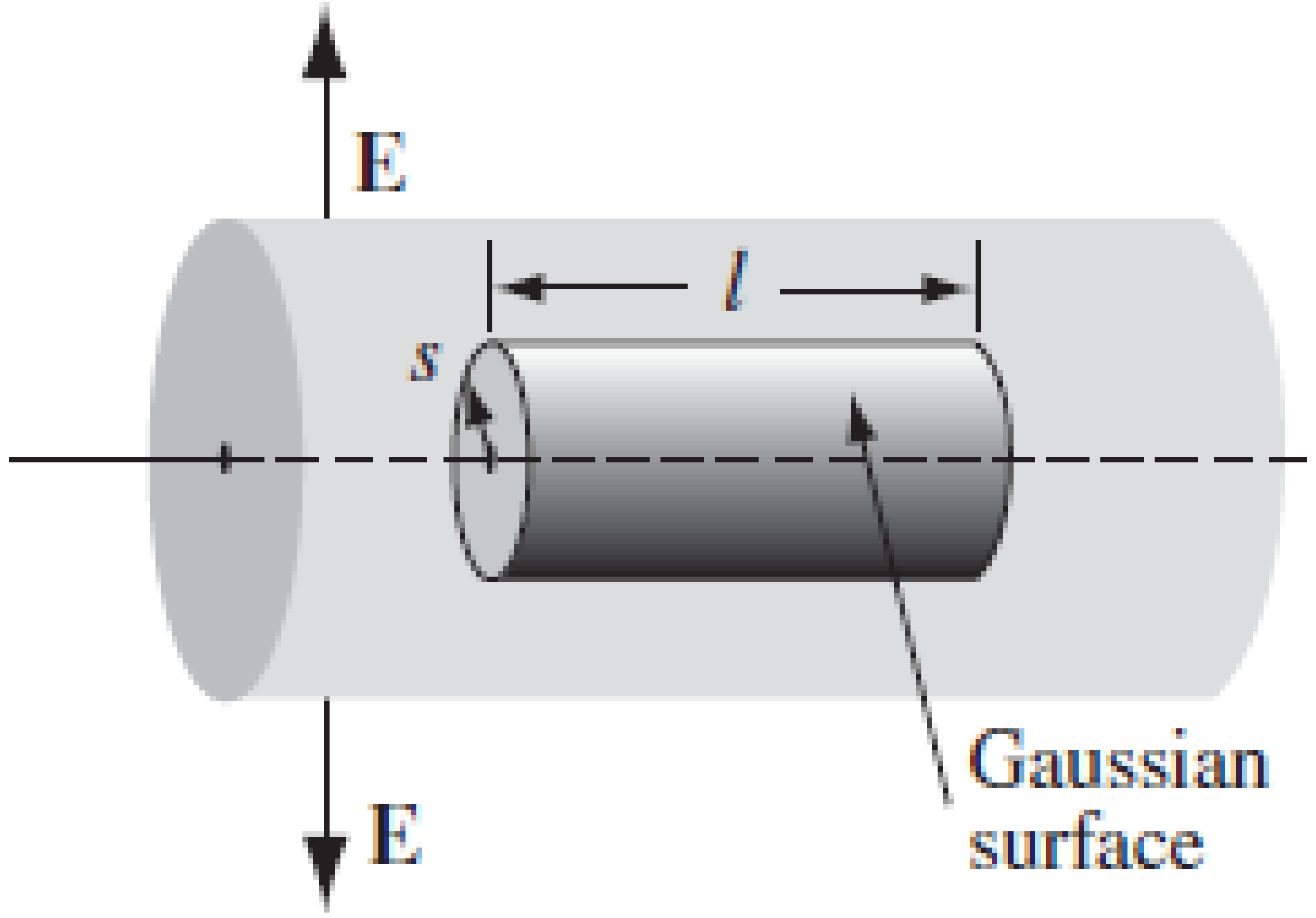
Example 2.3

A long cylinder carries a charge density that is proportional to the distance from the axis: $\rho = ks$, for some constant k. Find the electric field inside the cylinder?

Solution:

Pick up a Gaussian surface as shown in the figure.

The total charge enclosed is



$$Q_{enc} = \ell \int_0^s (ks')s'ds'd\phi = \frac{2}{3}\pi k \ell s^3$$

$$E = \frac{Q_{enc}}{\varepsilon_0 2\pi s \ell} = \frac{1}{3\varepsilon_0} ks^2 \text{ in } \hat{\mathbf{s}} \text{ direction}$$

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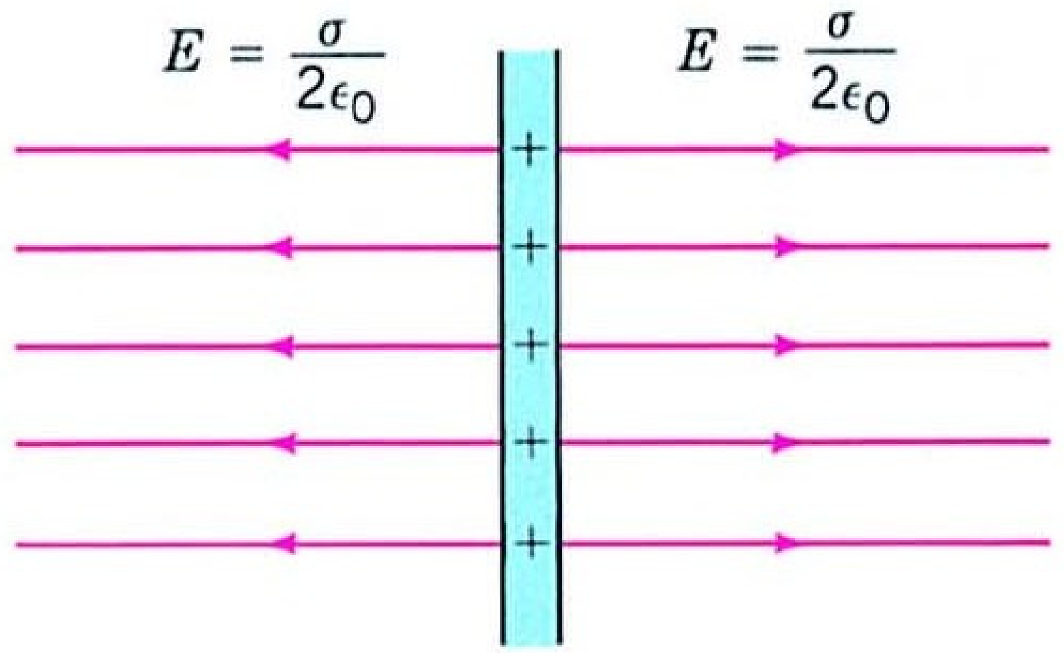
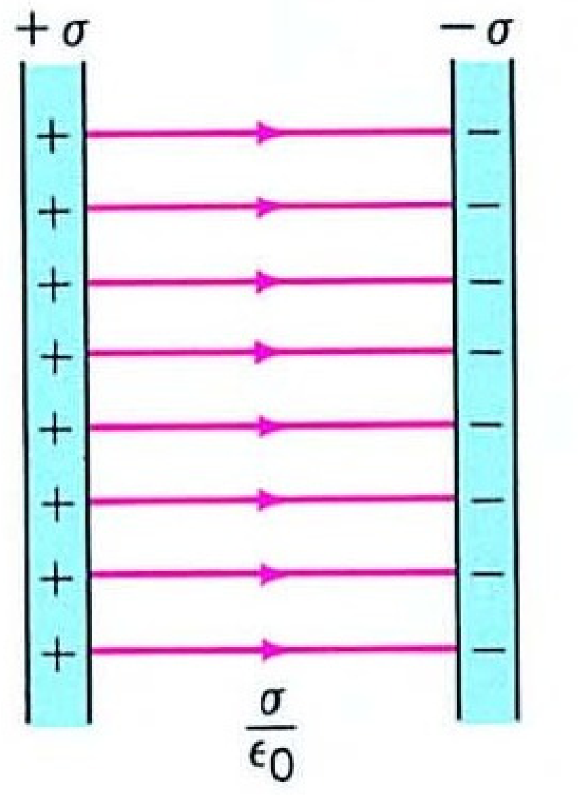
Example 2.5

Find the field due to the following: (a) an infinite sheet of charge with surface charge density $+\sigma$; (b) two parallel infinite sheets with charges density $+\sigma$ and $-\sigma$.

Solution:





How to Choose a Good Gaussian Surface?

Gauss's Law is always true, but it is not always useful.

Symmetry is crucial to the application of Gauss's law.

There are only three kinds of symmetry that work:

1. Spherical symmetry: Make your Gaussian surface a concentric sphere.
2. Cylindrical symmetry: Make your Gaussian surface a coaxial cylinder.
3. Plane symmetry: Use a Gaussian "pillbox", which straddles the surface.

2.2.4 The Curl of the Electric Field

The electric field can be expressed in the following form

$$\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\int_{\text{all space}}\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho(\mathbf{r}')d\tau'=\frac{-1}{4\pi\varepsilon_0}\int_{\text{all space}}(\nabla\frac{1}{\mathscr{r}})\rho(\mathbf{r}')d\tau'$$

Curl of the electric field is

Why doesn't the divergence operator apply on the $\mathbf{r}'$ coordinate?

$$\nabla\times\mathbf{E}=\frac{-1}{4\pi\varepsilon_0}\int_{\text{all space}}(\nabla\times(\nabla\frac{1}{\mathscr{r}}))\rho(\mathbf{r}')d\tau'$$

Curl of gradient is always zero. $\therefore\nabla\times\mathbf{E}=0$

The principle of superposition states that the total field is a vector sum of their individual fields $\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2+\ldots$

$$\nabla\times\mathbf{E}=\nabla\times(\mathbf{E}_1+\mathbf{E}_2+\cdots)=\nabla\times\mathbf{E}_1+\nabla\times\mathbf{E}_2+\cdots=0$$

2.3 Electric Potential
2.3.1&2 Introduction to and Comments on Potential

Can we apply the concept of ***potential***, first introduced in mechanics, to electrostatic system and find the law of ***conservation*** of energy?

We can define an electrostatic potential energy, analogous to gravitational potential energy, and apply the law of conservation of energy in the analysis of electrical problems.

Potential is not equal to the potential energy.

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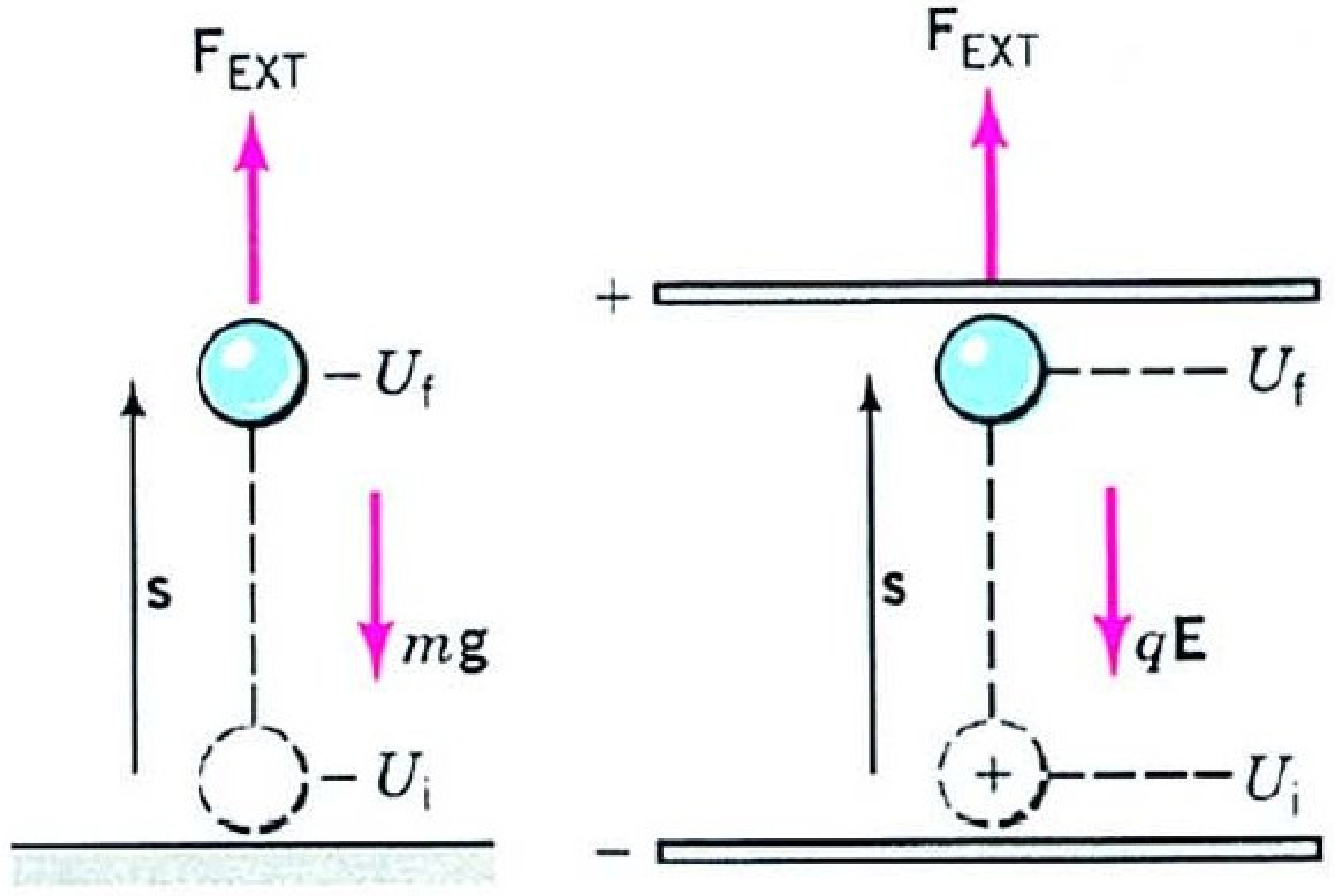
Mechanical Analogy of Potential

The motion of a particle with positive charge q in a uniform electric field is analogous to the motion of a particle of mass m in uniform gravitational field near the earth.

$$W_{EXT} = +\Delta U = U_f - U_i$$

If W_{EXT} >0, work is done **by** the external agent on the charges.
If W_{EXT} <0, work is done **on** the external agent by the field.



$\boxed{\text{Potential energy}}$ depends not only on the “source” but also on the “test” particle. Thus it will be more convenient if we can define a potential function which is function of “source” only.

The Unit of Potential: Volt

When a charge q moves between two points in the electrostatic field, the change in ***electric potential***, ΔV, is defined as the change in electrostatic potential energy per unit charge,

$$\Delta V = \frac{\Delta U}{q}$$

The SI unit of electric potential is the volt (V).

$$1\ \mathrm{V} = 1\ \mathrm{J/C} = 1\ \mathrm{N \cdot m/C}$$

The quantity ΔV depends only on the field set up by the source charges, not on the test charge.

$$W_{EXT} = q\Delta V = q(V_f - V_i)$$

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Only Changes in Potential are Significant

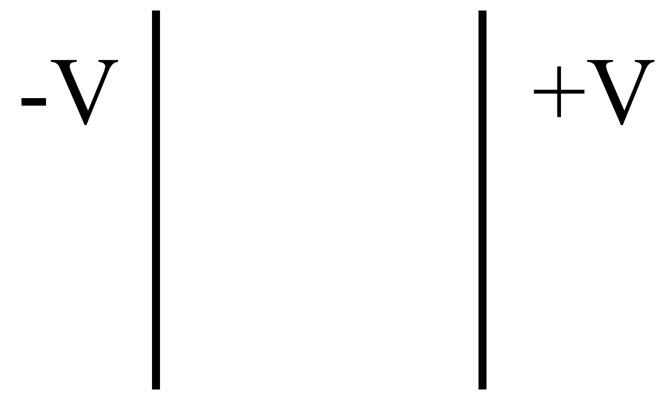
We see that only ***changes*** in potential ΔV, rather than the specific value of V_i and V_f, are significant.

It is convenient to choose the ground connection to earth as the zero of potential.

The potential at a point is the external work needs to bring a positive unit charge, at constant speed, from the position of zero potential to the given point.

In an external electric field, both positive and negative charges tend to ***decrease*** the electrostatic potential energy.

Which side will a charge particle drift if it is in the middle of two conducting plates with potential difference, higher or lower potential side?



Potential is Conservative

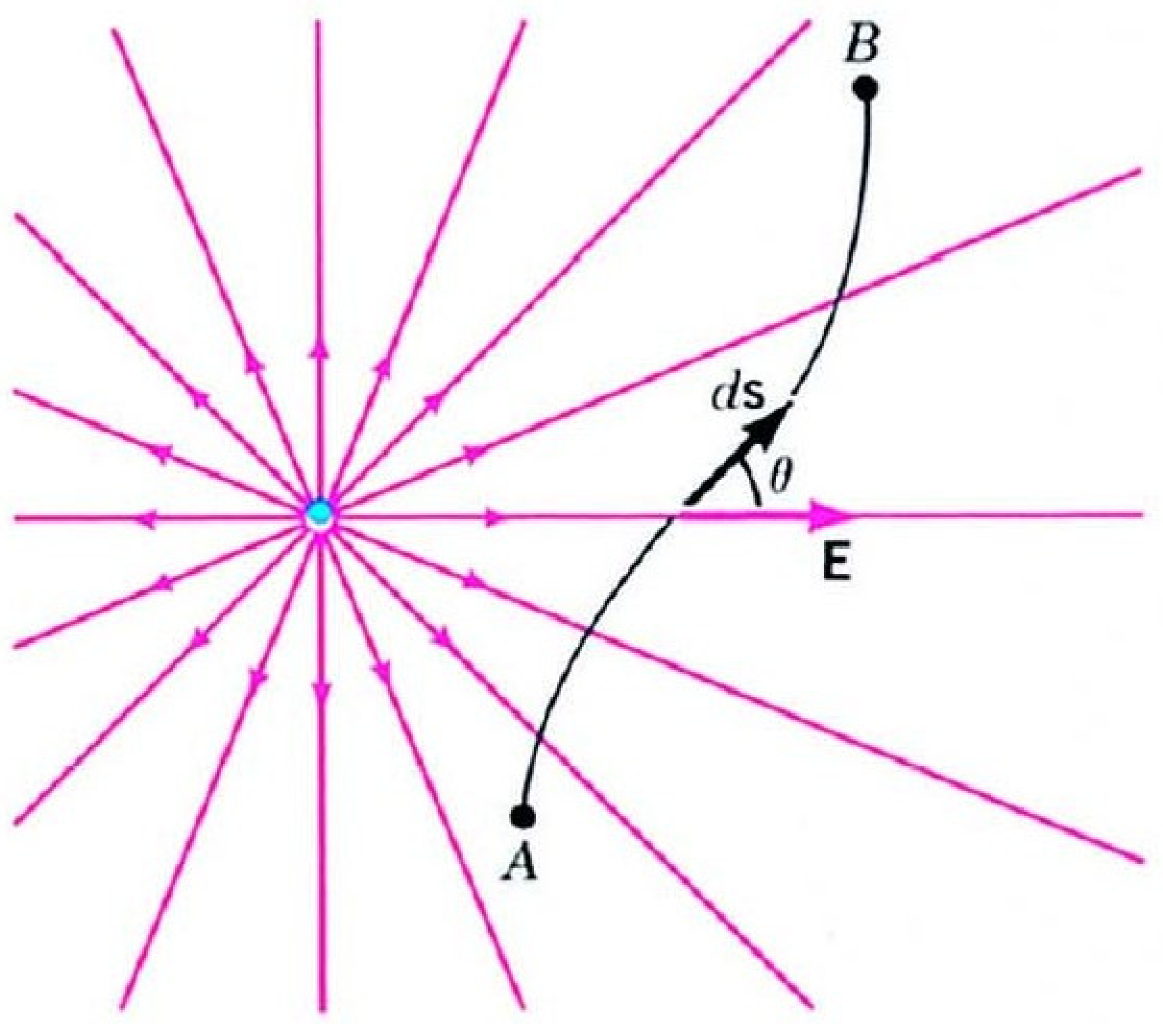
In mechanics, the definition of potential energy in terms of the work done by the conservative force is $\Delta U = -W_{c.}$. The negative sign tells us that positive work by the conservative force leads to a decrease in potential energy.

Therefore, the change in potential energy, associated with an infinitesimal displacement $d\mathbf{s}$, is

$$dU = -\mathbf{F_c} \cdot d\mathbf{s} = -q\mathbf{E} \cdot d\mathbf{s}$$

$$dV = \frac{dU}{q} = -\mathbf{E} \cdot d\mathbf{s}$$

$$V_B - V_A = -\int_A^B \mathbf{E} \cdot d\mathbf{s}$$



Since the electrostatic field is **conservative**, the value of this line integral depends only on the end points *A* and *B*, not on the path taken.

Differential form of Potential

The fundamental theorem of gradient states that

$$V_B - V_A = \int_A^B (\nabla V) \cdot d\mathbf{s}$$

and $V_B - V_A = -\int_A^B \mathbf{E} \cdot d\mathbf{s}$ so $\mathbf{E} = -\nabla V$

The electric field $\mathbf{E}$ is a very special kind of vector function whose curl is always zero.

$$\nabla \times \mathbf{E} = -(\nabla \times \nabla V) = 0$$

It is often easier to analyze a physical situation in terms of ***potential***, which is a ***scalar***, rather than the ***electric field strength***, which is a ***vector***.

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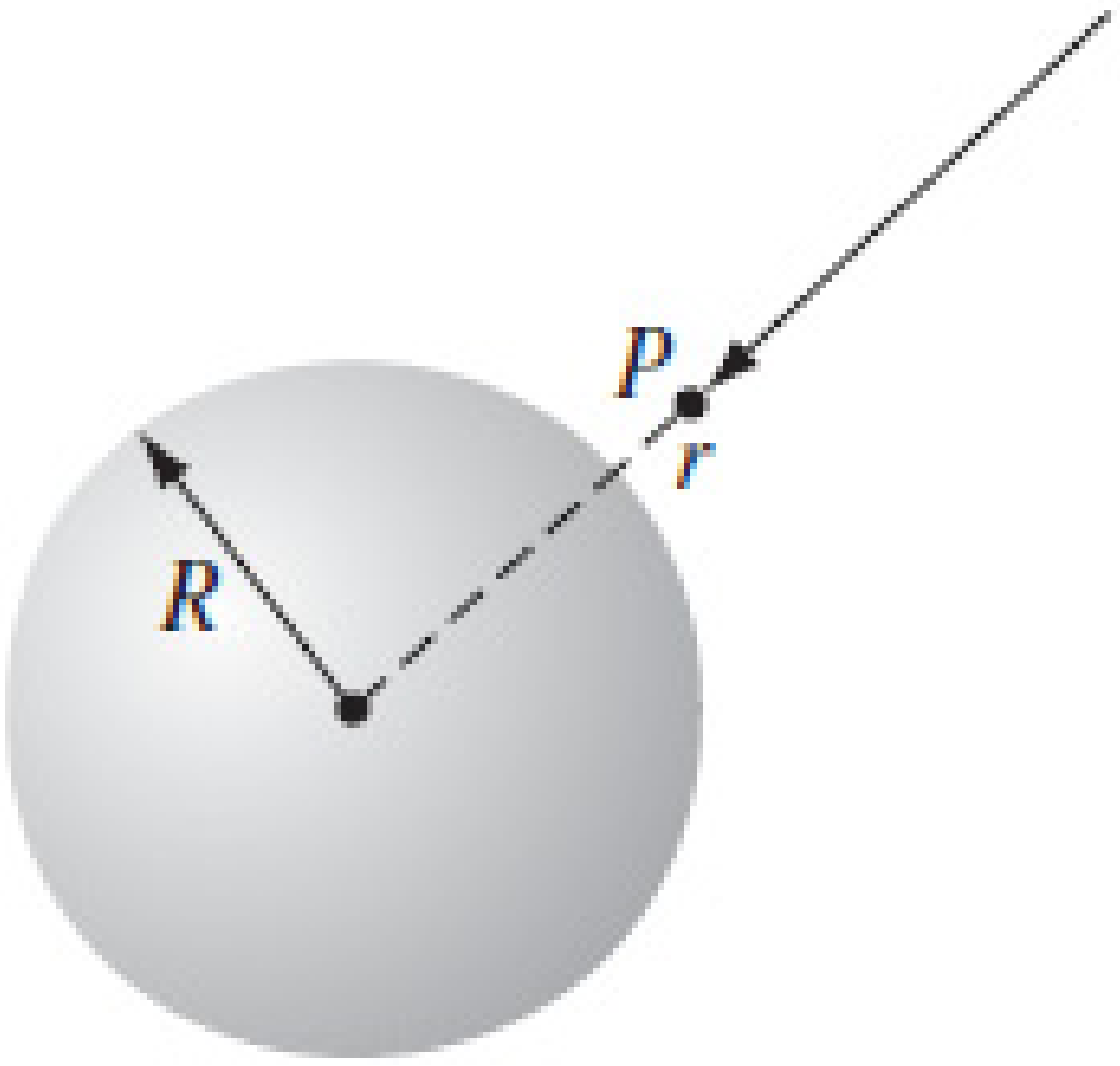
Example 2.6 Find the potential inside and outside a spherical shell of radius R, which carries a uniform surface charge. Set the reference point at infinity.

Sol: Use the Gauss's law to find the electric field and then use the electric field to calculate the potential.

$$\begin{cases} \text{Inside } (r < R) & E = 0 \\ \text{outside } (r > R) & E = \dfrac{q}{4\pi\varepsilon_0 r^2} \end{cases}$$



$$V(r) = -\int_\infty^r \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^r \frac{q}{4\pi\varepsilon_0 r'^2}(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})dr'$$

$$= \frac{q}{4\pi\varepsilon_0 r'}\bigg|_\infty^r = \frac{q}{4\pi\varepsilon_0 r} \quad (r > R)$$

and $V(r) = \dfrac{q}{4\pi\varepsilon_0 R} \quad (r \le R)$

2.3.3 Poisson's Equation and Laplace's Equation

The electric field can be written as the gradient of a scalar potential.

$$\mathbf{E} = -\nabla V$$

What do the fundamental equations for $\mathbf{E}$ looks like, in terms of V?

$$\text{Gauss's law } \nabla \cdot \mathbf{E} = -(\nabla \cdot \nabla V) = \boxed{-\nabla^2 V = \frac{\rho}{\varepsilon_0}}$$

$$\text{Curl law} \qquad \nabla \times \mathbf{E} = -(\nabla \times \nabla V) = 0$$

$$\nabla \times \mathbf{E} = 0 \;\; \text{permits} \;\; \mathbf{E} = -\nabla V;$$

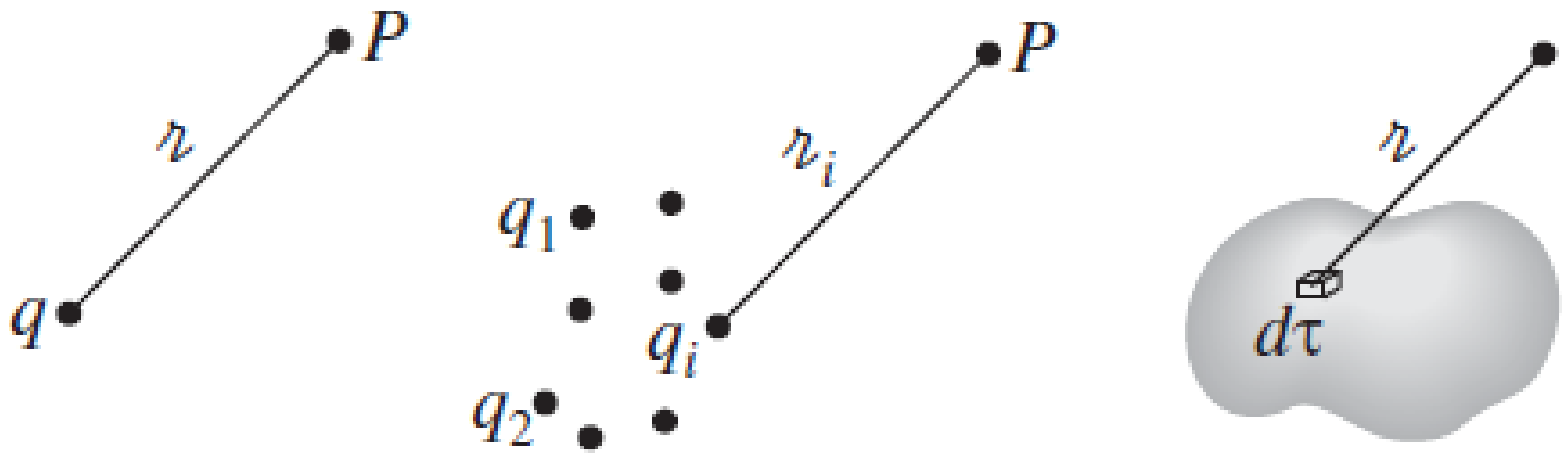
$$\text{in turn, } \mathbf{E} = -\nabla V \;\; \text{guarantees } \nabla \times \mathbf{E} = 0$$

2.3.4 The Potential of a Localized Charge Distribution

Setting the reference point at infinity, the potential of a point charge q **at the origin** is

$$V(r) = \frac{-1}{4\pi\varepsilon_0}\int_{\infty}^{r} \frac{q}{r'^2}\,dr' = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'}\bigg|_{\infty}^{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

The conventional minus sign in the definition of V was chosen precisely in order to make the potential of a positive charge come out positive.



The Potential of a Localized Charge Distribution

In general, the potential of a collection of charges is

$$V(r) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\left|\mathbf{r}-\mathbf{r}_i'\right|}$$

For a continuous distribution

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{\mathscr{r}} = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{\left|\mathbf{r}-\mathbf{r}'\right|}$$

For a volume charge ρ; a surface charge σ; a line charge λ.

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}')}{\mathscr{r}}d\tau' \qquad = \frac{1}{4\pi\varepsilon_0}\int\frac{\sigma(\mathbf{r}')}{\mathscr{r}}da' \qquad = \frac{1}{4\pi\varepsilon_0}\int\frac{\lambda(\mathbf{r}')}{\mathscr{r}}dl'$$

$$= \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}d\tau' \qquad = \frac{1}{4\pi\varepsilon_0}\int\frac{\sigma(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}da' \qquad = \frac{1}{4\pi\varepsilon_0}\int\frac{\lambda(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}dl'$$

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Example

A non-conducting disk of radius a has a uniform surface charge density σ C/m². What is the potential at a point P on the axis of the disk at a distance y from its center.

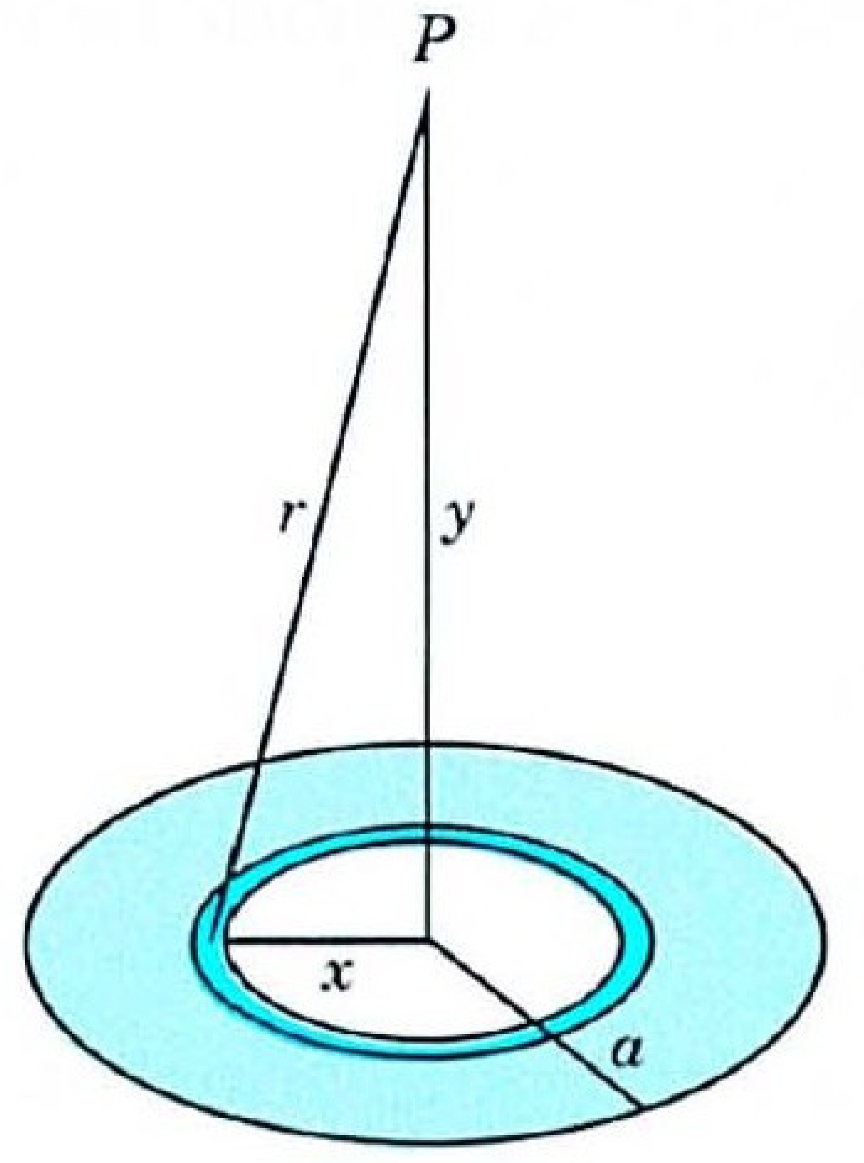
Solution:

$$dV = \frac{dq}{4\pi\varepsilon_0 r}, \quad dq = \sigma(2\pi x dx)$$

$$dV = \frac{\sigma\pi}{4\pi\varepsilon_0\sqrt{x^2+y^2}} dx^2$$

$$V = \int_0^a \frac{\sigma\pi}{4\pi\varepsilon_0\sqrt{x^2+y^2}} dx^2$$

$$= \frac{\sigma}{2\varepsilon_0}\left[\sqrt{x^2+y^2}\right]_0^{x^2=a^2} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{a^2+y^2} - |y|\right]$$



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Example

A shell of radius R has a charge Q uniformly distributed over its surface. Find the potential at a distance $r > R$ from its center.

Solution:

It is more straightforward to use the electric field, which we know from Gauss's law.

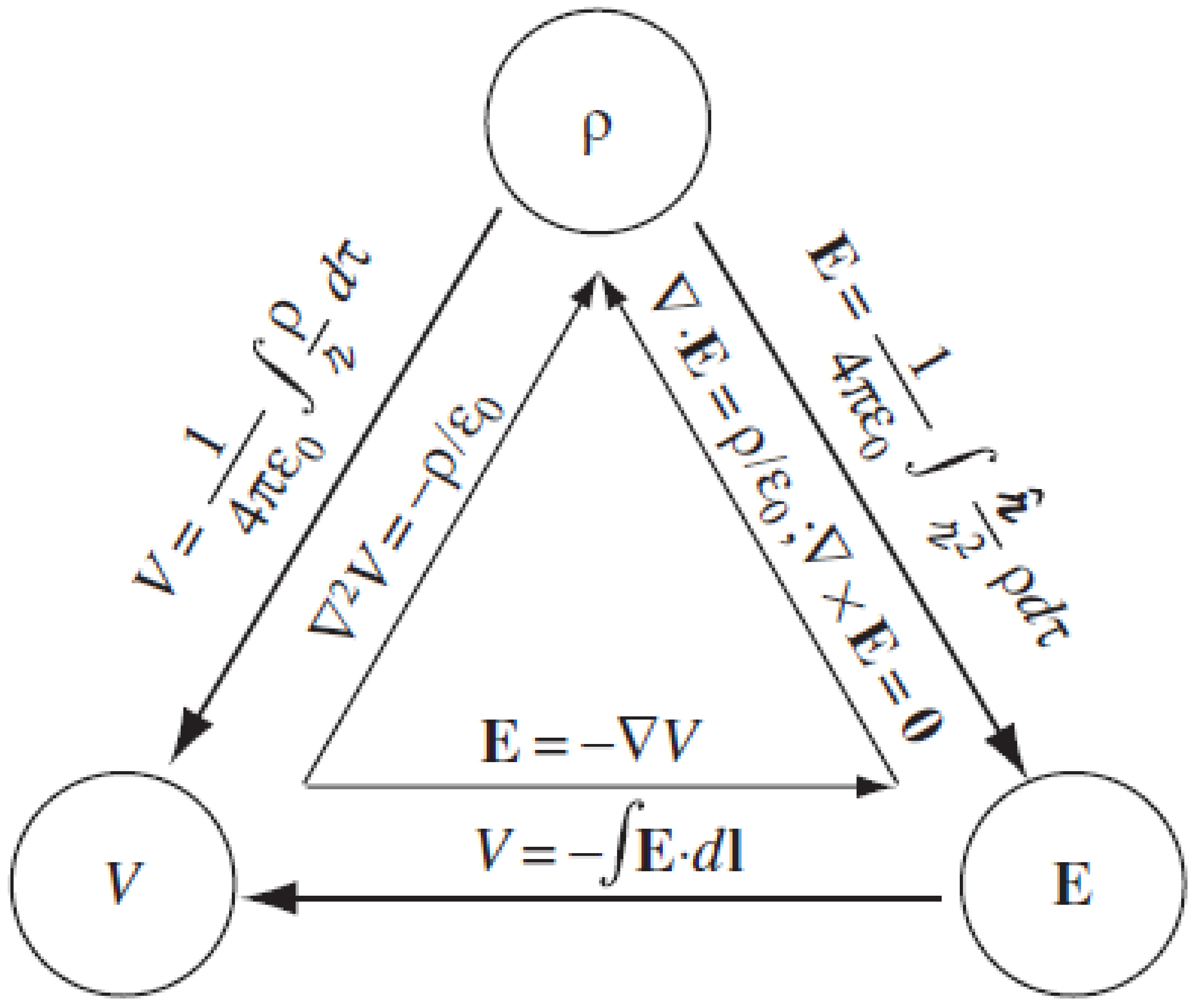
$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \qquad V(r) - V(\infty) = -\int_\infty^r \frac{Q}{4\pi\varepsilon_0 r^2} dr = -\frac{1}{4\pi\varepsilon_0} Q\left[-\frac{1}{r}\right]_\infty^r$$

$$V(r) = \frac{Q}{4\pi\varepsilon_0 r}$$

The potential has a fixed value at all points within the conducting sphere equal to the potential at the surface.

2.3.5 Summary; Electrostatic Boundary Conditions

We have derived six formulas interrelating three fundamental quantities: ρ, $\mathbf{E}$, and V.



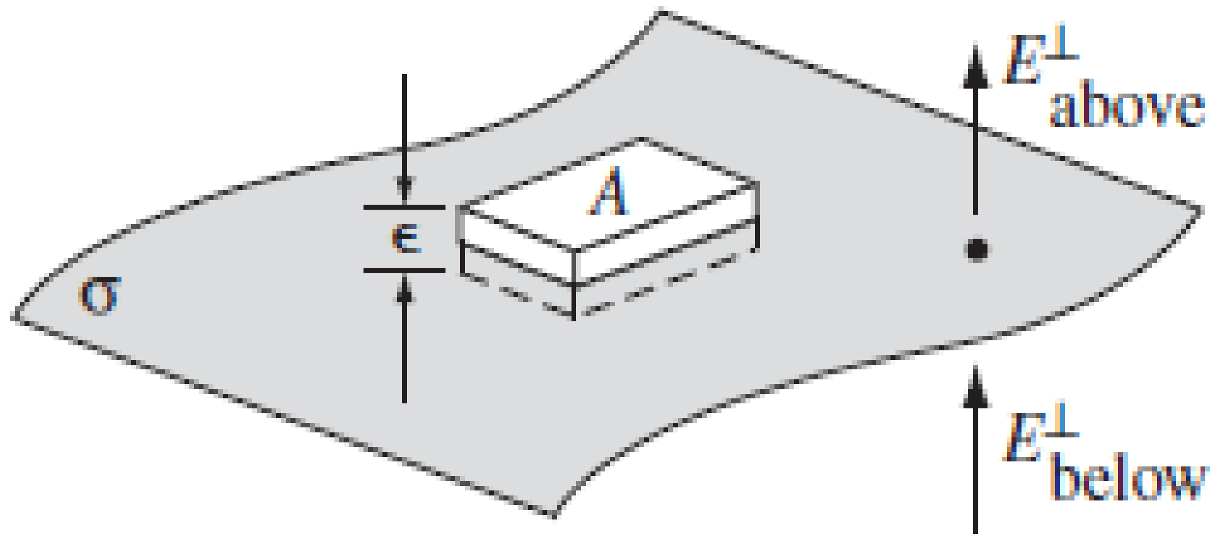
These equations are obtained from two observations:

- Coulomb’s law: the fundamental law of electrostatics
- The principle of superposition: a general rule applying to all electromagnetic forces.

Electrostatic Boundary Conditions: **Normal**

The electric field is not continuous at a surface with charge density σ. Why?

Consider a Gaussian pillbox.



Gauss's law states that $$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

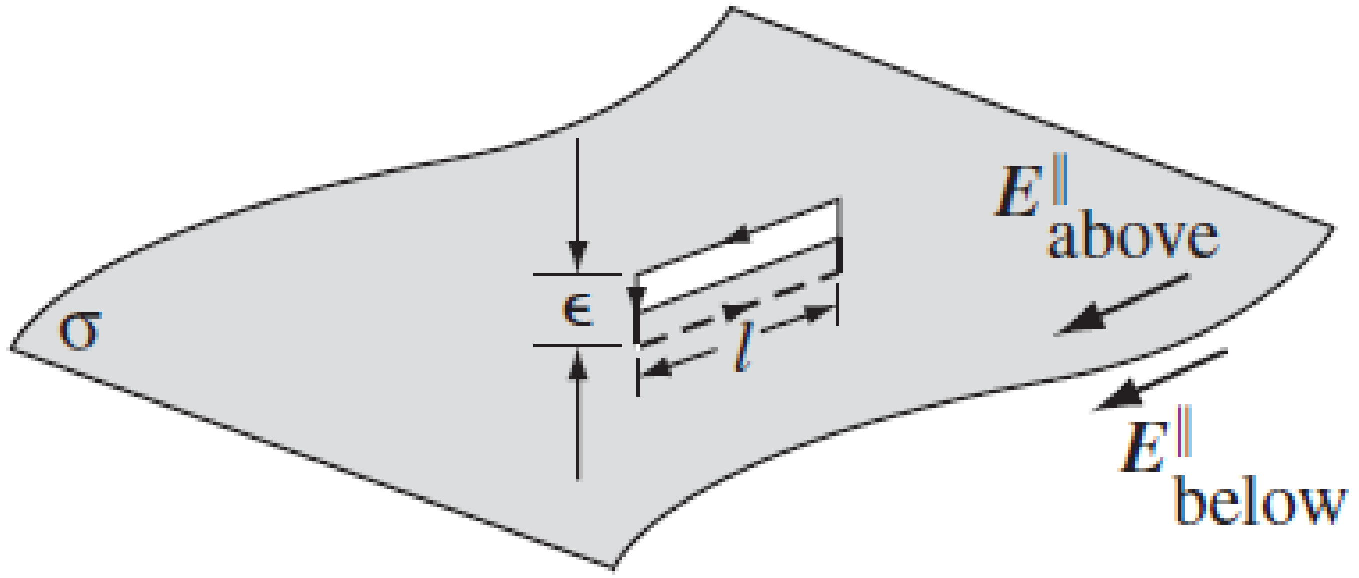
The sides of the pillbox contribute nothing to the flux, in the limit as the thickness ϵ goes to zero.

$$(E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}})A = \frac{\sigma A}{\varepsilon_0} \Rightarrow (E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}}) = \frac{\sigma}{\varepsilon_0}$$

Electrostatic Boundary Conditions: **Tangential**

The tangential component of $\mathbf{E}$, by contract,
is always continuous.

Consider a thin rectangular loop.



The curl of the electric field states that $\oint_P \mathbf{E} \cdot d\mathbf{l} = 0$

The ends give nothing (as $\epsilon \rightarrow 0$), and the sides give

$$(E^{//}_{\text{above}} - E^{//}_{\text{below}})\ell = 0 \quad \Rightarrow \quad E^{//}_{\text{above}} = E^{//}_{\text{below}}$$

In short, $\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \dfrac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$

Boundary Conditions in terms of potential

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}} \quad \Rightarrow \quad (\nabla V_{\text{above}} - \nabla V_{\text{below}}) = -\frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$$

or $$\left(\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n}\right) = -\frac{\sigma}{\varepsilon_0}$$

where $\frac{\partial V_{\text{above}}}{\partial n} (\equiv \nabla V \cdot \hat{\mathbf{n}})$ denotes the normal derivative of V.

$$V_{\text{above}} = V_{\text{below}}$$ Why?

If $V_{\text{above}} \neq V_{\text{below}}$, $\sigma = \infty$.

Homework of Chap. 2 (part I)

Problem 2.9 Suppose the electric field in some region is found to be $\mathbf{E} = kr^3\hat{\mathbf{r}}$, in spherical coordinates (k is some constant).
(a) Find the charge density ρ.
(b) Find the total charge contained in a sphere of radius R, centered at the origin.
(Do it two different ways.)

Problem 2.12 Use Gauss's law to find the electric field inside a uniformly charged solid sphere (charge density ρ). Compare your answer to Prob. 2.8.

Problem 2.15 A thick spherical shell carries charge density

$$\rho = \frac{k}{r^2} \quad (a \le r \le b)$$

(Fig. 2.25). Find the electric field in the three regions: (i) $r < a$, (ii) $a < r < b$, (iii) $r > b$. Plot $|\mathbf{E}|$ as a function of r, for the case $b = 2a$.

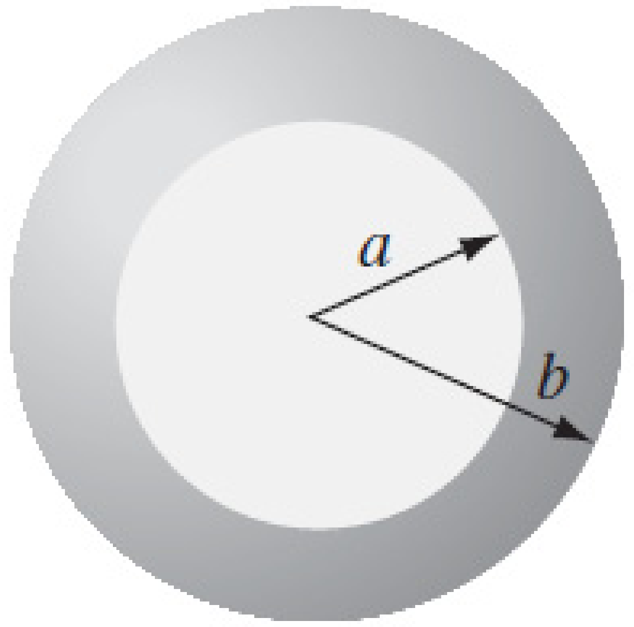


FIGURE 2.25

Homework of Chap. 2 (part I)

Problem 2.20 One of these is an impossible electrostatic field. Which one?

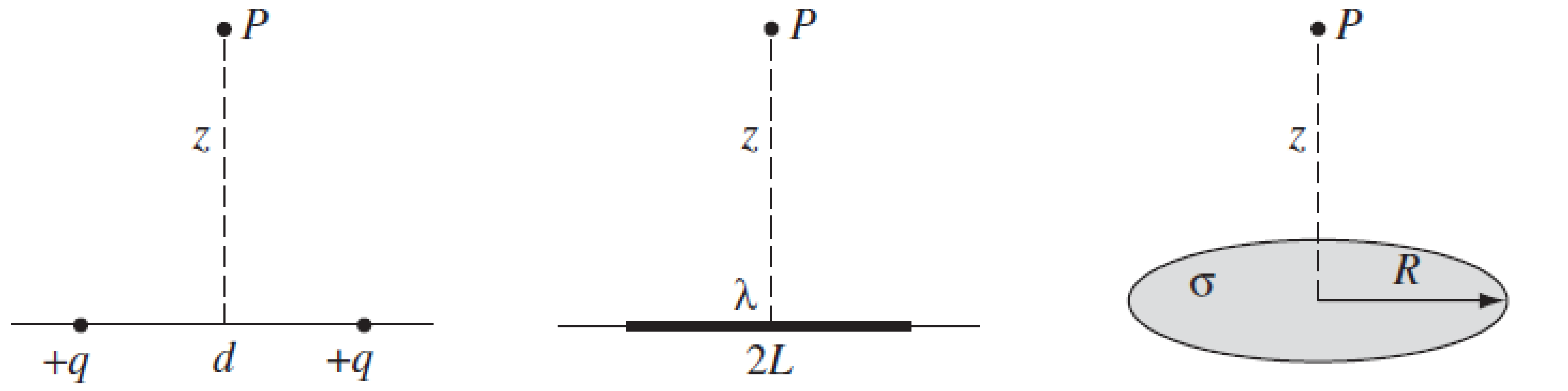
(a) $\mathbf{E} = k[xy\ \hat{\mathbf{x}} + 2yz\ \hat{\mathbf{y}} + 3xz\ \hat{\mathbf{z}}]$;

(b) $\mathbf{E} = k[y^2\ \hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\ \hat{\mathbf{z}}]$

Here k is a constant with the appropriate units. For the *possible* one, find the potential, using the *origin* as your reference point. Check your answer by computing $.\nabla V$.

[*Hint*: You must select a specific path to integrate along. It doesn't matter *what* path you choose, since the answer is path-independent, but you simply cannot integrate unless you have a definite path in mind.]

Problem 2.25 Using Eqs. 2.27 and 2.30, find the potential at a distance z above the center of the charge distributions in Fig. 2.34. In each case, compute $\mathbf{E} = -\nabla V$, and compare your answers with Ex. 2.1, Ex. 2.2, and Prob. 2.6, respectively. Suppose that we changed the right-hand charge in Fig. 2.34a to $-q$; what then is the potential at P? What field does that suggest? Compare your answer to Prob. 2.2, and explain carefully any discrepancy.

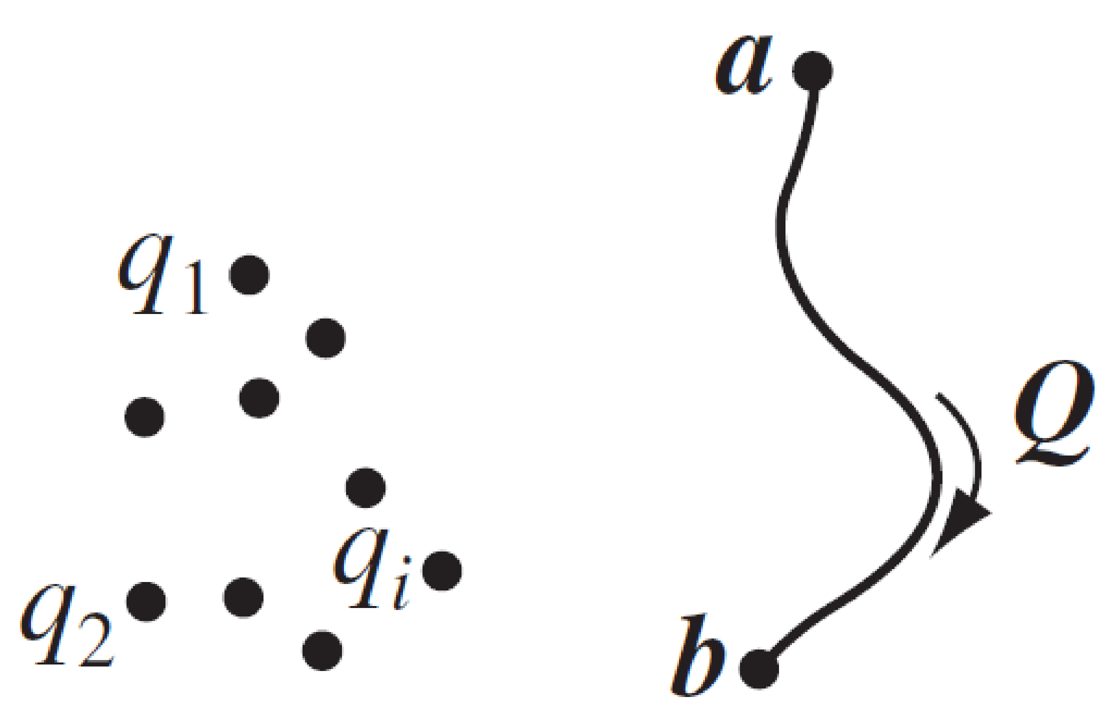


(a) Two point charges (b) Uniform line charge (c) Uniform surface charge

FIGURE 2.34

2.4 Work and Energy in Electrostatics
2.4.1 The Work Done to Move a Charge

How much work will you have to do, if you move a test charge Q from point $\mathbf{a}$ to point $\mathbf{b}$?



What we're interested is the *minimum* force you must exert to do the job.

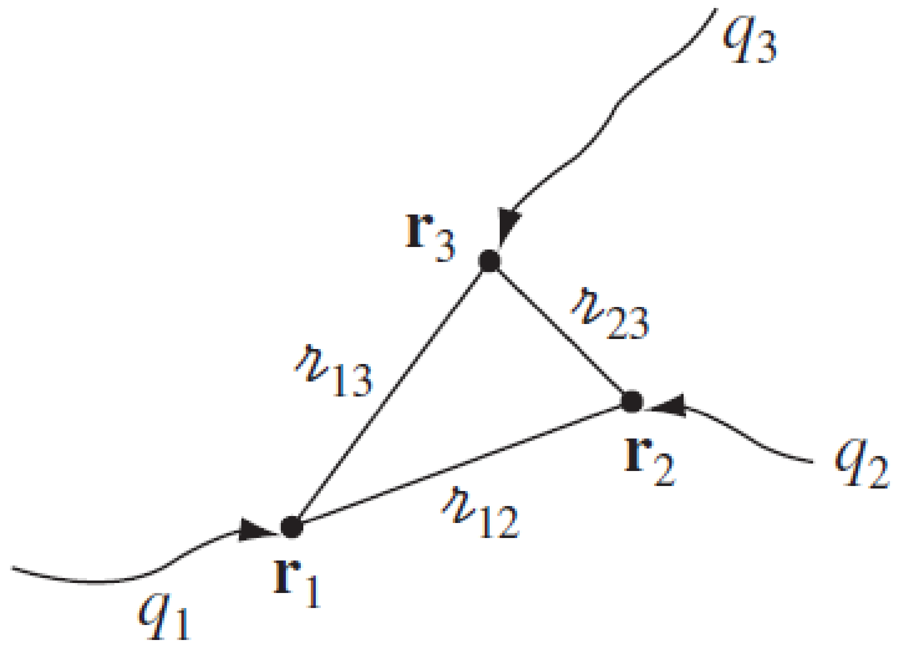
$$W = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = -Q\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = Q(V(\mathbf{b}) - V(\mathbf{a}))$$

$$\text{So} \quad V(\mathbf{b}) - V(\mathbf{a}) = W / Q$$

The potential difference between points $\mathbf{a}$ and $\mathbf{b}$ is equal to the work per unit charge required to carry a particle from $\mathbf{a}$ to $\mathbf{b}$.

2.4.2 The Energy of a Point Charge Distribution

How much work would it take to assemble an entire *collection* of point charges?



$$W_1 = 0, \qquad W_2 = \frac{1}{4\pi\varepsilon_0} q_2 \left(\frac{q_1}{\mathscr{r}_{12}}\right), \qquad W_3 = \frac{1}{4\pi\varepsilon_0} q_3 \left(\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}}\right)$$

$$W = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{q_2 q_3}{\mathscr{r}_{23}}\right)$$

The general rule: $$W = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j>i}}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}} = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}}$$

$$= \frac{1}{2}\sum_{i=1}^{n} q_i \left(\frac{1}{4\pi\varepsilon_0}\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{q_j}{\mathscr{r}_{ij}}\right) = \frac{1}{2}\sum_{i=1}^{n} q_i V_i(\mathbf{r}_i)$$

2.4.3 The Energy of a Continuous Charge Distribution

Generalizing the point charge distribution result:

$$dW_i = \frac{1}{2}(dq_i)V_i(\mathbf{r}_i) = \frac{1}{2}\rho_i V_i(\mathbf{r}_i)(d\tau)$$

$$W = \frac{1}{2}\int \rho V d\tau = \frac{1}{2}\int (\varepsilon_0 \nabla \cdot \mathbf{E}) V d\tau = \frac{\varepsilon_0}{2}\int (\nabla \cdot \mathbf{E}) V d\tau$$

Integration by parts: $\nabla \cdot (V\mathbf{E}) = (\nabla V) \cdot \mathbf{E} + (\nabla \cdot \mathbf{E}) V$

$$W = \frac{\varepsilon_0}{2}\int (\nabla \cdot \mathbf{E}) V d\tau = \frac{\varepsilon_0}{2}\left[\int (-\nabla V) \cdot \mathbf{E} d\tau + \oint (V\mathbf{E}) \cdot d\mathbf{a}\right]$$

$$= \frac{\varepsilon_0}{2}\left[\int E^2 d\tau + \cancel{\oint_S (V\mathbf{E}) \cdot d\mathbf{a}}\right]$$

divergence theorem

$$W = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2 d\tau$$

Potential and Potential Energy: Motion of Charges

The motion of a charge in an electric field may be discussed in terms of the conservation of energy, $\Delta K + \Delta U = 0$. In terms of potential, the conservation law may be written as

$$\Delta K = -q\Delta V$$

It is convenient to measure the energy of elementary particles, such as electrons and protons, in terms of a non-SI unit called the electronvolt (1 eV = 1.6x10^{-19} J).

According to Einstein famous $E = mc^2$, find the energy in terms of eV for an electron of rest mass 9.1×10^{-31} kg, where the speed of light is 3×10^8 m/s.

$E = 9.1\times10^{-31}\times(3\text{x}10^8)^2/1.6\text{x}10^{-19} = 0.511$ MeV

Example

A proton, of mass 1.67x10^{-27} kg, enters the region between parallel plates a distance 20 cm apart. There is a uniform electric field of 3x10^{5} V/m between the plates, as shown below. If the initial speed of the proton is 5x10^{6} m/s, what is its final speed?

Solution:

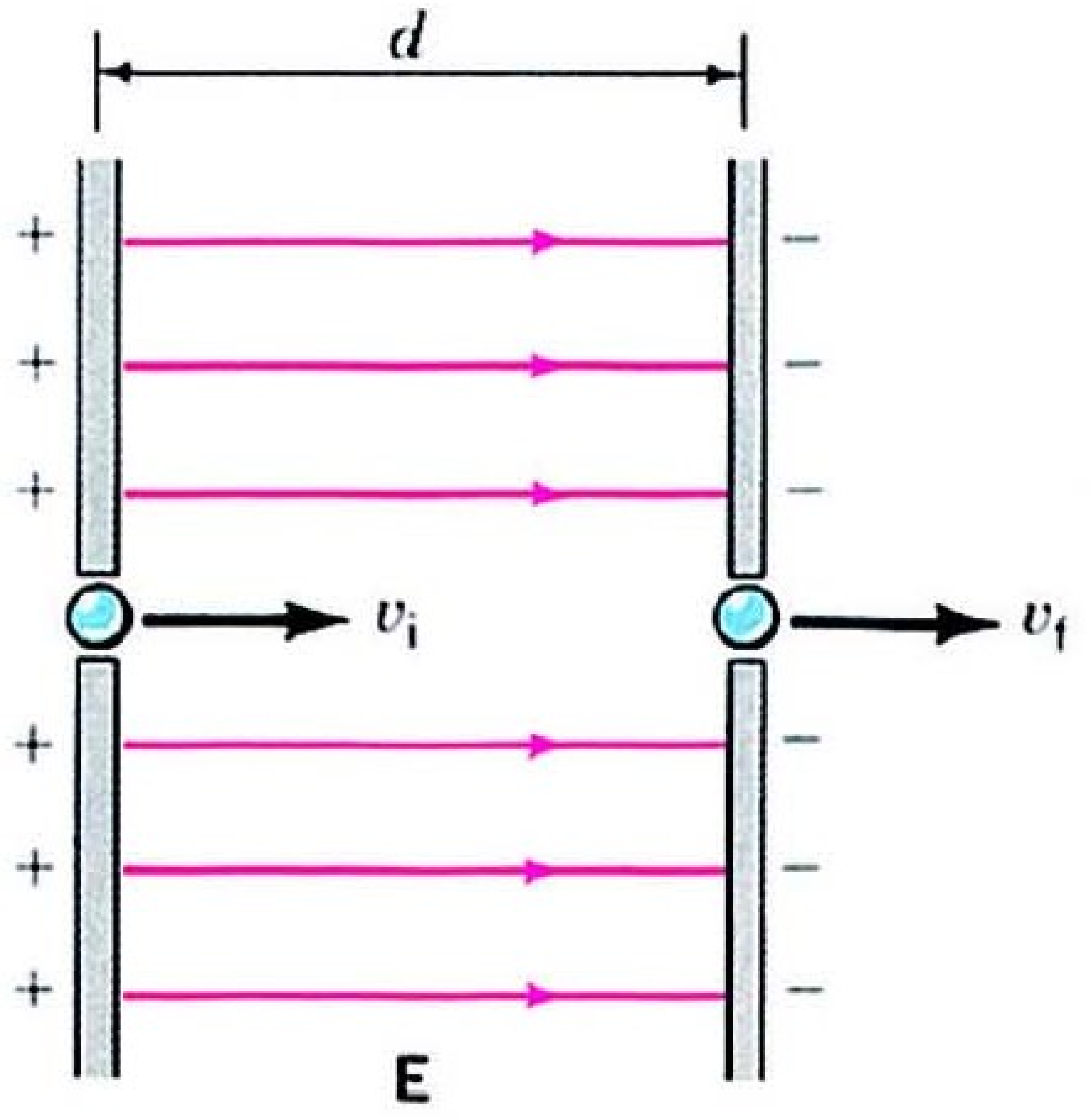
$$\Delta K = -q\Delta V = -q(-E\cdot d) = g\cdot(6\times10^{4})$$

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = -q\Delta V$$

$$v_f = \sqrt{v_i^2 - 2q\Delta V / m}$$

$$= \sqrt{(5\times10^{6})^2 + (2\times1.6\times10^{-19}\times6\times10^{4} / 1.67\times10^{-27})}$$

$$= 6\times10^{6}\ \text{m/s.}$$



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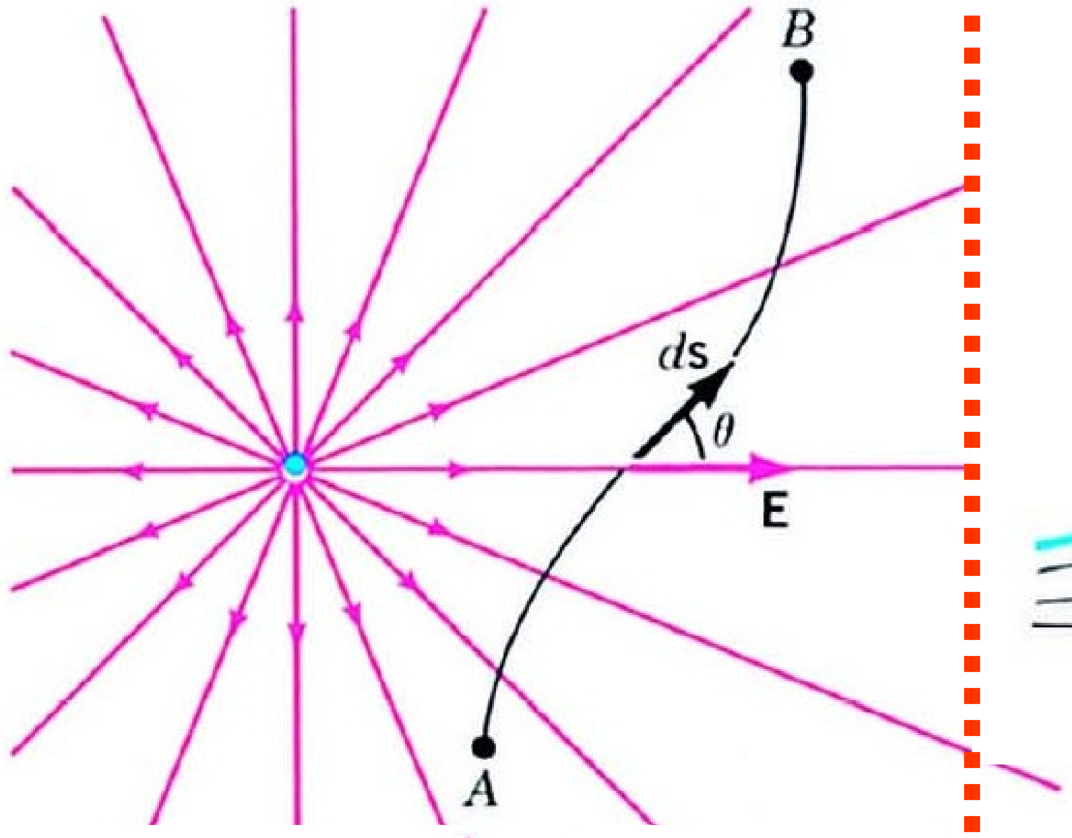
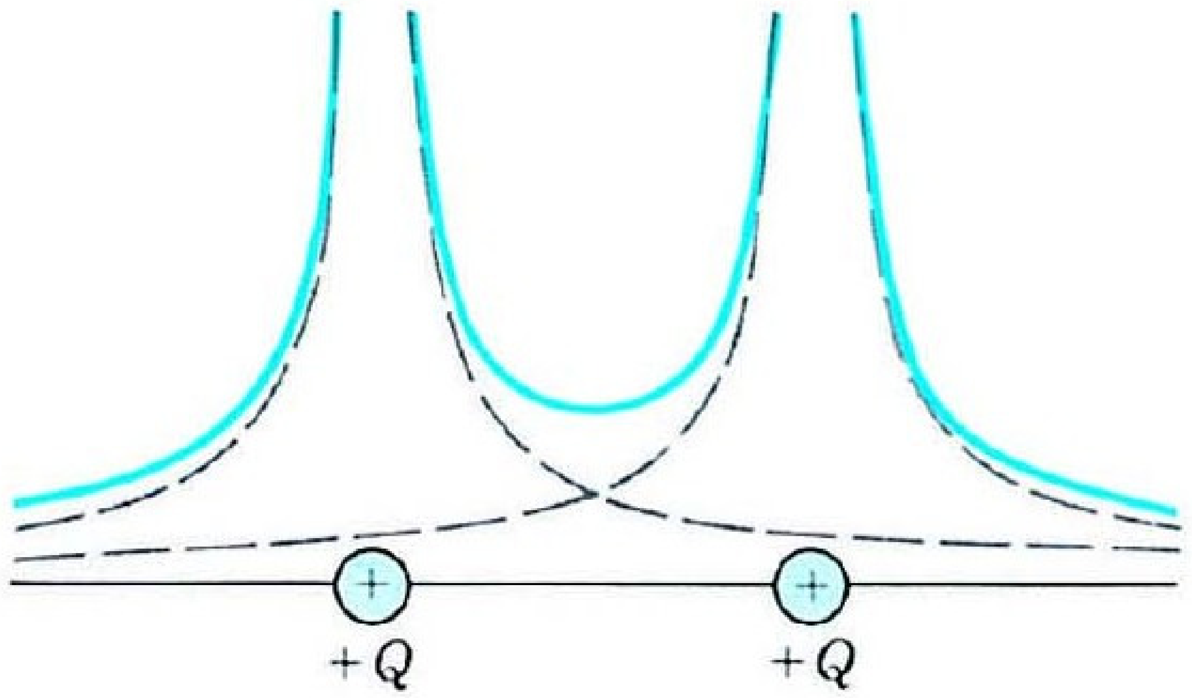
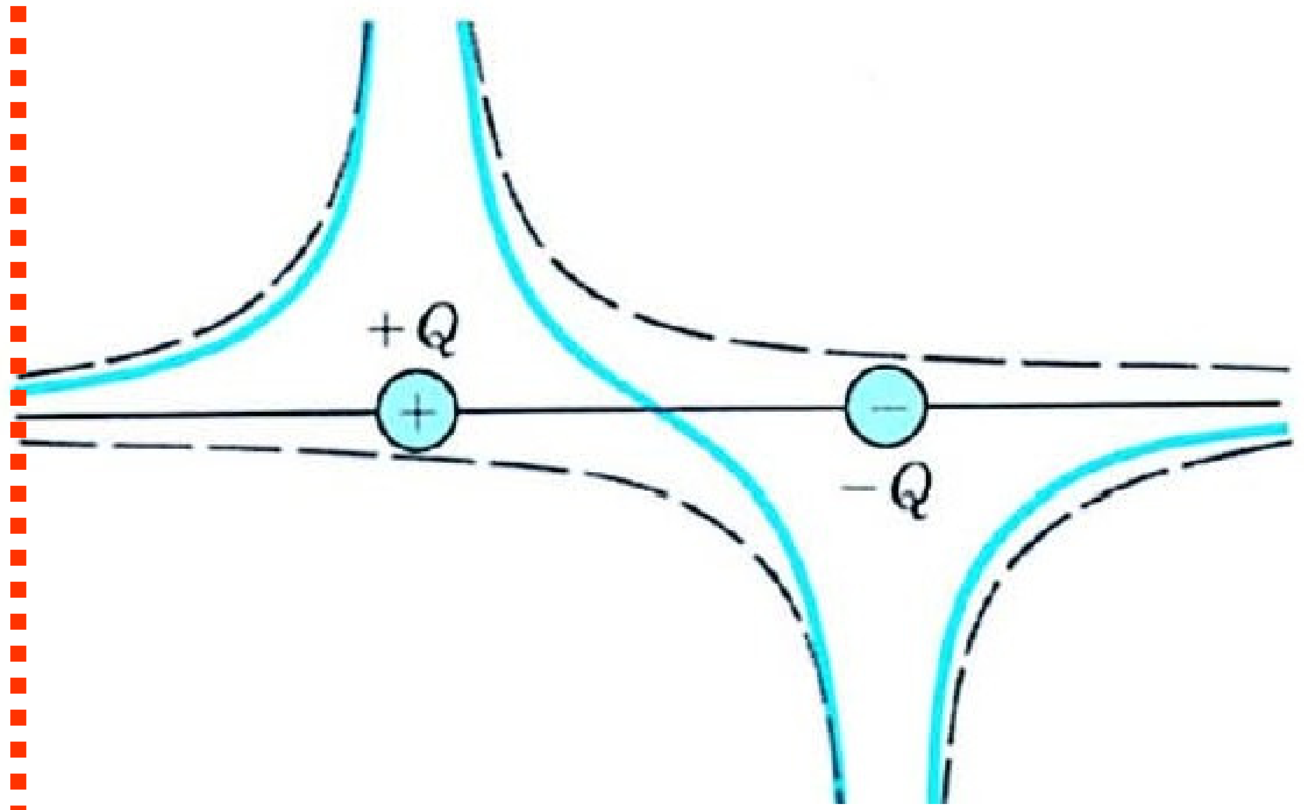
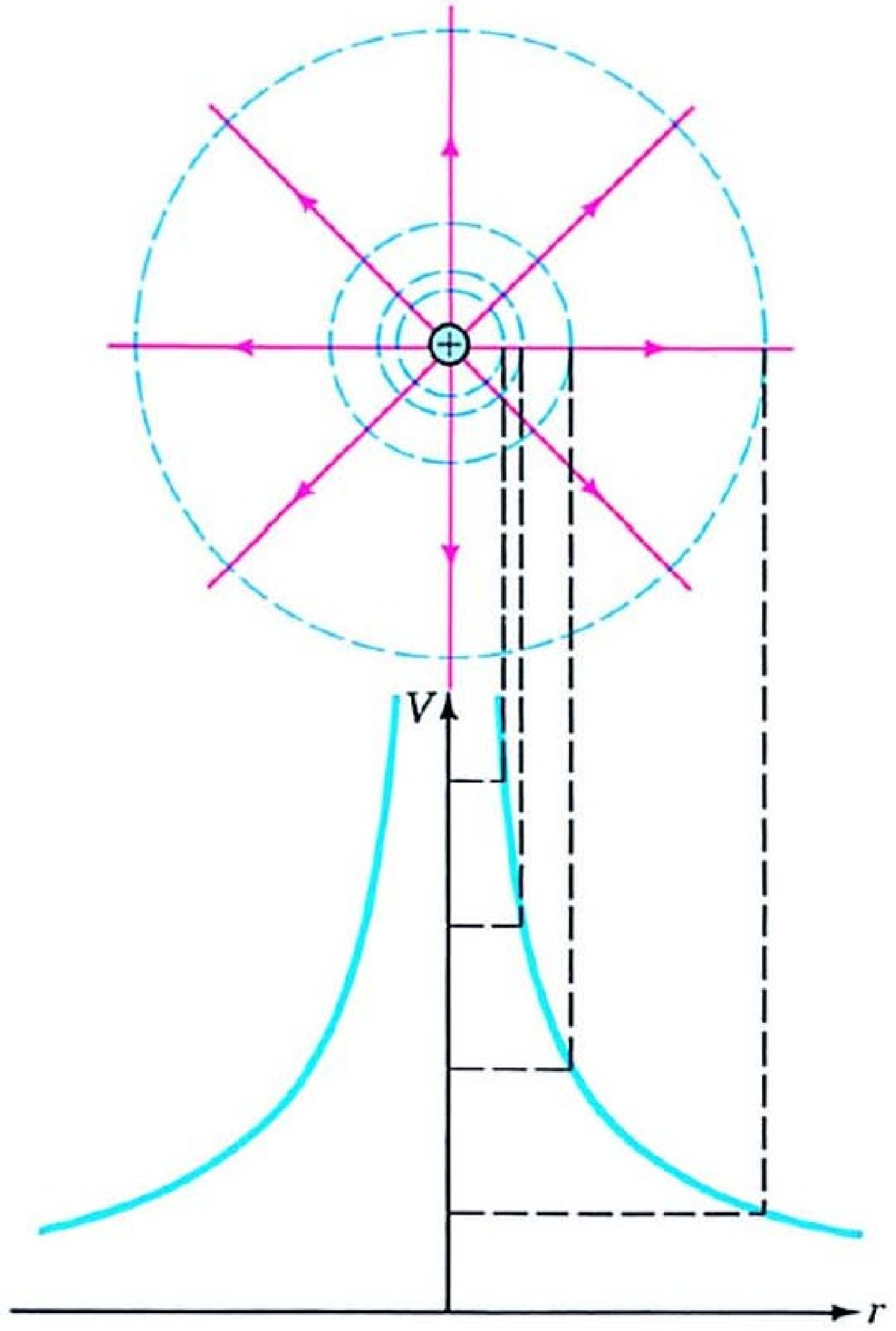
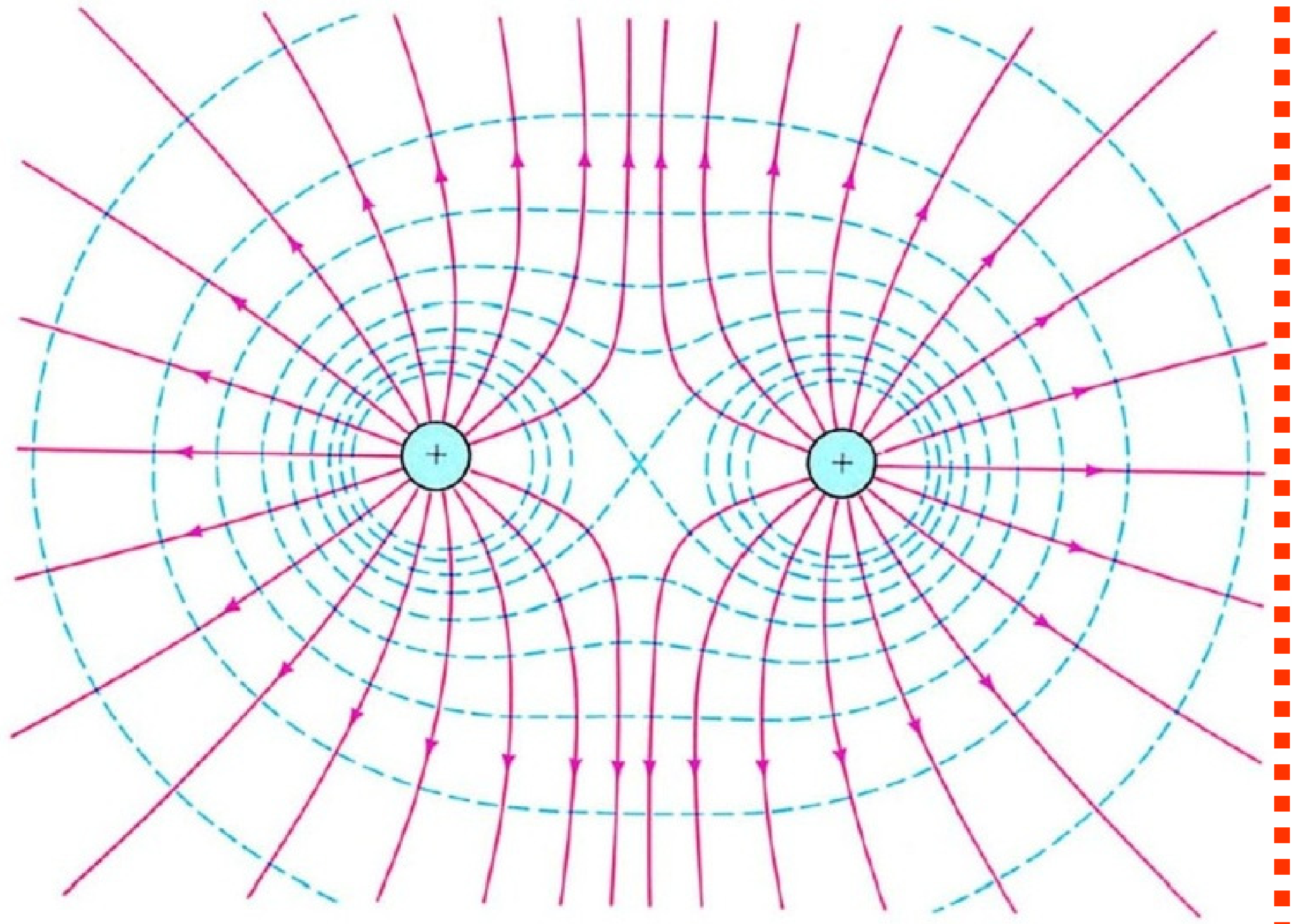
Potential and Potential Energy of Point Charges

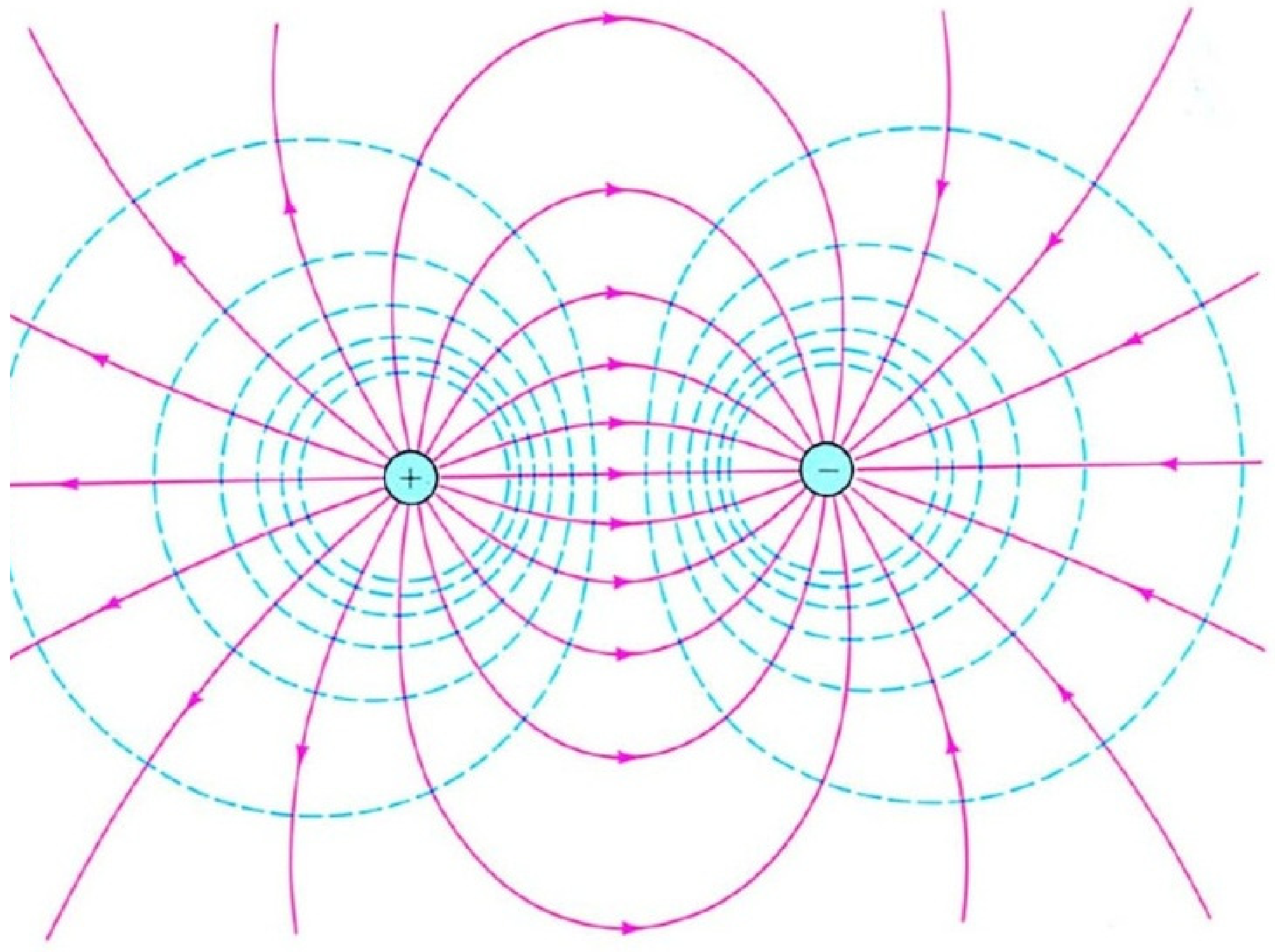












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Example

In 1913, Bohr proposed a model of the hydrogen atom in which an electron orbits a stationary proton in a circular path. Find the total mechanical energy of the electron given that the radius of the orbit is 0.53x10^{-10} m.

Solution:

The mechanical energy is the sum of the kinetic and potential energies, $E=K+U$. The centripetal force is provided by the coulomb attraction.

$$U = -\frac{e^2}{4\pi\varepsilon_0 r}$$

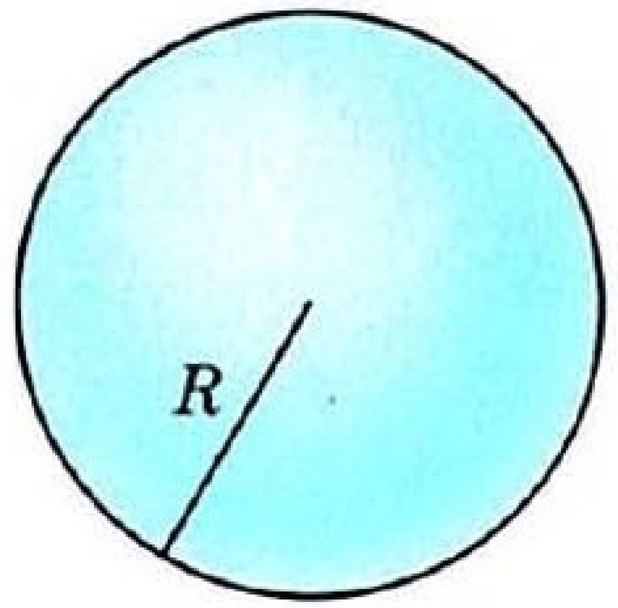
$$F = \frac{e^2}{4\pi\varepsilon_0 r^2} = \frac{mv^2}{r} \Rightarrow K = \frac{1}{2}mv^2 = \frac{e^2}{8\pi\varepsilon_0 r}$$

$$E = U + K = \frac{1}{2}U = -\frac{9\times10^9 \times (1.6\times10^{-19})^2}{2\times0.53\times10^{-10}} = -2.18\times10^{-18}\ \text{J} = -13.6\ \text{eV}$$

Example

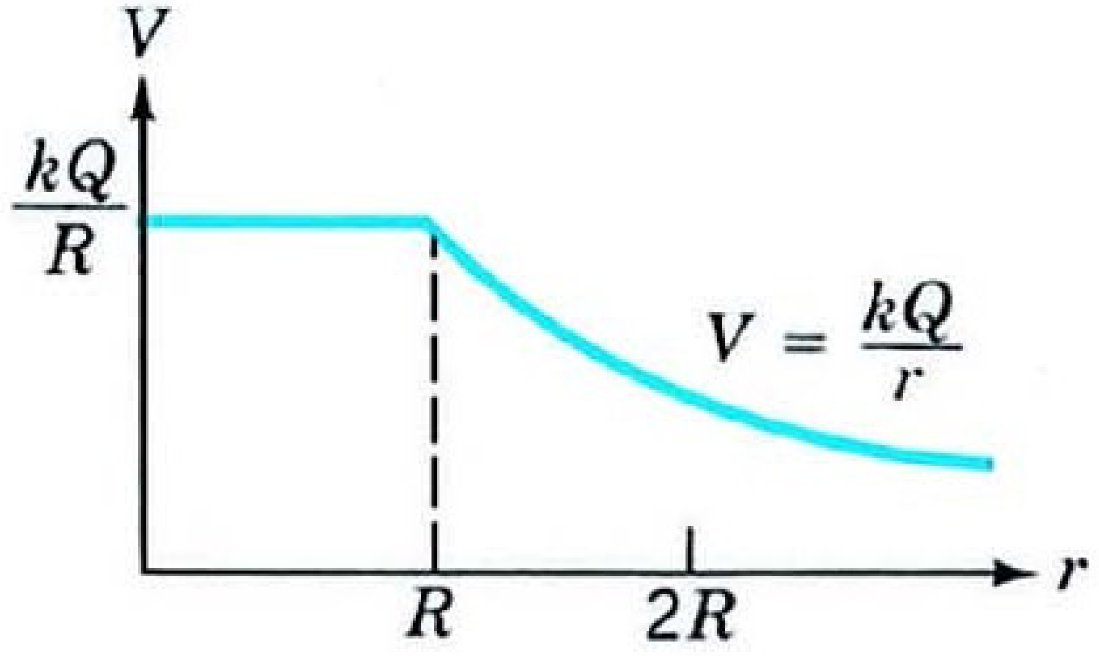
A metal sphere of radius R has a charge Q. Find its ***potential energy***.

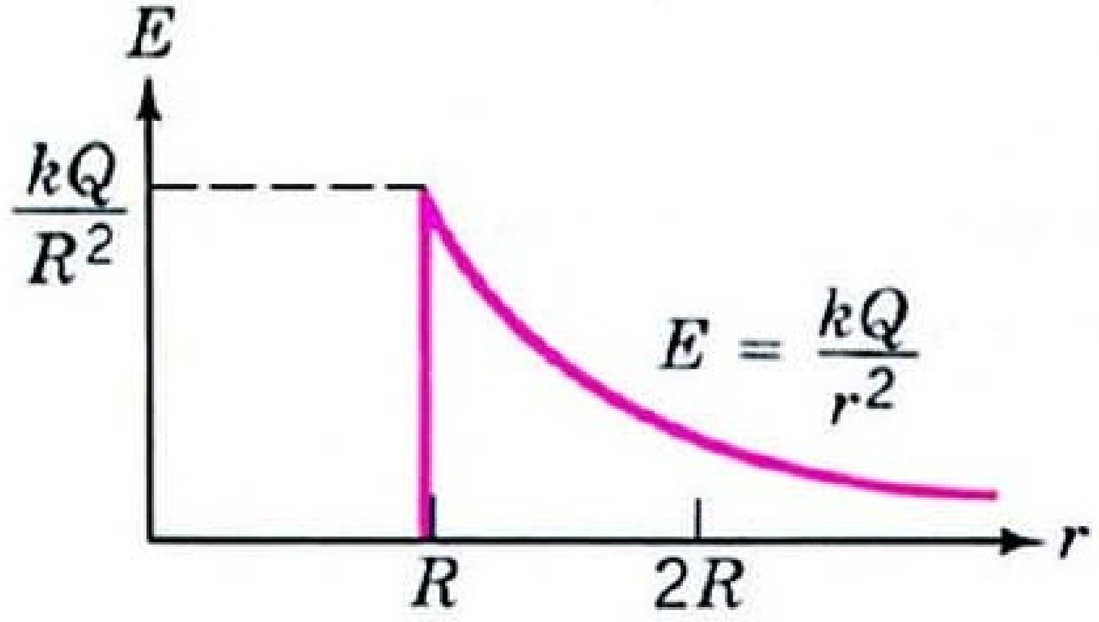
Solution:



$$dW = Vdq = \frac{q}{4\pi\varepsilon_0 R} dq$$

$$W = \int_0^Q \frac{q}{4\pi\varepsilon_0 R} dq = \frac{Q^2}{8\pi\varepsilon_0 R}$$





The potential energy $U=1/2QV$ is the work needed to bring the system of charges together.

2.4.4 Comments on Electrostatic Energy

(i) A perplexing "inconsistency"

$$\Delta K = -q\Delta V$$

$$W = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2 d\tau \quad \geq 0$$

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V_i(\mathbf{r}_i) \quad \geq \text{or} \leq 0$$

Which equation is correct?

Both equations are correct.

$*$The energy required to assemble the charges q_i.

Why is the energy of a point charge infinite?

$$W = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2 d\tau = \frac{\varepsilon_0}{2}\int_0^{\infty}\left(\frac{q}{4\pi\varepsilon_0 r^2}\right)^2 (r^2 \sin\theta dr d\theta d\varphi) = \infty$$

Does it make sense? No

Comments on Electrostatic Energy

(ii) Where is the energy stored?

$$W = \int_{\text{all space}} \left(\frac{\varepsilon_0}{2} E^2\right) d\tau \qquad\qquad W = \frac{1}{2}\sum_{i=1}^{n} q_i V_i(\mathbf{r}_i)$$

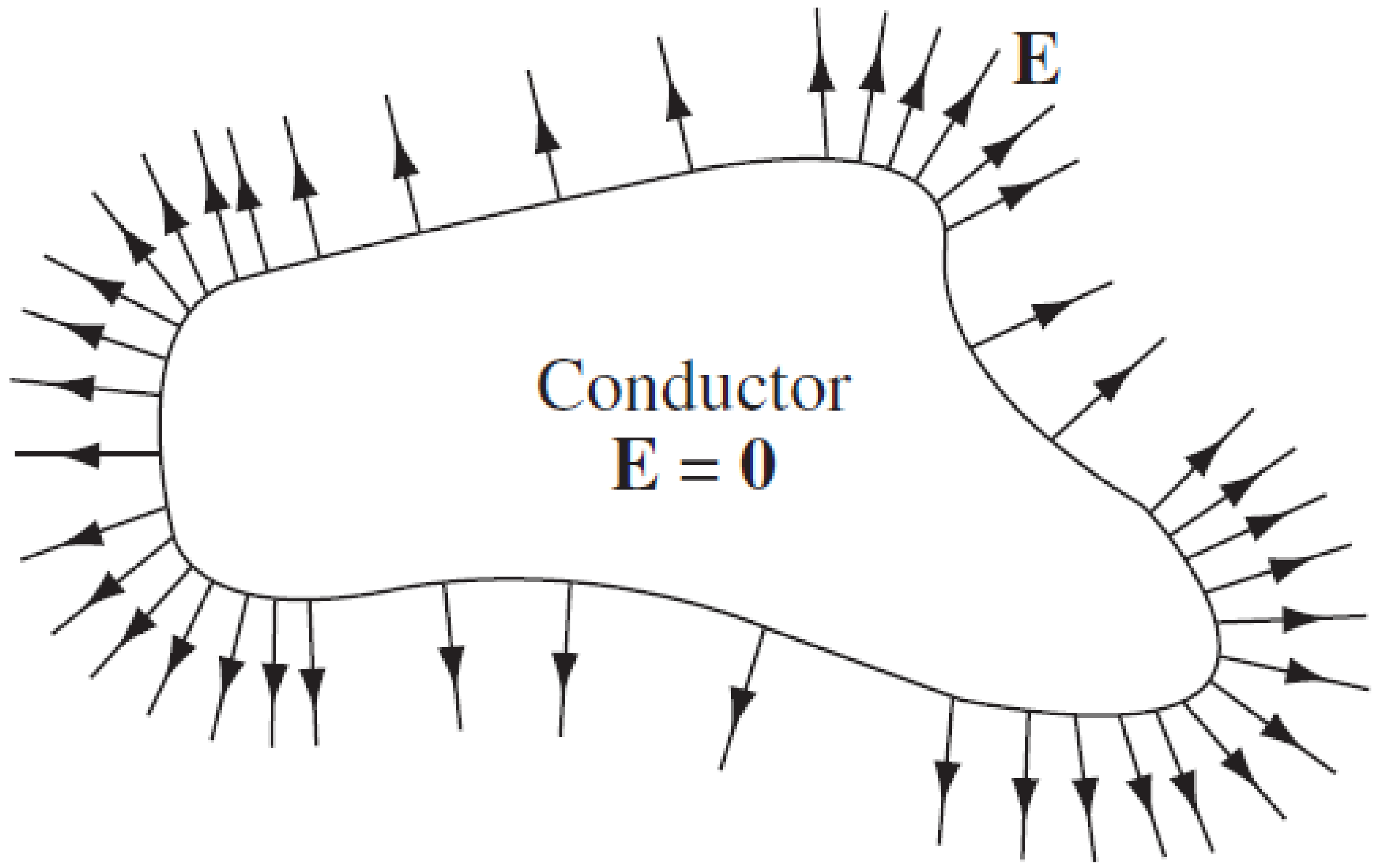
It is unnecessary to worry about where the energy is located.

(iii) Superposition principle is not valid, because the electrostatic energy is quadratic in the fields.

$$W = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2 d\tau = \frac{\varepsilon_0}{2}\int_{\text{all space}} (\mathbf{E}_1 + \mathbf{E}_2)^2 d\tau$$

$$= \frac{\varepsilon_0}{2}\int_{\text{all space}} (E_1^2 + E_2^2 + 2\mathbf{E}_1 \cdot \mathbf{E}_2) d\tau$$

2.5 Conductor
2.5.1 Basic Properties



$\mathbf{E} = 0$ inside a conductor

$\rho = 0$ inside a conductor

Any net charge resides on the surface

A conductor is equal-potential

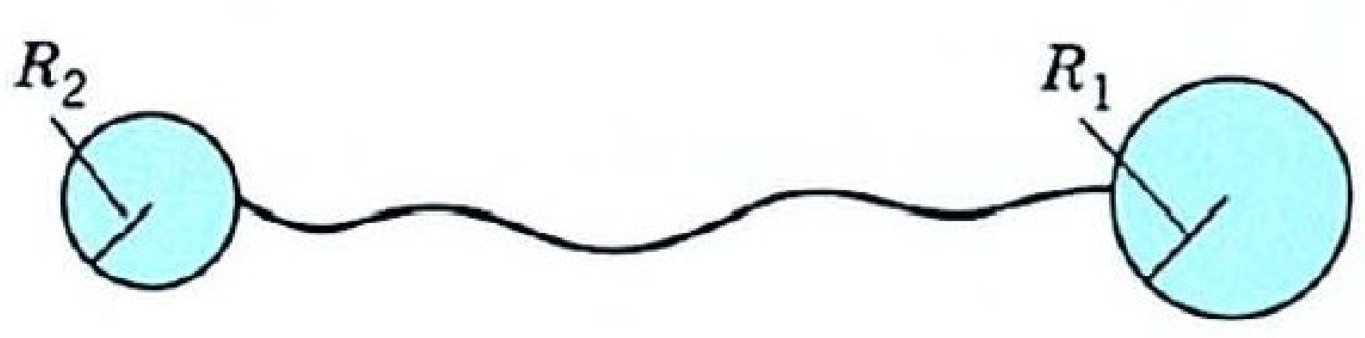
E is perpendicular to the surface, just outside a conductor.

※ The correctness of the above statements depends on the size and the conductivity of the metal, and the frequency of the wave.

Charge Redistribution

Suppose two charged metal spheres with radius R_1 and R_2 are connected by a long wire. Charge will flow from one to the other until their potential are equal. The equality of the potential implies that

$$\frac{Q_1}{R_1} = \frac{Q_2}{R_2}, \text{ since } Q = 4\pi R^2 \sigma$$



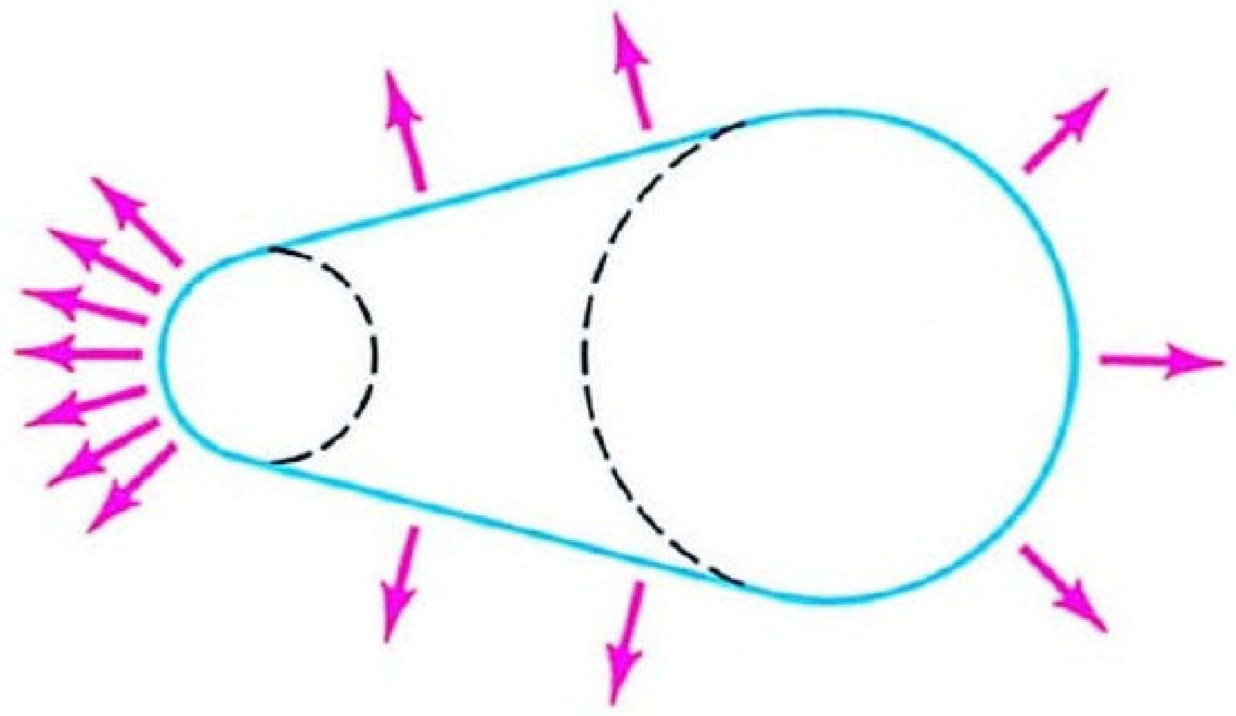
$$\sigma_1 R_1 = \sigma_2 R_2$$

We infer that the surface charge density on each sphere is inversely proportional to the radius.

The regions with the smallest radii of curvature have the greatest surface charge densities.

Discharge at Sharp Points on a Conductor

$$E = \frac{\sigma}{\varepsilon_0} \propto \frac{1}{R}$$



The above equation infers that the field strength is greatest at the sharp points on a conductor.

If the field strength is great enough (about $3 \times 10^6\ \mathrm{V/m}$ for dry air) it can cause an electrical discharge in air.

How does the breakdown occur in high voltage transmission line?

Dust Causing High Voltage Breakdown

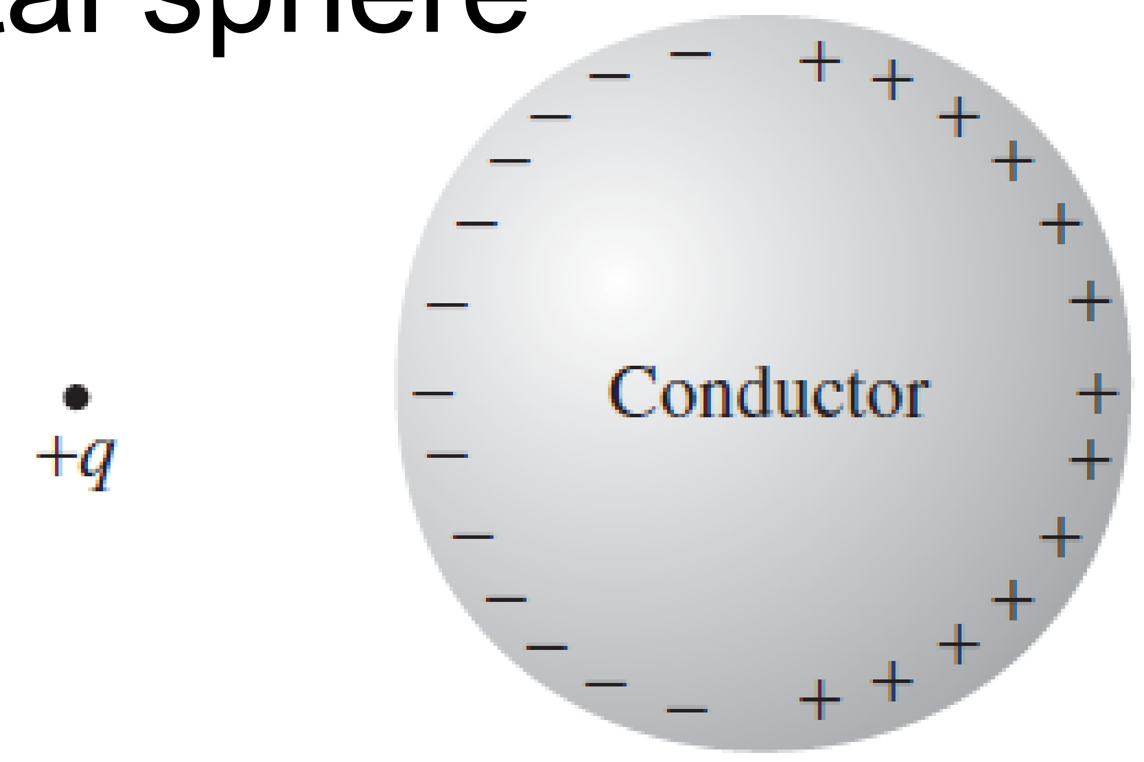
The potential at the surface of a charged sphere is $V = kQ/R$ and the field strength is $E = kQ/R^2$. So, for a given breakdown field strength, breakdown voltage is proportional to the radius, $V_B \propto R$.

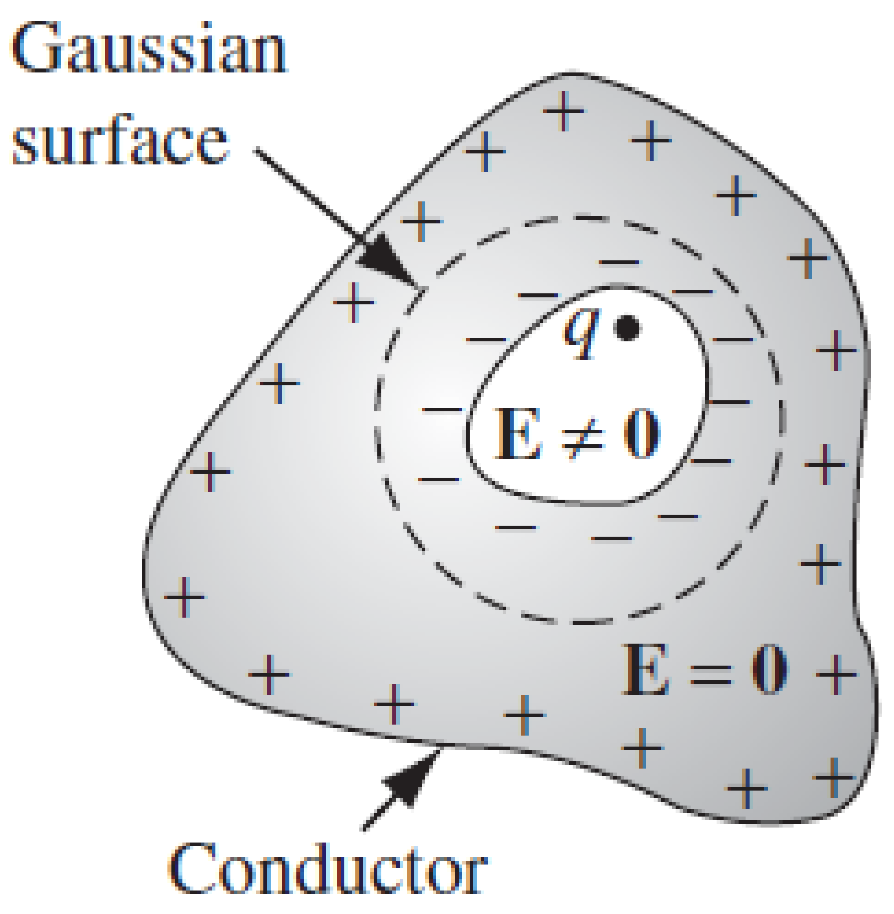
The potential of a sphere of radius 10 cm may be raised to $3\text{x}10^5$ V before breakdown. On the other hand, a 0.05 mm dust particle can initiate a discharge at 150 V.

A high voltage system must keep at very clean condition.

2.5.2 Induced Charges

Induced charge
on metal sphere





If there is some cavity in the conductor, and within that cavity there is some charge, then the field in the cavity will not be zero.

No external fields penetrate the conductor; they are canceled at the outer surface by the induced charge there.

2.5.3 Surface Charge and Force on a Conductor

Using energy density viewpoint

In the immediate neighborhood of the surface, the energy is

$$dW = (\frac{\varepsilon_0}{2}E^2)d\tau = (\frac{\varepsilon_0}{2}(\frac{\sigma}{\varepsilon_0})^2)dadx = fdadx$$

$$f = \frac{\sigma^2}{2\varepsilon_0} \longleftarrow \text{the force per unit area}$$

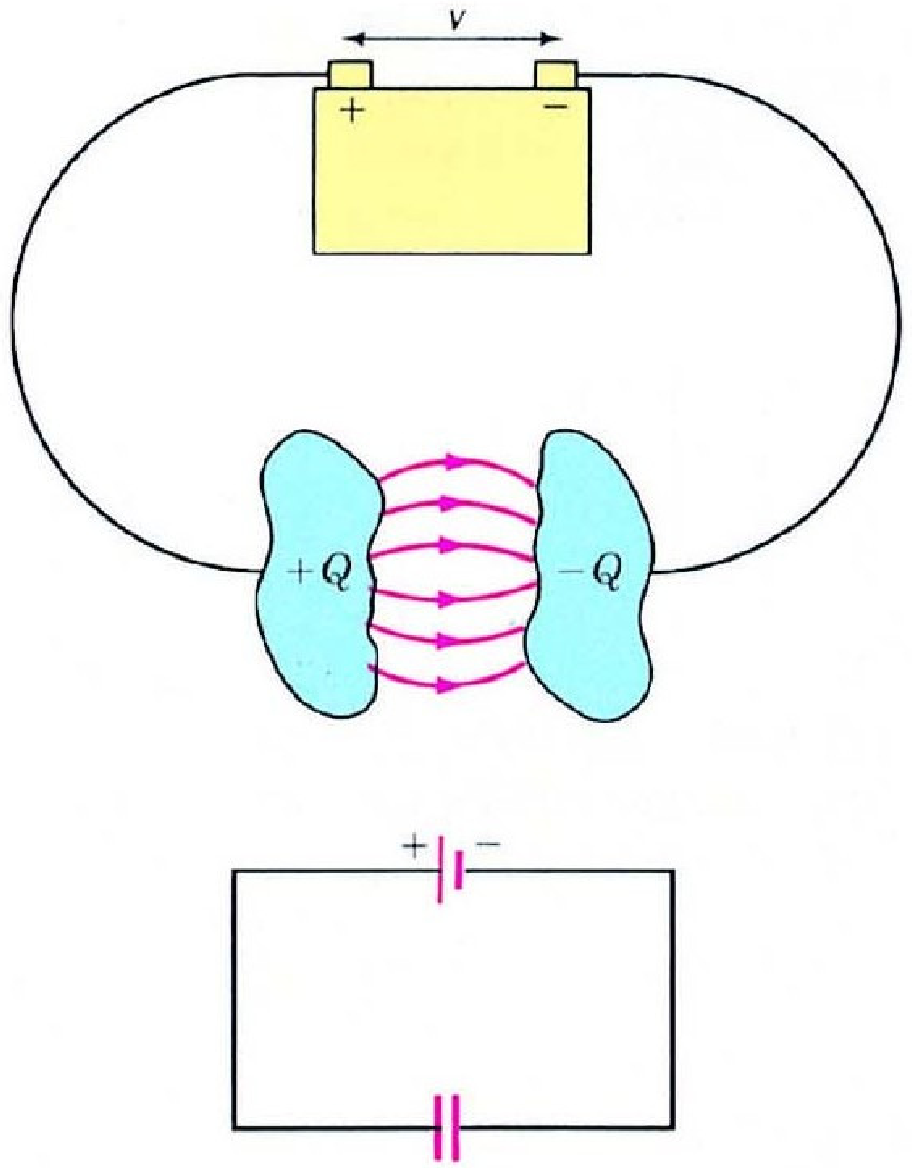
This amounts to an outward **electrostatic pressure** on the surface, tending to draw the charge into the field, regardless of the sign of σ.

$$P = \frac{\varepsilon_0}{2}E^2 = \frac{\sigma^2}{2\varepsilon_0}$$

2.5.4 Capacitors

The magnitude of the charge Q stored on either plate of a capacitor is directly proportional to the potential difference V between the plates. Therefore, we may write

$$Q = CV$$



Where C is a constant of proportionality called the capacitance of the capacitor.

The SI unit of a capacitance is the farad (F). 1 Farad =1 coulomb/volt

The capacitance of a capacitor depends on the ***geometry*** of the plates (their size, shape, and relative positions) and the ***medium*** (such as air, paper, or plastic) between them.

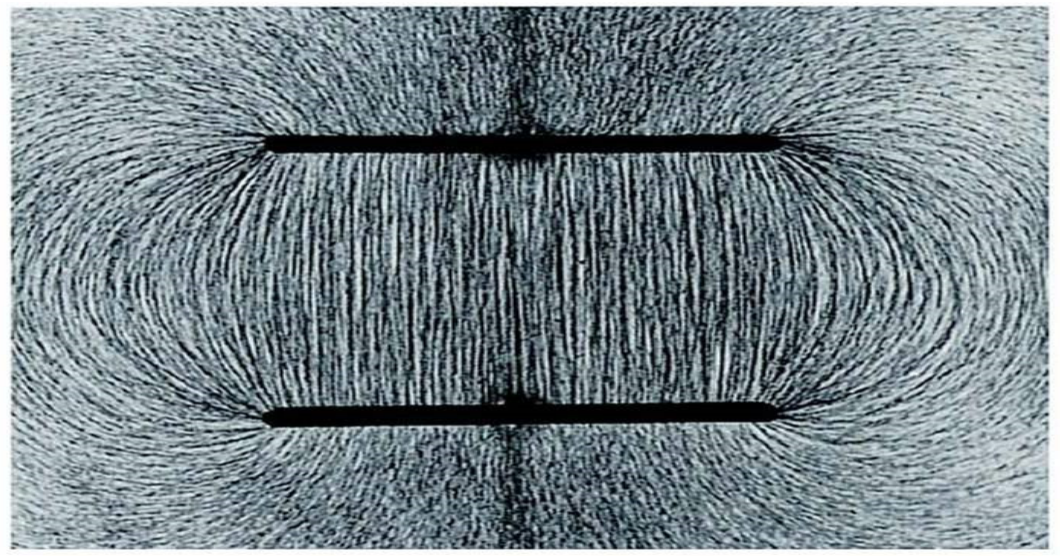
What are supercapacitors/ultracapacitors?

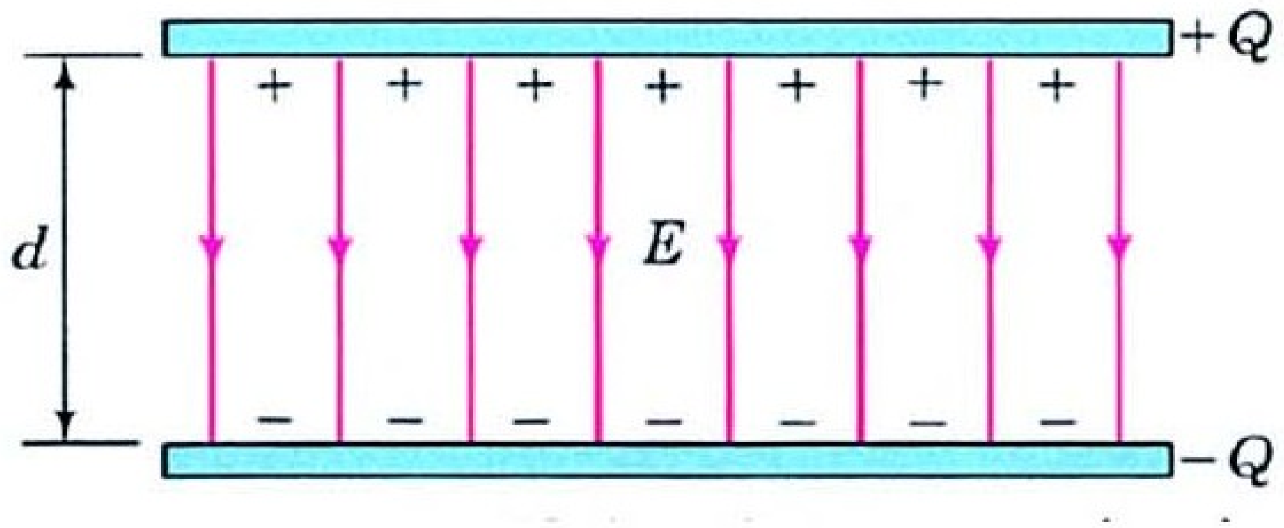
Parallel-plate capacitor

A common arrangement found in capacitors consists of two plates.

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A} \Rightarrow V = Ed = \frac{Qd}{\varepsilon_0 A} \qquad \therefore \quad C = \frac{\varepsilon_0 A}{d}$$

where ε_0 is 8.85×10^{-12} F / m.





Example 2.10 A parallel-plate capacitor with a plate separation of 1 mm has a capacitance of 1 F. What is the area of each plate?

$$A = \frac{Cd}{\varepsilon_0} = \frac{1\times10^{-3}}{8.85\times10^{-12}} = 1.13\times10^{8}\ \mathrm{m}^2$$

Benson

Example

What is the capacitance of an isolated sphere of radius R?

Solution:

$$V = \frac{Q}{4\pi\varepsilon_0 R} \Rightarrow C = 4\pi\varepsilon_0 R$$

If we assume that earth is a conducting sphere of radius 6370 km, then its capacitance would be 710 uF.

Is earth a good capacitor? No.

Benson

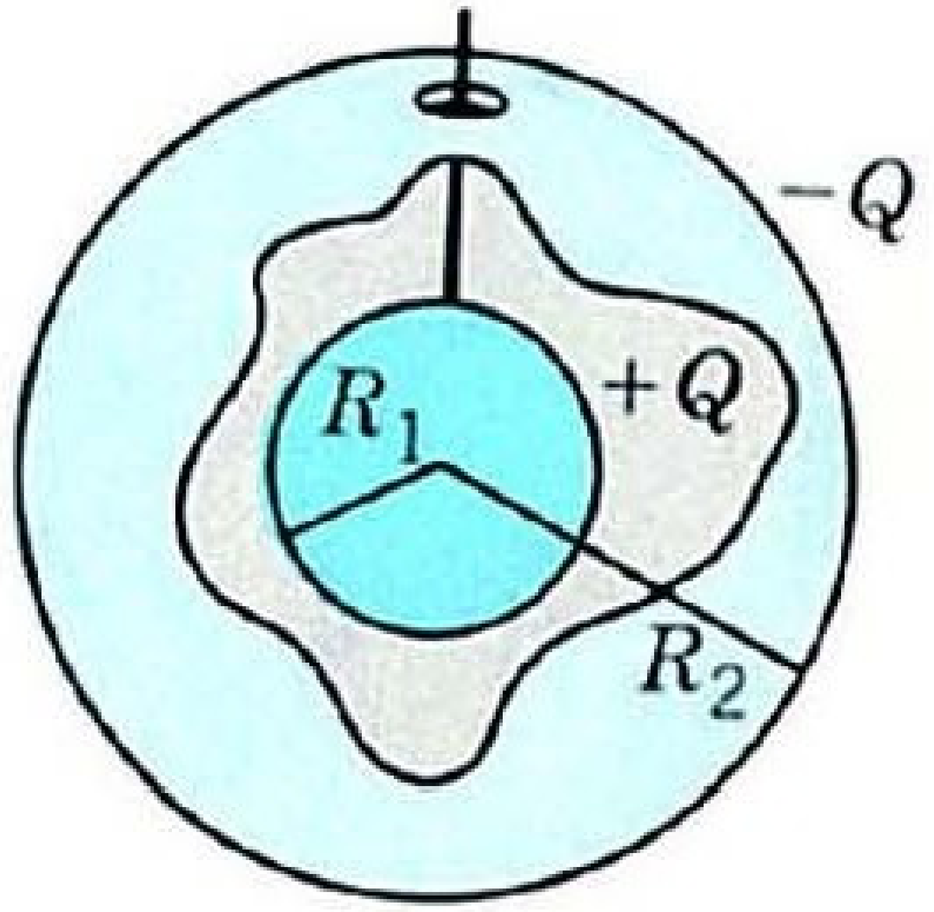
Example

A *spherical capacitor* consists of two concentric conducting spheres, as shown in the figure. The inner sphere, of radius R_1, has charge $+Q$, while the outer shell of radius R_2, has charge $-Q$. Find its capacitance.

Solution:

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} \Rightarrow V = -\int_{R1}^{R2} E dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$

$$C = -4\pi\varepsilon_0\left(\frac{R_1 R_2}{R_2 - R_1}\right)$$



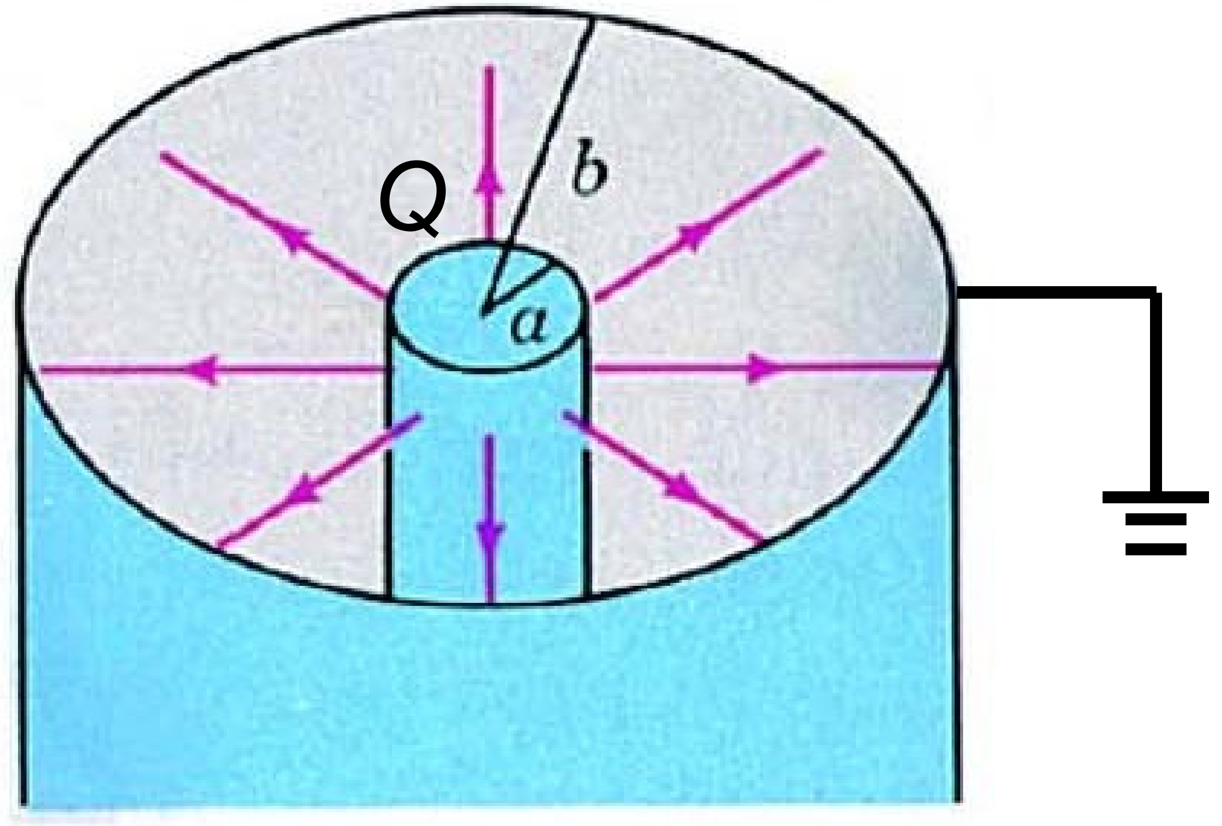
The capacitance happens to be negative quantity.

Why we are interested only in its magnitude?

Benson

Example

A *cylindrical capacitor* consists of a central conductor of radius a surrounded by a cylindrical shell of radius b, as shown below. Find the capacitance of a length L assuming that air is between the plates.



Solution:

$$E_r = \frac{\lambda L}{\varepsilon_0 2\pi r L} = \frac{\lambda}{2\pi\varepsilon_0 r}$$

$$V_r = -\int_a^b E_r dr = -\frac{\lambda}{2\pi\varepsilon_0}\ln(\frac{b}{a})$$

$$= -\frac{Q}{2\pi\varepsilon_0 L}\ln(\frac{b}{a})$$

$$C = -\frac{2\pi\varepsilon_0 L}{\ln(b/a)}$$

Again, we are interested only in the magnitude of the capacitance.

Energy Stored in a Capacitor

The energy stored in a capacitor is equal to the work done---for example, by a battery---to charge it.

The work needed to transfer an infinitesimal charge dq from the negative plate to the positive plate is $dW = Vdq = q/Cdq$.

The total work done to transfer charge Q is

$$W = \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C} = \frac{CV^2}{2}$$

What kind of the potential energy does this work convert?

Electric potential energy.

Homework of Chap. 2 (part II)

Problem 2.36 Consider two concentric spherical shells, of radii a and b. Suppose the inner one carries a charge q, and the outer one a charge $-q$ (both of them uniformly distributed over the surface). Calculate the energy of this configuration, (a) using Eq. 2.45, and (b) using Eq. 2.47 and the results of Ex. 2.9.

Problem 2.39 Two spherical cavities, of radii a and b, are hollowed out from the interior of a (neutral) conducting sphere of radius R (Fig. 2.49). At the center of each cavity a point charge is placed – call these charges q_a and q_b.

(a) Find the surface charge densities σ_a, σ_b, and σ_R.

(b) What is the field outside the conductor?

(c) What is the field within each cavity?

(d) What is the force on q_a and q_b?

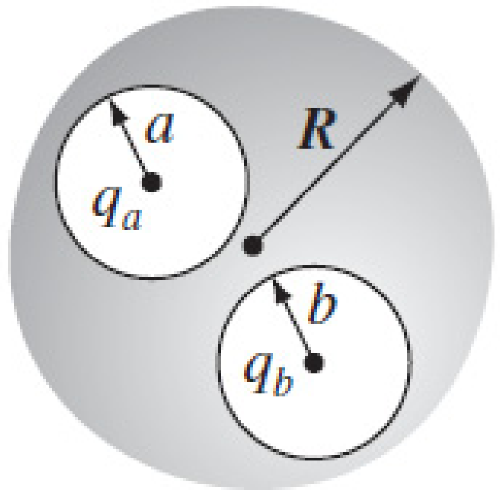


FIGURE 2.49

Problem 2.43 Find the capacitance per unit length of two coaxial metal cylindrical tubes, of radii a and b (Fig. 2.53).

FIGURE 2.53

Problem 2.50 The electric potential of some configuration is given by the expression

$$V(\mathbf{r}) = A\frac{e^{-\lambda r}}{r},$$

where A and λ are constants. Find the electric field $\mathbf{E}(\mathbf{r})$, the charge density $\rho(r)$, and the total charge Q. [*Answer*: $\rho = \varepsilon_0 A(4\pi\delta^3(\mathbf{r}) - \lambda^2 e^{-\lambda r}/r)$]

Homework of Chap. 2 (part II)

Problem 2.53 In a vacuum diode, electrons are "boiled" off a hot **cathode**, at potential zero, and accelerated across a gap to the **anode**, which is held at positive potential V_0. The cloud of moving electrons within the gap (called **space charge**) quickly builds up to the point where it reduces the field at the surface of the cathode to zero. From then on, a steady current I flows between the plates.

Suppose the plates are large relative to the separation ($A \gg d^2$ in Fig. 2.55), so that edge effects can be neglected. Then V, ρ, and v (the speed of the electrons) are all functions of x alone.

(a) Write Poisson's equation for the region between the plates.

(b) Assuming the electrons start from rest at the cathode, what is their speed at point x, where the potential is $V(x)$?

(c) In the steady state, I is independent of x. What, then, is the relation between ρ and v?

(d) Use these three results to obtain a differential equation for V, by eliminating ρ and v.

(e) Solve this equation for V as a function of x, V_0, and d. Plot $V(x)$, and compare it to the potential *without* space-charge. Also, find ρ and v as functions of x.

(f) Show that

$$I = KV_0^{3/2}, \qquad (2.56)$$

and find the constant K. (Equation 2.56 is called the **Child - Langmuir law**. It holds for other geometries as well, whenever space-charge limits the current. Notice that the space-charge limited diode is *nonlinear* – it does not obey Ohm's law.)

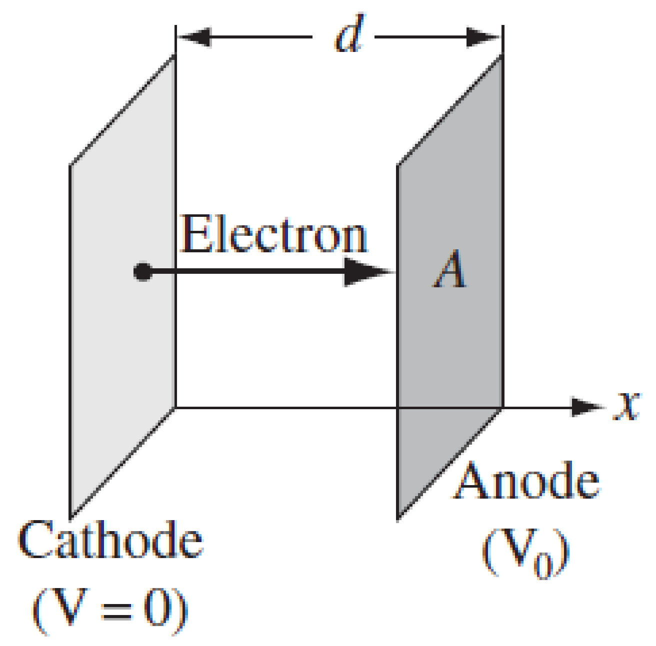


FIGURE 2.55