EINSTEIN'S THEORY OF SPECIAL RELATIVITY

Lecture 2, Introduction to Black Hole Astrophysics (PHYS480)

Hsiang-Yi Karen Yang, NTHU, 3/2/2021

ANNOUNCEMENTS

- Please sign up for the oral presentation ASAP via this link:

https://docs.google.com/spreadsheets/d/1_aYyMj1wf_uGheZ7zp_hvthmy4mdmPwIxFDdZOMG-nc/edit#gid=0

- For students who asked questions during or after class, please don't forget to tell the TA to receive the class participation points!

- HW1 will be posted today on iLMS and course website. Please hand in your solutions or submit online through iLMS by 13:20 next Tuesday (3/9).

PREVIOUS LECTURE…

- How black holes were predicted
  - Escape velocities and idea of dark stars in Newtonian gravity
  - GR's view of BHs – spacetime is so curved that light cannot escape within the Schwarzschild radius
  - Theoretical advancement in stellar evolution and relativity as well as observational discovery of pulsars sparkled interests in the search of astrophysical BHs

- How black holes were discovered
  - First candidate of stellar-mass BHs – the X-ray binary Cygnus X-1
  - BHs shine due to thermal radiation from heated gas within accretion disks
  - Quasars are found, later known as accreting supermassive black holes (SMBHs) or active galactic nuclei (AGN)
  - Every massive galaxy hosts a SMBH at the center

PREVIOUS LECTURE…

- Why study black holes?
  - They are simply cool!
  - They are important for understanding physics at extreme conditions
  - The radiation and jets from SMBHs can influence the formation and evolution of galaxies

- Another golden age for studying BHs
  - First image of BHs taken by the Event Horizon Telescope
  - Detections of gravitational waves due to BH mergers – the birth of multi-messenger astrophysics

THIS LECTURE…

- What people know before 1905
  - Newton's laws of motion and Galilean transformation
  - Newtonian gravity and the weak equivalence principle
  - The speed of light problem
- Two postulates of Einstein's theory of special relativity (SR)
- Consequences of SR
  - Time dilation
  - Length contraction
  - Lorentz transformation
  - Relativity of simultaneity and causality
  - Relativistic expressions for mass and energy
- Astrophysical effects of SR

WHAT WAS KNOWN BEFORE 1905

NEWTON'S THREE LAWS OF MOTION

- 1st law: **v** = constant if **F** = 0 (慣性定律：在無外力狀況下，靜者恆靜，動者恆動)
  - This is a major change of Galileo's thinking that "being at rest" is a natural state

- 2nd law: If a body of mass M is acted upon by a force **F**, then its acceleration **a** is given by **F** = M**a**
  - This law defines the "***inertial mass* (慣性質量)**", which represents the degree to which a body resists being accelerated by a force

- 3rd law: If a body A exerts a force **F** on body B, then body B exerts a force **–F** on body A (作用力與反作用力)

NEWTON'S LAWS REWRITTEN USING MOMENTUM

- Momentum: $\mathbf{p} = M\mathbf{v}$

  Total momentum of a system is conserved: $\mathbf{p}_{tot} = M_1\mathbf{v_1} + M_2\mathbf{v_2} + \ldots$

- Newton's 2^{nd} law: the rate of change of momentum of a body is equal to the force applied to that body
- Newton's 1^{st} law: a special case of the 2^{nd} law – the momentum of a body is unchanged if there is no force acting on the body
- Newton's 3^{rd} law: the momentum of an isolated system of objects is conserved

- Note that terms like **x, v, a**, need to be expressed by coordinates in some reference frame

WHAT HAPPENS IN A DIFFERENT REFERENCE FRAME?

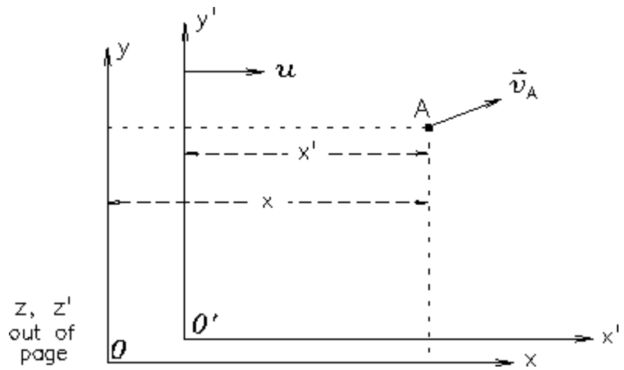
- Consider two ***inertial (non-accelerating) frames of reference* (慣性座標系)** that differ by some uniform velocity difference $\boldsymbol{u}$ (note that we are not considering accelerating frames of reference)
- This is called ***Galilean transformation***:

Simple velocity addition rule ->

$$x' = x - ut$$

$$v' = \frac{dx'}{dt} = v - u$$

$$a' = \frac{dv'}{dt} = a$$



- For a system with no external force, momentum is conserved under Galilean transformation
  - This is a result of spatial-translation ***symmetry***
  - Noether's theorem: if a system has a continuous symmetry property, then there is a corresponding conserved quantity

CONNECT NEWTON'S LAWS TO GALILEAN TRANSFORMATION

- ***Galilean invariance***: laws of physics are the same under Galilean transformation
- Newton's 1st law comes directly from Galilean invariance: there is no difference between a state of rest and a state of motion
- Newton's 2nd and 3rd laws are what's needed to make sure that momentum is conserved and thus is related to the symmetry of space

NEWTON'S LAW OF UNIVERSAL GRAVITATION

- A particle with mass m_1 will attract another particle with mass m_2 and distance r with a force F given by

$$F = \frac{Gm_1m_2}{r^2}$$

- G = gravitational constant = 6.67×10^{-8} $cm^3\ g^{-1}\ s^{-2}$
- It is universal and a long-range force (every particle in the universe attracts every other particle)
- Gravity often dominates in astrophysical systems
- This law defines the "***gravitational mass* (重力質量)**" of a body

INERTIAL MASS AND GRAVITATIONAL MASS

- Newton's 2nd law: $F = m_I a$

  m_I = inertial mass

- Newton's law of gravitation: $F = \frac{GMm_G}{r^2}$

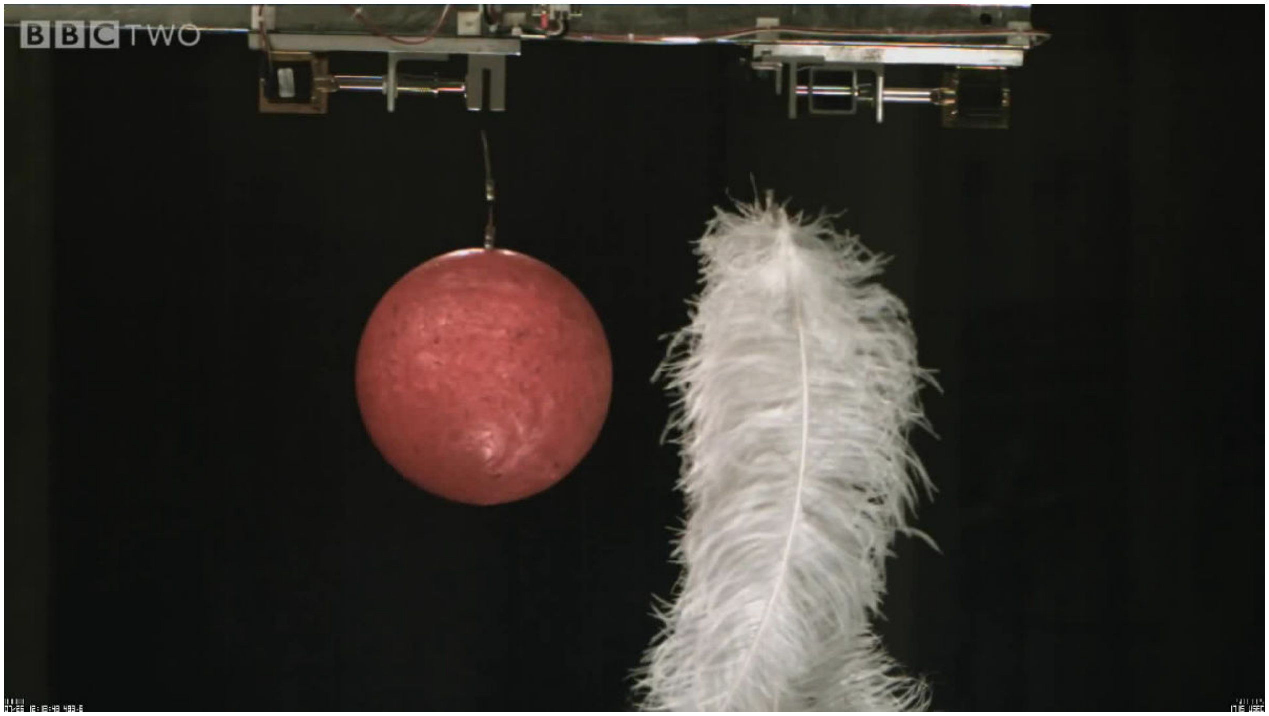
  m_G = gravitational mass

- So, acceleration due to gravity is:

$$a = \left(\frac{m_G}{m_I}\right)\frac{GM}{r^2}$$

- Note that m_I and m_G don't necessarily need to be equal
- If they are equal, it implies that acceleration due to gravity is independent of the object's mass

WHICH WOULD FALL FASTER?



Watch the modern version of experiment here:
https://www.youtube.com/watch?v=E43-CfukEgs

EXPERIMENT DONE ON THE MOON IN THE APOLLO 15 MISSION

- In July 1971, at the end of the last Apollo 15 moon walk, Commander David Scott performed a live test of m_I/m_G for the television cameras

https://en.wikipedia.org/wiki/File:Apollo_15_feather_and_hammer_drop.ogv

EQUIVALENCE OF INERTIAL AND GRAVITATIONAL MASS

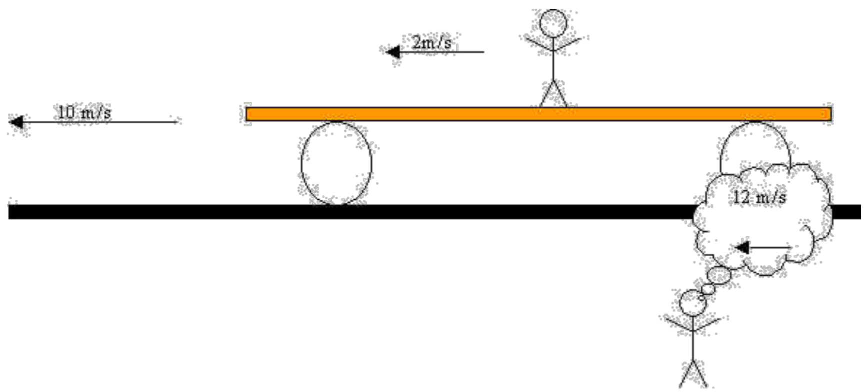
- Verified by various experiments (drop tower, pendulum…)
- $m_I = m_G$ for all bodies – the "***weak equivalence principle* (弱等效原理)**"

WHY DO ASTRONAUTS ON INTERNATIONAL SPACE STATION (ISS) FEEL WEIGHTLESS?

- The ISS orbits about 500km above Earth's surface. Radius of Earth is 6300km
- Gravitational acceleration at ISS is 86% of that on the Earth's surface!
- They feel weightless because the astronauts "***fall***" toward Earth at the same rate as the space station
- This is another example of the ***weak equivalence principle***

THE SPEED OF LIGHT PROBLEM

- Relativity: how to relate measurements in different frames of references
- Galilean relativity – simple velocity addition law



WHAT ABOUT LIGHT?

- In the 1860's, James Clerk Maxwell (1831-1879) developed theory of electromagnetic fields, i.e., the Maxwell's equations

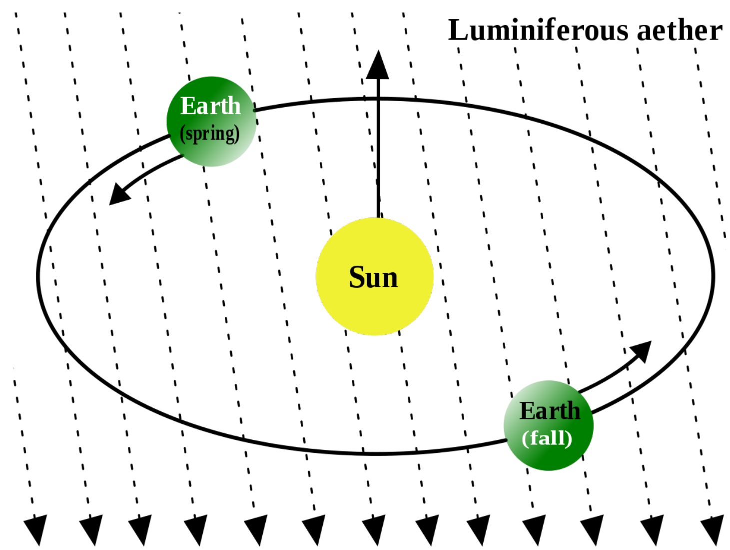
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

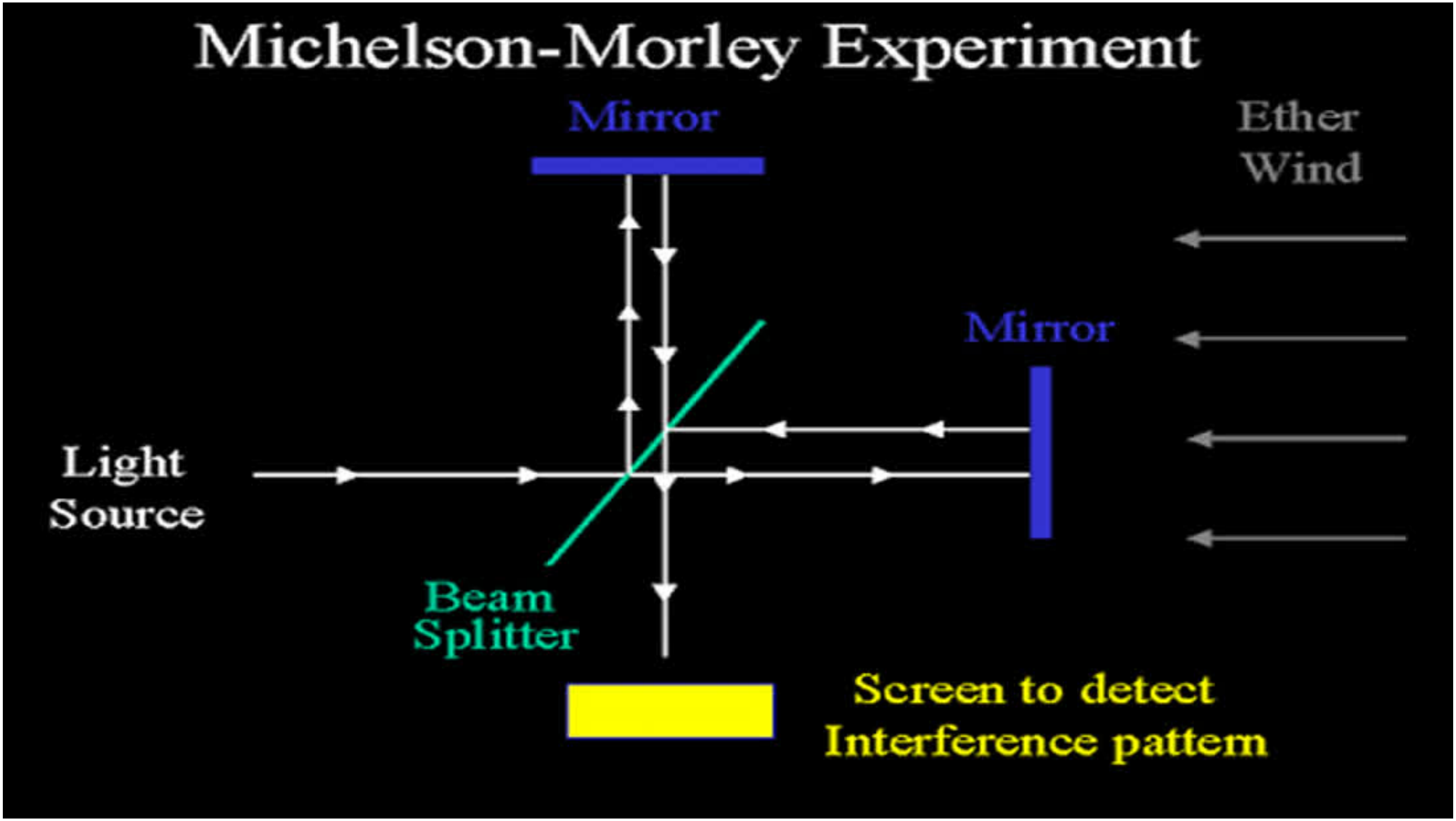
$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

- Light – waves of electromagnetic energy
- The speed of light "c" appears as a fundamental constant in the equations
- c = 299,792,458 km/s
- But, what frame of reference is this speed measured relative to?

LIGHT TRAVELS IN AETHER (乙太)?

- In the 19th century, it was believed that light travels in "Aether", a hypothetical substance that fills space through which light travels
- It was presumed that "c" should be measured w.r.t the rest frame of the Aether
- Albert Michelson & Edward Morley attempted to measure change of speed of light due to Earth's motion through Aether (1887)




Michelson-Morley Experiment
Mirror
Ether
Wind
Mirror
Light
Source
Beam
Splitter
Screen to detect
Interference pattern

MICHELSON-MORLEY RESULTS

- Travel time difference measured using interference fringes of light from two paths
- Repeated at different times of year (Earth's orbital speed around the Sun is ~30km/s)
- Results showed that speed of light is the same in any direction within 5 km/s
- Modern versions of the experiment show consistency better than 1 micron/s!

EINSTEIN'S POSTULATES OF SPECIAL RELATIVITY (1905)

- Albert Einstein abandoned:
  - The idea of Aether
  - Galilean relativity

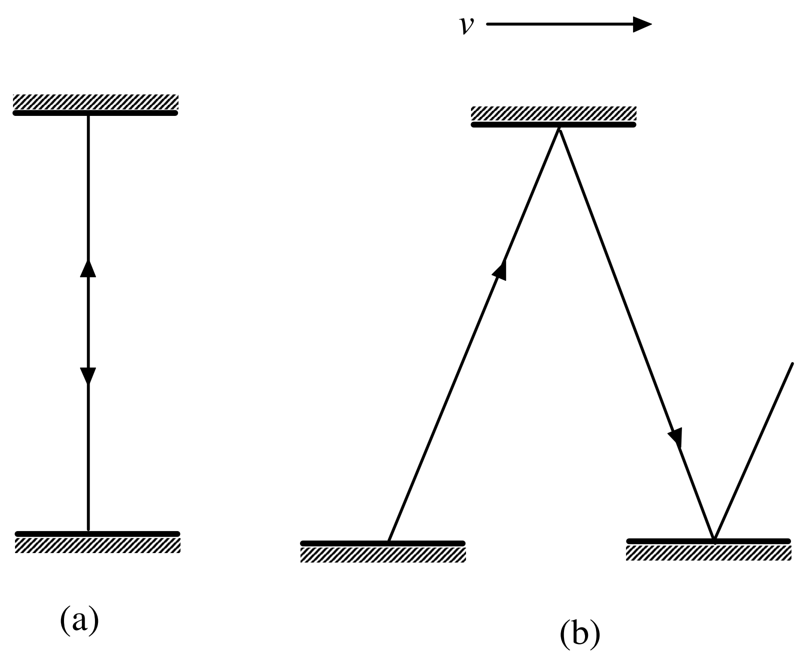
- ***Two postulates:***
  1) ***Laws of physics are invariant in all inertial (non-accelerating) frames of reference***
  2) ***The speed of light in a vacuum is the same for all observers***

CONSEQUENCES OF SR

WOULD TIME FREEZE IF WE MOVE AS FAST AS THE LIGHT??

EFFECT OF TIME DILATION

- Imagine a "light clock" which ticks each time when the light hits one of the mirrors
- In (a), $t_0 = l/c$
- In (b), $t = \dfrac{\sqrt{l^2 + v^2 t'^2}}{c}$
- => $t = \dfrac{t_0}{\sqrt{1 - v^2/c^2}}$



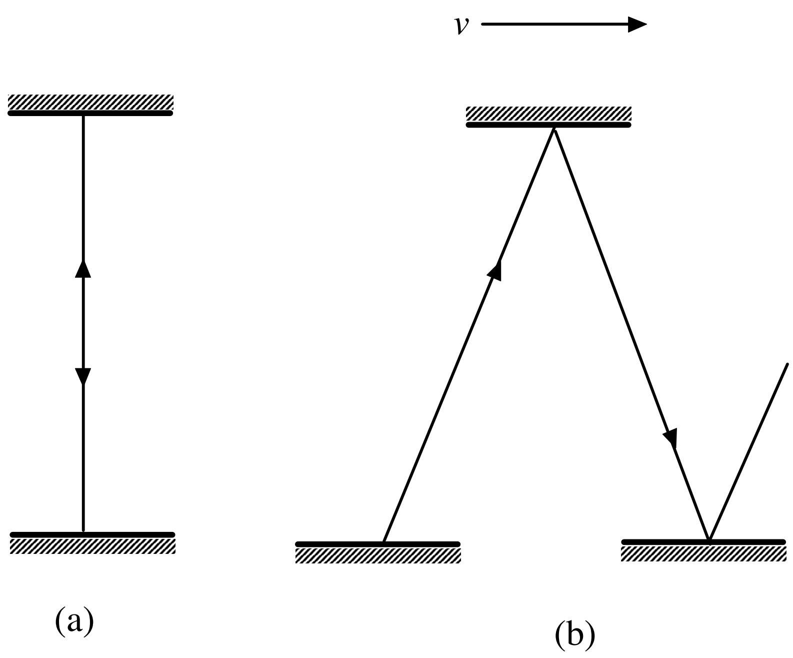
TIME DILATION

$$t = \frac{t_0}{\sqrt{1 - v^2/c^2}} = \gamma t_0$$

- Time is relative! ***The moving clock ticks more slowly by the Lorentz factor***

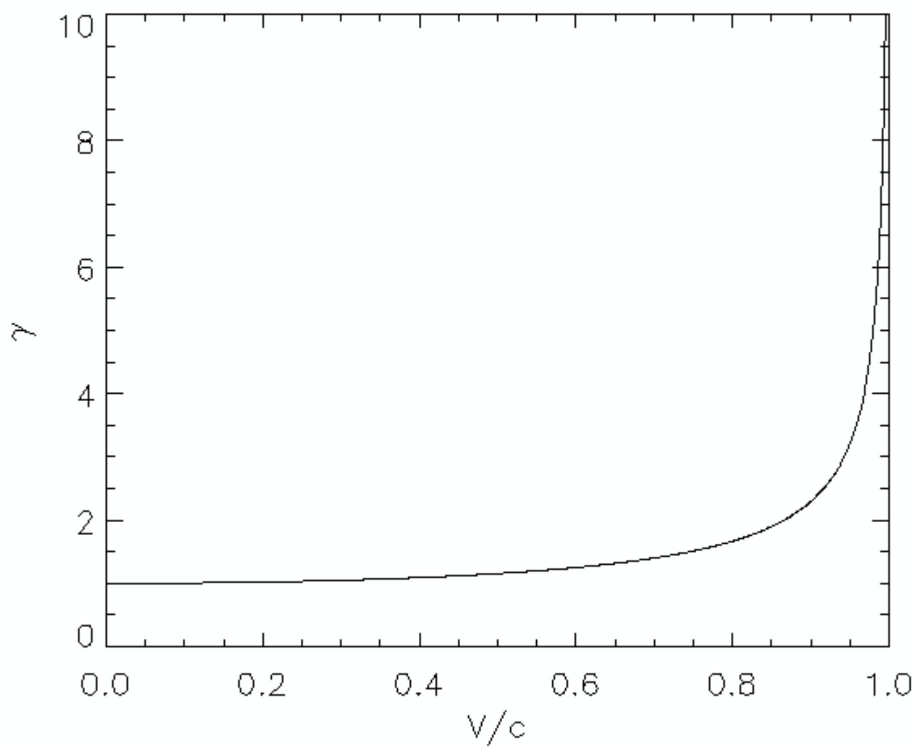
$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

- t_0 = proper time = time measured when clock is at rest



LORENTZ FACTOR

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

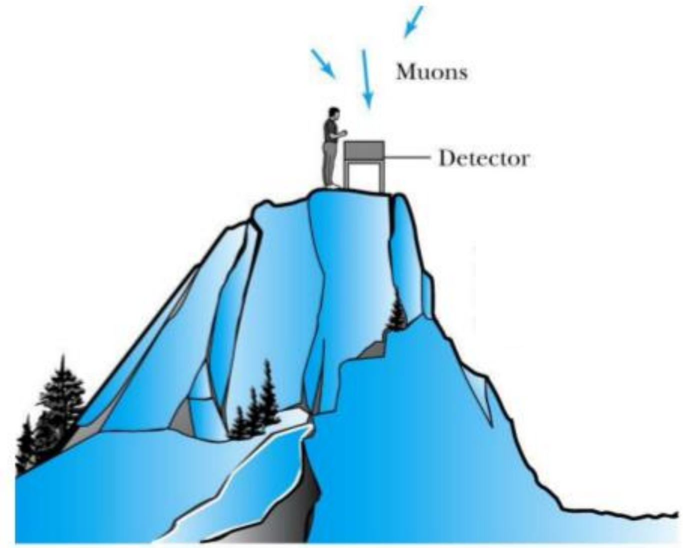


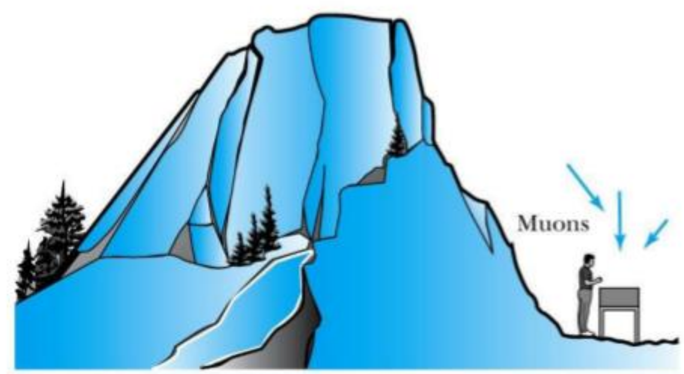
- Goes to infinity as γ -> 1
- Close to 1 for small v/c
- A 1% effect at v=0.14c or 42,000 km/s
- A 10% effect at v=0.42c or 126,000 km/s

MUON EXPERIMENT

- Muons are fundamental particles produced when cosmic rays (relativistic charged particles) hit the Earth's atmosphere
- Muons decay with a half life of 2.2 μs
- Muons travel at 0.99995c
- Rossi & Hall (1940) and Frisch & Smith (1963) compared number of muons at the top of mountain to that at sea level

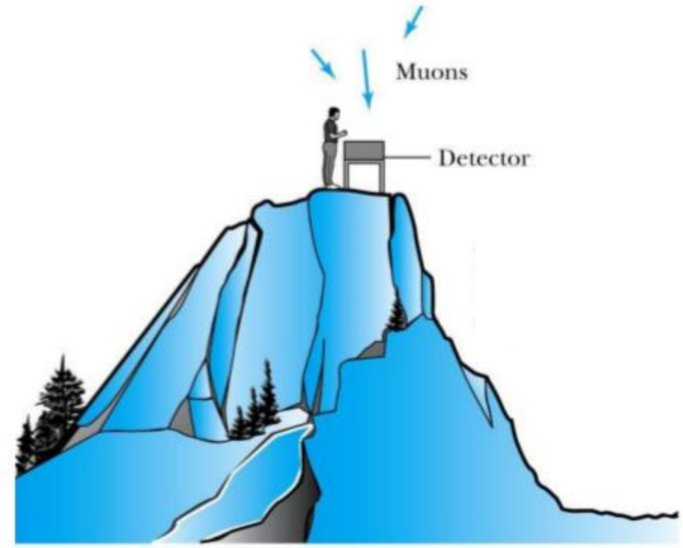
$$N = N_0 2^{-\frac{t}{t_{1/2}}}$$

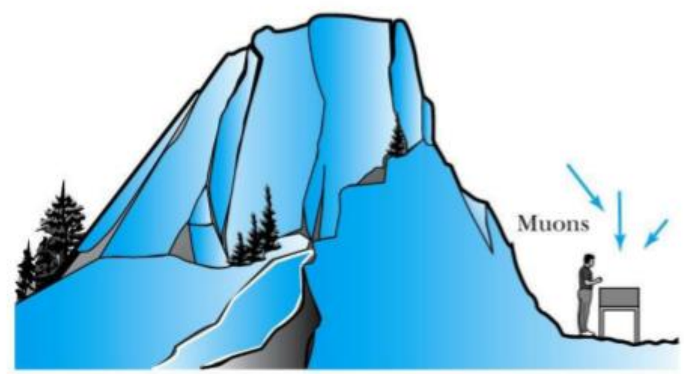




MUON EXPERIMENT

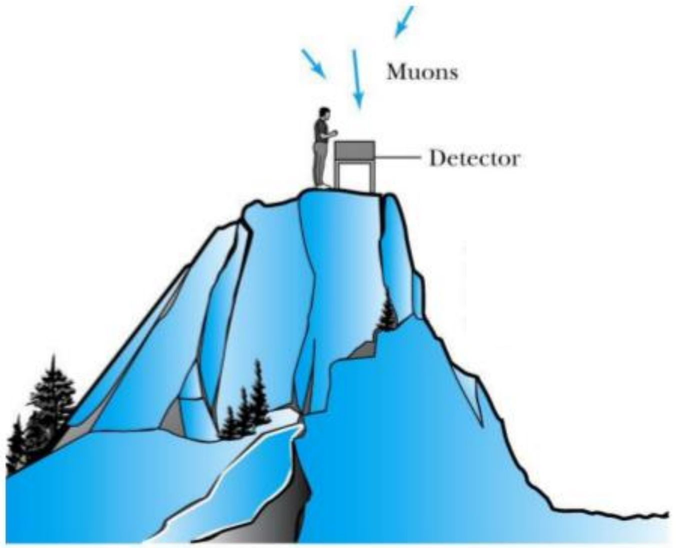
- They found 560 muons/hour at top of a 2000m mountain
- At v = 0.99995c, it takes 6.7μs for muons to travel 2000m
- More than 3 half lives, so less than 1/8 of particles should be left at sea level
- But they measured 543 muons/hour at bottom!

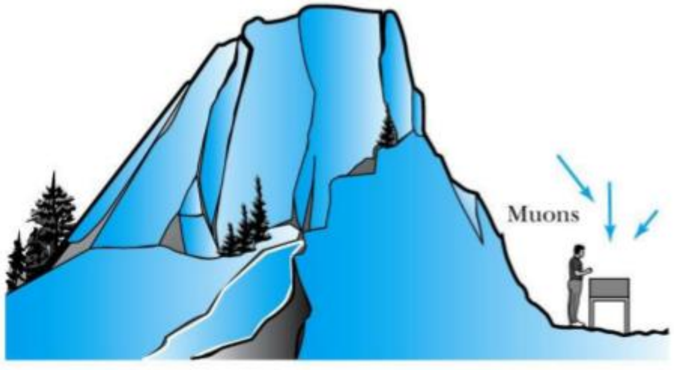




MUON EXPERIMENT

- Muons travel at v=0.99995c => γ = 100
- Muon's half life measured ***in its rest frame*** is 2.2 μs, which is ***dilated in Earth's frame*** to 2.2x10^{-4} s
- 6.7μs is only 0.03 x half life is passed as viewed by an observer on Earth
- Number consistent with time dilation predicted by special relativity!

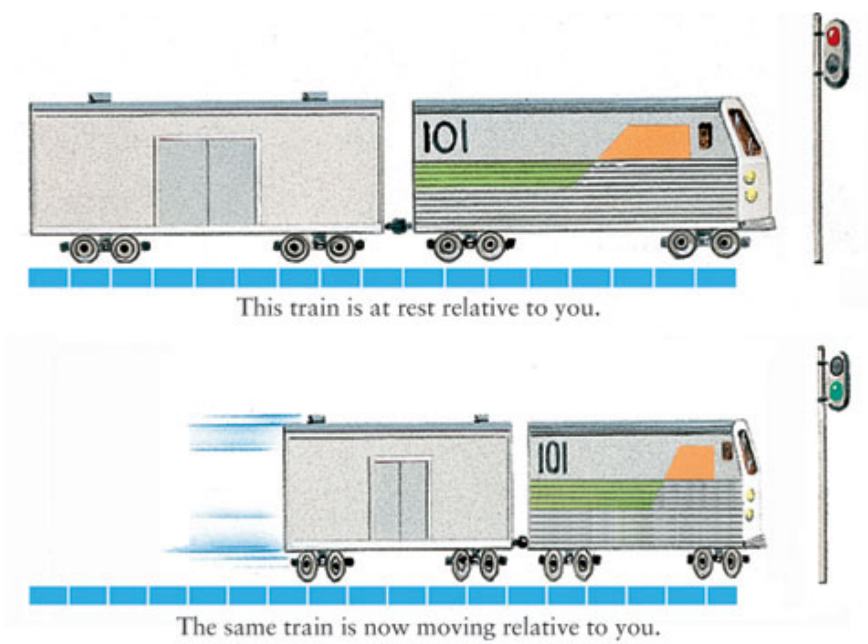




IN MUON'S REFERENCE FRAME?

- The muons see the mountain moving at v=0.99995c towards them
- For the muons, t = 0.03 x (half life of 2.2 μs) = 6.7 x 10^{-2} μs has passed
- It would think the height of the mountain is H = 0.99995c x t ~ 20m!

LENGTH CONTRACTION



- Space is also relative! ***A moving object contracts by a factor of γ in the direction of motion***

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}} = L_0/\gamma$$

- L_0 = proper length = length measured in a reference frame where the object is at rest

LORENTZ TRANSFORMATION

- Based on Einstein's two postulates, one can derive the Lorentz transformation:

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t - \frac{vx}{c^2})$$

- Note that space and time are interconnected to form a 4-dimensional ***spacetime***

LORENTZ TRANSFORMATION

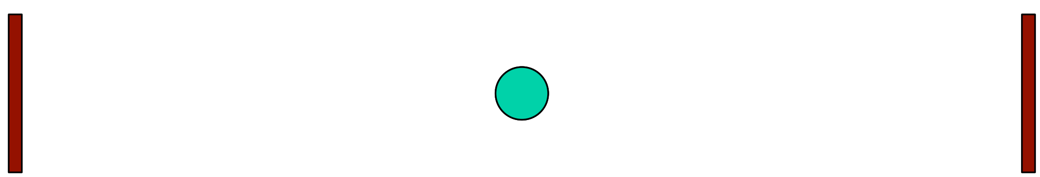
- In matrix representation:

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}$$

- (x, y, z, ct) is the position-time ***4-vector***
- $\boldsymbol{\beta = v/c}$

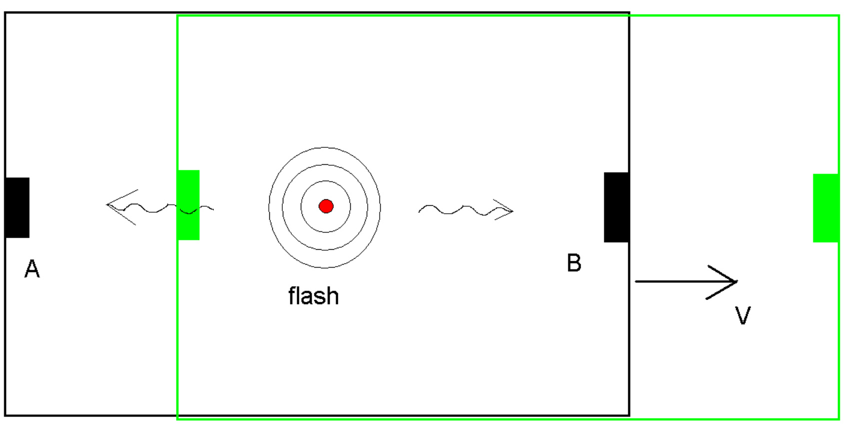
SIMULTANEOUS OR NOT?

- Consider an observer in a room. Suppose there is a flash bulb exactly in the middle of the room
- Sensors are put on the walls to record when the light rays hit the walls
- The light rays will hit opposite walls at precisely at the same time. Call these events A and B.



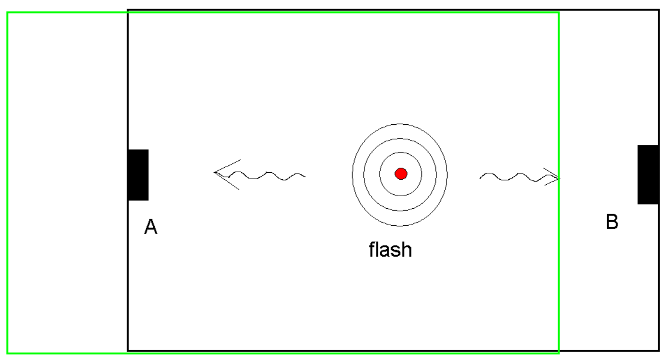
SIMULTANEOUS OR NOT?

- Now perform the same experiment on a moving spacecraft, and observe it from the ground
- Astronauts in spacecraft think events are simultaneous
- For the observer on the ground, the light rays will NOT strike the walls at the same time; A will happen before B.
- Therefore, the concept of "events being simultaneous" is different for different observers



LET'S CHANGE FRAMES AGAIN!

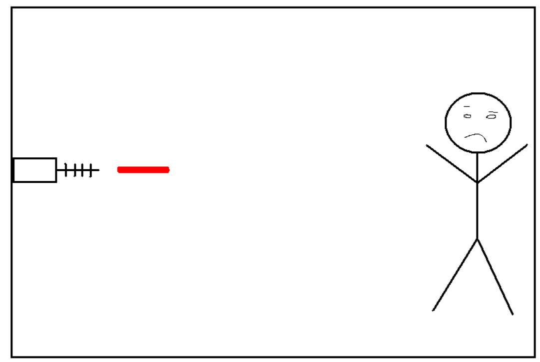
- What would a 3rd observer think who is moving faster than the spacecraft?

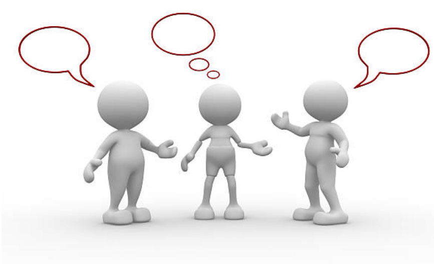


- He would see event B happens before event A
- Therefore, ***order*** of events can depend on the frame of reference

THE LASER GUN EXPERIMENT

- Suppose there is a laser gun at one end of spacecraft, targeted at a victim at the other end
- Laser gun fires (event A) and then victim gets hit (event B)
- Can we change the order of these events by changing the frame of reference? That is, can the victim get hit before the gun fires?



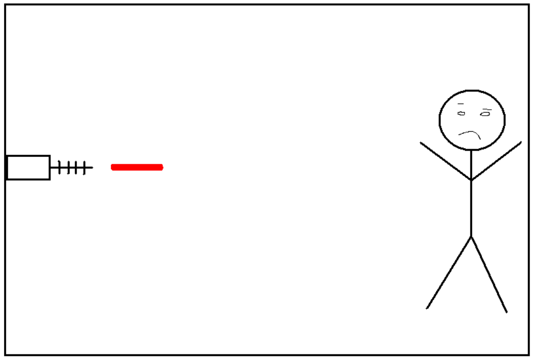


Group discussion:
Please break into groups of 3-4 people. Discuss your answers and provide an argument/proof. Write down your names and answers on a piece of paper and submit it to the TA.

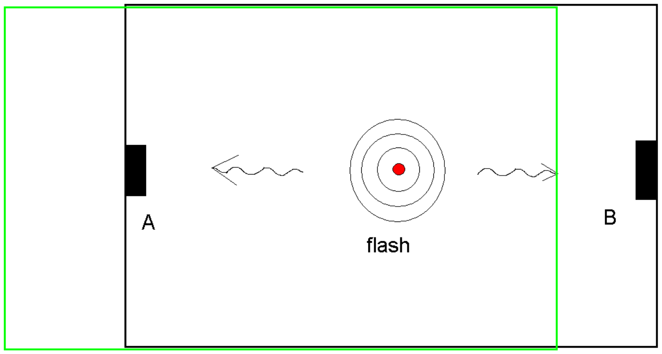
$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t - \frac{vx}{c^2})$$

THE LASER GUN EXPERIMENT

- One can show that, because events A and B are ***causally connected***, their orders cannot be changed by changing to any reference frames unless the observer is moving faster than c
- Since nothing could move faster than light, ***causality is preserved***

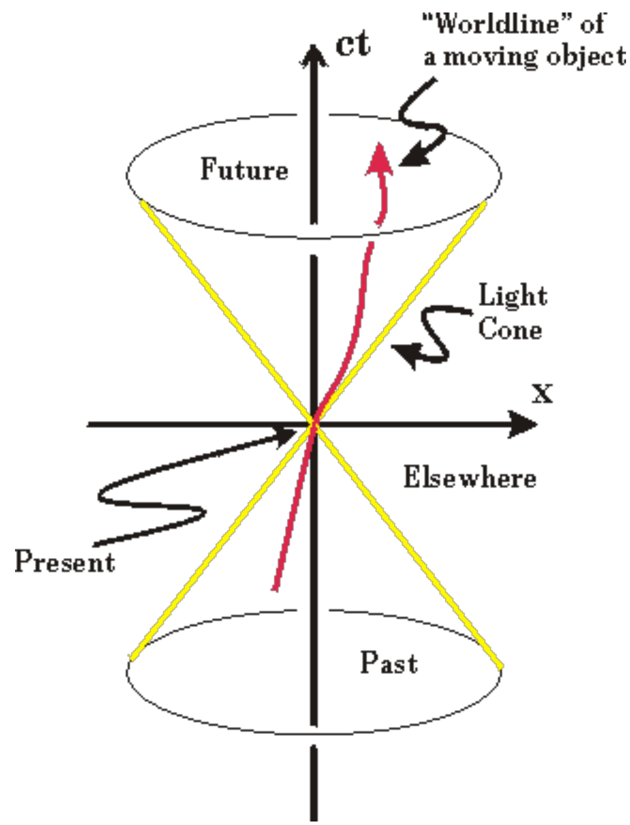


- *In contrast, the events A and B in the previous example (light hitting opposite walls) do NOT have any causal relationship, so it is possible to change their order*



SPACETIME DIAGRAM

- Diagram used to show when and where events occur in the 4-D spacetime
- For Minkowski (flat) spacetime, a light ray would travel along a trajectory with slope of 1
- ***Light cone*** -- path of light rays traveling in all directions emanating from a single event
- ***World line*** – path of an object in the 4-D spacetime



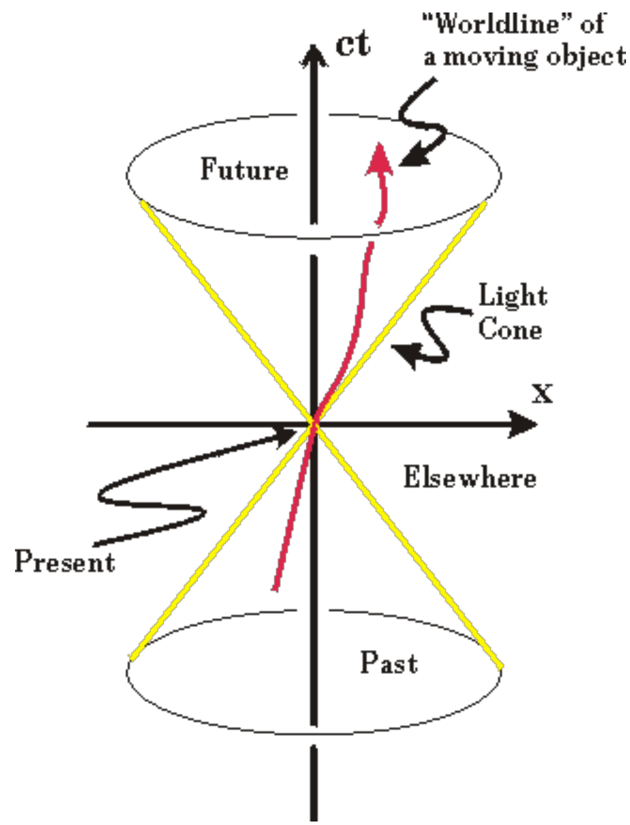
SPACETIME DIAGRAM

- Consider the quantity "***spacetime interval***":

$$S^2 = (ct)^2 - x^2 - y^2 - z^2$$

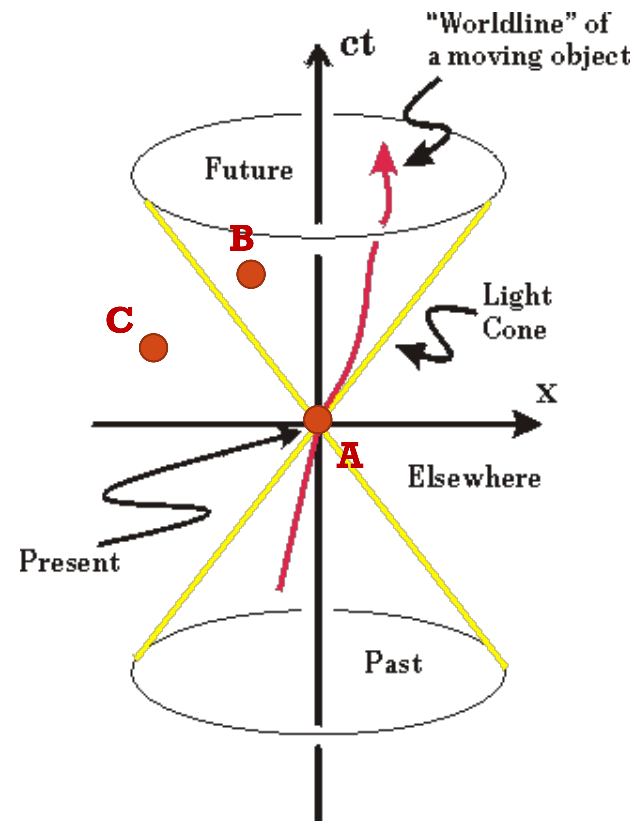
- It is a ***Lorentz invariant*** – the value does not change by changing frames

- Spacetime can be divided into three regions:
  - $S^2 = 0$: lightlike; this defines the light cones
  - $S^2 > 0$: timelike; region inside the light cones
  - $S^2 < 0$: spacelike; region outside the light cones



SPACETIME DIAGRAM

- Event B is within the light cone of event A and can communicate by sending a signal at, or less than, the speed of light -> ***causally connected***. Cannot change order by changing frames of reference
- Events C and A are ***causally disconnected.*** Possible to change their order by changing reference frames
- For any event, the light cones clearly define past, future, and elsewhere

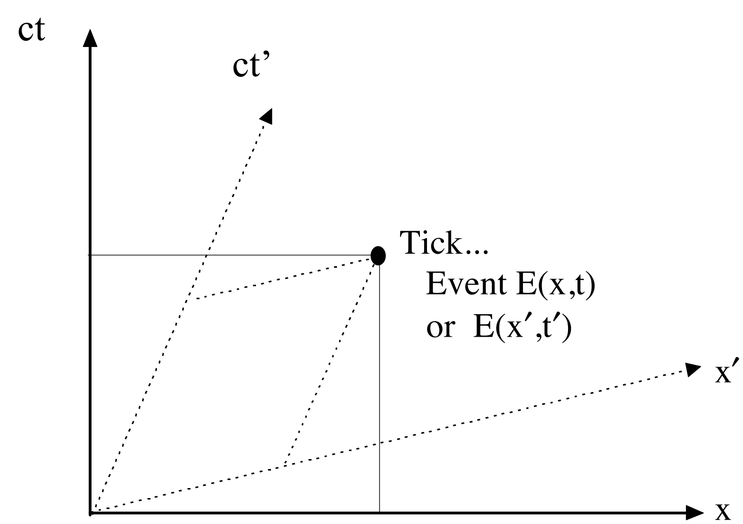


LORENTZ TRANSFORMATION ON SPACETIME DIAGRAMS

- Recall Lorentz transformation:

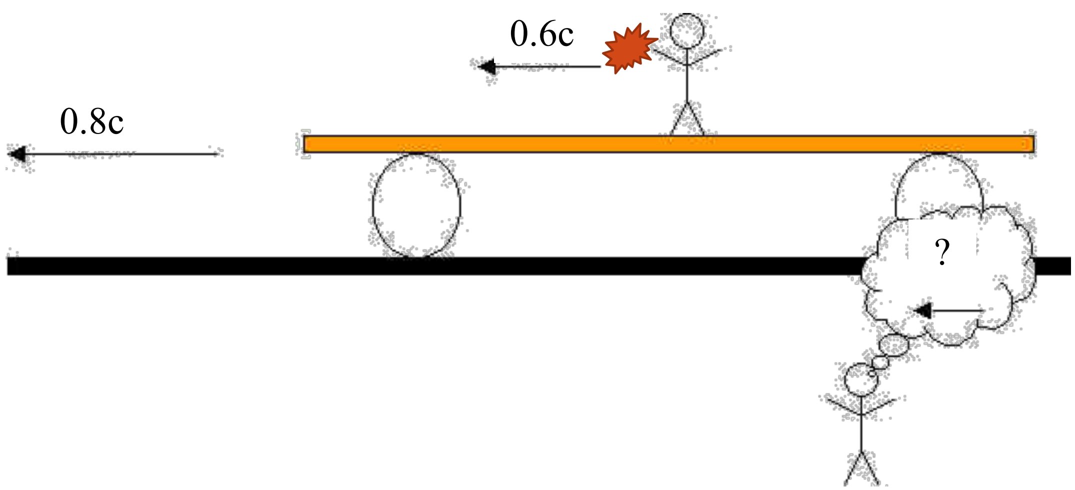
$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t - \frac{vx}{c^2})$$

- The primed axes could be found by setting x'=0 or ct'=0
- The axes are oblique
- This is useful when thinking about events from different reference frames



VELOCITY TRANSFORMATION

- A train moves at 0.8c. A person is firing a bullet with 0.6c relative to the train. How fast does it move relative to the ground?
- Galilean transformation: 0.8c + 0.6c = 1.4 c -> clearly wrong!



NEW VELOCITY ADDITION LAW IN SR

- From the Lorentz transformation, one can derive:

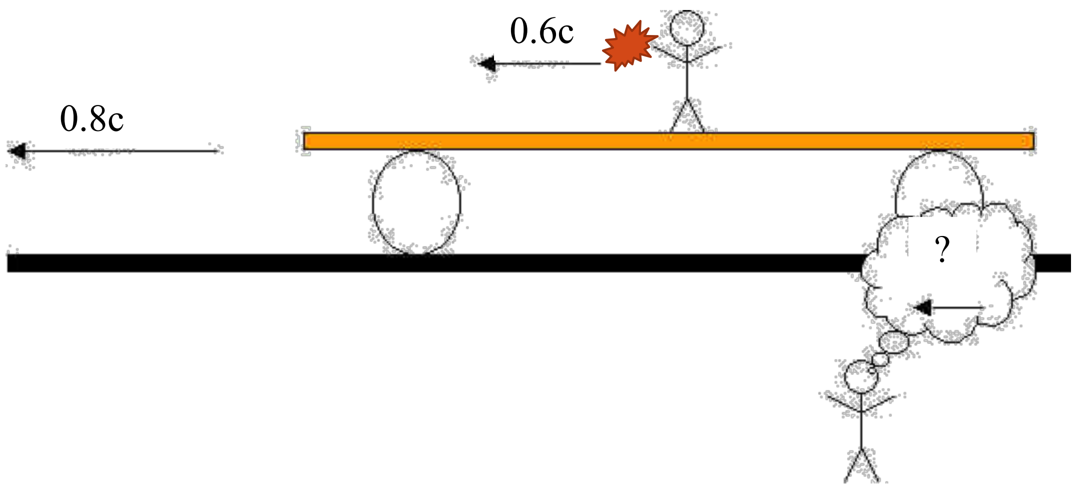
$$
\begin{aligned}
u'_x &= \frac{u_x - v}{1 - v.u_x/c^2} \\
u'_y &= \frac{u_y}{\gamma\left(1 - v.u_x/c^2\right)} \\
u'_z &= \frac{u_z}{\gamma\left(1 - v.u_x/c^2\right)}.
\end{aligned}
$$

- Note that the transverse velocities in y and z directions (if nonzero) are also altered!

BULLET ON THE TRAIN EXAMPLE

- Bullet speed seen from ground: $u_x = \dfrac{u'_x + v}{1 + v.u'_x/c^2} = \dfrac{(0.6 + 0.8)\,c}{1 + (0.6)\,(0.8)} = 0.95c$

- What about light itself, i.e., firing a laser gun? $u_x = \dfrac{(1.0 + 0.8)\,c}{1 + (1.0)\,(0.8)} = c.$



MASS AND ENERGY IN SR

- One can show that, in a collision of two particles, the following "mass" is conserved:

$$m_r = \frac{m}{\sqrt{1 - v^2/c^2}} = \gamma m$$

m_r: "relativistic mass"
m: "rest mass"

- Note that the relativistic mass goes to Infinity as v->c
- It takes infinite energy to pull an object as v->c => nothing can move past c!

MASS AND ENERGY IN SR

$$m_r = \frac{m}{\sqrt{1 - v^2/c^2}} = \gamma m$$

- Do a binomial expansion in v/c: $$\gamma m = m\left(1 - v^2/c^2\right)^{-1/2} = m + \frac{1}{2}m.v^2.\frac{1}{c^2} + \frac{3}{8}m\frac{v^4}{c^4}...$$
- We obtain the total energy: $$E = \gamma mc^2 = mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2}...$$

"rest mass energy"

"relativistic kinetic energy" $T = E - mc^2$

- Consequences:
  - ***Mass and energy are equivalent!!***
  - Conservation of mass is the same as conservation of energy

MASS AND ENERGY IN SR

- Define the relativistic momentum: $p = \gamma m v$

- Useful formula: $pc = \gamma m v c = E\beta$

  For photons: $E = pc$

- Another useful formula: $E^2 = p^2c^2 + m^2c^4$

  That is, $E^2 - p^2c^2$ is Lorentz invariant

LORENTZ TRANSFORMATION OF P AND E

- Lorentz transformation of momentum and energy:

$$\begin{aligned} p'_x &= \gamma\left(p_x - \beta.\frac{E}{c}\right) \\ p'_y &= p_y \\ p'_z &= p_z \\ \frac{E'}{c} &= \gamma\left(\frac{E}{c} - \beta p_x\right). \end{aligned}$$

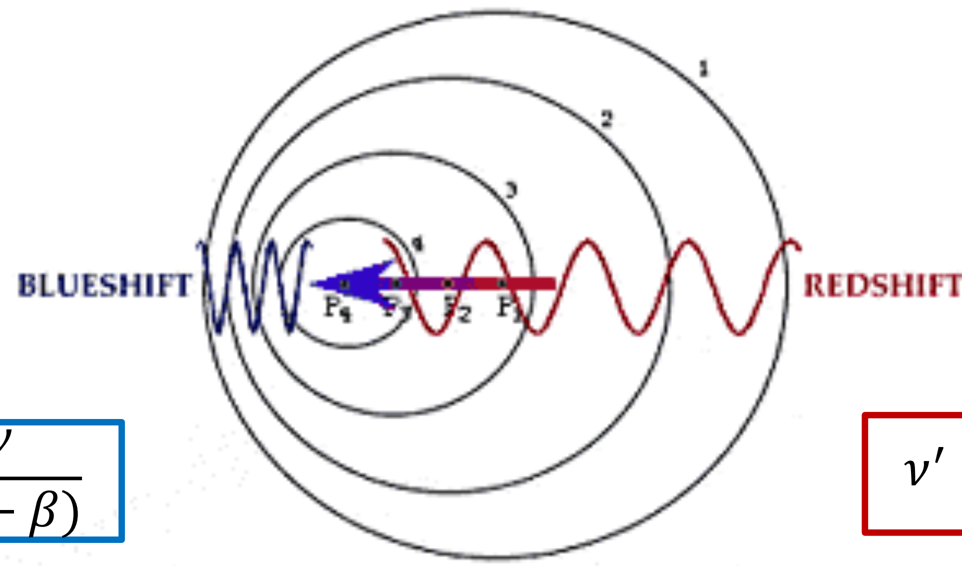
- Transformation of the momentum-energy 4 vector using matrix form:

$$\begin{pmatrix} p'_x \\ p'_y \\ p'_z \\ \frac{E'}{c} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ \frac{E}{c} \end{pmatrix}$$

ASTROPHYSICAL EFFECTS OF SR

RELATIVISTIC DOPPLER'S EFFECT

- Classical case:



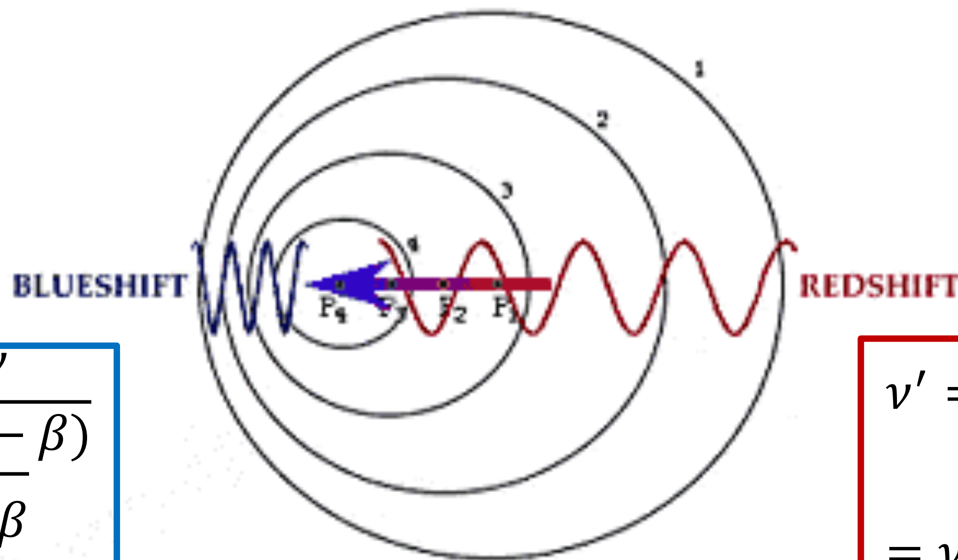
$$\nu' = \frac{\nu}{(1-\beta)}$$

$$\nu' = \frac{\nu}{(1+\beta)}$$

RELATIVISTIC DOPPLER'S EFFECT

- With SR, since the source is moving with v, its clock runs more slowly by a factor of

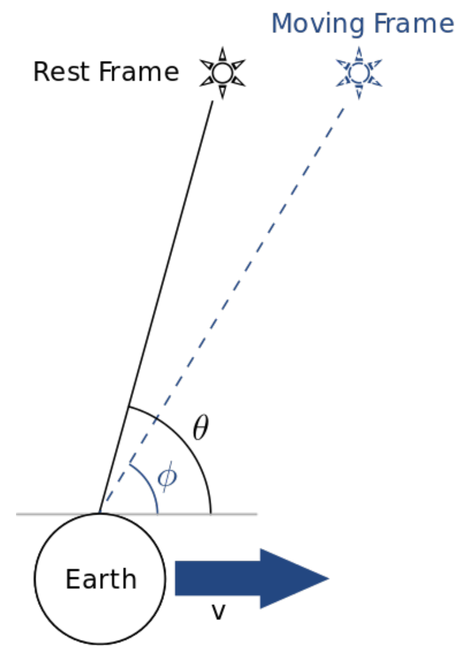
$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$



$$\nu' = \frac{\nu}{\gamma(1-\beta)} = \nu\sqrt{\frac{1+\beta}{1-\beta}}$$

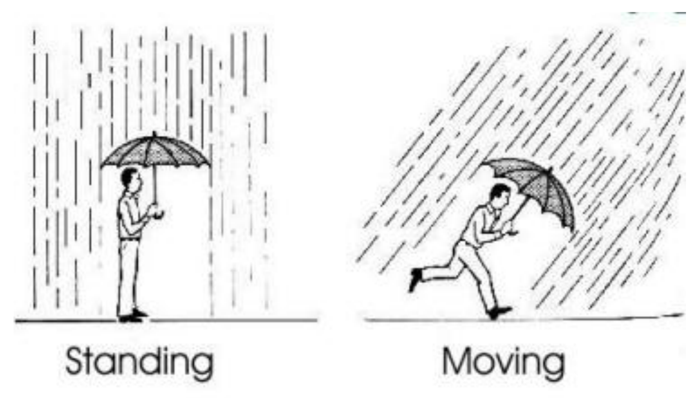
$$\nu' = \frac{\nu}{\gamma(1+\beta)} = \nu\sqrt{\frac{1-\beta}{1+\beta}}$$

ABERRATION OF LIGHT (光行差)



- Stellar aberration: apparent motion of stars about their true positions due to velocity of the observer
- This effect causes the stellar positions to vary periodically over the course of a year by ~20 arcseconds
- Galilean relativity:

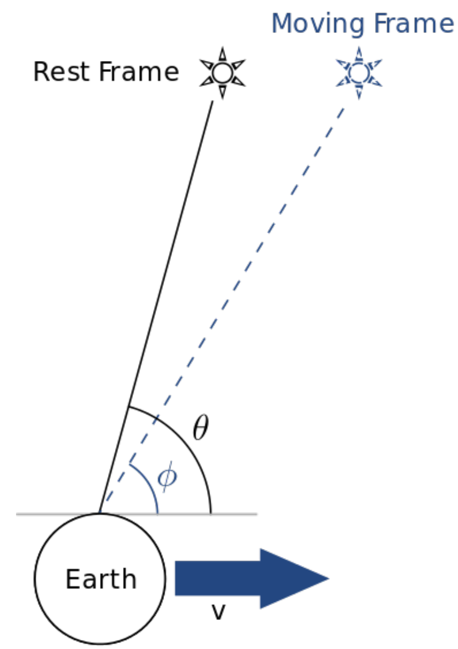
$$\tan(\phi) = \frac{u'_y}{u'_x} = \frac{u_y}{u_x + v} = \frac{\sin(\theta)}{v/c + \cos(\theta)}$$

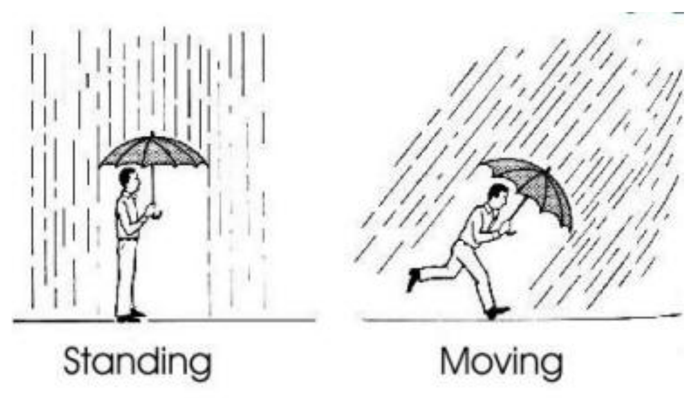


ABERRATION OF LIGHT (光行差)

- With the new velocity addition law in SR:

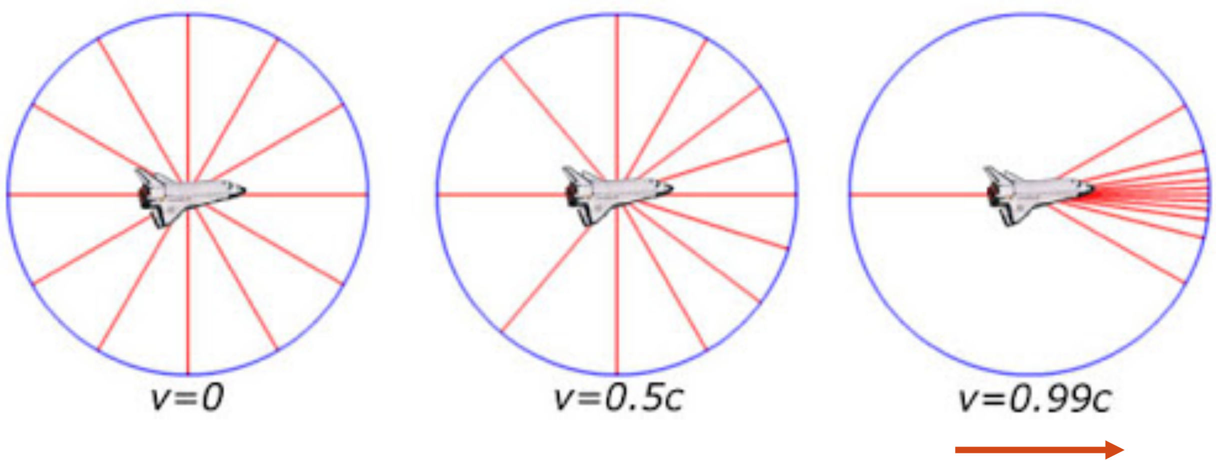
$$\tan(\phi) = \frac{u'_y}{u'_x} = \frac{u_y}{\gamma(u_x + v)} = \frac{\sin(\theta)}{\gamma(v/c + \cos(\theta))}$$





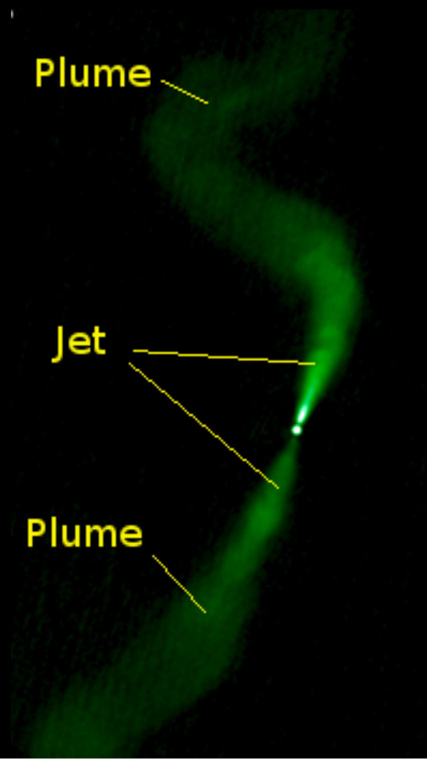
ABERRATION OF LIGHT (光行差)

- Because of aberration, light rays emitted by a source moving close to c would be ***beamed*** toward the observer
- This effect would make the source appear much brighter!



BLACK HOLE JETS

Chandra X-ray image of SMBH jets in Centaurus A galaxy



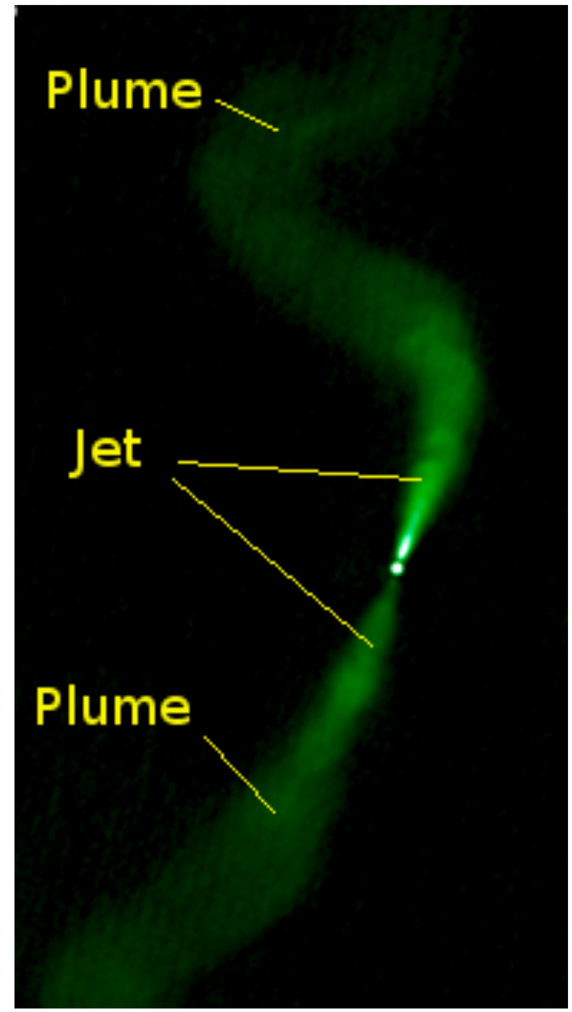
Radio galaxy 3C31

HST image of SMBH jets in M87 galaxy

RELATIVISTIC BEAMING

- Also called "***Doppler beaming***" or "***Doppler boosting***"
- Refers to how apparent luminosity of emitting matter moving close to c is modified by SR effects
- It includes the combined effects of ***relativistic Doppler's effect*** and ***relativistic aberration***
- This is why the approaching jet of the BH is much brighter than the receding jet and is more easily observable
- This is one of the hints that BH jets are relativistic!

Radio galaxy 3C31



SUMMARY

- Before 1905…
  - The transformation between inertial (non-accelerating) frames is described by Galilean transformation
  - Newton's three laws of motions come from symmetry of space and conservation of momentum
  - Weak equivalence principle: $m_I = m_G$, or that gravitational acceleration is independent of mass
  - Michelson-Morley experiment showed that speed of light is constant
- Two postulates of Einstein's theory of special relativity:
  1) Laws of physics are invariant in all inertial frames of reference
  2) The speed of light in a vacuum is the same for all observers

SUMMARY

- Consequences of SR:
  - Time dilation
  - Length contraction
  - Relativity of simultaneity
  - New velocity addition law
  - Mass and energy equivalence
- Astrophysical effects of SR:
  - Relativistic Doppler’s effect = Classical Doppler’s effect + time dilation
  - Relativistic aberration = Classical aberration + new velocity addition law
  - Relativistic beaming = relativistic Doppler’s effect + relativistic aberration
    => could explain why we often see one-sided/brighter BH jets