EINSTEIN'S THEORY OF GENERAL RELATIVITY

Lecture 3, Introduction to Black Hole Astrophysics (PHYS480)

Hsiang-Yi Karen Yang, NTHU, 3/9/2021

ANNOUNCEMENTS

- HW1 is due TODAY! Late submissions within a week still count with half credits.
- Homework solutions will be posted on iLMS and course website 1 week after due.
- For students who asked questions during or after class, please don't forget to tell the TA to receive the class participation points!
- You could start finding your teammates for the final report. Once you form a group of 3 people, choose a team leader and enter your names in iLMS.

PREVIOUS LECTURE…

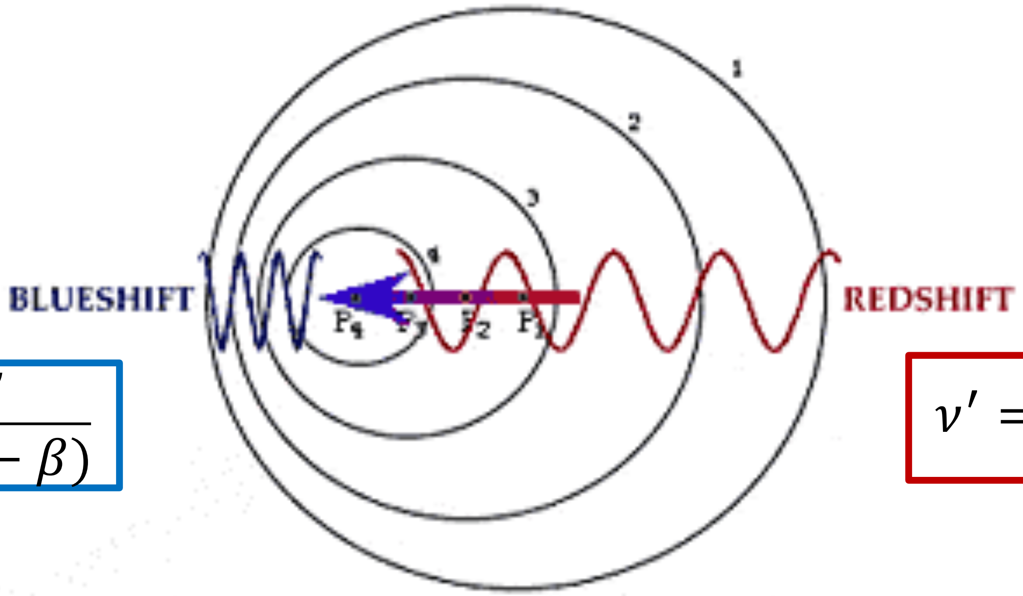
- Before 1905…
  - The transformation between inertial (non-accelerating) frames is described by Galilean transformation
  - Newton's three laws of motions come from symmetry of space and conservation of momentum
  - Weak equivalence principle: $m_I = m_G$, or that gravitational acceleration is independent of mass
  - Michelson-Morley experiment showed that speed of light is constant
- Two postulates of Einstein's theory of special relativity:
  1) Laws of physics are invariant in all inertial frames of reference
  2) The speed of light in a vacuum is the same for all observers

PREVIOUS LECTURE…

- Consequences of SR:
  - Time dilation
  - Length contraction
  - Relativity of simultaneity
  - New velocity addition law
  - Mass and energy equivalence
- Astrophysical effects of SR:
  - Relativistic Doppler's effect = Classical Doppler's effect + time dilation
  - Relativistic aberration = Classical aberration + new velocity addition law
  - Relativistic beaming = relativistic Doppler's effect + relativistic aberration => could explain why we often see one-sided/brighter BH jets

RELATIVISTIC DOPPLER'S EFFECT

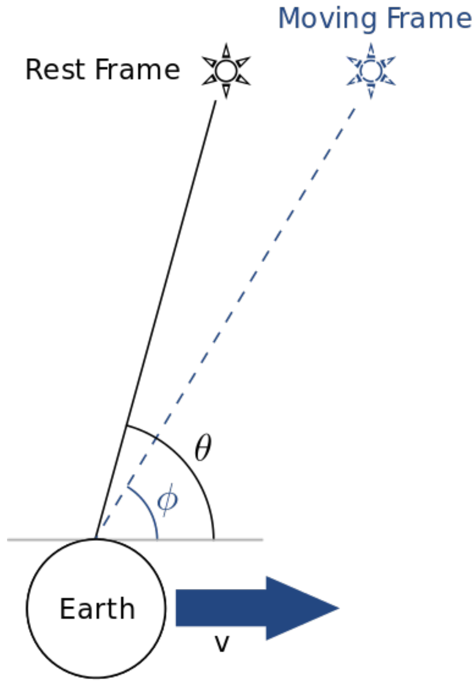
- Classical Doppler's effect + time dilation effect



$$\nu' = \frac{\nu}{\gamma(1-\beta)}$$

$$\nu' = \frac{\nu}{\gamma(1+\beta)}$$

ABERRATION OF LIGHT (光行差)

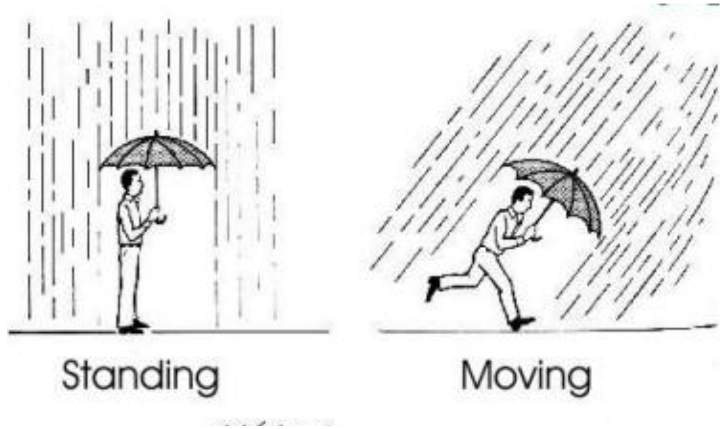


- Stellar aberration: apparent motion of stars about their true positions due to velocity of the observer
- Galilean relativity:

$$\tan(\phi) = \frac{u'_y}{u'_x} = \frac{u_y}{u_x + v} = \frac{\sin(\theta)}{v/c + \cos(\theta)}$$

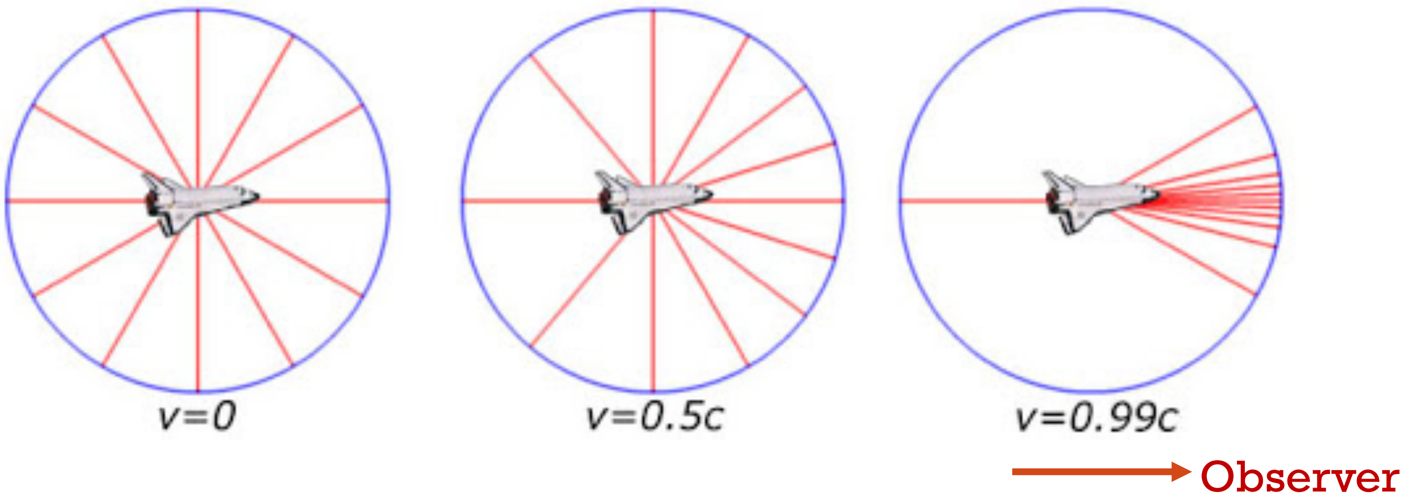
- With the new velocity addition law in SR:

$$\tan(\phi) = \frac{u'_y}{u'_x} = \frac{u_y}{\gamma(u_x + v)} = \frac{\sin(\theta)}{\gamma(v/c + \cos(\theta))}$$

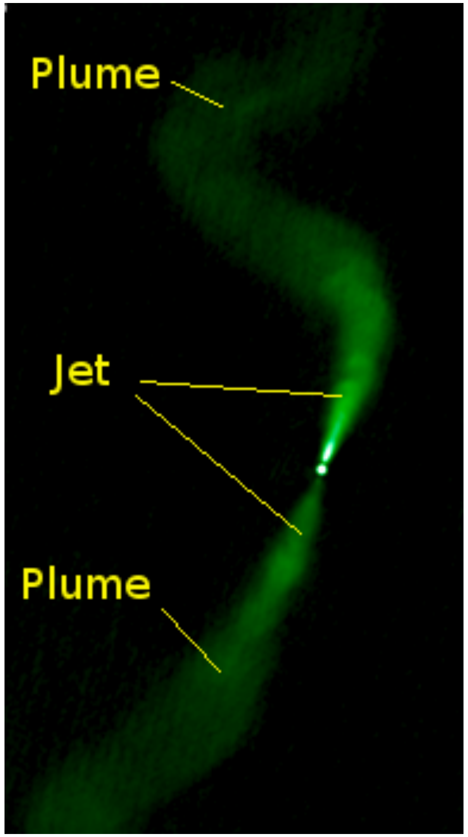


ABERRATION OF LIGHT (光行差)

- Because of aberration, light rays emitted by a source moving close to c would be ***beamed*** toward the observer
- This effect would make the source appear much brighter!



Radio galaxy 3C31

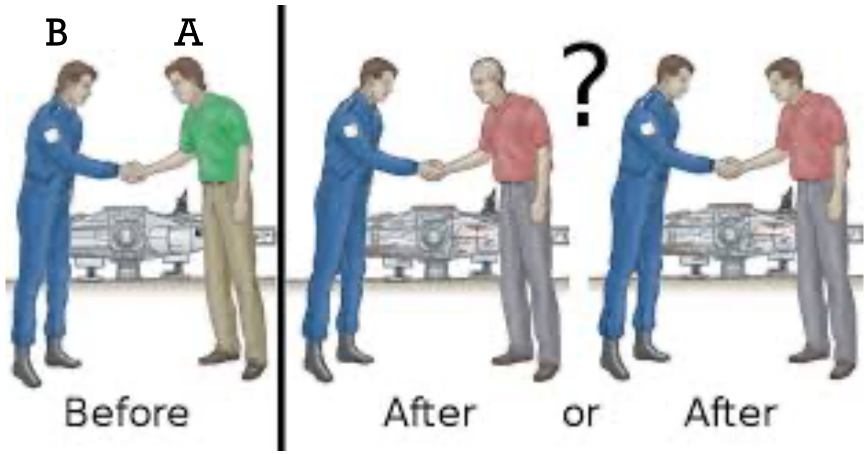


SPECIAL RELATIVITY (狹義相對論)

- "***Special***" means that this theory only applies to inertial (non-accelerating) frames of reference
- "***Relativity***" means that space and time are "relative", depending on which frames of reference the measurement is based on
  - Time is not an absolute entity
  - Only the speed of light is universal
  - Clocks would appear to tick slower for faster moving objects
  - All inertial reference frames are equivalent, so there is no way to tell if you are moving or other things are moving relative to you. However, everyone should agree with some basic facts (e.g., Newton's laws, who is older or younger)
  - Many paradoxes can be resolved by realizing the fact that "***simultaneity is relative***."

TWIN PARADOX

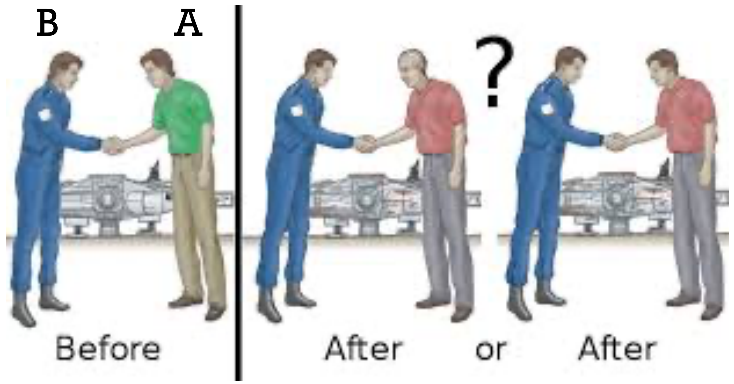
- A and B are twins. B leaves Earth at high speeds in a spaceship. A stays on Earth and watches B's clock slowing down. B ages more slowly than A, and therefore when B returns, he would be younger than A.
- But, from B's perspective, it is A who has moved away and come back again. So shouldn't A be the younger one?



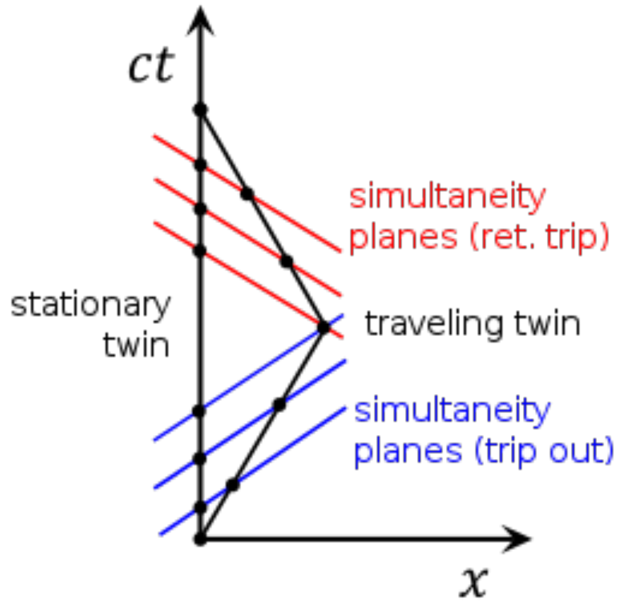
TWIN PARADOX

- **Solution 1**: In order for B to come back to Earth, his spaceship must change directions, i.e., ***accelerate,*** and therefore ***breaking the symmetry of the problem***.
- B would notice the acceleration and agree that he is the younger one when they meet.

TWIN PARADOX



- **<u>Solution 2</u>**: To avoid acceleration, let's introduce a third "twin" C who has velocity –v relative to A and who coordinates his clock with B as they cross.
- From A's perspective, both B and C move with high speeds to their clocks would appear to tick slower. Therefore, C should be younger than A when they meet.
- For the duration of both B's outbound trip and C's returning trip, they think A's clock tick slower and A ages slower ***during their trips***.
- The paradox can be resolved by realizing that, ***when B and C pass by each other, their readings of A's clock are different, due to the relative simultaneity***!
- If they record the difference between their readings and adjust their clocks, they would agree that A should be the older one when A and C meet.



THIS LECTURE…

- Finally talking about gravity
- Putting together SR and gravity
  - Effect of gravity on light
  - Effect of gravity on time
  - Strong equivalence principle
- Einstein's theory of general relativity
  - What it says
  - Predictions and verifications
  - How to express curved spacetime using math
  - How objects move in curved spacetime

PUTTING TOGETHER SR AND GRAVITY

FIRST PUZZLE

- SR says speed of light is a universal constant and nothing could travel faster than c
  - Consistent with Maxwell's EM equations -> but no gravity in it!
  - Newtonian gravity is a long-range force which exists between any two masses in the universe! -> but how to make it consistent with finite light speed?

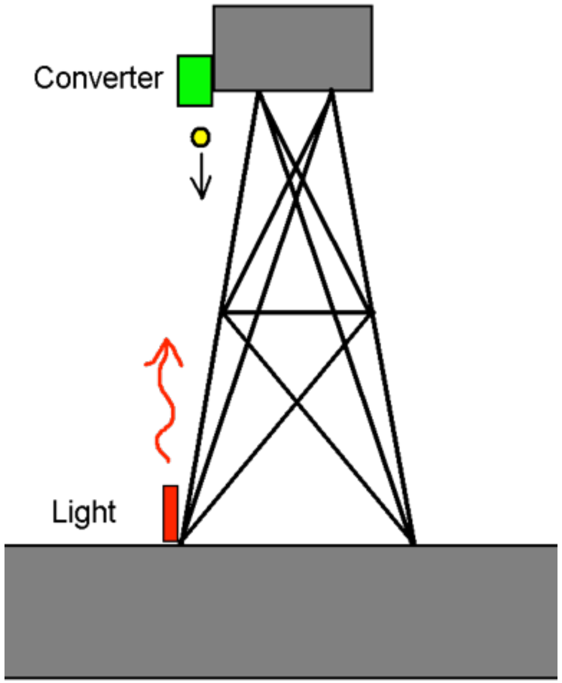
$$F = \frac{G m_1 m_2}{r^2}$$

THOUGHT EXPERIMENT – EINSTEIN'S TOWER

- Suppose light is NOT affected by gravity
- Consider a tower on Earth
  - Shine a light ray from bottom to top **E**
  - Convert light energy into mass **$m=E/c^2$**
  - Drop mass from top to bottom **$E_g = mgh$**
  - Convert mass back into energy

$$E_{new} = E + mgh = E\left(1 + \frac{gh}{c^2}\right)$$

We gained energy!!

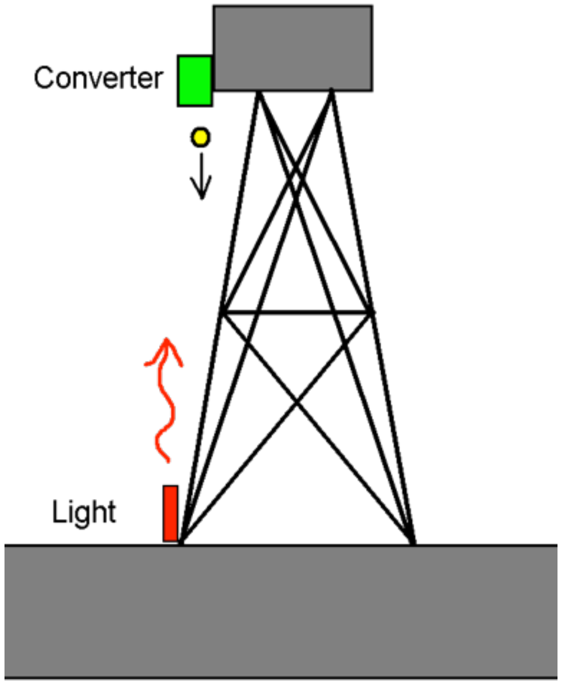


THOUGHT EXPERIMENT – EINSTEIN'S TOWER

- To conserve energy light MUST be affected by gravity
- The photon needs to lose energy as it climbs upwards:

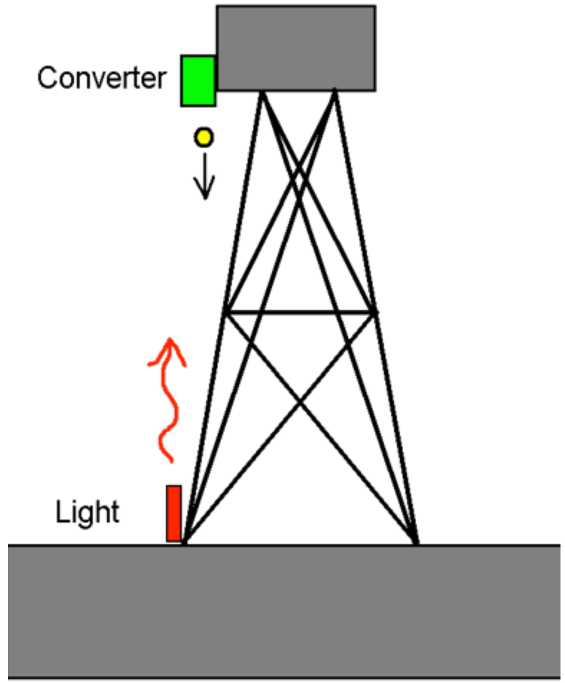
$$E_{top} = E\left(1 + \frac{gh}{c^2}\right)^{-1}$$

- This is known as ***gravitational redshift – gravity affects frequency of light!***



THOUGHT EXPERIMENT – EINSTEIN'S TOWER

- Imagine a clock based on frequency of light
- Place the clock at the base of tower and observe it from the top
- Photons lose energy and frequency decreases
- Thus, we see the clock running slowly
- ***Time would appear to run slower in a gravitational field!***
- This is called ***gravitational time dilation – gravity affects time!***



POUND-REBKA EXPERIMENT

- In 1959, Robert Pound and Glen Rebka conducted the "Einstein's Tower" experiment at the Jefferson laboratory at Harvard
- They measured the frequency shift of light shining from top to bottom
- Height of tower h = 22.5m, $gh/c^2 \sim 2.5\times10^{-15}$
- They shine 14keV gamma rays in order to measure the effect!
- Confirmed the blueshift of light rays at bottom

Physicist Glen Rebka, at the lower end of the Jefferson Towers, Harvard University, calling Professor Pound on the phone during setup of the famed Pound-Rebka experiment. By energetically driving the emitting or absorbing portion of the apparatus, scientists could directly test the energy loss/gain predictions of General Relativity for the correct energy shift of photons that experience gravitational redshifts and blueshifts. [-] CORBIS MEDIA / HARVARD UNIVERSITY

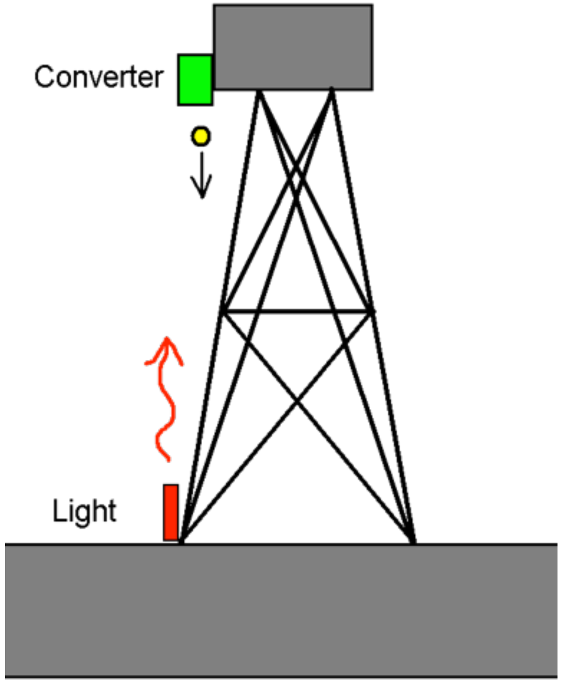
HAFELE-KEATING EXPERIMENT

- In 1971, Hafele & Keating flew around the world with atomic clocks and compare with clocks on the ground
- They flew both eastbound and westbound to test SR effects due to relative velocities
- Clocks on the ground run slower compared to clocks on the plane due to gravitational time dilation
- Results fully consistent with SR+GR predictions

	nanoseconds gained			
	predicted			measured
	gravitational (general relativity)	kinematic (special relativity)	total	
eastward	144±14	−184 ± 18	−40 ± 23	−59 ± 10
westward	179±18	96±10	275±21	273±7

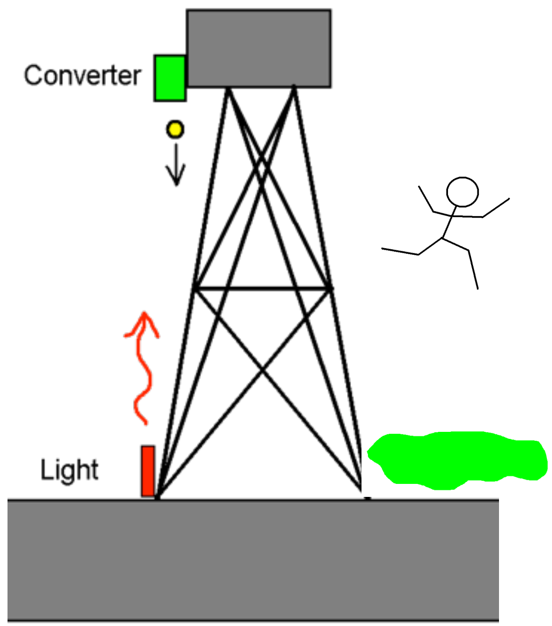
LIGHT IS AFFECTED BY GRAVITY?

- On Earth we see light affected by gravity, but gravity is not in Maxwell's equations, which works in inertial frames described by SR
- So Earth's surface is not an inertial frame of reference
- Could we change reference frames to remove this effect? That is, a reference frame where light frequency does not change and the laws of physics can be simply described by SR?



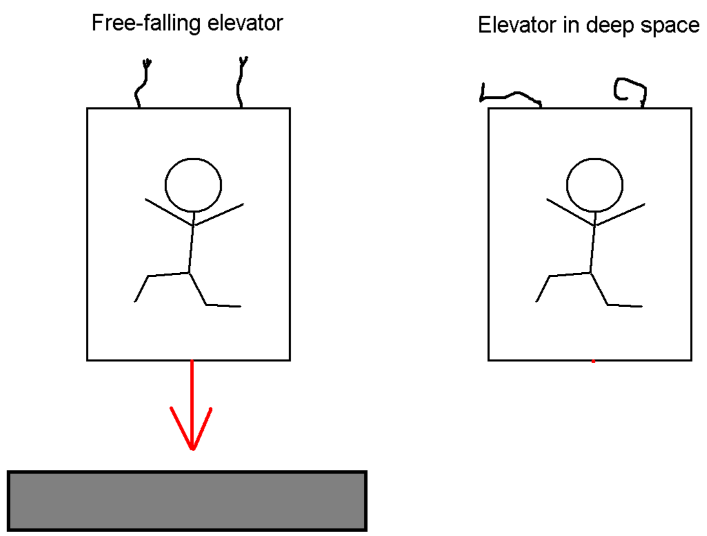
CONSIDER A FREE-FALLING OBSERVER

- From Earth's frame:
  - Free-falling observer travels faster and faster, corresponding to an increasing blueshift
  - Since frequency of light is redshifted as it climbs up, the free-falling observer would see light ray at a constant frequency, i.e., unaffected by gravity
- When gravity is present, the free-falling (accelerating) frame of reference would observe laws of physics described by SR same as any inertial reference frame
- In other words, the free-falling observers are (locally) free of effects of gravity



STRONG EQUIVALENCE PRINCIPLE

- Einstein: "A free-falling (accelerating) frame of reference in a gravitational field is equivalent to the inertial frames of special relativity."
- There is no way to tell the difference between a free-falling frame in a gravitational field and an inertial frame without gravitational field



RECALL THE WEAK EQUIVALENCE PRINCIPLE

- Newton's 2nd law: $F = m_I a$

  m_I = inertial mass

- Newton's law of gravitation: $F = \frac{GMm_G}{r^2}$

  m_G = gravitational mass

- ***Weak equivalence principle: $m_I = m_G$***
- So, ***gravitational acceleration is independent of the object's mass***

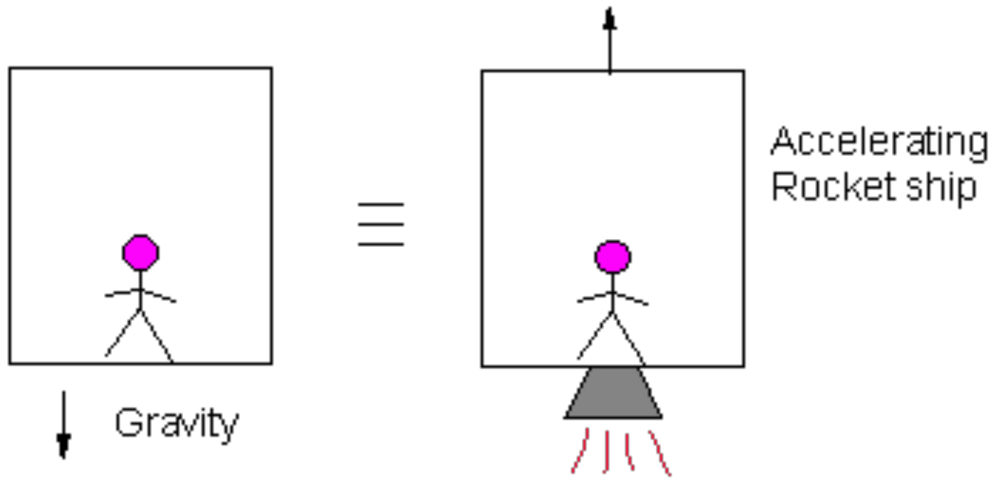
$$a = \left(\frac{m_G}{m_I}\right)\frac{GM}{r^2} = \frac{GM}{r^2}$$

WHY DO ASTRONAUTS ON INTERNATIONAL SPACE STATION (ISS) FEEL WEIGHTLESS?

- They feel weightless because the astronauts "***fall***" toward Earth at the same rate as the space station – this is due to the ***weak equivalence principle***
- They are in a free-falling frame in Earth's gravitational field
- The ***strong equivalence principle*** says that this free-falling frame is equivalent to an inertial frame without gravity
- This is exactly true! They feel weightless just like what they would feel floating in deep space away from the Earth!

STRONG EQUIVALENCE PRINCIPLE

- We measure "weight" standing on a scale due to Earth's gravity
- We would in fact measure the same weight if we were standing on scale inside an accelerating rocket
- There is also no way to tell the difference between gravity and an accelerating frame of reference
- ***Gravity is an illusion caused by the fact that we are in an accelerating frame!!!***

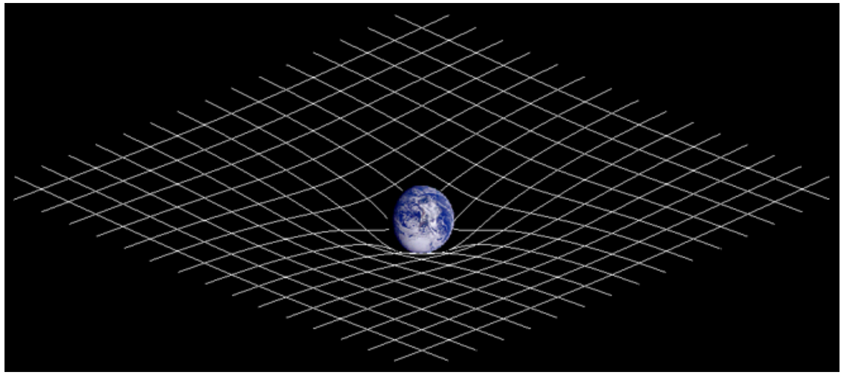


HOW TO REPLACE GRAVITY WITH ACCELERATING FRAMES?

- Hint 1:
  - Taking Earth for example, there is no ***single*** accelerating reference frame that works
  - Somehow we need to stick together reference frames accelerating in different directions
- Hint 2 -- inspecting the weak equivalence principle:
  - (inertial mass)*(acceleration) = (gravitational mass)*(gravitational field)
  - $a = GM/r^2$
  - It suggests that "acceleration" seems to be a ***geometrical*** property of M!

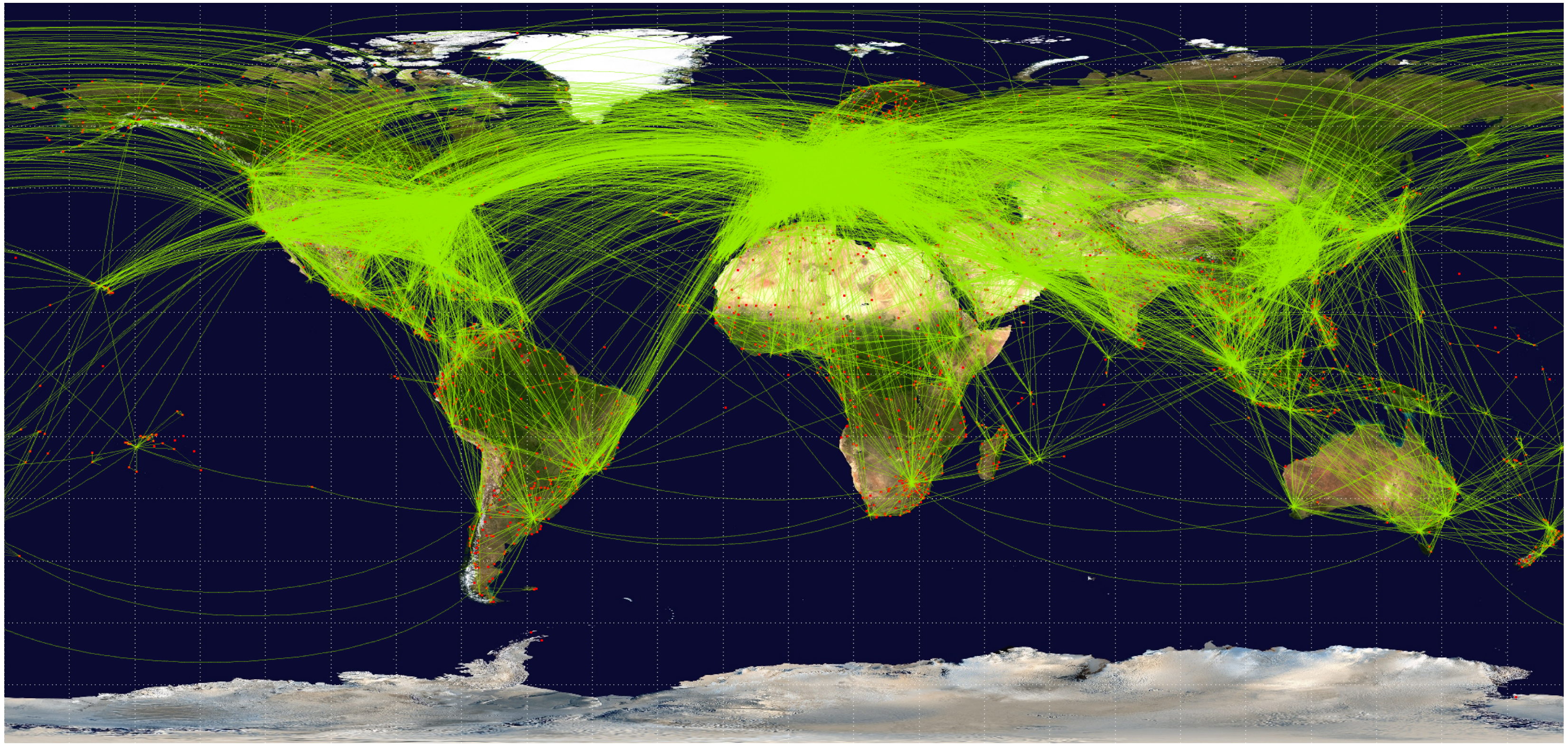
CURVED SPACETIME

- Matter and energy causes the 4-D spacetime to curve
- Free-falling objects move on "***geodesics***" (straight lines) through the 4-D spacetime
- Geodesics is the shortest path between two points on a surface



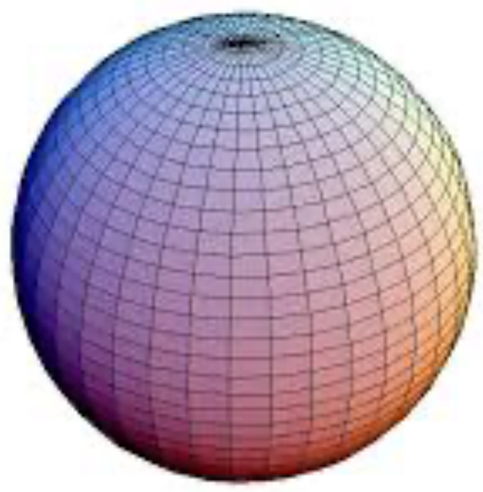
GEODESICS -- EXAMPLES

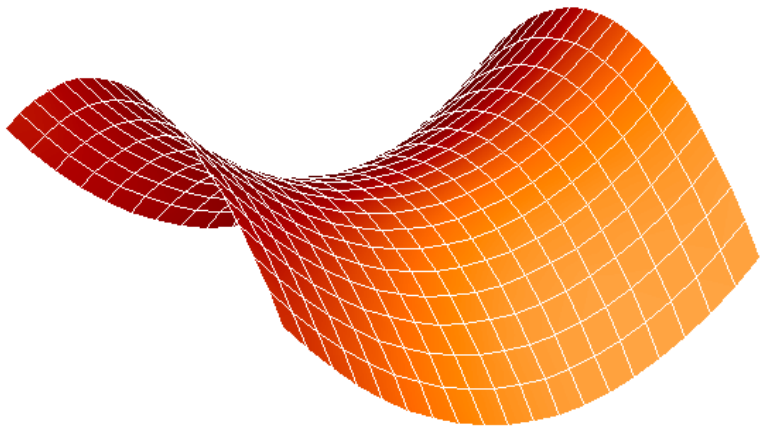
- Path flown by aircraft – great circles on the surface of a sphere



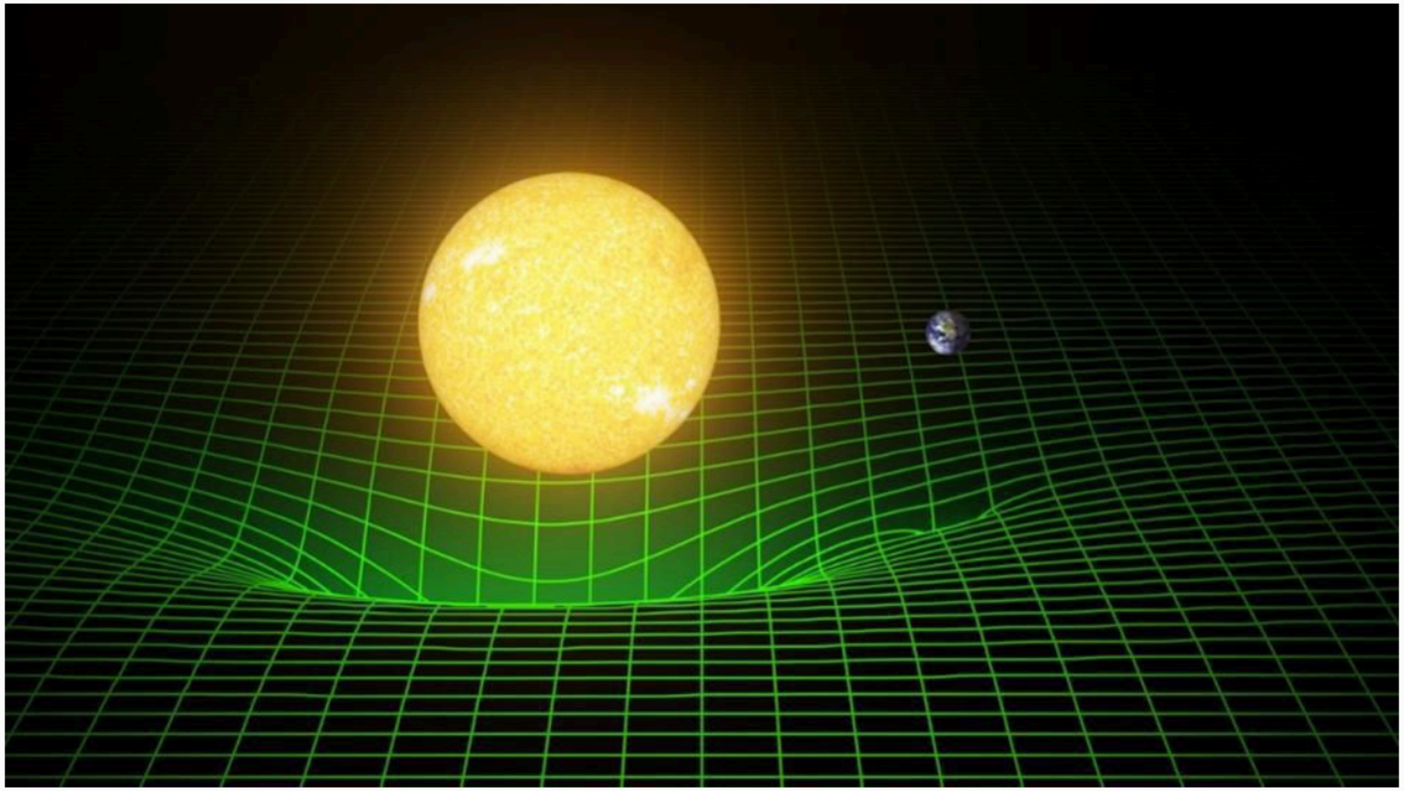
GEODESICS -- EXAMPLES

- Geodesics that start parallel can converge or diverge (or even cross)



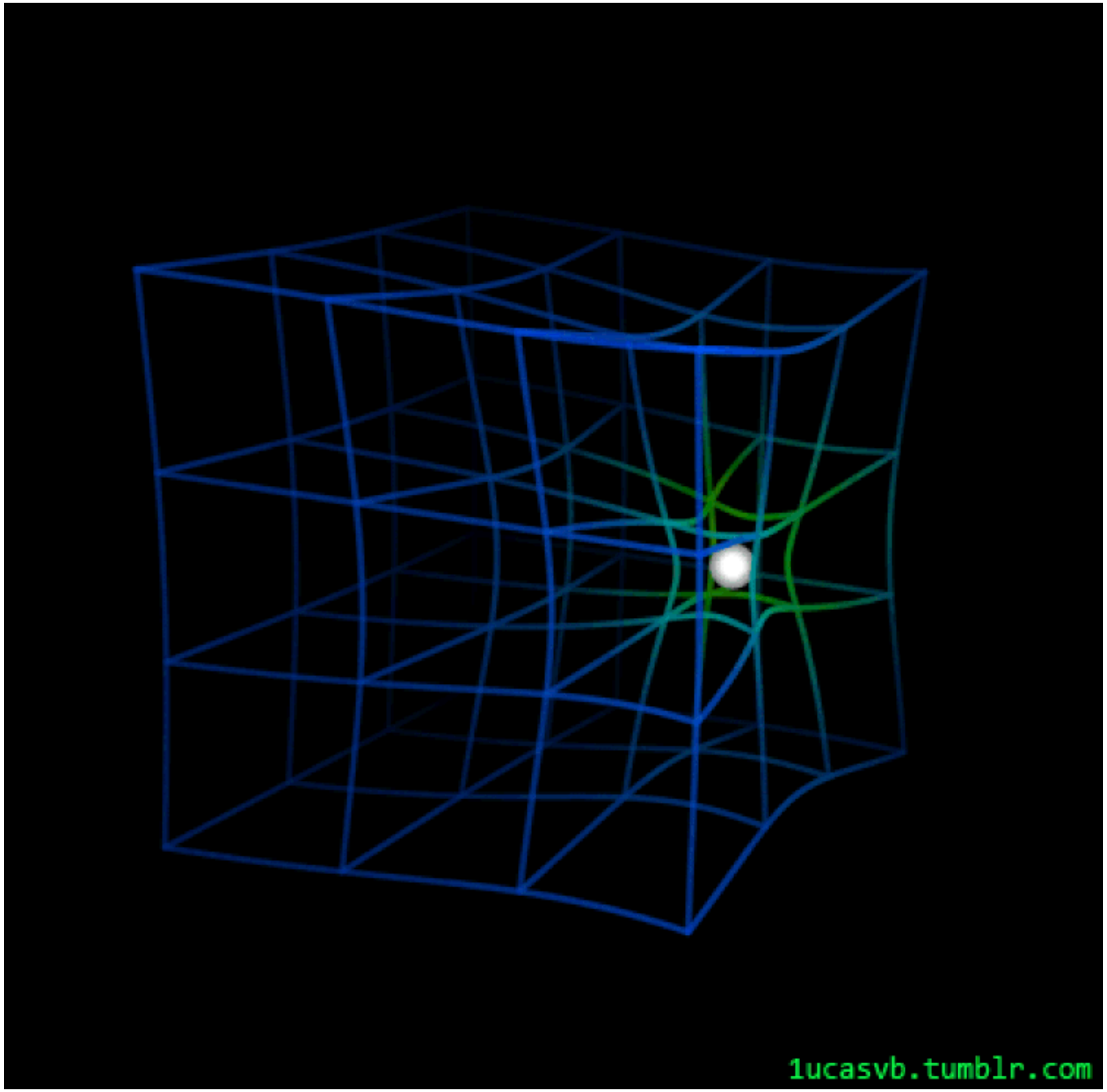


CURVED SPACETIME



The spacetime curvature around any massive object is determined by the combination of mass and distance from the center-of-mass. However, this two-dimensional grid-like depiction of spacetime isn't necessarily the most accurate way to perceive it. [-] T. PYLE/CALTECH/MIT/LIGO LAB

CURVED SPACETIME IN 3D



THEORY OF GENERAL RELATIVITY

- Within a free-falling frame, the SR applies
- Spacetime is curved around objects with mass and energy
- Objects move on geodesics (測地線) through curved spacetime
- ***"Matter/energy tells spacetime how to curve; spacetime curvature tells matter/energy how to move"***

THEORY OF GENERAL RELATIVITY

Einstein field equation:

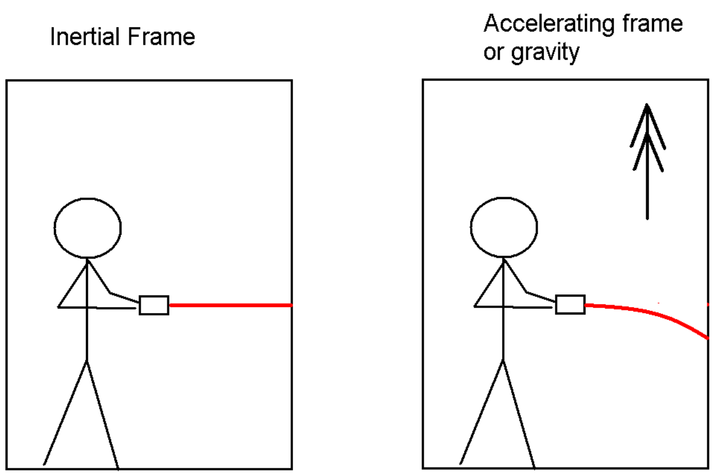
$$\underline{\underline{\mathbf{G}}} = \frac{8\pi G}{c^4}\underline{\underline{\mathbf{T}}}$$

- G = ***Einstein curvature tensor***, which describes curvature of 4-D spacetime
- T = ***Stress-energy tensor***, which describes distribution of mass/energy
- 10 coupled non-linear partial differential equations!
- Reduce to Newtonian gravity for weak gravitational fields

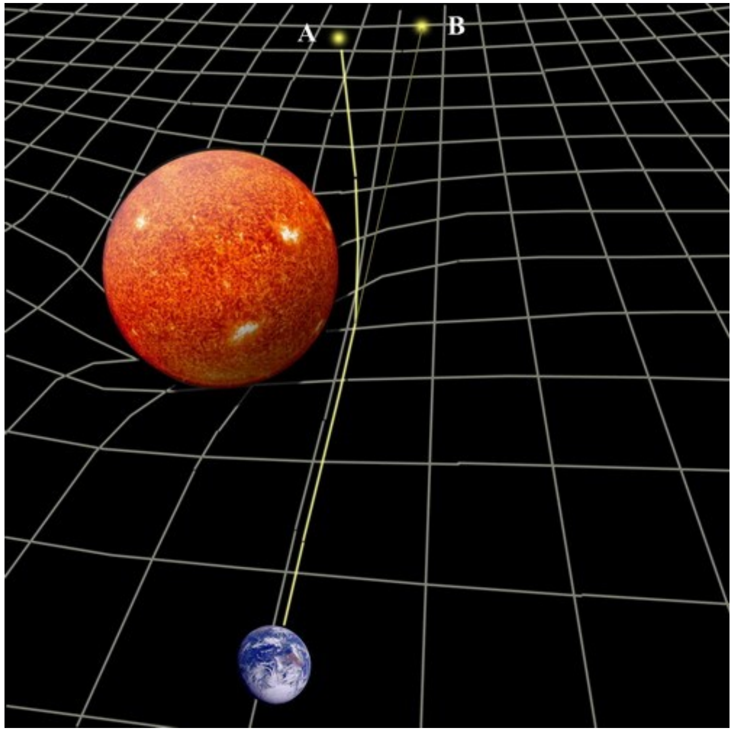
PREDICTIONS OF GR

LIGHT FALLS!

- For an astronaut in an inertial frame with flash light, light goes in straight lines
- If we look at the same light path from an accelerating reference frame, light beam would bend and appear to "***fall***"
- According to the strong equivalence principle, light would also fall due to gravity!

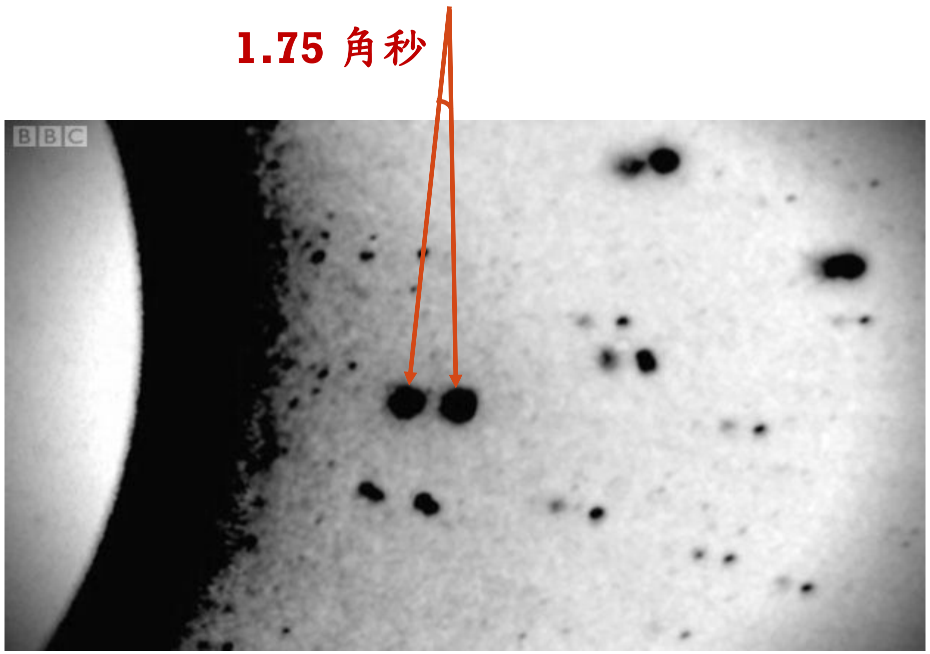


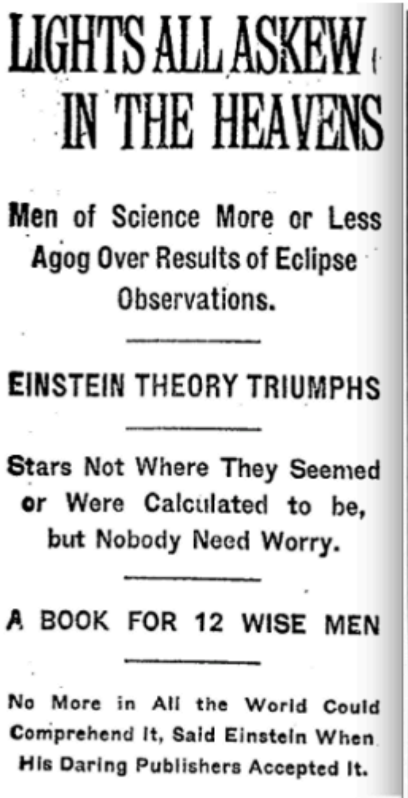
PREDICTIONS OF GR – LIGHT BENDING



- Light follows geodesics in curved spacetime => light bends when traveling by a massive object ("***gravitational lensing***" effect)

THE EDDINGTON TEST (5/29/1919)





LIGHTS ALL ASKEW IN THE HEAVENS

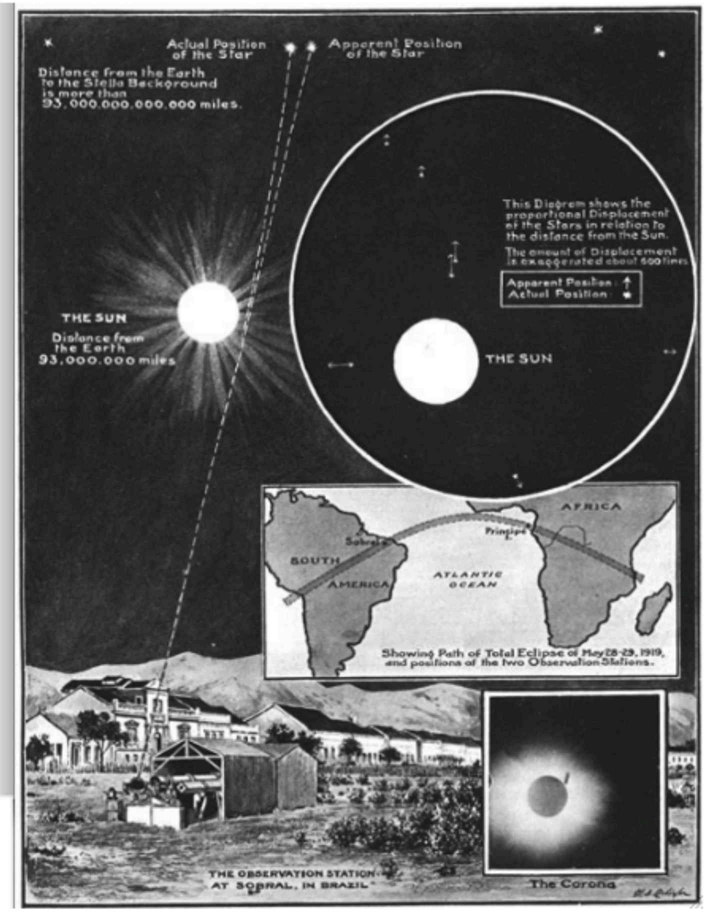
Men of Science More or Less Agog Over Results of Eclipse Observations.

EINSTEIN THEORY TRIUMPHS

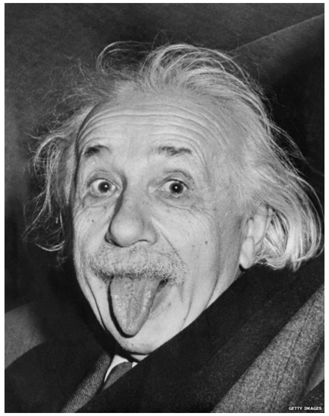
Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

A BOOK FOR 12 WISE MEN

No More in All the World Could Comprehend It, Said Einstein When His Daring Publishers Accepted It.



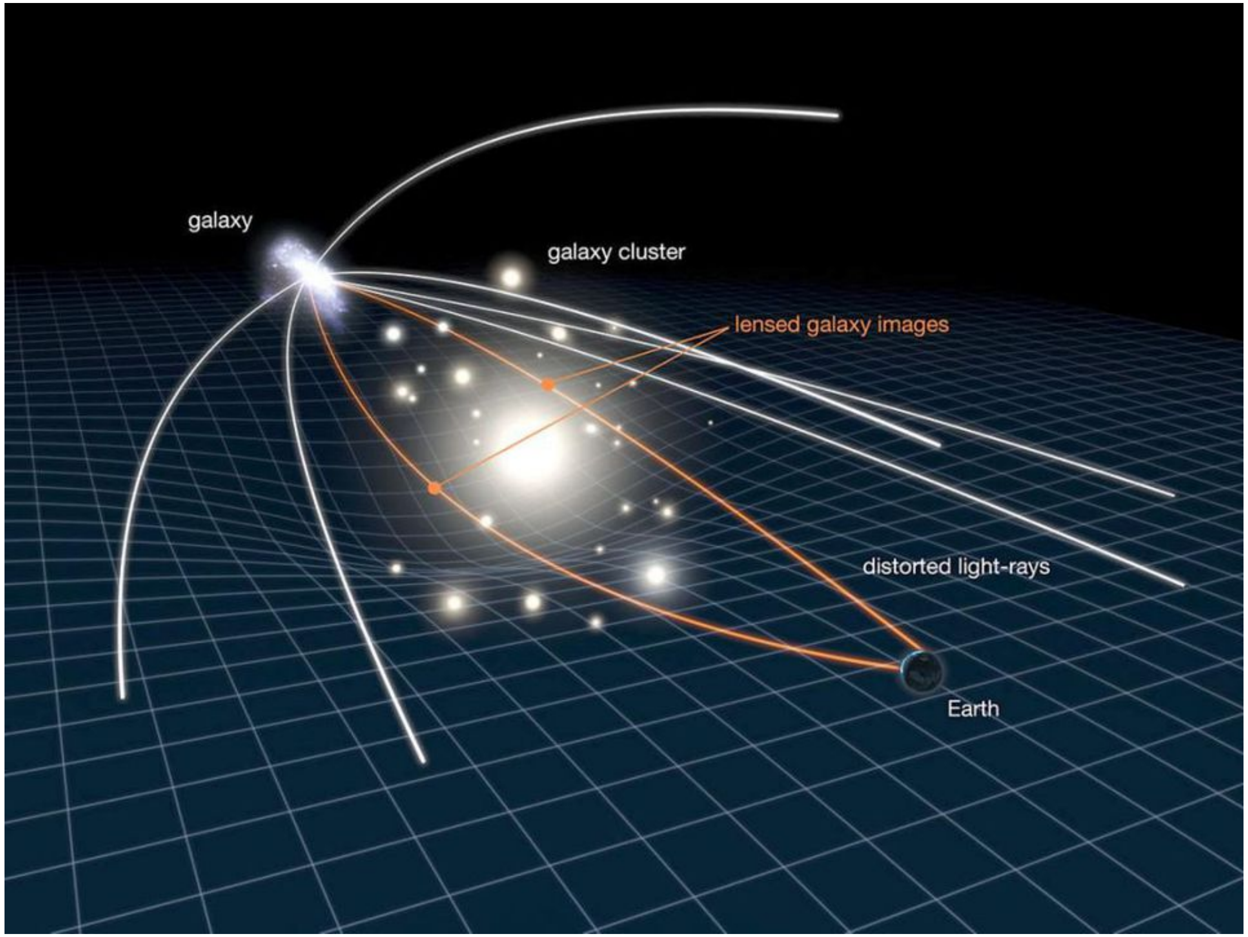
Images credit: New York Times, 10 November 1919 (L); Illustrated London News, 22 November 1919 (R).



VERIFIED PREDICTIONS OF GR

- ✓Gravitational lensing (this lecture)
- ✓Orbital precession of Mercury (this lecture)
- ✓Gravitational redshift (discussed)
- ✓Gravitational time dilation (discussed, more in this lecture)
- ✓Expansion of the universe (not covered)
- ✓Gravitational waves (Week 15)
- ✓Existence of black hole event horizon (Week 14)

GRAVITATIONAL LENSING

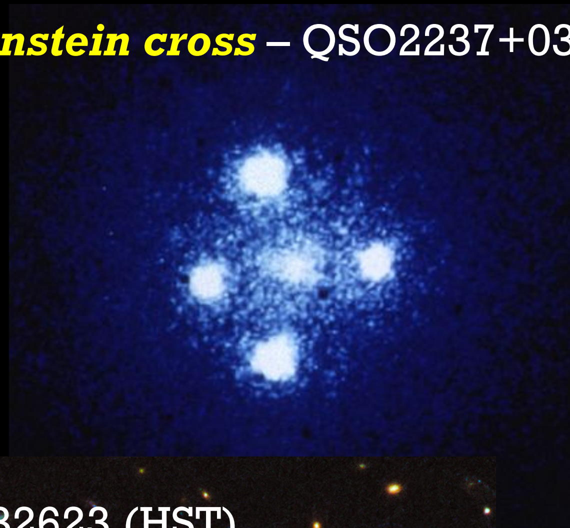


- Light from distant sources bends when passing through a massive object to an observer
- Deflection angle can be computed using GR:

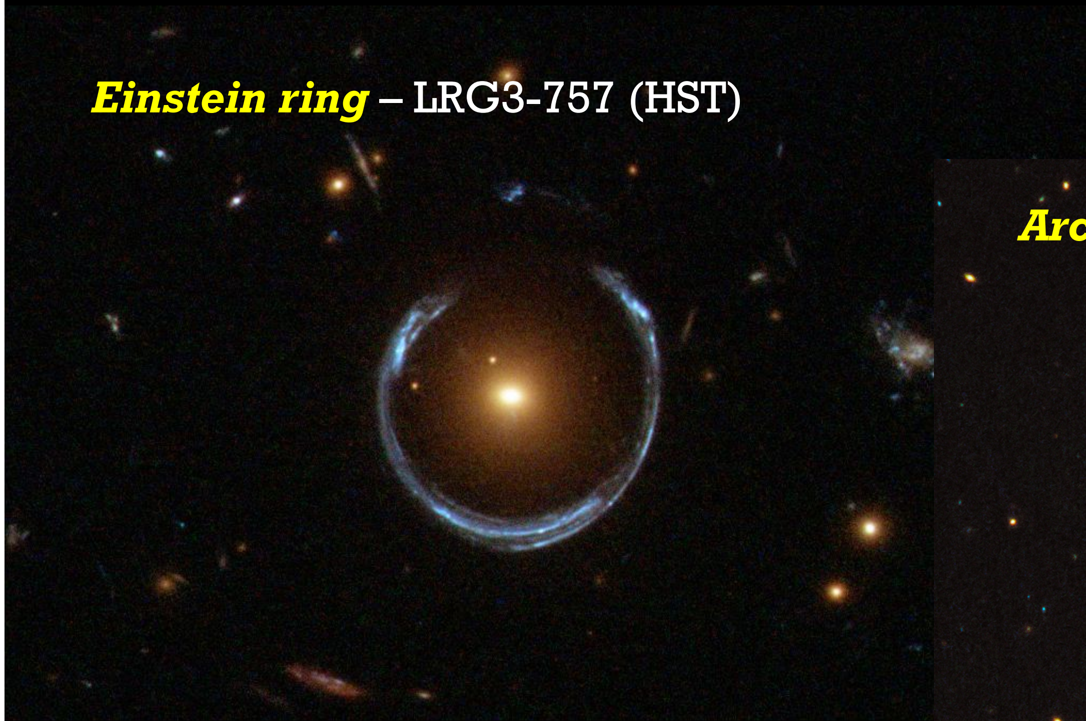
$$\Delta\phi = \frac{4GM}{c^2 R}$$

- R = impact parameter = projected distance between source and lens axis
- The effect is typically small
  - ~arcseconds by a galaxy with 10^{11} M_{sun}
  - ~arcminutes by a galaxy cluster with $10^{14\text{-}15}$ M_{sun}

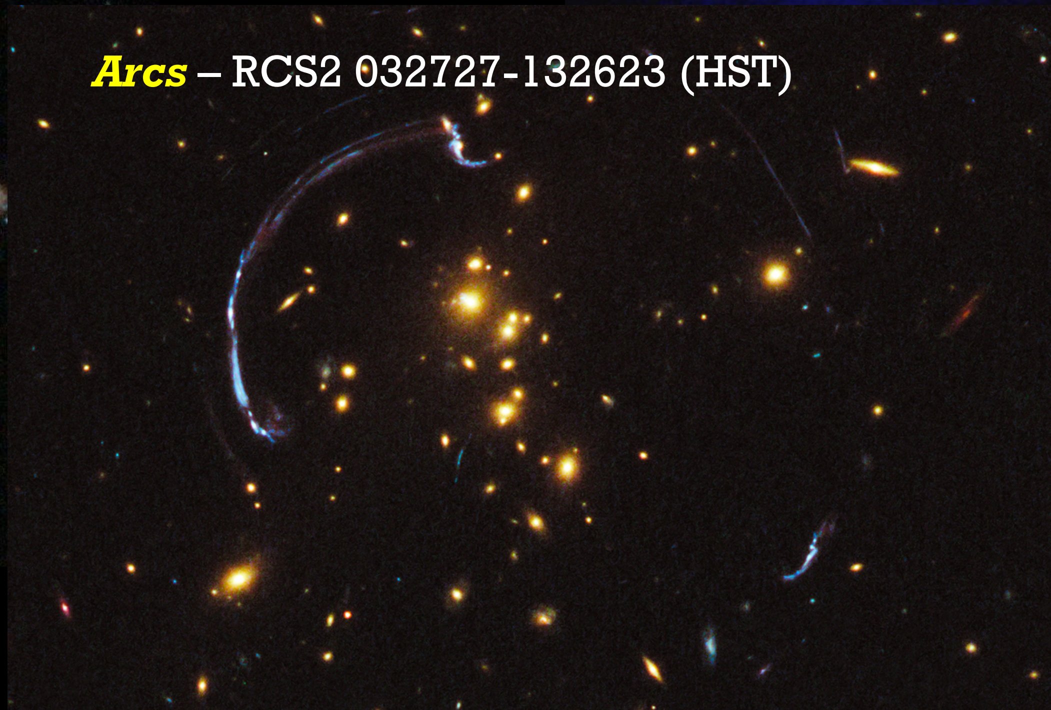
DIFFERENT MORPHOLOGIES BY STRONG LENSING



Einstein cross – QSO2237+0305



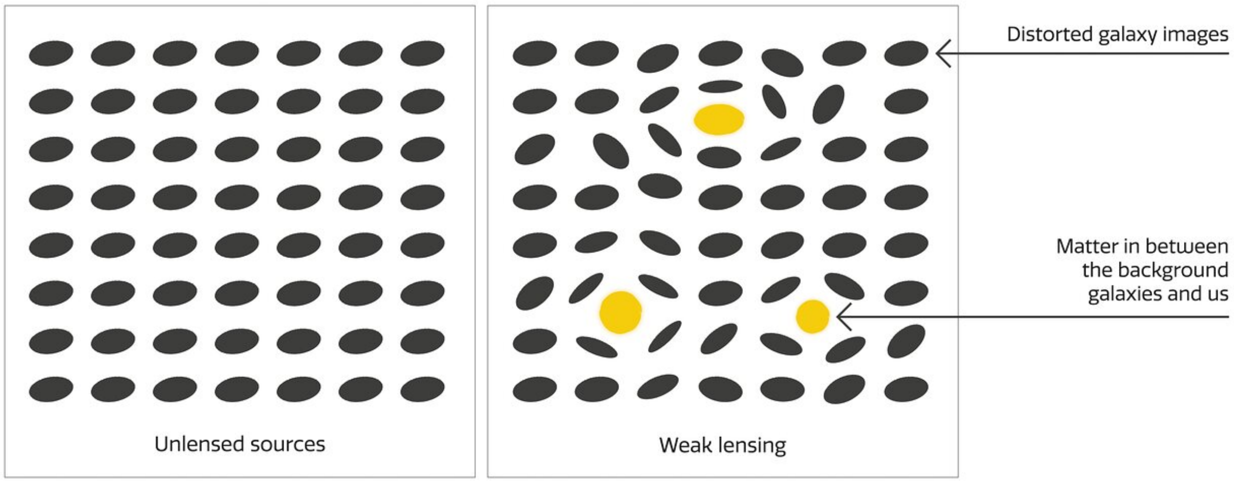
Einstein ring – LRG3-757 (HST)



Arcs – RCS2 032727-132623 (HST)

WEAK LENSING

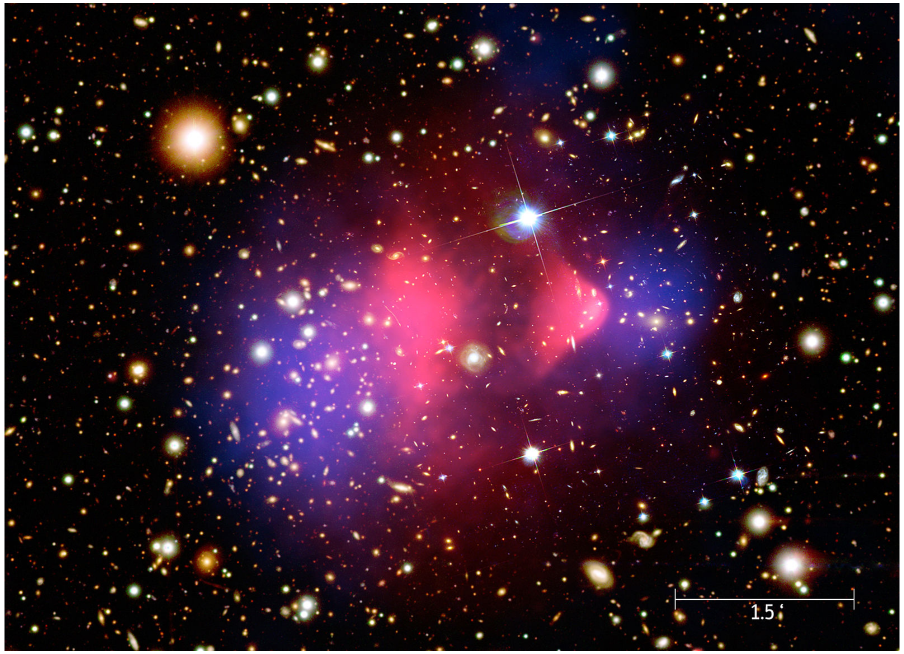
- Deflection is so weak that it is impossible to detect using a single background source
- The foreground mass can still be detected by ***systematic alignment of background sources around the lensing mass***
- By ***statistically*** measuring shear in galaxy shapes, mass distribution of the lens can be determined



WEAK LENSING

- Measurable when the lens is massive, e.g., by galaxy clusters ($M \sim 10^{14\text{-}15}\ M_{sun}$)
- Important way to measure mass in clusters for constraining cosmological parameters of cosmic expansion
- Famous example – the "***Bullet cluster***"
  - Merger of two galaxy clusters
  - Mass distribution inferred from weak lensing strongly suggest the existence of dark matter!

X-ray: gas
Lensing: dark matter

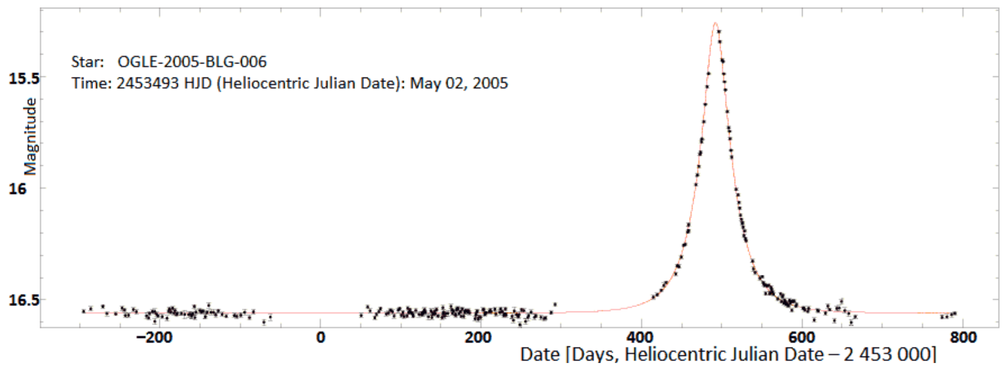


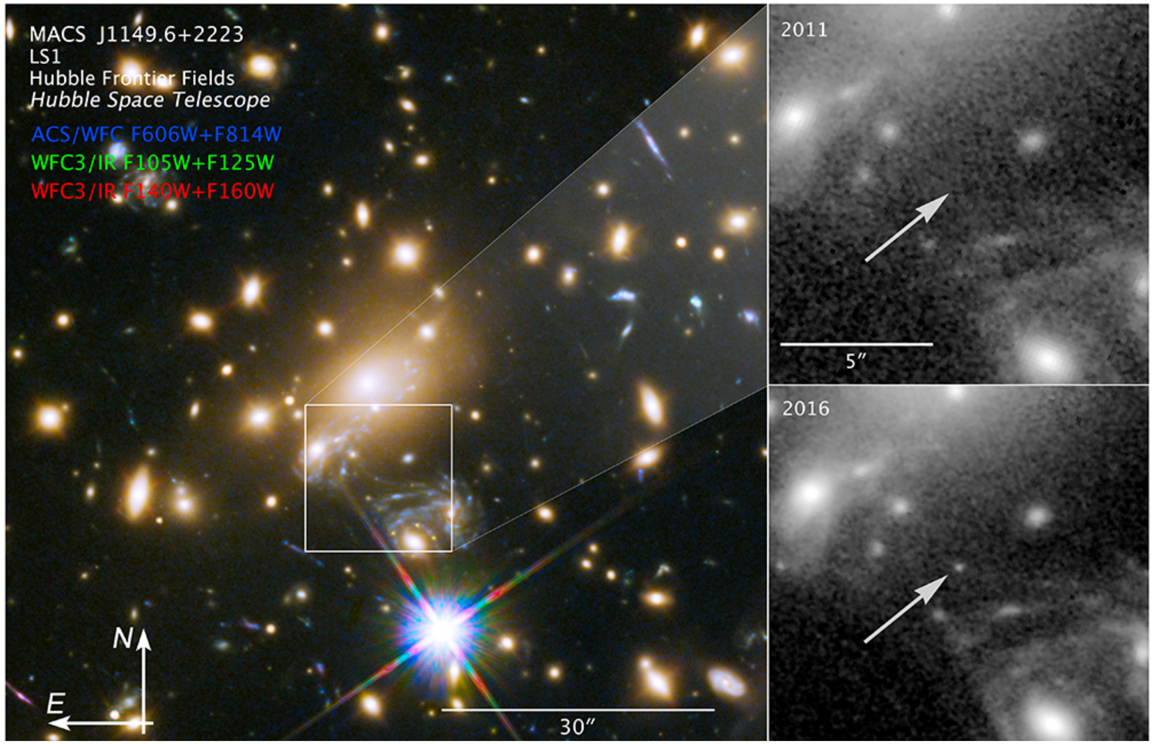
MICROLENSING

- No distortion in shape can be seen but ***brightness*** of background objects changes in time
- Often seen as the lens move across background star/quasar
- Extreme example: MACS J1149 or "***Icarus***" – the most distant star observed (*14 billion light years away or z=1.49)

*This refers to "comoving distance" in cosmology

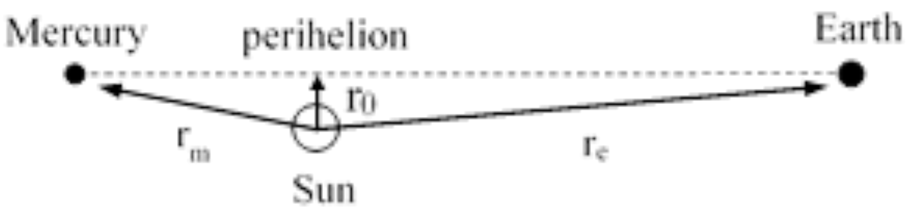
Typical ***light curve*** of a microlensing event

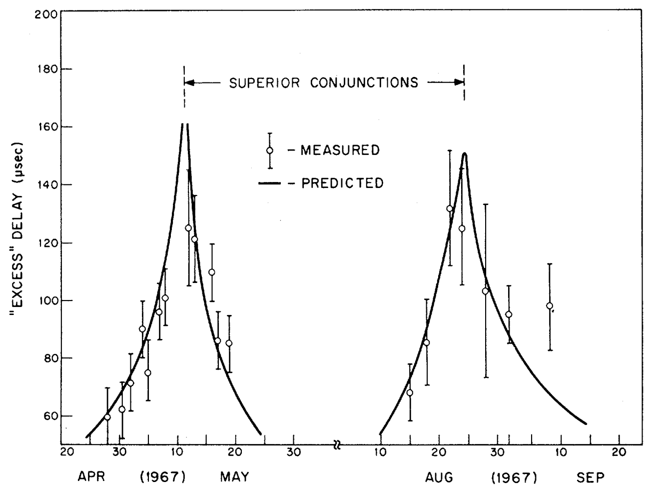




SHAPIRO TIME DELAY

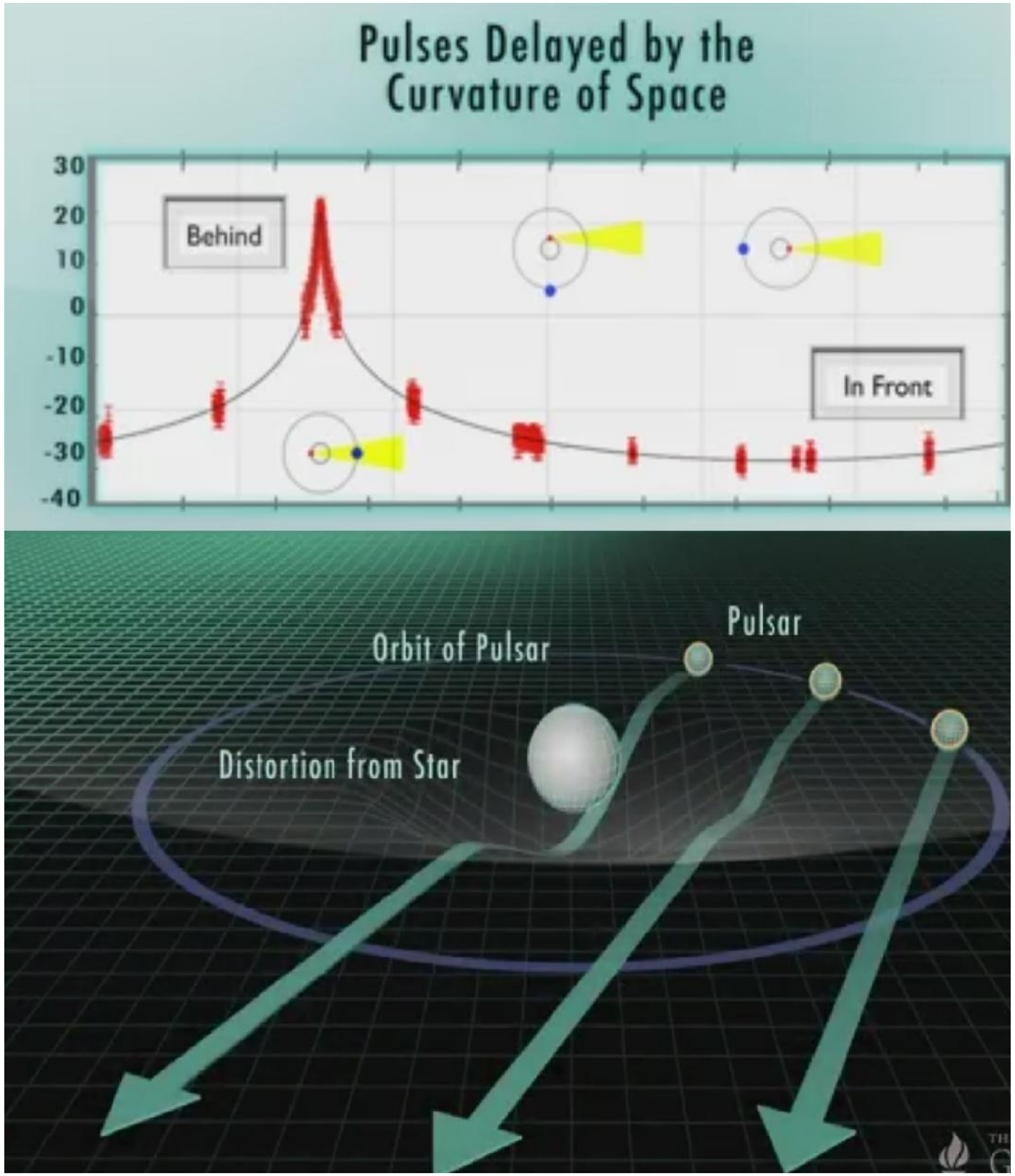
- One of the examples of ***gravitational time dilation***
- Refers to the ***delay of photon arrival time*** when light passes by a massive object
- Proposed by Irwin Shapiro in 1964
- First detected using radars bouncing off Mercury and Venus when Sun passes in between





SHAPIRO TIME DELAY IN PULSAR BINARIES

- Pulsars are rotating neutron stars whose pulses are natural clocks
- In pulsar binary systems, light arrival time would be delayed when one pulsar orbits behind its companion due to the curved spacetime



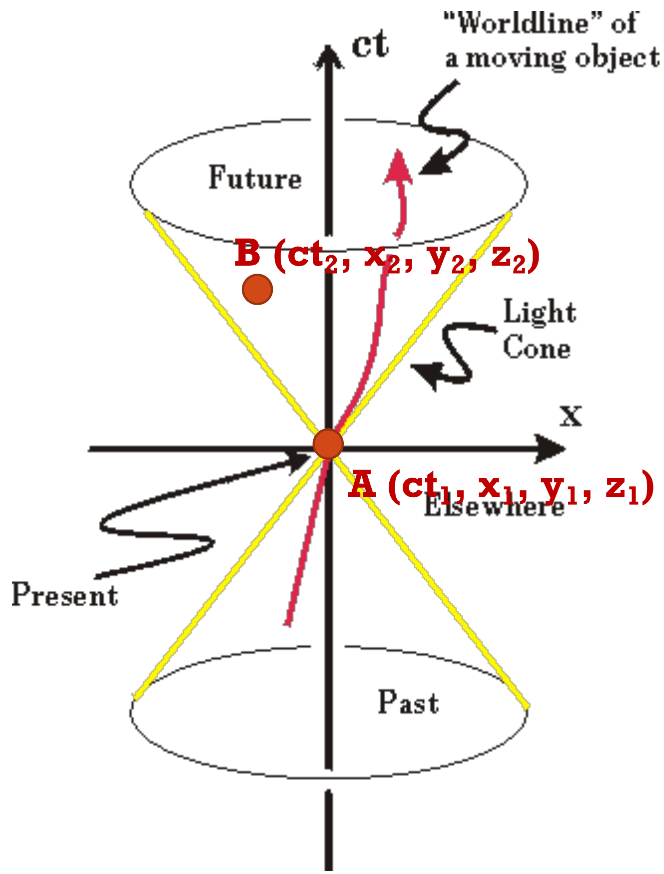
MATHEMATICAL DESCRIPTIONS OF GR

GEOMETRY OF SPACETIME

- In 4-D spacetime, any event can be specified by 4 real numbers, $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$
- For Minkowski (flat) spacetime, the interval (distance) between two events is

$$\Delta S^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \equiv (c\Delta \tau)^2$$

- It is a ***Lorentz invariant*** – the value does not change by changing frames
- τ is the ***proper time***
- Spacetime can be divided into three regions:
  - $\Delta S^2 = 0$: ***lightlike***; this defines the light cones
  - $\Delta S^2 > 0$: ***timelike***; region inside the light cones
  - $\Delta S^2 < 0$: ***spacelike***; region outside the light cones
- In Minkowski spacetime, light cones always point upward with slope 1



GEOMETRY OF SPACETIME

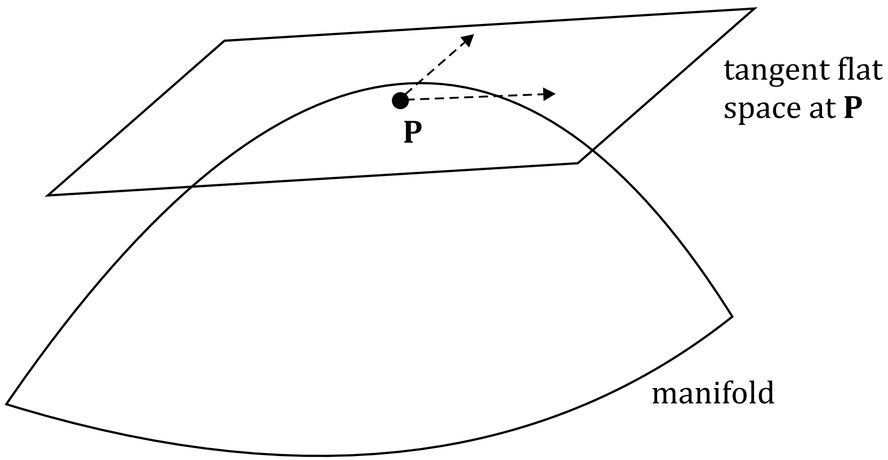
- Position-time 4 vector: $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$
- For Minkowski (flat) spacetime, the interval between two events ***with close separation*** is

$$ds^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$
$$= \sum_\mu \sum_\nu \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} dx^\mu dx^\nu. \qquad (\mu, \nu = 0,1,2,3)$$

where the Minkowski ***metric tensor*** is:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

CURVED SPACETIME



- When the spacetime is curved, we could use a mathematical object called "***4-D manifold* (流形)**" to describe its geometry
- We could generalize the differential interval on the manifold into:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

- Locally, the manifold is flat, i.e.,

$$g_{\mu\nu}(P) = \eta_{\mu\nu}$$

MATTER/ENERGY TELLS SPACE HOW TO CURVE

- Einstein Field equation describes how matter/energy curves spacetime:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- G = ***Einstein curvature tensor***, which is related to $g_{\mu\nu}$
- T = ***Stress-energy tensor***, which describes distribution of mass/energy:

$$T_{\mu\nu} = (\rho + P/c^2) u_\mu u_\nu - P g_{\mu\nu}$$

where $u^\mu = dx^\mu / d\tau$ is the 4-velocity

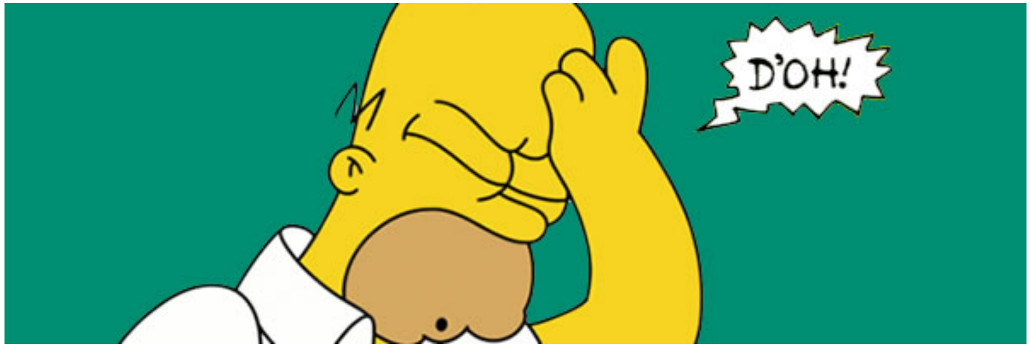
- The equation entails
  - ***Energy-momentum conservation***
  - ***Equation of motion of free particles***

THE COSMOLOGICAL CONSTANT

- The set of Einstein's equations is not unique – we can add any constant multiple of $g_{\mu\nu}$ to the L.H.S. and still have a consistent set of equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- Λ = ***cosmological constant***
- Einstein added it to have a static universe, which he called his "***biggest blunder***" after cosmic expansion was discovered



THE COSMOLOGICAL CONSTANT

- In fact, the cosmological constant can even explain why expansion of our universe is ***accelerating***!
- The accelerated expansion was discovered using observations of Type Ia supernovae (2011 Physics Nobel Prize)
- This can be seen from the ***Friedmann's equation*** (which can also be derived from Einstein field equation), which describes expansion of the universe:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}$$

- $a(t) = d(t)/d_0$ is called the "***scale factor***", which characterizes the expansion of the universe

HOW OBJECTS MOVE IN CURVED SPACETIME

SPACETIME CURVATURE TELLS MATTER/ENERGY HOW TO MOVE

- Particles move on ***geodesics*** of the metric
- For a particle with mass, geodesic is the path with an ***extremal lapse of interval or proper time***
- Recall the differential interval in curved spacetime: $ds^2 = c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$
- The total interval between two points A and B is

$$s_{AB} = \int_A^B ds = \int_A^B [g_{\mu\nu} dx^\mu dx^\nu]^{\frac{1}{2}} = \int_A^B \left[g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right]^{\frac{1}{2}} ds$$

$$= \int_A^B G(x^\mu, \dot{x}^\mu)\, ds \quad \textit{where} \quad G(x^\mu, \dot{x}^\mu) = [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{\frac{1}{2}}$$

where $\dot{x}_\mu \equiv dx_\mu / ds$

SPACETIME CURVATURE TELLS MATTER/ENERGY HOW TO MOVE

- Finding the path which extremizes s_{AB} then leads to the ***Euler-Lagrange equations***:

$$\frac{d}{ds}\left(\frac{\partial G}{\partial \dot{x}^{\mu}}\right) - \frac{\partial G}{\partial x^{\mu}} = 0$$

- There are 4 equations (one for each $\mu = 0, 1, 2, 3$)
- This set of equations are analogous to the Euler-Lagrange equations in classical mechanics, which allows us to write down the ***equations of motion*** of particles
- From the definition of G (last page), we could also see that

$$G(x^{\mu}, \dot{x}^{\mu}) = 1$$

EXAMPLE: SCHWARZSCHILD METRIC

- In 1915, Karl Schwarzschild found a solution to Einstein's equations for a ***static, spherically symmetric point mass***
- The Schwarzschild metric:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2$$

- Let's use a simpler, more general form (where A and B are only functions of r and independent of t):

$$ds^2 = Ac^2 dt^2 - Bdr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2$$

$$A = B^{-1} = (1 - \frac{2GM}{c^2 r})$$

- Now we could use the Euler-Lagrange equations to derive particle motions in the Schwarzschild metric (see derivation on blackboard)
- These equations will be crucial for later discussions on what happens to materials around the black holes!

SUMMARY OF DERIVATION OF ORBITAL MOTIONS IN SCHWARZSCHILD METRIC

***Please note the correction for the definition of A in the Schwarzschild metric

From G = 1, assume θ = pi/2 => $A c^2 \dot{t}^2 - B\dot{r}^2 - r^2\dot{\phi}^2 = 1$

From the 0 and 3 components of the Euler-Lagrange equations: $A\dot{t} \equiv \frac{k}{c}, r^2\dot{\phi} \equiv \frac{h}{c}$

One could get the equation for the radial motion:

$$\dot{r}^2 = \frac{1}{B}\left(\frac{k^2}{A} - \frac{h^2}{c^2 r^2} - 1\right)$$

Plug in A and B for Schwarzschild metric:

$$\dot{r}^2 = (k^2 - 1) - \frac{h^2}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right) + \frac{2GM}{c^2 r}$$

or

$$\left(\frac{dr}{d\tau}\right)^2 = c^2(k^2 - 1) - \frac{h^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) + \frac{2GM}{r}$$

ORBITAL EQUATIONS IN SCHWARZSCHILD METRIC

- Given the Schwarzschild metric, we have derived from the Euler-Lagrange equations the orbital equations for particles:

$$\left(\frac{dr}{d\tau}\right)^2 = c^2(k^2 - 1) - \frac{h^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) + \frac{2GM}{r}$$

where $k=E/mc^2$ measures the particle energy, and $h=l/m$ represents the angular momentum

- Let $u = 1/r$, the above equation can be written as

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2$$

ORBITAL PRECESSION DUE TO GR

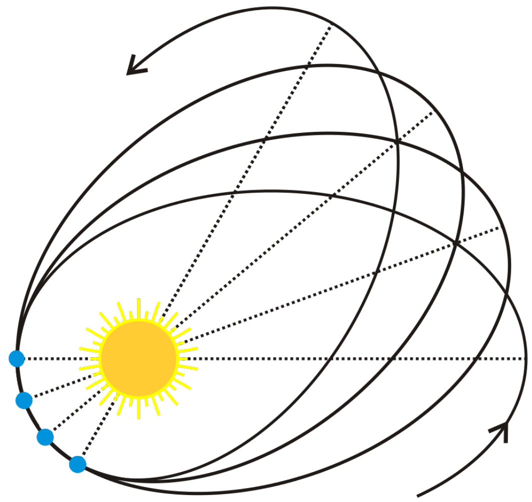
$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2$$

- In the Newtonian limit, u = 1/r is small so the last term -> 0, and the solution is closed ellipses:

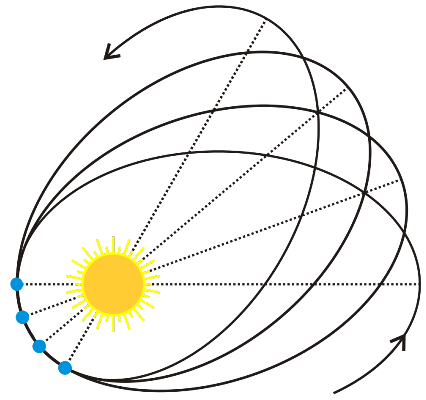
$$u = \frac{GM}{h^2}(1 + e\cos\phi)$$

- With the correction term due to GR, the ellipses would not be closed when the particle comes back to perihelion
- The ellipses would precess with an angle:

$$\Delta\phi = \frac{6\pi GM}{a(1-e^2)c^2}$$



PRECESSION OF MERCURY



$$\Delta\phi = \frac{6\pi GM}{a(1-e^2)c^2}$$

- Take Mercury as an example, a = 5.8 x 10^{12} cm, eccentricity e = 0.2, M_{sun} = 2x10^{33} g, so

$$\Delta\phi = 0''.1\ per\ orbit$$
$$= 43''\ per\ century\ (1\ orbit = 88\ days)$$

- The total precession of Mercury is ~5600'' per century
- Precession from other planets calculated from Newtonian physics gives ~5557'' per century
- The discrepancy can be resolved by including the additional precession caused by GR
- This solved a long-standing problem in Newtonian physics and is one of the first confirmations of GR!

SUMMARY

- Where GR came from:
  - ***Strong equivalence principle***: "a free-falling (accelerating) frame of reference in a gravitational field is equivalent to an inertial reference frame without gravity"
  - Gravity is an illusion due to that we are living in accelerating frames
- Theory of general relativity: ***"Matter/energy tells spacetime how to curve; spacetime curvature tells matter/energy how to move"***
- Verified predictions of GR
  - Light bending (Eddington test, gravitational lensing)
  - Gravitational redshift (Pound-Rebka experiment)
  - Gravitational time dilation (Hafele-Keating experiment, Shapiro time delay for planets & pulsar binaries)
  - Orbital precession (Mercury)

SUMMARY

- How to describe GR using math
  - The interval between events can be described by the ***metric tensor*** $g_{\mu\nu}$
  - The metric tensor can be obtained given the distribution of mass/energy $T_{\mu\nu}$ by solving the ***Einstein Field equation***

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

  - Particles follow ***geodesics*** on curved spacetime. Their motions can then be solved using the ***Euler-Lagrange equations***:

$$\frac{d}{ds}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right) - \frac{\partial G}{\partial x^\mu} = 0$$

- Example: ***Schwarzschild metric*** for a static, spherically symmetric point mass
  - Orbital equations derived
  - Orbital precession of Mercury