SCHWARZSCHILD AND KERR BLACK HOLES

Lecture 4, Introduction to Black Hole Astrophysics (PHYS480)

Hsiang-Yi Karen Yang, NTHU, 3/16/2021

ANNOUNCEMENTS

- HW2 will be posted on iLMS and course website today. Due next class!
- HW1 solutions will also be posted on iLMS and course website. Please check it out!
- Please start searching for black hole news for the oral presentation. Once you decide on the topic, please paste the news link here:

https://docs.google.com/spreadsheets/d/1_aYyMj1wf_uGheZ7zp_hvthmy4mdmPwIxFDdZOMG-nc/edit?usp=sharing

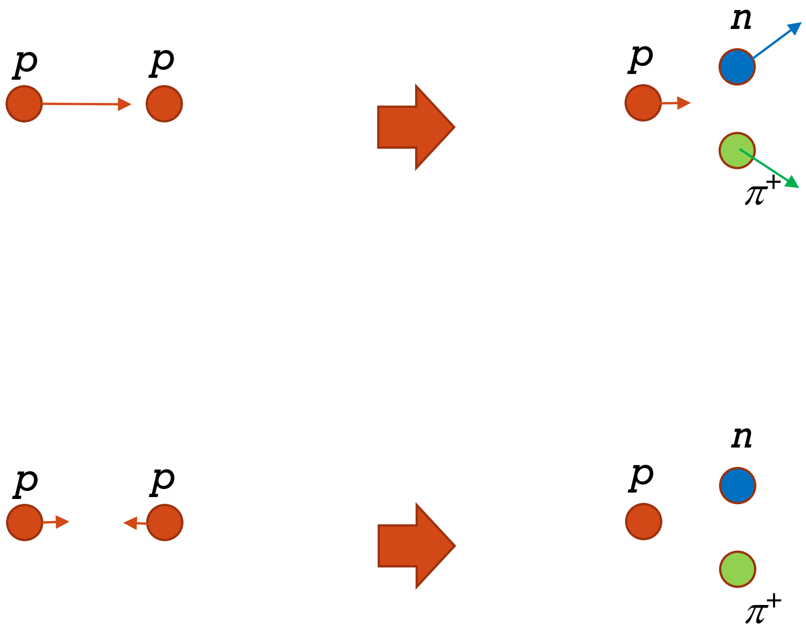
WHERE TO FIND NEWS

- I will point to some recent news related to today's class; check them out!
- You could also search by typing keyword "black hole" on the news page of major astronomy media
- Both English or Mandarin news are okay, but English websites typically contain more detailed information
- Some examples (not a complete list)
  - Science: https://search.sciencemag.org/?searchTerm=black hole&order=newest&limit=title&pageSize=10&articleTypes=News&
  - Sky & Telescope: https://skyandtelescope.org/astronomy-news/black-holes/
  - Science News: https://www.sciencenews.org/page/1?s=black+hole
  - Space: https://www.space.com/search?searchTerm=black+hole
  - 科技新報: https://technews.tw/google-search/?googlekeyword=黑洞
- Please post the news link in the spreadsheet:

https://docs.google.com/spreadsheets/d/1_aYyMj1wf_uGheZ7zp_hvthmy4mdmPwIxFDdZOMG-nc/edit#gid=0

HW1 -- EXERCISE 4

- Hadronic collisions:



PREVIOUS LECTURE…

- Where GR came from:
  - ***Strong equivalence principle***: "a free-falling (accelerating) frame of reference in a gravitational field is equivalent to an inertial reference frame without gravity"
  - Gravity is an illusion due to that we are living in accelerating frames
- Theory of general relativity: ***"Matter/energy tells spacetime how to curve; spacetime curvature tells matter/energy how to move"***
- Verified predictions of GR
  - Light bending (Eddington test, gravitational lensing)
  - Gravitational redshift (Pound-Rebka experiment)
  - Gravitational time dilation (Hafele-Keating experiment, Shapiro time delay for planets & pulsar binaries)
  - Orbital precession (Mercury)

PREVIOUS LECTURE...

- How to describe GR using math
  - The interval between events can be described by the ***metric tensor*** $g_{\mu\nu}$
  - The metric tensor can be obtained given the distribution of mass/energy $T_{\mu\nu}$ by solving the ***Einstein Field equation***

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

  - Particles follow ***geodesics*** on curved spacetime. Their motions can then be solved using the ***Euler-Lagrange equations***:

$$\frac{d}{ds}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right) - \frac{\partial G}{\partial x^\mu} = 0$$

- Example: ***Schwarzschild metric*** for a static, spherically symmetric point mass
  - Orbital equations derived (note some changes of variable definitions; see Lecture 3 slides)
  - Orbital precession of Mercury

THIS LECTURE…

- We will discuss properties of black holes predicted by GR given a given metric / spacetime geometry
- The no-hair theorem
- Schwarzschild (non-spinning) black holes
  - Properties of the event horizon and singularity
    - Gravitational time dilation and gravitational redshift
    - Perspectives from infalling observer vs. external observer
  - Orbits of massive particles
    - Innermost stable circular orbit (ISCO)
    - Maximum radiative efficiency of accretion disks
  - Orbits of photons
- Kerr (spinning) black holes
  - Structures: singularity, event horizon, and the ergosphere
  - ISCO & maximum efficiency of accretion disks

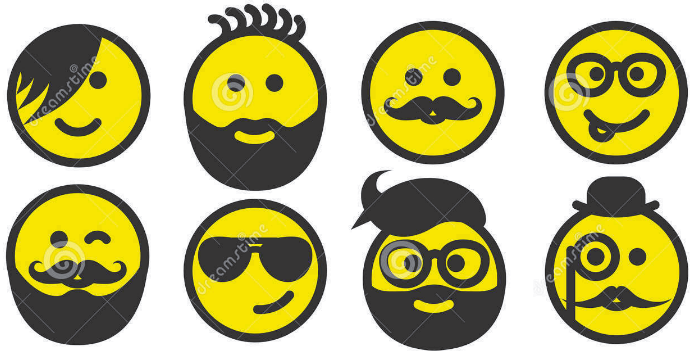
RECALL THE SCHWARZSCHILD METRIC

- It is a solution to Einstein field equations for a ***static, spherically symmetric point mass M***
- The Schwarzschild metric:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2$$

- Is this solution ***unique***? That is, given M, is it possible to find another solution of spacetime geometry that also satisfies Einstein field equations?
- In 1967, Werner Israel showed that the answer is NO -- the Schwarzschild metric is a unique solution
- Similarly, it was shown that the solutions are also unique for ***charged*** or ***spinning*** black holes

NO HAIR THEOREM



- It says that ***"Stationary BHs after formation can be uniquely described given mass (M), angular momentum (J), and charge (Q)."***
- Once (M, J, Q) are given, the spacetime is determined
- It means that BHs are in fact very simple objects!

Maximum set of parameters:

$\{M, J, Q\}$

NOT ALL VALUES ARE ALLOWED!

- They need to satisfy this relation: $Q^2 + (J/M)^2 <= M^2$ (in Planck unit, G=c=1)
- Otherwise, there would be no event horizon to hide the singularity! This is called "***naked singularity***" **(裸奇點)**
  - Singularity is where laws of physics and predictions break down
  - It would be bad if it happens
- Therefore, some physicists (including Penrose and Hawking) proposed that the "***cosmic censorship hypothesis***" **(宇宙監督假說)** should hold: "Singularities should be hidden by the event horizon" and "GR is a deterministic theory."
  - Note that this is a hypothesis
- Some studies have shown that solutions of naked singularities exist in some special cases, but they are not proven to be physically realistic

CAN BLACK HOLES HAVE CHARGE?

- Mathematically, BHs can have nonzero charge and have a valid solution to the Einstein field equation
- However, astrophysical BHs, which are formed from gravitational collapse of massive stars, are not isolated from their surroundings
- If the collapsing material started with nonzero charge initially, they would easily attract opposite charges and neutralize themselves during formation
- Note that the electric force between two charges is typically >> gravity
- Therefore, generally speaking, ***Q = 0 for astrophysical BHs***
- ***Astrophysical BHs can be uniquely characterized by two parameters: mass M and spin J***

CONSTRAINTS ON BLACK HOLE SPIN

- For Q = 0, the previous inequality becomes ***$|J| <= M^2$***
- Define ***a = J/M^2 as the BH spin parameter***, then ***$0 <= |a| <= 1$***
- Two types of BHs:
  - ***Schwarzschild BHs (non-spinning, a=0)***
  - ***Kerr BHs (spinning, a/=0)***

Q: Can BHs have a > 1 such that the above inequality is violated?

- Imagine a case where you would like to violate the cosmic censorship hypothesis by creating a BH with $|J|>M^2$
- You would need to throw a material with a very large angular momentum
- However, such a material would not be able to accrete onto the BH due to the centrifugal force, which becomes larger for materials with higher angular momentum

LIMITS ON BLACK HOLE MASSES?

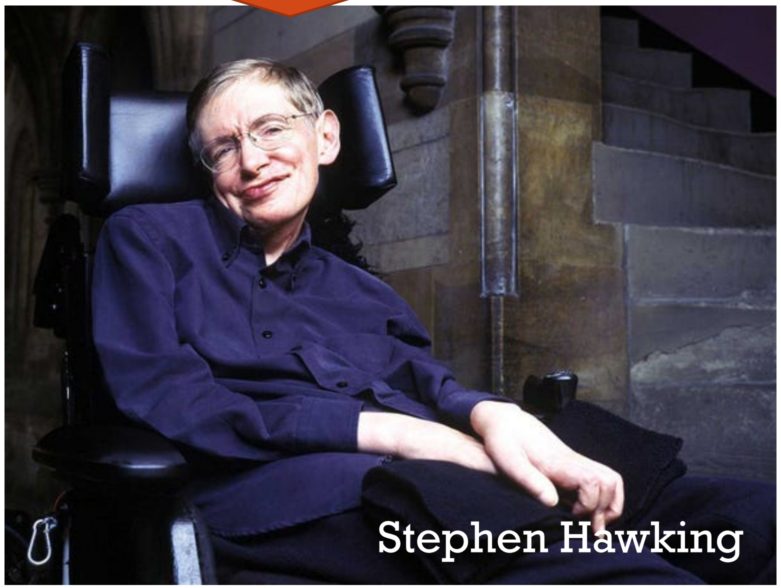
- The lower limit on BH masses come from the upper limit of masses of neutron stars (~2-3 M_{sun}), but there are no theoretical limits for how massive a BH can get!
- Current record holder:
  - Smallest stellar-mass black hole: ~3.3 M_{sun} (**check out this news**)
  - Largest SMBH: quasar TON 618 -- 66 billion M_{sun} (**check out this news**)

IS THE NO-HAIR THEOREM TRUE?

- The no-hair theorem of course is still a "theorem"; many physicists are trying to come up with tests to see if it is true
- It's still debated whether BHs have hairs or not

THE THORNE-PRESKILL-HAWKING BET

I bet the cosmic censorship hypothesis is correct and black holes are bald



No no, BHs might have hair and naked singularities could exist!

THE BET IS STILL OPEN

- Some people are trying to test if BHs have hairs or not
- Examples of recent news – yes, BHs are bald!
  - **Black hole reverberations suggest the cosmic beasts are as 'bald' as cue balls**
  - **Spitzer Telescope Reveals the Precise Timing of a Black Hole Dance**
- News – no, BHs could have some fine hair!
  - **Extreme black holes have hair that can be combed**

SCHWARZSCHILD BLACK HOLE

Karl Schwarzschild (1873-1916)

SCHWARZSCHILD BLACK HOLES

- In 1915, Schwarzschild solved the Einstein's field equations for a spherically-symmetric point mass
- First exact solution of Einstein's equations
- Describes a non-spinning, non-charged black hole
- Schwarzschild metric:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2$$

- Two singularities:
  - r = 0: where mass concentrates and spacetime curvature -> Infinity
  - $r = r_S = 2GM/c^2$: coordinate singularity (can be removed by changing coordinates)

HOW LARGE IS THE SCHWARZSCHILD RADIUS?

Object	**Mass**	**Schwarzschild radius**
TON 618 (largest known BH)	6.6×10^{10} M_{sun}	~1300 AU (~10^{16} cm)
SMBH in M87	6.5×10^{9} M_{sun}	~130 AU
SMBH in Milky Way	4×10^{6} M_{sun}	~0.1 AU
Stellar-mass BHs	~3 - 100 M_{sun}	~10-300 km
Neutron stars	~1.4 - 3 M_{sun}	~5-10 km (real size ~ 20km)
Sun	$M_{sun} = 2 \times 10^{33}$ g	***3 km***
Earth	3×10^{-6} M_{sun}	~ 1 cm

- ***$r = r_S = 2GM/c^2 = 3$ km (M/M_{sun})***
- For objects other than BHs and NSs, their sizes are much larger than r_S
- Even for SMBHs, their event horizon is still much smaller than sizes of their host galaxies (~10 kpc ~ 10^{22} cm)

EVENT HORIZON -- WHERE LIGHT CAN'T ESCAPE

- Refers to the region inside which ***even light cannot escape*** because the spacetime is very curved

$$ds^2 = c^2 d\tau^2 = Ac^2dt^2 - Bdr^2 - r^2d\theta^2 - r^2 sin^2\theta d\phi^2$$

$$A = B^{-1} = (1 - \frac{r_S}{r})$$

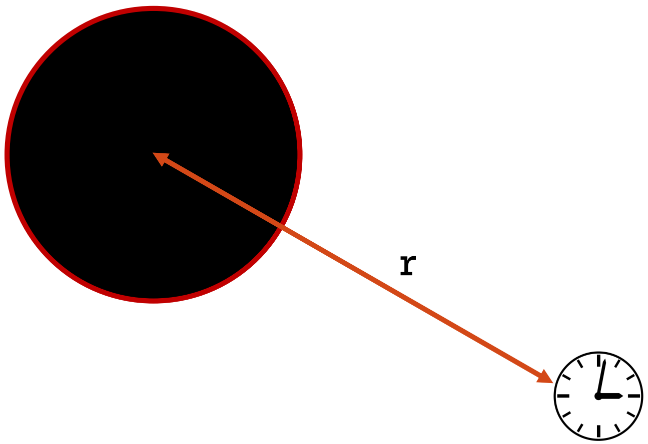
- ***For light rays, $ds^2 = 0$***, assuming a light ray is moving outward in the radial direction, then its speed is

$$ds^2 = Ac^2dt^2 - Bdr^2 = 0$$

$$\Rightarrow \frac{dr}{dt} = c\sqrt{\frac{A}{B}}$$

- When r -> r_S, A -> 0 and B -> Infinity, so dr/dt ->0, meaning that even light cannot move outward!

EVENT HORIZON – WHERE THINGS FREEZE

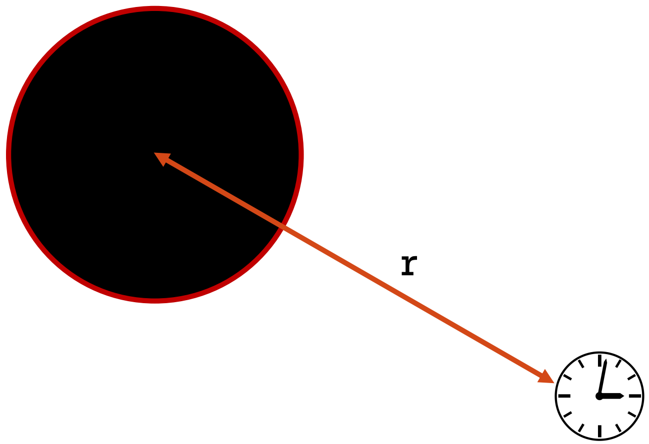


- From the Schwarzschild metric:

$$dt = \frac{d\tau}{\sqrt{1 - \frac{r_S}{r}}}$$

- When r -> infinity, $dt = d\tau$, meaning **dt is the proper time measured from an infinite distance**
- When r gets closer to the massive object, dt gets longer, i.e., clocks appear to tick slower in deeper gravitational well – "***gravitational time dilation***"
- ***When r -> r_S, dt -> infinity*** (the clock seems to stop ticking!)

EVENT HORIZON – WHERE THINGS FADE AWAY



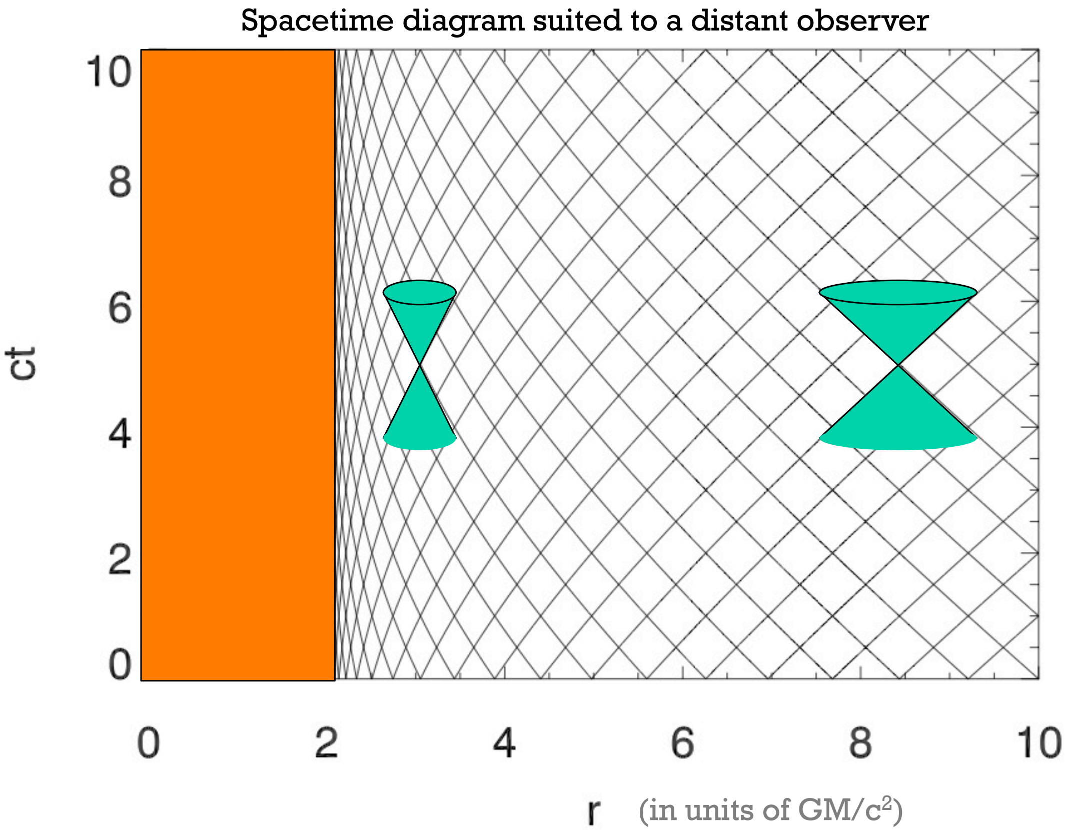
- Because of the gravitational time dilation effect, frequency of light would decrease and appear redder as it gets closer to the central object
- This is the "***gravitational redshift***" effect
- We could define this redshift as follows:

$$1 + z = \frac{\lambda_\infty}{\lambda} = \frac{1}{\sqrt{1 - \frac{r_S}{r}}}$$

EVENT HORIZON – VIEW FROM A DISTANT OBSERVER

- Due to ***gravitational time dilation***, from point of view of distant observer, infalling objects will appear to freeze at the event horizon
- ***Gravitational redshift***: light would appear to shift to lower frequencies (longer wavelengths, or "redshifted") and appear dimmer when close to a massive object
  - Event horizon is the surface of infinite gravitational redshift
  - Infalling object will appear to fade away as it freezes

SPACETIME DIAGRAM OF LIGHT RAYS AS SEEN BY A DISTANT OBSERVER



- For photons, dr/dt -> 0 as r -> r_S
- Light emitted near r_S takes an infinite amount of time to reach a distant observer
- For a distant observer, there is no way to peek inside the event horizon

WHAT HAPPENS IF YOU FALL INTO THE EVENT HORIZON?

- Imagining that you are falling radially into the event horizon
- When $r < r_S$, A and B become negative

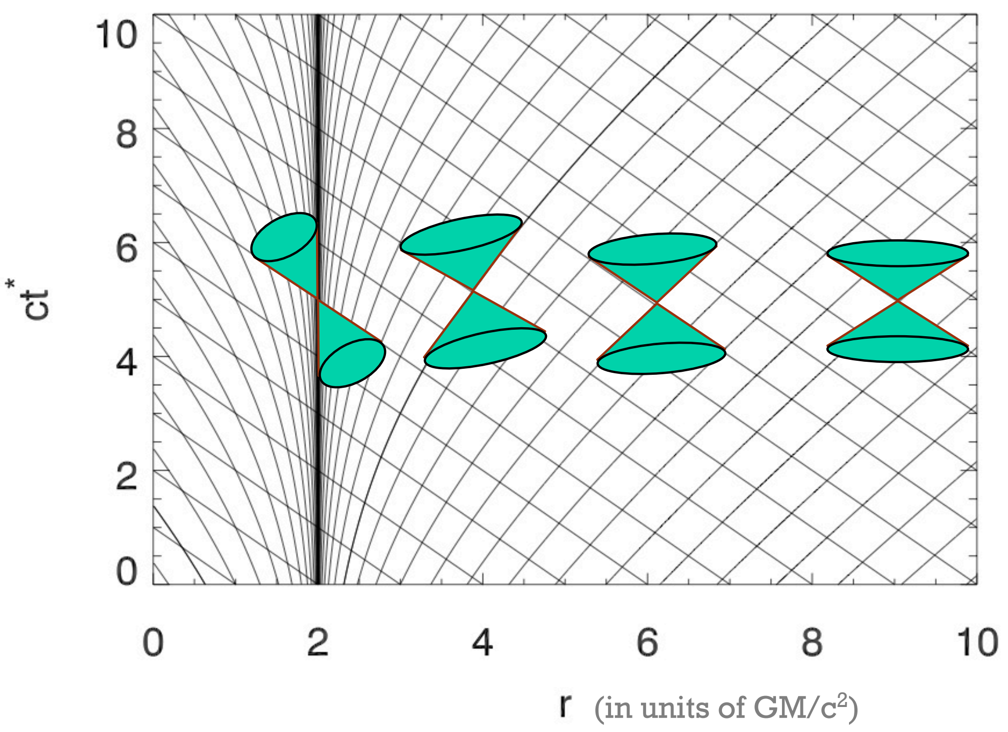
$$ds^2 = Ac^2dt^2 - Bdr^2 \qquad A = B^{-1} = (1 - \frac{r_S}{r})$$

- It means that inside the event horizon, the roles of time and space are reversed!
  - The r coordinate becomes uni-directional just like time can only move forward
  - To move outward in r would be as difficult as traveling back in time
  - Everything inside the event horizon would have no choice but to fall toward the singularity at the center!

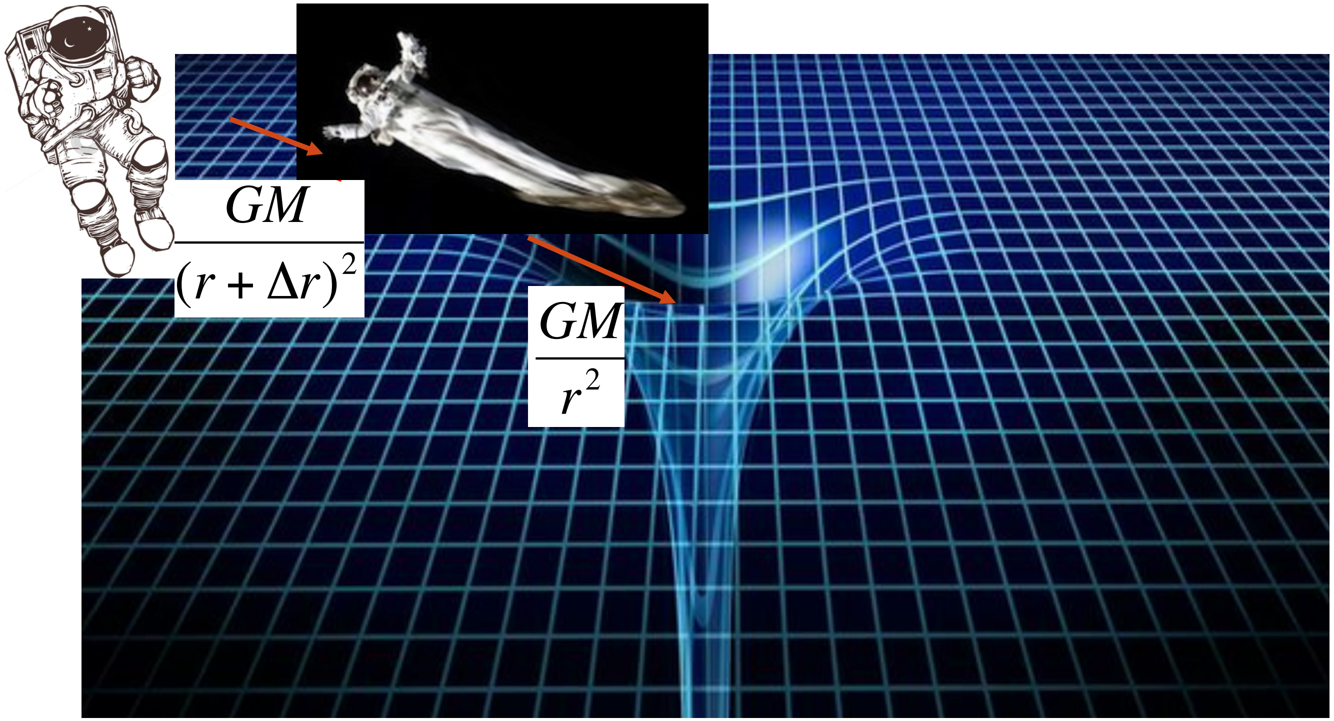
VIEW OF AN INFALLING OBSERVER

- For an infalling observer, his light cone would be gradually tilted as he gets closer to the BH
- He would pass through the event horizon without fuss
- Once entering the event horizon, his light cone is so tilted that even the outgoing light ray cannot leave the event horizon
- He (and everything else) will eventually reach the singularity at the center (r=0)

Spacetime diagram suited to an infalling observer
(ingoing Eddington-Finkelstein coordinate)



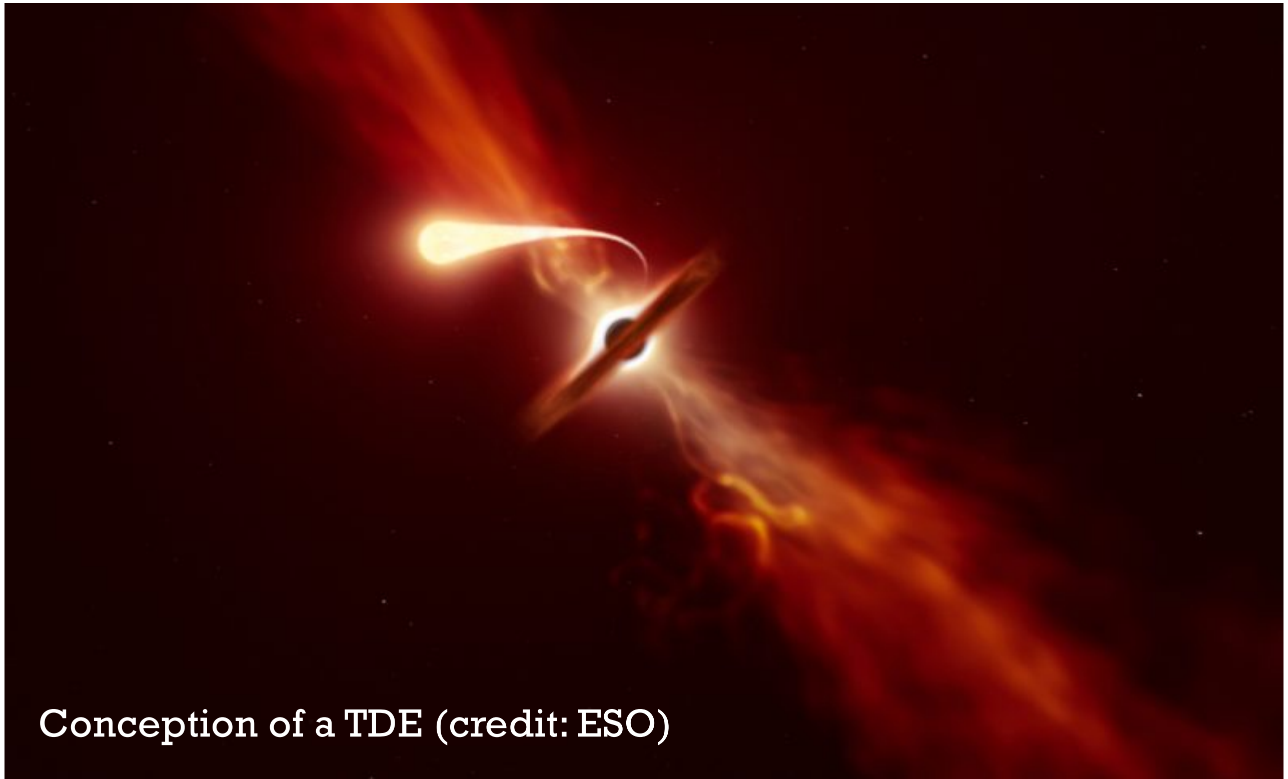
SPAGHETTIFICATION



- We would never make it to the center intact because the tidal force ($F_{tidal} \sim M/r^3$) would tear us apart

SPAGHETTIFICATION OF A STAR

- A star, if getting too close to a BH, could be spaghettified due to the strong tidal forces
- This is called "***tidal disruption events***" (TDE)
- **Recent news of such an event**



Conception of a TDE (credit: ESO)

HOW OBJECTS ORBIT AROUND SCHWARZSCHILD BLACK HOLES

SPACETIME CURVATURE TELLS MATTER/ENERGY HOW TO MOVE

- Particles move on ***geodesics*** of the metric
- For a particle with mass, geodesic is the path with an ***extremal lapse of interval or proper time***
- Recall the differential interval in curved spacetime: $ds^2 = c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$
- The total interval between two points A and B is

$$s_{AB} = \int_A^B ds = \int_A^B [g_{\mu\nu} dx^\mu dx^\nu]^{\frac{1}{2}} = \int_A^B \left[g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right]^{\frac{1}{2}} ds$$

$$= \int_A^B G(x^\mu, \dot{x}^\mu)\, ds \quad where \quad G(x^\mu, \dot{x}^\mu) = [g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu]^{\frac{1}{2}}$$

where $\dot{x}^\mu \equiv dx^\mu / ds$

SPACETIME CURVATURE TELLS MATTER/ENERGY HOW TO MOVE

- Finding the path which extremizes s_{AB} then leads to the ***Euler-Lagrange equations***:

$$\frac{d}{ds}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right) - \frac{\partial G}{\partial x^\mu} = 0$$

- There are 4 equations (one for each $\mu = 0, 1, 2, 3$)
- This set of equations are analogous to the Euler-Lagrange equations in classical mechanics, which allows us to write down the ***equations of motion*** of particles
- From the definition of G (last page), we could also see that

$$G(x^\mu, \dot{x}^\mu) = 1$$

EXAMPLE: SCHWARZSCHILD METRIC

- In 1915, Karl Schwarzschild found a solution to Einstein's equations for a ***static, spherically symmetric point mass***
- The Schwarzschild metric:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2$$

- Let's use a simpler, more general form (where A and B are only functions of r and independent of t):

$$ds^2 = Ac^2 dt^2 - B dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2$$

$$A = B^{-1} = (1 - \frac{2GM}{c^2 r})$$

- Now we could use the Euler-Lagrange equations to derive particle motions in the Schwarzschild metric (see derivation on blackboard)
- These equations will be crucial for later discussions on what happens to materials around the black holes!

SUMMARY OF DERIVATION OF ORBITAL MOTIONS IN SCHWARZSCHILD METRIC

***Please note the correction for the definition of A in the Schwarzschild metric

From G = 1, assume θ = pi/2 => $Ac^2\dot{t}^2 - B\dot{r}^2 - r^2\dot{\phi}^2 = 1$

From the 0 and 3 components of the Euler-Lagrange equations: $A\dot{t} \equiv \frac{k}{c}, r^2\dot{\phi} \equiv \frac{h}{c}$

One could get the equation for the radial motion:

$$\dot{r}^2 = \frac{1}{B}(\frac{k^2}{A} - \frac{h^2}{c^2r^2} - 1)$$

Plug in A and B for Schwarzschild metric:

$$\dot{r}^2 = (k^2-1) - \frac{h^2}{c^2r^2}(1 - \frac{2GM}{c^2r}) + \frac{2GM}{c^2r}$$

or

$$(\frac{dr}{d\tau})^2 = c^2(k^2 - 1) - \frac{h^2}{r^2}\left(1 - \frac{2GM}{c^2r}\right) + \frac{2GM}{r}$$

ORBITAL EQUATIONS IN SCHWARZSCHILD METRIC

- Given the Schwarzschild metric, we have derived from the Euler-Lagrange equations the orbital equations for particles:

$$\left(\frac{dr}{d\tau}\right)^2 = c^2(k^2 - 1) - \frac{h^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) + \frac{2GM}{r}$$

where $k=E/mc^2$ measures the particle energy, and $h=l/m$ represents the angular momentum

- Let u = 1/r, the above equation can be written as

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2$$

ORBITAL EQUATIONS IN NEWTONIAN PHYSICS

- For a particle orbiting a central mass M, the total energy per unit mass is:

$$E = \frac{1}{2}\dot{r}^2 + \frac{1}{2}(r\dot{\phi})^2 + U(r) = \frac{1}{2}\dot{r}^2 + \frac{1}{2}\frac{h^2}{r^2} + U(r)$$

$$(h = \frac{l}{m} = r^2\dot{\phi})$$

- Therefore,

$$\dot{r}^2 = 2(E - U) - \frac{h^2}{r^2}$$

- Define an ***effective potential***

$$V(r) = U(r) + \frac{h^2}{2r^2}$$

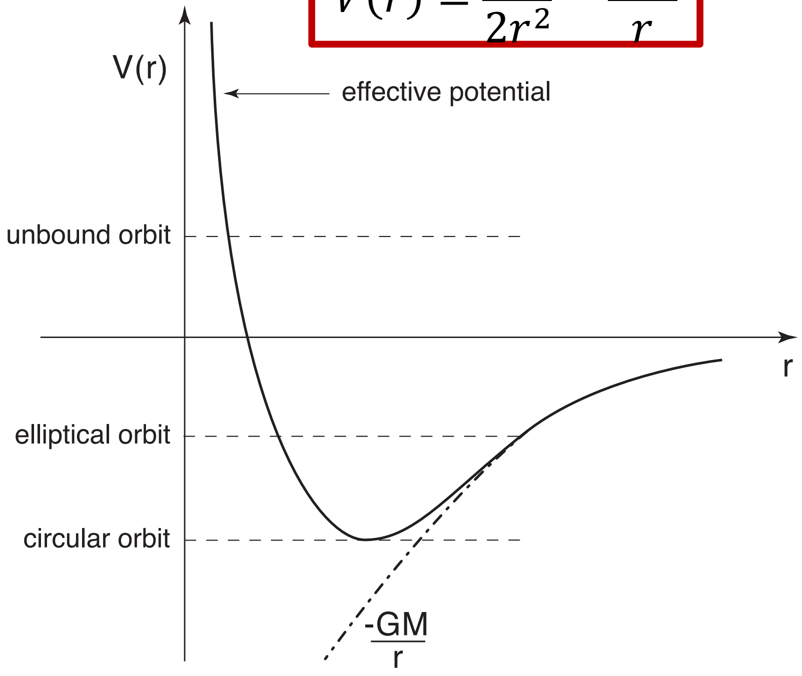
- Then $\dot{r}^2 = 2(E - V)$

ORBITAL EQUATIONS IN NEWTONIAN PHYSICS

$$\dot{r}^2 = 2(E - V)$$

$$V(r) = \frac{h^2}{2r^2} - \frac{GM}{r}$$

- Given the particle energy E and angular momentum h, we could use the above equation to find solutions of $v_r = dr/dt$ and analyze the particle's motion
- If h = 0 (no orbital velocity, only radial velocity)
  - If E<0, the particle would accelerate toward the center
  - If E>0, the particle can escape the pull of the massive object

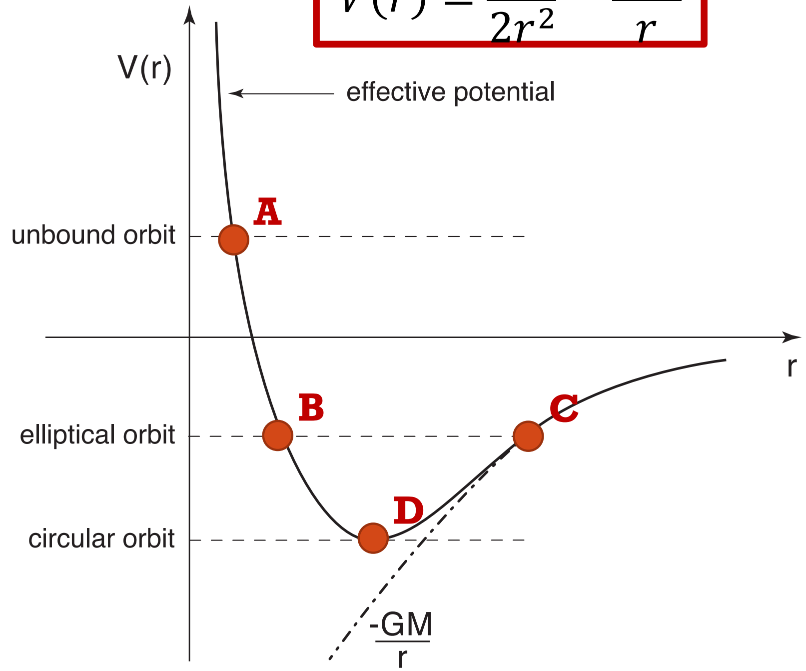


ORBITAL EQUATIONS IN NEWTONIAN PHYSICS

$$\dot{r}^2 = 2(E - V)$$

$$V(r) = \frac{h^2}{2r^2} - \frac{GM}{r}$$

- When h>0, the ***angular momentum barrier*** (the $1/r^2$ term) prevents a particle from reaching r=0
- If E > 0, solution exists between A and Infinity -> particle is unbound
- If E < 0, particle is bound
  - If $E > V_{min}$, solution exists between B and C -> elliptical orbit
  - If $E = V_{min}$, only one solution exists
    - circular orbit at $r = r_D(h,E,M)$
    - This orbit is a ***stable circular orbit***, i.e., the particle would return to V_{min} given small perturbations



ORBITS IN SCHWARZSCHILD METRIC

- Similarly, we could analyze the stability of orbits in Schwarzschild metric using the effective potential
- Recall the orbital equation:

$$\left(\frac{dr}{d\tau}\right)^2 = c^2(k^2 - 1) - \frac{h^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) + \frac{2GM}{r}$$

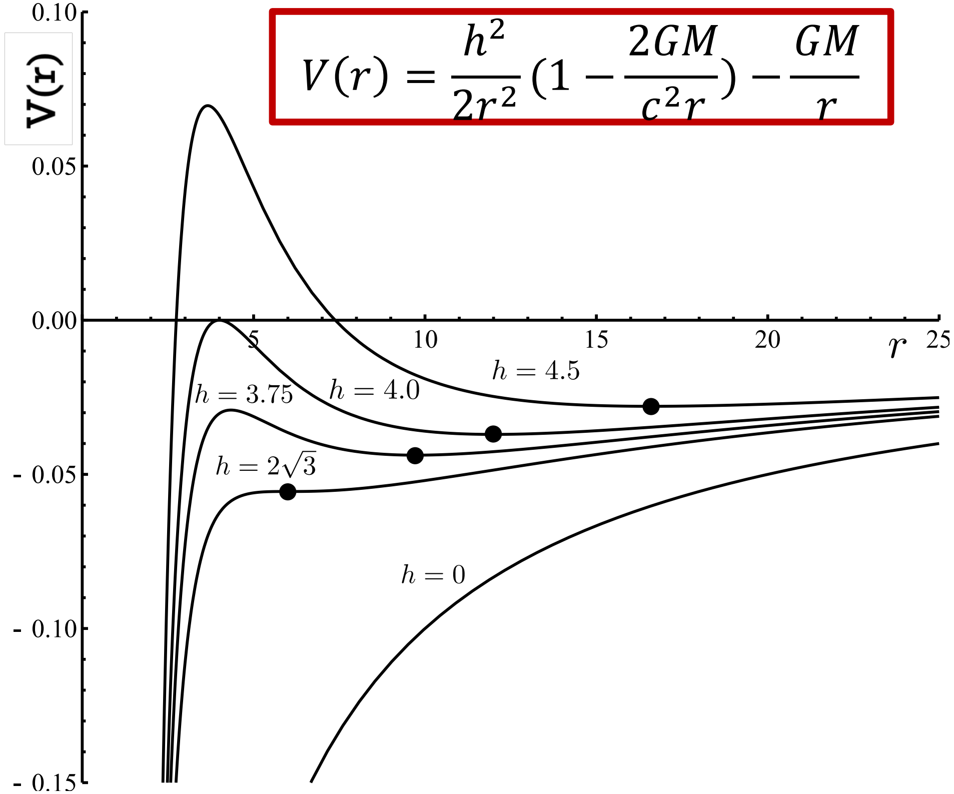
- We could define the effective potential as

$$V(r) = \frac{h^2}{2r^2}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{GM}{r}$$

- By setting V'(r)=0, one can find the extrema of the effective potential at radius:

$$r = \frac{h^2}{2GM}\left\{1 \pm \sqrt{1 - 12\left(\frac{GM}{hc}\right)^2}\right\}$$

STABILITY OF CIRCULAR ORBIT IN GR



- For a given h, one can find the shape of V(r)
- The relativistic ($-1/r^3$) term weakens the angular momentum barrier at small r
- Dots show the locations of ***stable circular orbits***
- For h < 2√3, no stable circular orbits
- ***Innermost stable circular orbit (ISCO)*** for Schwarzschild black holes (h= 2√3):

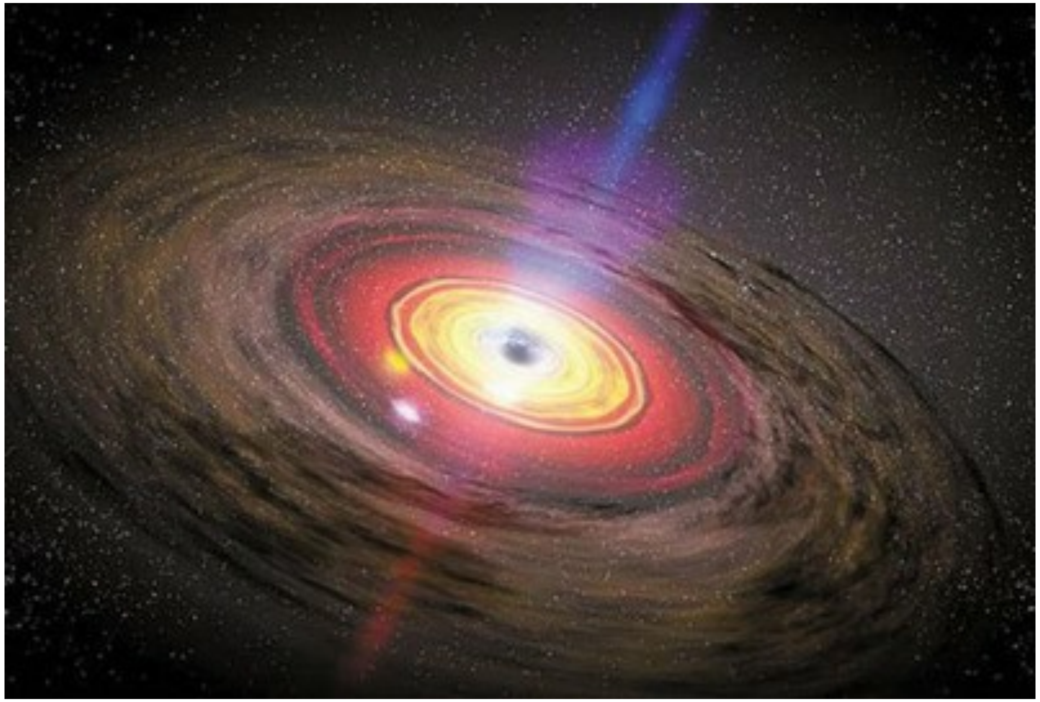
$$R_{ISCO} = \frac{6GM}{c^2} = 3r_s$$

INNERMOST STABLE CIRCULAR ORBIT (ISCO)

- ***R_{ISCO} = the smallest distance from a BH for a particle to stably maintain a circular orbit***
- Inside $R_{ISCO,}$ any small perturbation would lead to inspiral into the BH
- ***$R_{ISCO} = 6GM/c^2 = 3R_s$ for Schwarzschild (non-spinning) black holes***
  - R_{ISCO} is different for spinning BHs
- Importance of ISCO for ***accretion disks***:
  - Often approximated as the inner radius of BH accretion disks (where disks are hottest and most luminous)
  - The "***radiative efficiency***" of accretion disks can be estimated by calculating how much gravitational energy could be converted into radiation at ISCO

RECALL BLACK HOLE ACCRETION DISKS

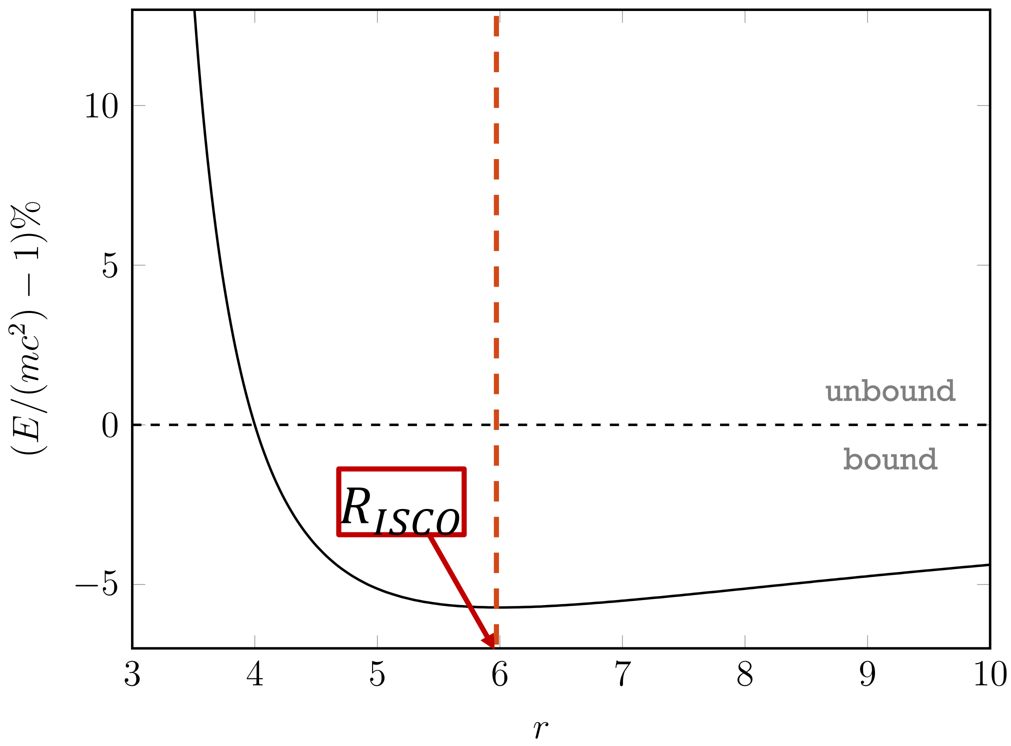
Conception of an AGN



- As gas falls in, gravitational energy is turned into heat
- Disk is heated and emits thermal radiation
- (Amount of energy lost to radiation) ~ (energy of particles at Infinity) – (energy of particles at ISCO) = $mc^2 - E = mc^2\,(1\text{-}k)$
- Radiative efficiency = (radiative energy loss) / $mc^2 = 1 - k$

RADIATIVE EFFICIENCY OF ACCRETION DISKS

Radiative efficiency ε



(r in units of GM/c²)

- ***Max radiative efficiency ε ~ 5.7% at ISCO for Schwarzschild BHs***
- Efficiency ε ~ 0.7% for nuclear fusion
- Accretion disks are powerful energy sources in the universe!!

HOW FAST DO THE DISKS ROTATE AT ISCO?

- From the orbital equations, one could derive the orbital velocity of particles:

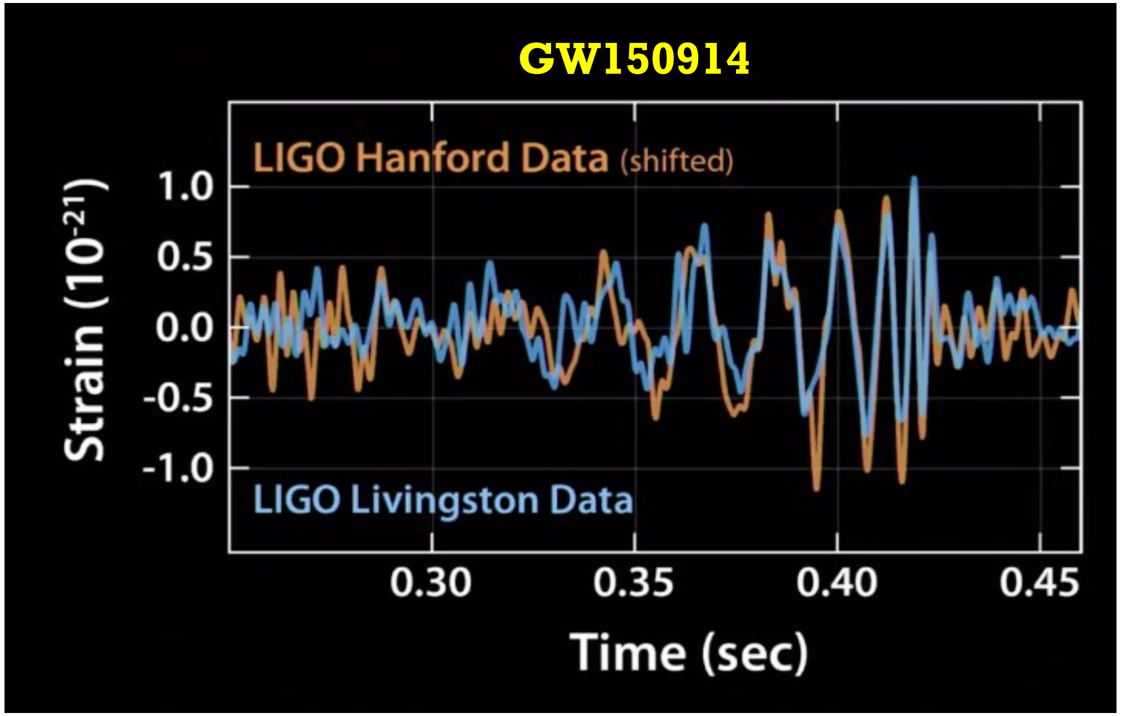
$$v^2 = \frac{GM}{r - \frac{2GM}{c^2}}$$

- Therefore, at R_{ISCO} = 6 GM/c^2 => ***v = 0.5c!!***
- In terms of orbital frequency, one finds that

$$\nu_{orb} = \frac{1}{2\pi}\sqrt{\frac{GM}{r^3}}$$

- Therefore, at R_{ISCO}, orbital frequency is proportional to 1/M
  - ***~ 218 Hz for a 10 M_{sun} stellar-mass BH***
  - ***~2.18 mHz for a 10^6 M_{sun} SMBH***

WHAT IS THE SIGNIFICANCE OF THE ORBITAL FREQUENCY?

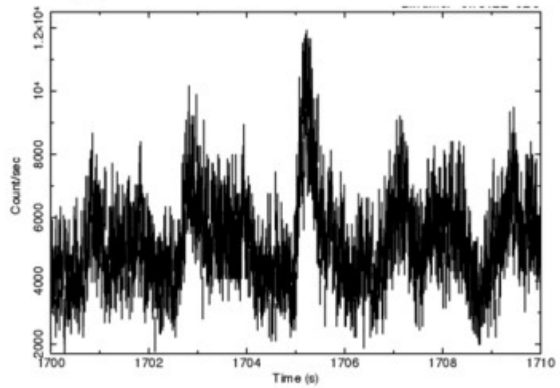


- GW150914: 35-250 Hz
- This event is produced by merger of stellar-mass BHs with ~30 M_{sun}
- The frequency of GWs emitted by BH binary mergers is directly proportional to the orbital frequency
- LIGO observatory is sensitive to the frequency range 30Hz-7kHz => range relevant for stellar-mass BHs

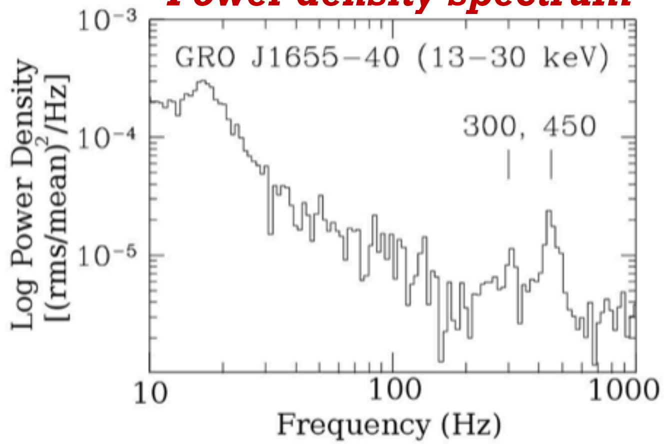
WHAT IS THE SIGNIFICANCE OF THE ORBITAL FREQUENCY?

- It might be related to "***quasi-periodic oscillations***" ***(QPO's/似週期振盪)*** seen in X-ray binaries and AGNs
- Recall that X-ray binaries are accreting stellar-mass BHs or NSs, where the X-ray comes from hot accretion disks
- Their X-ray emission is often variable, oscillating with certain characteristic frequencies
- From their power density spectrum (FFT of the light curve), some X-ray binaries show QPO's at high frequencies, close to the orbital frequency at ISCO
- It allows us to probe properties of accretion disks close to the BHs!

Light curve of an X-ray binary



Power density spectrum



HOW PHOTONS ORBIT AROUND SCHWARZSCHILD BLACK HOLES

SPACETIME CURVATURE TELLS PHOTONS HOW TO MOVE

- ***Photons*** move on ***null geodesics*** of the metric: $ds^2 = c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu = 0$
- Recall that the total interval between A and B is:

$$s_{AB} = \int_A^B ds = \int_A^B \left[g_{\mu\nu} dx^\mu dx^\nu\right]^{\frac{1}{2}} = \int_A^B \left[g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}\right]^{\frac{1}{2}} ds$$

$$= \int_A^B G(x^\mu, \dot{x}^\mu)\, ds \quad where \quad G(x^\mu, \dot{x}^\mu) = \left[g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu\right]^{\frac{1}{2}}$$

- To find the orbital equations for photons, we could use the same Euler-Lagrange equations and the fact that

$$G(x^\mu, \dot{x}^\mu) = 0$$

PHOTON ORBITS IN SCHWARZSCHILD METRIC

- Similar to the orbital equations for particles, the equations for photon orbits in the Schwarzschild metric are:

$$\left(\frac{dr}{d\tau}\right)^2 = c^2k^2 - \frac{h^2}{r^2}\left(1 - \frac{2GM}{c^2r}\right)$$

where again $k=E/mc^2$ measures the particle energy, and $h=l/m$ represents the angular momentum

- Let u = 1/r, the above equation can be written as

$$\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2}u^2$$

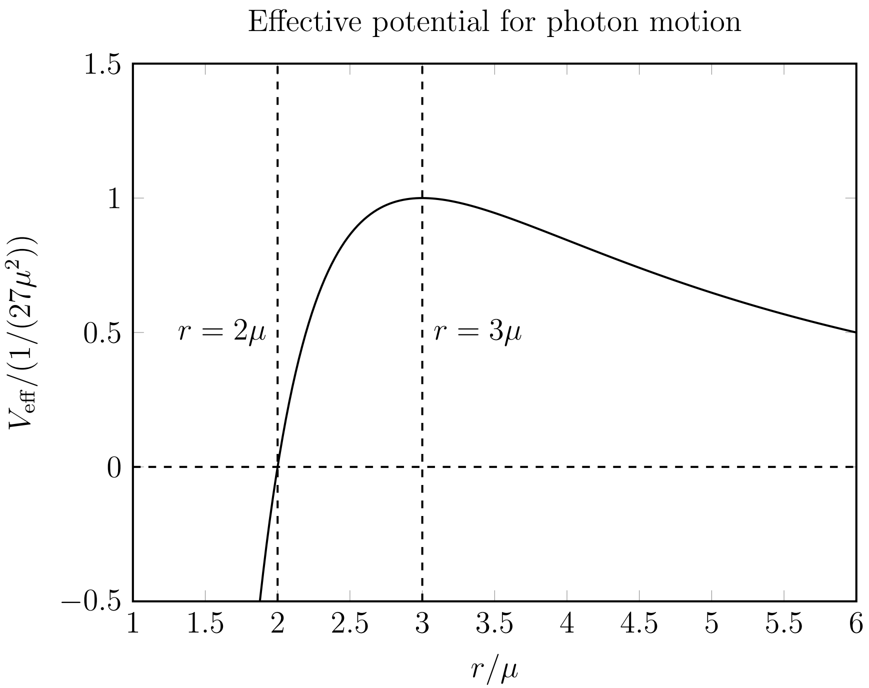
CIRCULAR ORBITS OF PHOTONS

$$\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2}u^2$$

- For circular orbits, du = dr = 0 => $r = \frac{3GM}{c^2} = 1.5\, r_S$

- The spacetime around BHs is so curved that even photons could travel in circular orbits!
- This is called the "***photon sphere***"
- Imagine a person shining flash light backward along the photon sphere, the photon would orbit around the BH and return to his front, such that this person could see the back of his head!

THIS PHOTON ORBIT IS UNSTABLE THOUGH



($\mu = GM/c^2$)

- We could again analyze the stability of photon orbits using the effective potential:

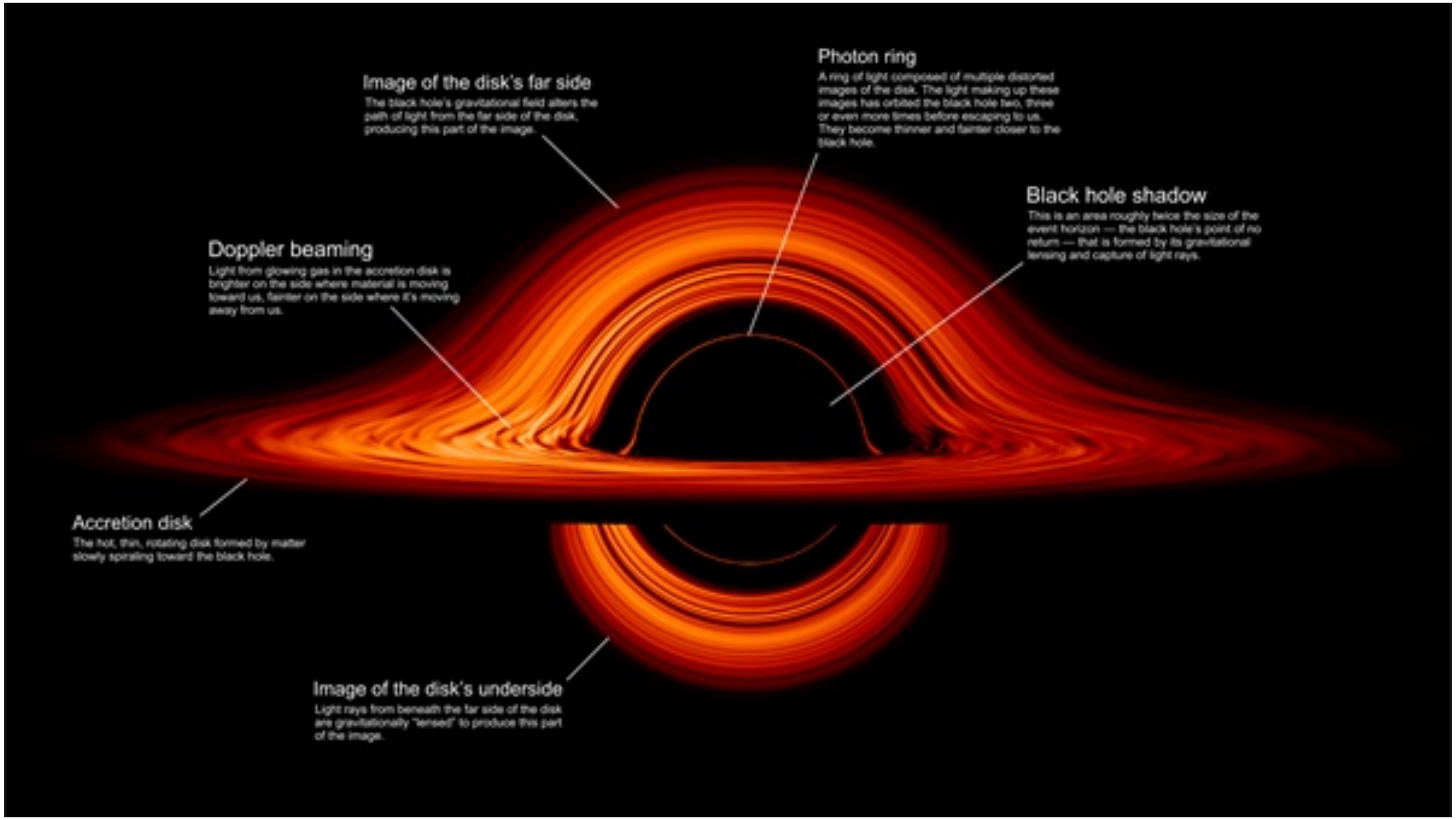
$$V_{\text{eff}}(r) = \frac{1}{r^2}\left(1 - \frac{2\mu}{r}\right)$$

- There is ***no stable orbit for photons***
- For photons in the photon sphere, given a perturbation, they would either fall into the BH or escape to infinity
- The escaped photons from the photon sphere would be seen as a ring from a distant observer – this is the "***photon ring***"

REMEMBER THE INTERSTELLAR BLACK HOLE?

[redacted]

DECIPHER THE INTERSTELLAR BLACK HOLE



Video link: https://www.youtube.com/watch?v=o-Psuz7u5OI

SOME NEWS RELATED TO PHOTON RINGS

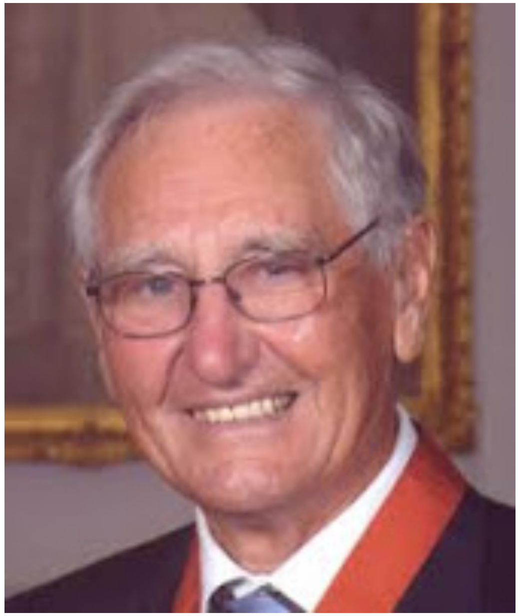
- There Are Infinite Rings of Light Around Black Holes. Here's How We Could See Them

- All black holes should sport light rings

PROPERTIES OF KERR BLACK HOLES

KERR/SPINNING BLACK HOLES

- In 1963, Roy Kerr found the exact solution of a charge-less, spinning black hole
- This is the solution for all astrophysical BHs!
- ***BH spin parameter a = J/M^2 , where 0 <= |a| <=1***



Roy Kerr (1934-)

METRIC OF KERR BLACK HOLES

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi - g_{\phi\phi}d\phi^2 - \Sigma\Delta^{-1}dr^2 - \Sigma d\theta^2$$

$$g_{tt} = \left(c^2 - 2GMr\Sigma^{-1}\right)$$

$$g_{t\phi} = 2GMac^{-2}\Sigma^{-1}r\sin^2\theta$$

$$g_{\phi\phi} = \left[\left(r^2 + a^2c^{-2}\right)^2 - a^2c^{-2}\Delta\sin^2\theta\right]\Sigma^{-1}\sin^2\theta$$

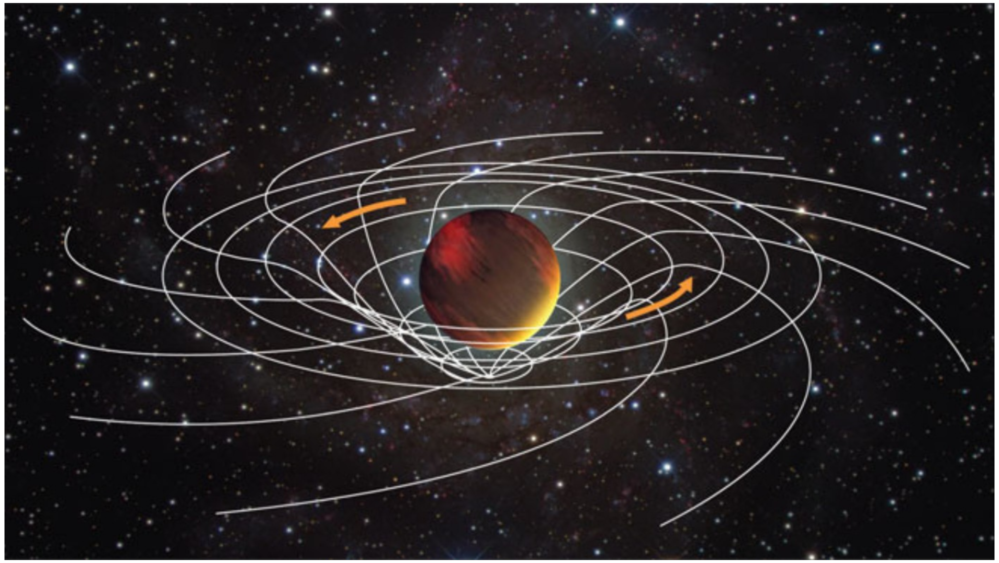
$$\Sigma \equiv r^2 + a^2c^{-2}\cos^2\theta$$

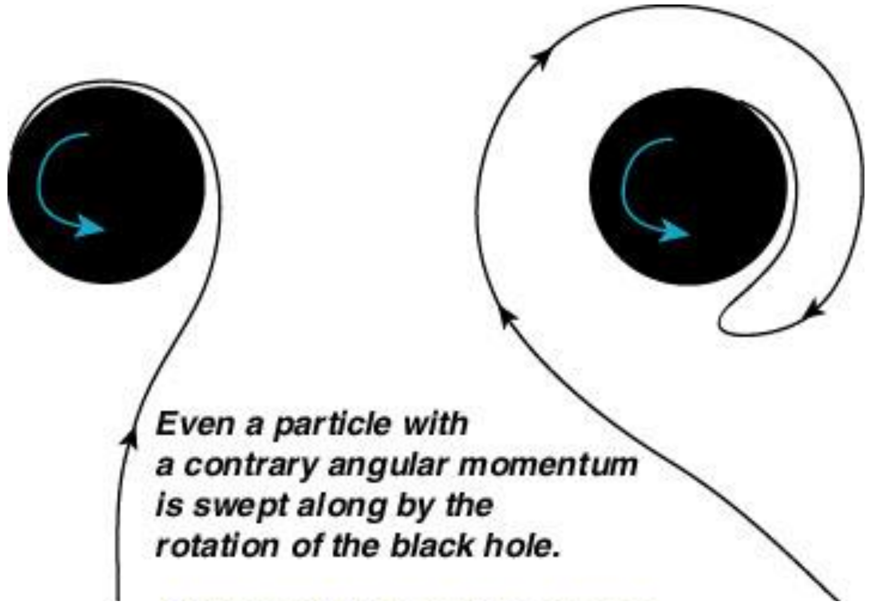
$$\Delta \equiv r^2 - 2GMc^{-2}r + a^2c^{-2}.$$

*In Boyer-Lindquist coordinate

**a = J/M in this particular expression

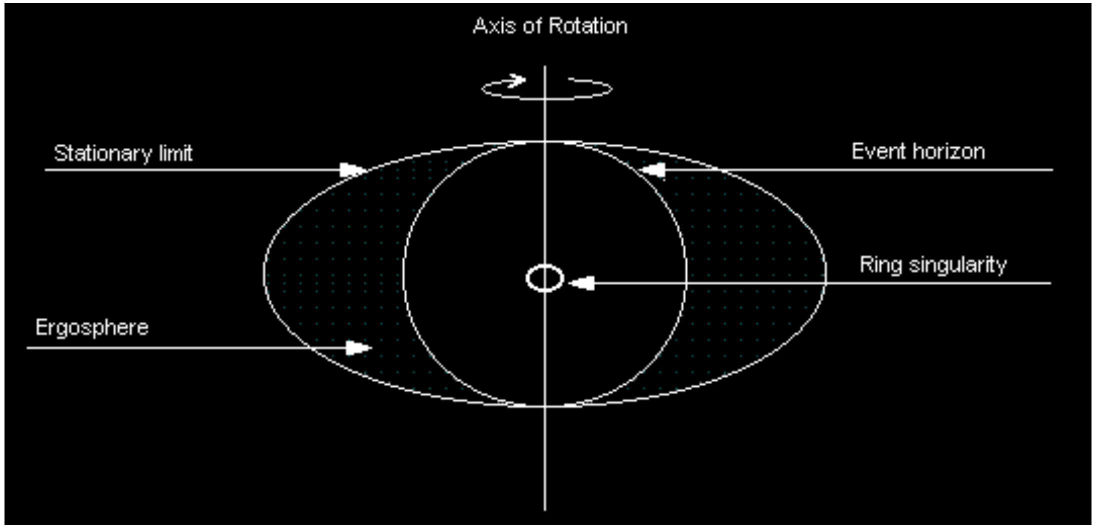
EFFECT OF FRAME DRAGGING





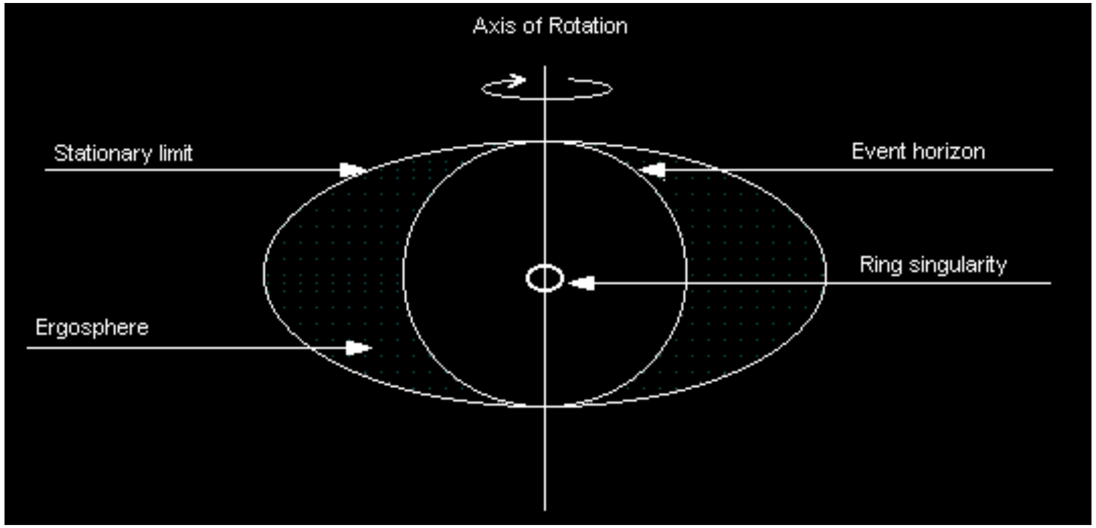
- ***Frame-dragging (Lense-Thirring) effect***: rotation of a massive object would distort spacetime, causing a change in the rotational axis of a nearby test particle

STRUCTURE OF A KERR BLACK HOLE



- ***Ring singularity***
  - Singularity is a ring on the equatorial plane, not a point
  - Radius of the ring = a * (GM/c^2)
- Unlike the unavoidable Schwarzschild singularity, geodesics exist that could avoid the ring singularity!
  - There exists possibilities to exit the BH to another universe via a wormhole, though the solution is highly unstable
  - It also appears possible to follow closed timelike curves around the ring singularity, such that one could return to his past!
  - Note that all the above is purely hypothetical

STRUCTURE OF A KERR BLACK HOLE

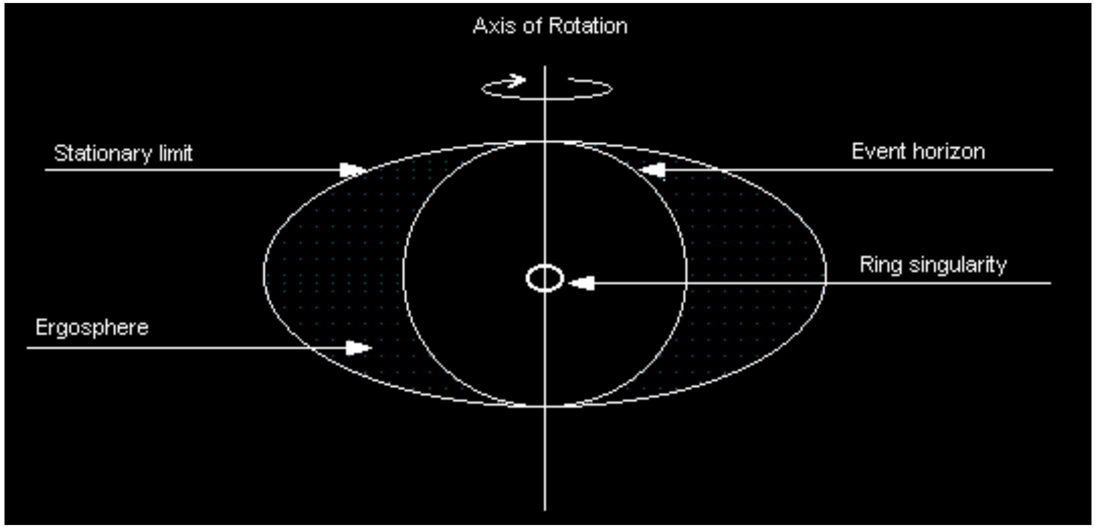


- ***Event horizon***
  - A sphere with radius depending on the spin parameter a

$$r = \frac{GM}{c^2}\left[1 + \sqrt{1 - a^2}\right]$$

  - For a = 0, $r = 2GM/c^2 = r_S$
  - For a = 1, $r = GM/c^2 = 0.5\ r_S$

STRUCTURE OF A KERR BLACK HOLE

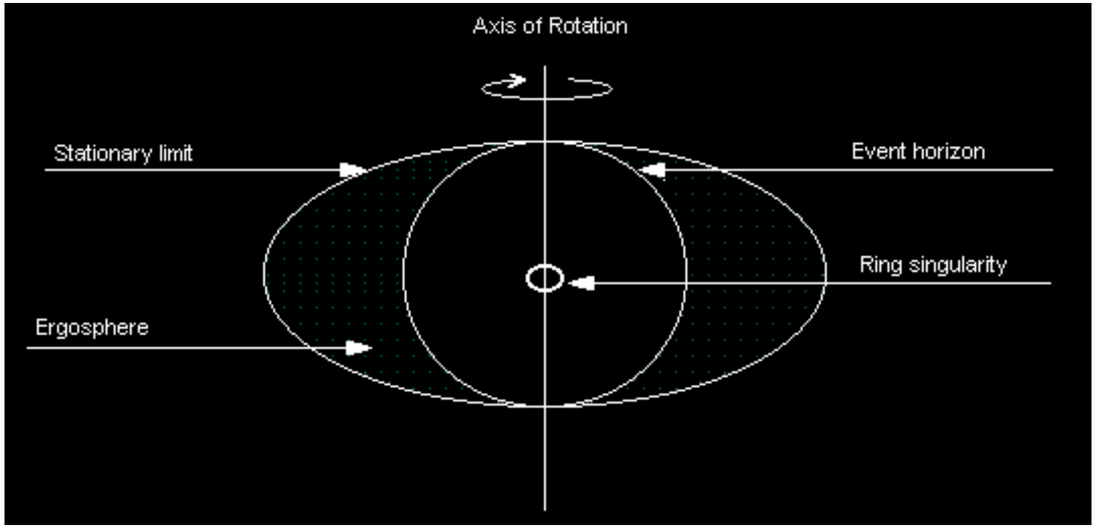


- ***Stationary limit surface***
  - The surface within which it is impossible to stand still due to the frame-dragging effect
  - This surface is an oblate ellipsoid with radius:

$$r = \frac{GM}{c^2}\left[1 + \sqrt{1 - a^2cos^2\theta}\right]$$

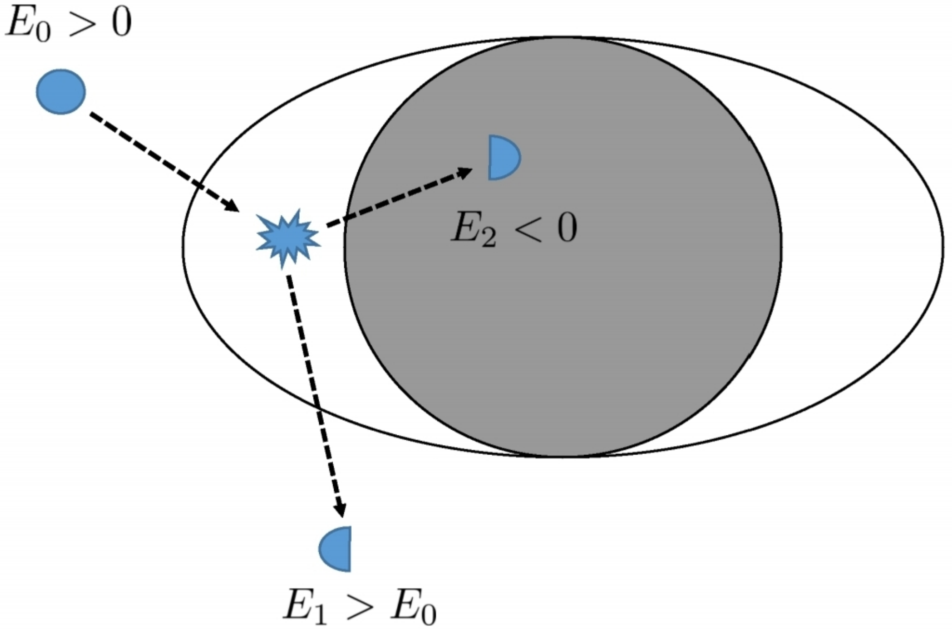
  - For a = 1, r = $2GM/c^2$ at equatorial plane and r = GM/c^2 at the pole

STRUCTURE OF A KERR BLACK HOLE



- ***Ergosphere***
  - The region between the event horizon and the stationary limit surface
  - It is impossible to stand still, but still possible to escape
  - In principle, one could extract energy of the BH from this region
  - This is why it is called ergosphere, where "ergo" means "work" in Greek

PENROSE PROCESS



- In 1971, Roger Penrose proposed a mechanism to ***extract BH rotational energy from the ergosphere***
- An incoming particle decays into two within the ergosphere; one goes into the event horizon while the other escapes
- With proper arrangement, it is possible to make the trapped particle to have negative energy
- By energy conservation, the ongoing particle would gain energy compared to the incoming particle
- ***This is one of the proposed mechanisms to launch BH jets!***

SUMMARY

- No hair theorem: BHs can be uniquely described by M, a, and Q
  - Astrophysical BHs can be described by M and a (Q=0)
  - Properties of ***Schwarzschild BHs*** (a=0) and ***Kerr BHs*** (a/=0) predicted by GR

- Meaning of event horizon from different perspectives
  - For a distant observer, objects approaching the event horizon would appear frozen and dimmer due to ***gravitational time dilation*** and ***gravitational redshift***
  - An infalling observer would pass the event horizon, being spaghettified by strong tidal forces, and eventually fall into the singularity

SUMMARY

- Orbits for particles around BHs:
  - ***ISCO = innermost stable circular orbit***, R_{ISCO} = (1~9) GM/c^2
  - ***Maximum radiative efficiency of accretion disks***, ε ~ 5-40%

- Orbits for photons around BHs:
  - For Schwarzschild BHs, photons could have circular orbits at $r=3GM/c^2$ -> "***photon sphere***"
  - There is ***no stable orbits for photons*** (for both spinning and non-spinning BHs)