Homework 2

Introduction to Black Hole Astrophysics (PHYS480)

(Due at the start of class on March 23, 2021)

Exercise 1

[Video summary of general relativity (0.5 pt)] Watch a 15-minute video summarizing key concepts of Einstein's theory of general relativity: `https://youtu.be/tzQC3uYL67U`. Write down one thing you learned from this video, or one question you have about general relativity.

Exercise 2

[Effects of relativity on GPS (1 pt)] Through this exercise you will learn how the theory of relativity is not just an abstract theory but can have significant consequences in our daily lives! The Global Positioning System (GPS) nowadays could identify your position on Earth to an accuracy of 5-10 meters. The technology relies on 30 or so satellites at an altitude of about 20,000 km from the ground, and has an orbital speed of about 14,000 km/hour. Each satellite carries an atomic clock that ticks with an accuracy of 1 nanosecond. A GPS receiver determines its current position by comparing the time signals it receives from the currently visible satellites and trilaterating on the known positions of each satellite. To achieve such a high precision, the clocks must be known to an accuracy of 20-30 nanoseconds, and the effects of Special Relativity and General Relativity must be taken into account.

(1) According to Special Relativity, clocks on the satellites would run faster or slower for an observer on the ground due to their orbital speeds? By how much per day?
(2) According to gravitational time dilation predicted by General Relativity, should the clocks on the satellites tick faster or slower compared to the ground due to Earth's gravity? By how much per day? You will need the full expression for gravitational time dilation (without assuming constant gravitational acceleration):

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{2GM}{c^2 r}}}$$

(3) What is the combined effect from both Special Relativity and General Relativity, i.e., how much do the clocks on the satellites run faster or slower than clocks on Earth? Since the GPS requires a 20-30 nanosecond accuracy, do you think the relativistic corrections are important for GPS navigation in our daily lives?

Exercise 3

[Spaghettification (1 pt)] Consider an astronaut with height $h = 180$ cm approaching a black hole. The tidal force due to the difference of gravitational pull on his head and toes can be expressed as

$$F_{tidal} = \frac{2GMmh}{r^3},$$

where M is the mass of the black hole, m is the mass of the astronaut, and r is the distance between them.

(1) Assuming the astronaut is approaching a stellar-mass black hole with $M = 10\ M_{\odot}$, how strong the acceleration due to tidal force (i.e., $a \equiv F_{tidal}/m$) would be when he gets to the Schwarzschild radius of the black hole?
(2) A human body can sustain a maximum acceleration of $9g$, where $g = 980$ cm/s^2 is the gravitational acceleration on Earth's surface. Do you think the astronaut could possibly survive at the Schwarzschild radius of the stellar-mass black hole or would he be spaghettified? How close to the Schwarzschild radius could he get when the tidal acceleration is $9g$?
(3) Now consider a supermassive black hole (SMBH) of $M = 10^6\ M_{\odot}$ instead. What is the acceleration due to tidal forces at its Schwarzschild radius?
(4) Same as (2), comment on whether the astronaut could safely pass the event horizon of this SMBH, and estimate how close he could approach the SMBH in units of the Schwarzschild radius.

Exercise 4

[Photon orbits around Schwarzschild black holes (2 pt)] In this exercise, we will derive the orbital equations for photons around a Schwarzschild black hole and analyze the stability of photon orbits.

(1) Recall that the Schwarzschild metric is

$$ds^2 = c^2 d\tau^2 = Ac^2 dt^2 - Bdr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) = g_{\mu\nu} dx^\mu dx^\nu,$$

where $x^\mu = [ct, x, y, z]$ is the position-time 4-vector, and $A = B^{-1} = 1 - 2GM/(c^2 r)$. Using the definition $G(x^\mu, \dot{x}^\mu) \equiv [g_{\mu\nu} d\dot{x}^\mu d\dot{x}^\nu]^{1/2}$ (where $\dot{x}^\mu \equiv dx^\mu/ds$) and the fact that $G(x^\mu, \dot{x}^\mu) = 0$ for photons together with the Euler-Lagrange equations

$$\frac{d}{ds}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right) - \frac{\partial G}{\partial x^\mu} = 0,$$

derive the orbital equation for photons:

$$\left(\frac{dr}{d\tau}\right)^2 = k^2 c^2 - \frac{h^2}{c^2}\left(1 - \frac{2GM}{c^2 r}\right), \tag{4.1}$$

where $A\dot{t} \equiv k/c$ and $r^2\dot{\phi} \equiv h/c$.

(2) Let $u \equiv 1/r$, derive the following form of the orbital equation for photons:

$$\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2}u^2. \tag{4.2}$$

(3) By setting $du = dr = 0$ in Eq. 4.2, we could see that circular orbits could exist at $r = 3GM/c^2$ for photons. Using the first and second derivaties of the effective potential,

$$V_{eff}(r) \equiv \frac{1}{r^2}\left(1 - \frac{2GM}{c^2 r}\right),$$

show that the circular orbit is an unstable solution.

Exercise 5

[Visualization of effects around the black hole (0.5 pt)] Watch a 6-minute video visualization the effects you would see if you get close to a black hole, including the Einstein ring, gravitational blueshift/redshift effects, ISCO, photon sphere, and accretion disks. The link can be found here: `https://youtu.be/JDNZBT_GeqU`. Write down one thing you learned or find fascinating from this video, or one question you have about effects mentioned in this video.