2025 Homework #2

Introduction to Quantum Theory of Black Holes and Holography (I)
(National Tsing Hua University Course)

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Due date: Nov 4, 2025
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Background

In class, we derived the Penrose diagram for the Minkowski spacetime and Schwarzschild black holes in asymptotically flat spacetime, as well as the Kruskal coordinates for black holes.

Keep this discussion in mind as you work on the following problems.

Main Task (Required)

In this homework, we will analyze the global structure of Anti-de Sitter (AdS) spacetime using its metric in two coordinate systems, **global coordinates** and **Poincaré coordinates**.

- Global coordinates

$$ds^2 = -\left(\frac{r^2}{L^2} + 1\right) dt^2 + \frac{dr^2}{\left(\frac{r^2}{L^2} + 1\right)} + r^2 d\Omega_{D-2}^2 \tag{1}$$

- Poincaré coordinates

$$\frac{ds^2}{L^2} = \frac{1}{z^2}\left(-dt^2 + \sum_{i=1}^{D-2}(dx^i)^2 + dz^2\right) \tag{2}$$

You can check that both solutions satisfy the following vacuum Einstein equation;

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0 \tag{3}$$

where the cosmological constant is given (as we did in class) by

$$\Lambda = -\frac{(D-1)(D-2)}{2L^2}\,. \tag{4}$$

Note that in class we already derived the Schwarzschild AdS (Anti-de Sitter) black hole solution. Setting its mass parameter to zero reduces the metric to (1).

1. **Penrose diagram of global AdS**

   (a) As a warm up, from the global metric (1), perform the following coordinate change

   $$r = L \sinh\rho\,, \quad \tilde{t} = \frac{t}{L} \tag{5}$$

   and write down the metric in terms of ρ and $\tilde{t}$. Here $\tilde{t}$ is the rescaled time.

   (b) Next, let us compactify the coordinate r by introducing a new coordinate χ. Propose a coordinate transformation

   $$r = r(\chi) \tag{6}$$

   such that finite range of χ can cover all the range of r. (Hint: we already did very similar coordinate transformation for the Minkowski spacetime in class.) Then what metric one can obtain? Again write down the metric in terms of $\tilde{t}$ and χ. Also specify the parameter range for the new coordinate χ.

   (c) Where is the boundary? Is the boundary timelike? or null? or spacelike?

   (d) Draw the Penrose diagram of global AdS.

   (e) Is there a horizon in global coordinates? If so, where is it and how do you see that?

2. **Poincaré 'patch' compared with global 'patch'**

In fact, AdS_D is defined as a hyperboloid embedded in a flat $(D+1)$-dimensional space with signature $(2, D-1)$:

Definition of AdS_D in Lorentzian Signature

$$ds^2 = -(dX^0)^2 - (dX^D)^2 + \sum_{i=1}^{D-1}(dX^i)^2, \tag{7}$$

$$(X^0)^2 + (X^D)^2 - \sum_{i=1}^{D-1}(X^i)^2 = L^2 \tag{8}$$

(a) **(Verification 1)**

- Write down the relationship between X^0, X^i, X^D with $\tilde{t}$, ρ, Ω^i.
- Check that this relationship is consistent with the hyperboloid constraint eq. (8).
- With that, show eq. (7) leads to eq. (1).

(b) **(Verification 2)** Define Poincaré coordinates (t, x^i, z) (or equivalently $U = 1/z$) by

$$\begin{aligned} X^0 &= L\,\frac{t}{z}, \qquad X^i = L\,\frac{x^i}{z}, \quad (i = 1, \ldots, D-2), \\ X^{D-1} &= \frac{L}{2z}\left(-t^2 + \sum_{i=1}^{D-2}(x^i)^2 + z^2 - 1\right), \\ X^D &= \frac{L}{2z}\left(-t^2 + \sum_{i=1}^{D-2}(x^i)^2 + z^2 + 1\right). \end{aligned} \tag{9}$$

- Show that this satisfies the hyperboloid condition (8).
- With that, show eq. (7) leads to eq. (2).

(c) For $z > 0$, show that $X^D - X^{D-1}$ takes definite sign.
Similarly for $z < 0$, show that $X^D - X^{D-1}$ takes definite sign.
Hence, the AdS hyperboloid is divided symmetrically into two regions by the null surface $X^D - X^{D-1} = 0$, and the Poincaré patch corresponds to only one of them ($z > 0$ side). Thus, we can interpret the limit $z \to 0^+$ as the boundary.

(d) Is there a horizon in Poincaré coordinates? If so, where is it and how do you see that?