Quantum Mechanics (I): Homework 3
Due: November 5

Ex.1 (a) 10 Show that if $\hat{A}$ has no explicit time dependence and $[\hat{H}, \hat{A}] = 0$, then $< \hat{A}^n, >, n = 1, 2, \cdots$ are time-independent constants. Here $\hat{H}$ is the Hamiltonian.

(b) 10 Suppose $< \hat{A}^n, >$ in (a) are known and the eigenvalues of $\hat{A}$ are real and continuous (denoted by $a$). Devise a method to calculate the probability density of finding $a$.

Ex.2 As mentioned in the class, the WKB approximation is not valid near the turning point $x_0$ defined by $E = V(x_0)$. If we expand $V(x)$ around $x_0$ and keep only to the linear term, we may write $V(x) = E + A(x - x_0)$ where $A = V'(x_0)$.

Let us consider the turning point where $A > 0$.

(a) 5 Change the variable to $y = (2mA/\hbar^2)^{1/3}(x - x_0)$, find the differential equation that $\psi(y)$ satisfies.

(b) 10 The solution to the above equation is the Airy function $Ai(y)$. The asymptotic form of large negative $y$ for the Airy function is $Ai(y) \sim 1/\sqrt{\hbar|y|^{1/4}} \sin \left(\frac{y}{6} \right)$. Show that this implies that for $x < x_0$

$$\psi(x) \sim \frac{1}{\sqrt{p(x)}} \cos \left(\frac{1}{\hbar} \int_{x_0}^x p(x') dx' + \frac{\pi}{4}\right),$$

Ex.3 Consider a two-state system characterized by the Hamiltonian

$$\hat{H} = \Delta \begin{pmatrix} 2(1) & 2(1) \end{pmatrix} - \begin{pmatrix} 2(1) & 2(1) \end{pmatrix},$$

where $\Delta$ is a real number with the dimension of energy. $|1>$ and $|2>$ are normalized and are also orthogonal to each other.

(a) 5 If one measures the energy of this system, what possible values one would get?

(b) 10 Suppose the ket of the system is $|\psi(t)\rangle = \alpha(t)|1\rangle + \beta(t)|2\rangle$, what are the differential equations that $\alpha(t)$ and $\beta(t)$ obey?

(c) 10 If at $t = 0$, $|\psi(0)\rangle = |1\rangle$, what is the probability of finding the system at $|2\rangle$ at $t > 0$?

Ex.4 Quark potential Consider a wave function $\psi_E(x)$ that satisfies the one dimensional time-independent Schrödinger equation $\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)\right) \psi(x) = E \psi(x)$, where $V(x)$ is a linear potential (which is known reasonably good for describing quarks near each other)

$$V(x) = Cx$$

with $C > 0$, for $x > 0$.

(a) 10 Let the Fourier transformation of $\psi_E(x)$ be $\Phi_E(p)$. Show that

$$\Phi_E(p) = A \exp \left[\frac{i}{\hbar} \left(\frac{p^2}{6m} - Ep\right)\right]$$

where $A$ is a integration constant.

(b) 10 Using the normalization condition for $\Phi_E(p)$: $\int_{-\infty}^{\infty} \Phi_x^*(p) \Phi_E(p) dp = \delta(E - E')$, find $A$ (assuming $A$ is real).

(c) 10 The boundary condition for $\psi_E(x)$ is $\psi_E(0) = 0$, find the equation that the energy eigenvalue $E$ satisfies.

(d) 10 From this equation, one can solve for the discrete energy spectrum. The ground state energy is $E_0 = 2.338 \left(\frac{c^2 \hbar^2}{2m}\right)^{1/3}$. Use WKB method to get approximated ground state energy and compare it with this value.

Ex 5 10 The evolution operator $\hat{U}(t, t')$ for a system described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$$

given by $\hat{U}(t, t') = \exp[-i/\hbar \hat{H}(t - t')]$. If we define $G^+(x; x'; t')$ to be $\langle x | \hat{U}(t, t') | x' \rangle \cdot \theta(t - t')$, show that $G$ satisfies the differential equation:

$$i\hbar \frac{\partial}{\partial t} G^+ - \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] G^+ = i\hbar \delta(x - x') \delta(t - t'),$$

in other words, $G^+$ is a Green's function of the Schrödinger equation. Note that without imposing appropriate initial condition, the above equation also admits solutions that do not vanish when $t < t'$. For this reason, this equation has to be supplemented with the condition $G^+(x; x'; t') = 0$ for $t < t'$.
Ex 6 (a) 10 Consider a particle of mass $m$ incident on a one-dimensional step function potential $V(x) = V\theta(x)$ with energy $E > V$. Calculate the reflection and transmission coefficients when the particle is incident from $x = -\infty$.

(b) 10 Find the transfer matrix in problem (a). Using the formalism of transfer matrix to calculate the transmission coefficient for a particle of mass $m$ incident on the following potential with $E > 0$:

\[
V(x) = 0, \text{ for } |x| > a \\
V(x) = -V, \text{ for } -a \leq x \leq a.
\]

(c) Bonus (+1) Following Ex. 14 in Homework 1, consider a particle that is confined to move on the one-dimensional lattice: $x = na$ with $n$ be integers and $a$ being a positive constant. If the wavefunction at site $x = na$ is denoted by $\Psi_n$, for the step potential on the lattice, $\Psi_n$ of fixed energy $E$ satisfies

\[-t\Psi_{n-1} - t\Psi_{n+1} + V_n \Psi_n = E\Psi_n,
\]

where $t$ is a positive constant, $V_n = 0$ for $n < 0$ and $V_n = V > 2t$ for $n \geq 0$. Find the reflection and transmission coefficients when the particle is incident from $n = -\infty$ with energy $E > V - 2t$.

Ex 7 (a) 15 Construct the S-matrix for the potential $V(x) = -A\delta(x)$ . From the S-matrix, find the energies for the bound states.

(b) Bonus (+1) Following (a), let us denote the bound state by $\phi_b(x)$ and the scattering states by

\[
\phi_{k,R}(x) \to e^{-ikx} + R e^{ikx} \text{ for } x \to \infty \\
Te^{-ikx} \text{ for } x \to -\infty \\
\phi_{k,L}(x) \to e^{ikx} + R e^{-ikx} \text{ for } x \to -\infty \\
Te^{ikx} \text{ for } x \to \infty.
\]

Show that the following completeness relation is satisfied.

\[
\sum_b \phi_b(x)\phi_b^*(x') + \frac{1}{2\pi} \sum_s \int_0^\infty dk \phi_{k,s}(x)\phi_{k,s}(x') = \delta(x-x').
\]

Ex. 8 10 Consider a particle of mass $m$ in an asymmetric potential well given by $V(x) = V_2$ if $x < 0$, $V(x) = 0$ if $0 < x < a$, and $V(x) = V_1$ if $x > a$. Find the condition when there is no bound state.

Ex. 9

(a) 10 Consider a particle of mass $m$ incident with energy $E$ from $x = -\infty$ upon a potential barrier $V(x)$ shown in FIG. 1. The asymptotic behaviors of $V(x)$ are: $V(x) \to 0$ when $x \ll a$ and $x \gg b$. Find the transmission coefficient by using the WKB approximation.

(b) 10 Splitting of energy due to the potential barrier

Consider a particle of mass $m$ in a potential $U(x)$ shown in FIG. 2. $U(x)$ consists of two symmetrical potential wells. Let $\psi_0(x)$ being the energy eigenstate of one of the energies $E_0 < U_0$ to the right potential well with the normalization $\int_0^\infty \psi_0^2(x) = 1$. Due to the possibility of tunneling, the particle will also occupy the left potential well.

As a result, the energy is splitted into $E_1$ and $E_2$ as shown in FIG. 2. Show that $E_2 - E_1 = (2\hbar^2/m)\psi_0(0)\psi_0(0)$. By using the WKB approximation with connection formula, express $E_2 - E_1$ in terms of an appropriate integral of $|p(x)| = \sqrt{2m(U(x) - E_0)}$ and the frequency $\omega$ of the classical motion of the energy $E_0$.

Ex. 10 Suppose that the wavefunction of a particle at $t = 0$ is given by

\[
\psi(x,0) = (\pi\Delta^2)^{-1/4} \exp(-x^2/2\Delta^2).
\]

The particle is free for $t \geq 0$.

(a) 7 Find the position operator in the Heisenberg picture $\hat{x}_H(t)$ ( for $t > 0$) in terms of the operators $\hat{x}$ and $\hat{p}$ defined in the Schrodinger’s picture. Calculate the commutator $[\hat{x}_H(t),\hat{x}_H(t')]$.

(b) 8 Using results from (a), calculate $\Delta x(t)$ for $t > 0$.

Ex. 11 Bonus problem (+1) Combining S-matrices coherently

(a) Consider two one-dimension local potentials described by the scattering matrices

\[
S_1 = \begin{pmatrix} r_1 & t_1' \\ t_1 & r_1' \end{pmatrix}, \quad S_2 = \begin{pmatrix} r_2 & t_2' \\ t_2 & r_2' \end{pmatrix}.
\]

Show that the total transmission coefficient is

\[
T = \frac{T_1T_2}{1 - 2\sqrt{T_1R_2 \cos \theta + R_1R_2}},
\]
where \( T_i = |t_i|^2 \), \( R_i = |r_i|^2 \), and \( \theta \) is the phase shift acquired in one round-trip between two potentials.

(b) Apply the above formulation to find the total transmission coefficient as function as the total incident energy \( E \) for the potential

\[
V(x) = U_0(\delta(x) + \delta(x - d)).
\]

Sketch the plot the total transmission coefficient versus energy \( E \) and show how to find the resonant energy when \( T = 1 \).

(c) Argue that if instead, we combine the probabilities directly for two potentials, we would get the so-called incoherent total transmission coefficient

\[
T = \frac{T_1 T_2}{1 - R_1 R_2},
\]

in this case the variable \( \theta \) disappears.