Ex. 1 Consider the setup of the AB effect as shown in Fig. 1. Instead of setting up a solenoid behind the slits, we now have a strip of magnetic field behind the slits. If $L$ is very larger than $d$, we may approximate any two paths that come out from different slits as parallel lines.

(a) Find the shift $\Delta x$ of the diffraction pattern.

(b) The shift $\Delta x$ can be viewed as deflection of all trajectories by the small angle $\alpha = \Delta x/L$. Show that by assuming all incident particles have momentum parallel to $y$, $\alpha$ can be also understood in terms of the classical Lorentz force $F = qv \times B/c$.

Ex. 2 Consider a two-state system characterized by the Hamiltonian

$$H = \Delta [2|1><1| - |2><2| + 2(|1><2| + |2><1|)],$$

where $\Delta$ is a real number with the dimension of energy. $|1>$ and $|2>$ are normalized and are also orthogonal to each other. Consider a collection of such two-state systems. Suppose at $t = 0$, the percentage of systems in $|1\rangle$ is $p_0$, the remaining systems are all in the state of $|2\rangle$. Find the density matrix $\rho(t)$ for $t > 0$.

Ex. 3 Consider an operator $\hat{A} = a|1\rangle\langle 1| + b|1\rangle\langle 2| + c|2\rangle\langle 1| + d|2\rangle\langle 2|$. Here $\langle 1|1\rangle = \langle 2|2\rangle = 1$ and $\langle 1|2\rangle = \chi$. Show that the eigenvalues $\lambda$ of $\hat{A}$ satisfy

$$\left| a + b\chi^* + c\chi + d|\chi|^2 - \lambda \sqrt{1 - |\chi|^2 (b + d\chi)} \right| \left| \frac{\sqrt{1 - |\chi|^2 (b + d\chi)}}{d(1 - |\chi|^2) - \lambda} \right| = 0.$$

Ex. 4 Consider the $n$th eigenstate of a harmonic oscillator, what is $\Delta x \Delta p$?

Ex. 5 By re-expressing the creation operator $a^\dagger$ in terms of $p$ and $\frac{\partial}{\partial p}$, show that the Fourier transformation of the $n$th eigenstate of a harmonic oscillator is

$$\langle p|n\rangle = (-i)^n \left[ \frac{1}{\pi \hbar \omega 2^n (n!)^2} \right]^{1/4} \exp(-p^2/2\hbar \omega) H_n \left( \frac{p}{\sqrt{\hbar \omega}} \right).$$

Ex. 6 Charged Harmonic Oscillator

(a) Consider a charged 1D harmonic oscillator (mass = $m$, charge = $q$, frequency = $\omega$) placed in a constant electric field $E = -\frac{d\phi}{dx}$. If we set $\phi(0) = 0$, what is the Hamiltonian?

(b) Rewrite the Hamiltonian in terms of the creation and annihilation operators (i.e., $a^\dagger$ and $a$).

(c) The Hamiltonian obtained in (b) is not ready to be solved. Can you redefine new creation and annihilation operators ($b^\dagger$ and $b$) in terms of $a^\dagger$ and $a$ so that the Hamiltonian is purely quadratic in $b^\dagger$ and $b$?

(d) Identify the zero-point energy from the results of (c).

(e) The physics behind the results of (c) is actually quite simple: a constant electric field only shifts the equilibrium position of a harmonic oscillator to a new position. What is the new equilibrium position?

Ex. 7 From the expression

$$H_n(x) = \exp(x^2/2) \left( x - \frac{d}{dx} \right)^n \exp(-x^2/2),$$

show that the Hermite polynomials can be also rewritten as
\[ H_n(x) = (-1)^n \exp(x^2) \left( \frac{d}{dx} \right)^n \exp(-x^2), \]
and the coefficient of the maximal power is \( 2^n \).

Ex. 8 (10) Page 212, ex. 7.4.3.
Ex. 9 (a) (10) Ex.8.6.2. (b) (10) Ex.8.6.3.
Ex. 10 The Landau Levels
When a particle of mass \( m \) and charge \( q \) is placed in a uniform magnetic field \( B \), its energy spectrum will change from continuous one to discrete spectrum. The resulting discrete spectrum is known as the Landau levels.

(a) If we align \( B \) along \( z \) direction, show that the vector potential \( A \) can be chosen to be
\[ A = \frac{B}{2} (-y\hat{x} + x\hat{y}). \]
This is known as the symmetry gauge. You can verify that other choices such as \( A = Bx\hat{y} \) (known as the Landau gauge) are also good.

(b) When we align \( B \) along \( z \) direction, the particle is free along \( z \) direction. Therefore, we can ignore the \( z \) direction. Let us thus consider the Schrödinger equation in two dimension (\( x \) and \( y \)). Using the symmetry gauge, show that the Hamiltonian can be rewritten as that of a simple harmonic oscillator
\[ H = \frac{p^2}{2m} + \frac{1}{2} m\omega_c^2 Q^2. \]
Here \( \omega_c = qB/mc \) is the cyclotron frequency, and \( Q = (c\hat{p}_x + qyB/2)/qB = cmv_x/qB \) and \( P = c\hat{p}_y - qBx/2c = mv_y \). \( P \) and \( Q \) satisfy the commutation relation \([Q,P] = i\hbar \) (i.e., \( v_x \) and \( v_y \) no longer commute in the presence of magnetic field). Therefore, the energy spectrum of the particle is the same as that of a simple harmonic oscillator. The resulting spectrum is known as the Landau Levels.

(c) Follow (b), find the normalized ground state wavefunction.

(d) Suppose now that in addition to the above magnetic field, one applies an electric field \( E \) in the \( x \) direction. By using the Landau gauge, find the energy spectrum.

Ex. 11 (a) The density matrix operator at temperature \( T \) is defined by
\[ \hat{\rho} = \frac{e^{-\hat{H}/k_BT}}{Z}, \]
where \( Z \) is the partition function and is equal to \( Tr(e^{-\hat{H}/k_BT}) \). To find the matrix element of \( \hat{\rho} \), one needs to find \( \rho(x',x) = \langle x'|e^{-\hat{H}/k_BT}|x \rangle \). Consider a special case in which \( \hat{H} = \hat{p}^2/2m + V(\hat{x}) \) describes a single particle in the potential \( V(x) = ax \), find the corresponding \( \rho(x',x) \).

(b) Neutron interferometry due to gravity

Using results of (a), consider now a monoenergetic beam of neutrons whose wavelength of matter wave is \( \lambda = 1.445 \text{Å} \). As shown in the above figure, the beam is incident at point \( A \) and is splitted into two parts that travel along two different paths \( ABD \) and \( ACD \) and finally meet at \( D \). Here \( ABCD \) is a rectangular with \( AB \) being 2.709cm and \( AC \) being 3.885cm. It is found that by changing the angle \( \phi \), the intensity of neutron at \( D \) changes as shown in the right figure. Deduce from these data the value of acceleration \( g \) due to gravity.

Ex. 12 (a) Suppose that the propagator of a particle in the presence of a vector potential \( \vec{A}(\vec{r}) \) is given by \( K(\vec{r},\vec{r}',t) \). How does \( K(\vec{r},\vec{r}',t) \) change if \( \vec{A}(\vec{r}) \) undergoes the gauge transformation, \( \vec{A}(\vec{r}) \rightarrow \vec{A}(\vec{r}) + \nabla \Lambda(\vec{r}) \) ?

(b) Suppose that the particle in (a) carries mass \( m \) and charge \( q \). If the particle now moves in the outside of a cylinder of radius \( R \). The magnetic field \( B \) is uniform inside the cylinder and vanishes outside the cylinder. Furthermore, the electron is blocked from moving into the cylinder. What are allowed magnitudes of \( B \)?
**Bonus (+1)** In this problem, I will guide you to find the propagator for the harmonic oscillator.

(a) Let us first re-analyze the free particle case. Show that one can rewrite
\[ \langle x_b t_b | x_a t_a \rangle = e^{\frac{i \bar{h} S_{cl}}{2 \Delta t}} \lim_{N \to \infty} \int dx_1 \int dx_2 \cdots \int dx_N \left( \frac{m}{2 \pi \bar{h} \Delta t} \right)^{(N+1)/2} \exp \left( -\psi^T M \psi \right) \]
\[ \equiv A(t_b - t_a) e^{\frac{i \bar{h} S_{cl}}{\Delta t}}, \]
where \( S_{cl} \) is the classical action that we found in class \([= im(x_b - x_a)^2/2\hbar(t_b - t_a)], \Delta t = (t_b - t_a)/(N + 1),\)

\[ \psi = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}, \]
and \( \psi^T \) is the transpose of \( \psi \). \( M \) is a \( N \times N \) matrix. Find \( M \) and show that for any general real symmetric matrix \( M \)
\[ \int dx_1 \int dx_2 \cdots \int dx_N \exp \left( -\psi^T M \psi \right) = \frac{\pi^{N/2}}{\sqrt{\det M}}, \]
(Hint: diagonalize \( M \) first.) Let \( I_N \equiv \det M \) when \( M \) is a \( N \times N \) matrix. By finding the recursion relation for \( I_N \),
calculate \( I_N \). From here, one concludes
\[ A(t_b - t_a) = \sqrt{\frac{m}{2 \pi \bar{h} (t_b - t_a)}}, \]

(b) Following the same procedure outlined in (a), show that for the harmonic oscillator
\[ A(t_b - t_a) = \sqrt{\frac{m\omega}{2 \pi \bar{h} \sin \omega(t_b - t_a)}}. \]
Here \( \omega \) is the natural frequency of the harmonic oscillator. Note that \( S_{cl} \) was already found in Hw.4 Ex.4 and \( \omega \to 0 \),
one recovers the result for free particles.
(c) Now, consider a particle described by \( \hat{H} = \hat{p}^2/2m - \hat{F} \hat{x} \). By rewriting the path integral in the momentum space,
find the evolution operator \( U(x, t; x', t') \equiv \langle x | \exp[\frac{\bar{h}}{i} \hat{H}(t - t')]|x' \rangle \).