Quantum Mechanics (II): Homework 4
Due: May 1, 2013

Ex. 1 20 Find all related Clebsch-Gordan coefficients for adding $j_1 = 1/2$ and $j_2 = 3/2$.

Ex. 2 10 Exercise 15.2.2.(2)

Ex. 3 10 Exercise 15.3.1.(1)

Ex. 4 10 Exercise 15.3.4.(1).

Ex. 5 5 Exercise 15.3.5.

Ex. 6 5 Exercise 15.2.3.

Ex. 7 (a) 10 Write $xy$, $xz$, and $(x^2 - y^2)$ as components of a spherical (irreducible) tensor of rank 2.

(b) 10 The expectation value

$$Q \equiv e(\alpha, j, m = j| (3z^2 - r^2)| \alpha, j, m = j)$$

is known as the quadrupole moment. Evaluate

$$e(\alpha, j, m'|(x^2 - y^2)| \alpha, j, m = j),$$

(where $m' = j, j - 1, j - 2, \ldots$) in terms of $Q$ and appropriate Clebsch-Gordan coefficients.

Ex. 8 10 Suppose two spin-1/2 particles are known to be in the spin-singlet state. Let $S_b^{(1)}$ be the component of spin for one of the particles along $\hat{a}$ direction and $S_b^{(2)}$ be the component of spin for the remaining particle along $\hat{b}$ direction. Show that

$$\langle S_a^{(1)}S_b^{(2)} \rangle = -\frac{\hbar^2}{4} \hat{a} \cdot \hat{b}.$$ 

Ex. 9 10 Consider two electrons (denoted by 1 and 2) interact with each other via the Coulomb interaction $U = \frac{e^2}{|r_1 - r_2|}$, where $r_i$ is the position operator for the $i$th electron. Suppose that the orbital states of electrons can be either $\phi_a(r)$ or $\phi_b(r)$. Find the difference of the average Coulomb energy (\Delta $U$) between spin singlet and spin triplet states. If one tries to attribute $\Delta U$ as the difference of spin-spin interaction $-J\sigma_1 \cdot \sigma_2$, find the expression of $J$ and show that it is always positive. Here $\sigma_i$ is the Pauli spin matrix operator for the $i$th electron.

Ex. 10 10 In classical physics, to find, say, $(S_1 - S_2)^2$ is equivalent to find $|S_1 + (-S_2)|^2$. In other words, both $(S_1 - S_2)^2$ and $(S_1 + S_2)^2$ fall into the range between $|S_1 - S_2|^2$ and $|S_1 + S_2|^2$. Therefore, if this is also true for Quantum Mechanics, then if both $S_1$ and $S_2$ are spin 1/2, the eigenvalues obtained for $(S_1 - S_2)^2$ from this argument should be 0 or $2\hbar^2$. Show that this is not correct by finding the correct eigenvalues.

Ex. 11 10 A system of two particles with spins $s_1 = 3/2$ and $s_2 = 1/2$ is described by the approximated Hamiltonian $H = \alpha S_1 \cdot \tilde{S}_2$, with $\alpha$ being a given constant. At $t = 0$, the system is in the simultaneous eigenstate of $S_1^2$, $S_2^2$, $S_{1z}$, and $S_{2z}$: $|\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\rangle$. Evaluate the probability of finding the system in the state $|\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}\rangle$ at $t > 0$.

Ex. 12 10 Consider the addition of angular momenta $\vec{J}_1$ and $\vec{J}_2$. Let $|j_1,j_2,jm\rangle$ be the eigenstate to the total angular moment $J^2$ and $J_z$. Calculate the matrix elements $\langle j_1,j_2,jm|\vec{J}_1|j_1,j_2,jm\rangle$ and $\langle j_1,j_2,jm|\vec{J}_2|j_1,j_2,jm\rangle$.

Ex. 13 10 Consider the dipole-dipole interaction between two magnetic moments, $\vec{m}_1$ and $\vec{m}_2$,

$$V(r) = \frac{\vec{m}_1 \cdot \vec{m}_2 - 3(\vec{m}_1 \cdot \vec{r})(\vec{m}_2 \cdot \vec{r})}{r^3},$$

where $r$ is the relative distance between two magnetic moments. Since the magnetic moment is proportional to the spin of the particle, the interaction can be expressed in terms of Pauli matrices $(\vec{\sigma}_1$ and $\vec{\sigma}_2)$ as $V(r) = -\frac{\gamma^2 \mu^2}{4} \frac{v(r)}{r^3}$, with $v(r)$ being given by

$$v(r) = 3 \frac{(\vec{\sigma}_1 \cdot \vec{r})(\vec{\sigma}_2 \cdot \vec{r})}{r^2} - \vec{\sigma}_1 \cdot \vec{\sigma}_2.$$ 

Find the expression of $v(r)$ in terms of the irreducible 2nd-rank tensor operators constructed by the total spin $\vec{S} = (\vec{\sigma}_1 + \vec{\sigma}_2)/2$ and the position operator $\vec{r}$.

Ex. 14 10 Consider the addition of two angular moment of same magnitude, $j_1 = j_2 = j$. Show that the state with zero angular momentum can be put in the following form

$$|0, 0\rangle = \frac{1}{\sqrt{2j + 1}} \sum_{m = -j}^{j} (-1)^{m - 1/2}|jm; j - m\rangle \quad j = \text{half - integer},$$

$$|0, 0\rangle = \frac{1}{\sqrt{2j + 1}} \sum_{m = -j}^{j} (-1)^{m}|jm; j - m\rangle \quad j = \text{integer}.$$
**Ex. 15** Consider two particles governed by the Hamiltonian $H = \frac{J^2}{2I}$, where $J = J_1 + J_2$ is the total angular moment of two particles and $I$ is the moment of inertia. Show that the angle between $J$ and $J_1$ or $J$ and $J_2$ is a constant. Demonstrate that $J_1$ and $J_2$ precess about $J$.

**Ex. 16** Consider the coupling of three spin-$1/2$ particles. Let $|\alpha\beta\gamma\rangle$ denote the state when the first particle is in the state $|\alpha\rangle$, the 2nd in $|\beta\rangle$ and the 3rd in $|\gamma\rangle$, where $\alpha$, $\beta$, and $\gamma$ are either $+$ (spin up) or $-$ (spin down).

(a) Construct all states with total angular momentum $J = 3/2$.

(b) Construct all states with $J = 1/2$. (Hint: add two spin-$1/2$ particles first, and then include the 3rd particle.)