Quantum Mechanics (II): Homework 6
Due: June 10, 2013

Ex 1 (a) 15 In this problem, we verify the adiabatic theorem in the spin 1/2 system. Consider an electron in a time-dependent magnetic field

\[ \mathbf{B} = B (\sin \theta \cos \omega t \hat{x} + \sin \theta \sin \omega t \hat{y} + \cos \theta \hat{z}) . \]

Let us consider only the spin degree of freedom. Suppose that at \( t = 0 \) the spin state \( |\psi(t = 0)\rangle \) of the electron is spin-up along the \( \mathbf{B} \) direction. By finding the exact solution \( |\psi(t)\rangle \) to the time-dependent Schrödinger equation, show that \( |\langle -t|\psi(t)\rangle|^2 \rightarrow 0 \) in the adiabatic limit when \( T \rightarrow \infty \), where \( T = 2\pi/\omega \) and \( |\pm, t\rangle \) are instantaneous spin up/down states at time \( t \). From here, find \( |\psi(T)\rangle \) in the adiabatic limit and compare it with the prediction made by the adiabatic theorem.

(b) 15 A particle of mass \( m \) is in an infinite potential well with

\[ V(x) = 0 \text{ for } 0 \leq x \leq a + vt, \]
\[ = \infty \text{ otherwise.} \]

Show that the following wavefunction is a solution

\[ \Psi_n(x, t) = \sqrt{2 \over \omega} \sin \left( {n\pi x \over \omega} \right) e^{i(mv x^2 - 2E_n at)/(2\hbar \omega)}, \]

where \( \omega = a + vt \) and \( E_n = n^2 \pi^2 \hbar^2 / 2ma^2 \). Find the dynamical phase and the Berry phase for this solution. From here, check how good the adiabatic limit is.

Ex 2 10 As we have shown in the class that the solution to \( (H_0 + V)|\psi\rangle = E|\psi\rangle \) is formally given by the Lippmann-Schwinger equation

\[ |\psi^\dagger\rangle = |\phi\rangle + \frac{V}{E - H_0 + ie} |\psi^\dagger\rangle, \]

where \( |\phi\rangle \) is a solution to \( H_0|\phi\rangle = E|\phi\rangle \). Derive this equation by the method of adiabatic switch-on. Hint: Turn on \( V \) from \( t = -\infty \) to 0 by replacing \( V \) by \( V \exp(\varepsilon t/\hbar) \) with \( \varepsilon \rightarrow 0^+ \). Write the Schrödinger equation in the interaction picture and convert it to an integral equation.

Ex 3 10 Consider a two-state system with the Hamiltonian given by

\[ H = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| + \gamma e^{i\omega t}|1\rangle\langle 2| + \gamma e^{-i\omega t}|2\rangle\langle 1|. \]

Suppose that initially, the system is in \(|1\rangle\), find the probability of finding it in the state of \(|2\rangle\) at a later time \( t \).

Ex 4 10 Exercise 18.2.1
Ex 5 10 Exercise 18.2.6

Ex 6 (a) 20 Find the one dimensional Lippmann-Schwinger equation. Suppose that the Hamiltonian for scattering is given by

\[ H = \frac{p^2}{2m} + V(x), \]

where \( V(x) \) vanishes for \(|x| > a\). Consider the case when the particle incidents from \(-\infty \) to \( \infty \) with momentum \( \hbar k \), show that the reflection \((R)\) and transmission coefficients \((T)\) are given by

\[ R = \left| \frac{\sqrt{2\pi}}{2ik} \int_{-\infty}^{\infty} e^{ikx} U(x) \psi(x) \, dx \right|^2, \]
\[ T = \left| 1 + \frac{\sqrt{2\pi}}{2ik} \int_{-\infty}^{\infty} e^{-ikx} U(x) \psi(x) \, dx \right|^2, \]

where \( U(x) = 2mV(x)/\hbar^2 \) and \( \psi(x) \) is the exact solution to \( H \) with eignevalue \( E = \hbar^2 k^2 / 2m \).

(b) 20 If \( U(x) = -\alpha \delta(x) \), use the above result to find \( R \) and \( T \). Verify that \( R + T = 1 \). Find the exact Green’s function.

Ex 7 10 Using the Born approximation, calculate the total cross sections as functions of the wavenumber \( k \) of the incident particles for the potentials

\[ V(r) = -V_0 \text{ for } r < a \text{ and } V(r) = 0 \text{ for } r > a, \]
\[ V(r) = V_0 \exp(-r^2/2a^2). \]
Ex 8 Galilean invariance of the Schrödinger equation
Consider the Galilean transformation, \( x' = x - vt, \ t' = t \), on a system with the Hamiltonian \( H = \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x,t) \). In the unprimed system, we denote the wavefunction as \( \psi(x,t) \), while in the primed system, the wavefunction is \( \psi'(x',t') \).

(a) \( 10 \) Verify that
\[
\psi'(x',t') = \exp \left[ -i \frac{\hbar}{\bar{m}} \left( mux - \frac{m\nu^2}{2}t \right) \right] \psi(x,t)
\]

(b) \( 5 \) Give an argument why the above is correct without direct verification.

(c) \( 5 \) What is the relation between the probability currents \( J'(x',t') \) and \( J(x,t) \)?

(d) \( 10 \) What would the electromagnetic potentials (\( A, \phi \)) change under Galilean transformation?

Ex 9 \( 10 \) Verify the matrices in the Dirac equation, \( \alpha_i \) and \( \beta \) satisfy the relations
\[
\begin{align*}
\alpha_i \alpha_j + \alpha_j \alpha_i &= 2 \delta_{ij} \\
\alpha_i \beta + \beta \alpha_i &= 0 \\
\beta^2 &= 1
\end{align*}
\]

Ex 10 \( 10 \) Deduce the following integral formula
\[
e^{i\delta} \sin \delta(k) = -\frac{2m}{\hbar^2} k \int_0^\infty j_l(kr)V(r)R_{l,k}(r)r^2 \, dr,
\]
where \( \delta(k) \) is the phase shift for the spherical symmetrical potential \( V(r) \) (with angular momentum \( l \) and energy \( E = \hbar^2 k^2 / 2m \)), \( j_l(x) \) is the spherical Bessel function and \( R_{l,k} \) is the radial part of the wavefunction.

Ex 11 \( 10 \) Ex.19.5.5.

Ex 12 Consider a relativistic spin-1/2 particle in a central potential \( V(r) \) governed by the Dirac Hamiltonian
\[
H = c\bar{\alpha} \cdot \frac{\hbar}{i} \bar{\nabla} + \beta mc^2 + V(r),
\]
where \( \bar{\alpha} = \alpha_1, \alpha_2, \alpha_3. \)

(a) \( 5 \) Let the orbital angular moment be \( \bar{L} = \bar{r} \times \frac{\hbar}{2} \bar{\nabla} \). Find \( [H, L_i] \) with \( i = 1, 2, 3. \) From here, it is seen that unlike the non-relativistic case, the Hamiltonian does not commute with the orbital angular moment.

(b) \( 10 \) The total angular momentum in the Dirac equation is found to be \( \bar{J} = \bar{L} + \frac{\hbar}{2} \bar{\Sigma} \) with \( \Sigma_i = \frac{i}{2} \sum_{jk} \epsilon_{ijk} \beta \alpha_j \beta \alpha_k. \) Find the matrix form of \( \bar{\Sigma} \) and calculate the expression for \( [H, \bar{J}]. \)

Ex 13 \( 10 \) The tritium atom, \( ^3\text{H} \), is radioactive isotope of hydrogen and decays with a half-life of 12.3 years to \( ^3\text{He}^+ \) by the emission of an electron from its nucleus (\( \beta \) decay). The electron departs with 16 keV of kinetic energy. Find the probability that the newly-formed \( ^3\text{He}^+ \) atom is in an excited state. Justify the approximation regarding dynamics involved in finding the probability.

Ex 14 \( 10 \) Consider the scattering of a particle by a distribution of charges due to Coulomb interaction. The charge of the particle is \( e \) (\( e > 0 \)) while the charge of each scatterer is also \( e. \) Suppose the momentum of incident particle is \( \hbar k \) (along \( z \) direction), find the differential cross section in the Born approximation for the following configurations.

(a) \( 10 \) The scatterers are placed at eight vertices of cube with length of side being \( a \) and one side being parallel to \( z \) direction.

(b) \( 5 \) Do the same for an infinite cubic lattice of lattice spacing \( b. \)

Bonus

Problem 1. \( +1 \) Two objects, A and B, have irregular forms and are identical in size and shape. Object A is filled with a positively infinite (impenetrable) potential and scatters particles of mass \( m. \) Object B is made of metal and has a certain electrostatic capacity \( C \) when isolated from other objects. Derive expressions for the differential and total cross sections for elastic scattering of particles of mass \( m \) by object A in the limit \( k \to 0 \), in terms of the capacity \( C \) of object B.

Problem 2. \( +1 \) The Hamiltonian for a two-level system is time-dependent and is given by
\[
H = \frac{\nu t}{2} \sigma_z + \Delta \sigma_x,
\]
where \( t \) represents the time and \( \nu \) and \( \Delta \) are positive real numbers. If at \( t = -\infty \), the state of the system is \( \psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), find the exact probability of finding the system to be \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) at \( t = \infty \).

**Problem 3.** +1 Consider a particle of mass in the following potentials: (a) \( V(r) = -\alpha r \) (b) \( V(r) = \frac{1}{2}m\omega^2 r^2 \). If the particle is in the \( n \)th energy eigenstate, find the recursion relation among \( \langle r^m \rangle \), \( \langle r^{m-1} \rangle \), and \( \langle r^{m-2} \rangle \) for (a) and the recursion relation among \( \langle r^m \rangle \), \( \langle r^{m+2} \rangle \), and \( \langle r^{m-2} \rangle \) for (b). Here \( m \) is an non-negative integer.