Quantum Mechanics (I): Homework 6
December 28, 2011

Ex.1 **10** Show that for the nth eigenstate $|n\rangle$ of a simple harmonic oscillator, the following equality holds for any positive integer k

$$\langle n\,|\,(m\omega\,\hat{\mathrm{x}})^{2k}\,|\,n\rangle = \langle n\,|\,\hat{\mathrm{p}}^{2k}\,|\,n\rangle.$$

Ex.2 Consider a coherent state $|\lambda, t\rangle$.
(a) **5** The probabilities of finding $|\lambda\rangle$ ($= |\lambda, 0\rangle$) in $|n\rangle$ is $|\,\langle n|\lambda\rangle\,|^2$, what kind of distribution is it?
(b) **5** What is the average number of *quanta* (i.e, n) in $|\lambda\rangle$?
(c) **10** What is the standard deviation (i.e., Δn) of n? Show that the so-called factorial cumulants $\kappa_p \equiv \langle n(n-1)(n-2)\cdots(n-p+1)\rangle - \langle n\rangle^p$ vanish for any p.
(d) **10** Let $\psi_\lambda(x,t) = \langle x\,|\,\lambda, t\rangle$, find $\psi_\lambda(x,t)$.
(e) **15** Consider $t = 0$, let λ be real and $\lambda \equiv \sqrt{2}\,s$. Using the results of (d), show that

$$e^{-s^2+2ys} = \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(y)$$

i.e., $\exp(-s^2 + 2ys)$ is the generating function of the Hermite polynomials $H_n(y)$.
(f) **15** Following the derivatin of the coherent state in the class, if we displace the average position and the momentum of the nth energy eigenstate by x_0 and p_0 at $t = 0$, find the wavefunction at $t > 0$. Verify that the average position moves exactly in the same way as that of the classical particle.
Ex.3 **10** Page 212, ex. 7.4.4.
Ex.4 **10** Page 212, ex. 7.4.5.
Ex.5 Consider a particle of mass m subject to a one-dimensional potential of the following form:

$$V = \begin{cases} \frac{1}{2}kx^2 & \text{for } x > 0 \\ \infty & \text{for } x < 0 \end{cases}$$

(a) **5** What is the ground-state energy?
(b) **5** What is $\langle x^2\rangle$ for the ground state?
(c) **5** Find $\langle\,\frac{1}{x^2}\,\rangle$ for the ground state.
(d) **7** Use the WKB approximation to estimate the ground-state energy and compare it with your answer of (a).
(e) **5** Find the orthonormal energy eigenfunctions for this potential.
Ex.6 **10** Page 258, ex. 10.1.3.
Ex.7 **10** Page 259, ex. 10.2.1.
Ex.8 **10** Consider two identical particles. Suppose that one is in the state $\psi_a(x)$, and the other is in the state $\psi_b(x)$. Let us denote the average of the square of their relative distance as $\langle(x_1 - x_2)^2\rangle_\pm$, where $+$ indicates the case when particles considered are bosons, and $-$ indicates the case when particles considered are fermions. Show that averagely speaking, fermions have larger relative-distance; it differs from that of bosons by

$$\langle(x_1 - x_2)^2\rangle_- - \langle(x_1 - x_2)^2\rangle_+ = 4\left(\int dx x\psi_a^*(x)\psi_b(x)\right)^2.$$

Ex. 9 **10** Page 278, ex.10.3.4.
Ex. 10 Consider a particle characterized by the Hamiltonian

$$\hat{\mathrm{h}} = \Delta\,[2|1><1| - |2><2| + 2(|1><2| + |2><1|)]\,.$$

Here Δ is a real number with the dimension of energy; $|1>$ and $|2>$ are normalized and are also orthogonal to each other. We shall assume that statistics of such particles is bosonic. Suppose that a system is composed of three such particles.
(a) **10** Find the matrix of Hamiltonian ($\equiv \hat{\mathrm{H}}$) for this system in the Fock space. Here the index conventions are: 1 for $|3,0\rangle$, 2 for $|2,1\rangle$, 3 for $|1,2\rangle$, and 4 for $|0,3\rangle$. We use $|n,m\rangle$ to represent the state with n particles in the state $|1\rangle$ and m particles in the state $|2\rangle$.
(b) **15** Let the creation and annihilation operators for $|1\rangle$ and $|2\rangle$ be $a_1^\dagger$, $a_2^\dagger$ and a_1, a_2. We can define another set of creation and annihilation operators for $|\mathrm{I}\rangle$ and $|\mathrm{II}\rangle$ be $a_{\mathrm{I}}^\dagger$, $a_{\mathrm{II}}^\dagger$ and a_{I}, a_{II}, where $|\mathrm{I}\rangle$ and $|\mathrm{II}\rangle$are eigenkets of $\hat{\mathrm{h}}$ with

largest and smallest eigenvalues respectively. Show that the Hamiltonian $\hat{\mathrm{H}}$ is diagonalized in terms of $a_{\mathrm{I}}^{\dagger}$, $a_{\mathrm{II}}^{\dagger}$, a_{I}, and a_{II} if

$$a_{\mathrm{I}} = \frac{1}{\sqrt{5}}(2a_1 + a_2),$$
$$a_{\mathrm{II}} = \frac{1}{\sqrt{5}}(a_1 - 2a_2),$$

and ($a_{\mathrm{I}}^{\dagger}$, $a_{\mathrm{II}}^{\dagger}$, a_{I}, a_{II}) satisfy "standard" commutation relations such as $[a_{\mathrm{I}},\, a_{\mathrm{II}}] = 0$ and $[a_{\mathrm{I}},\, a_{\mathrm{I}}^{\dagger}] = 1$.

Ex. 11 Crystal momentum versus momentum As we learned in the class, if the Hamiltonian of a system $\hat{\mathrm{H}}(x)$ is invariant under infinitesimal translations ε, i.e., $\hat{\mathrm{H}}(x+\varepsilon) = \hat{\mathrm{H}}(x)$, the momentum p of this system is conserved. In particular, the wavefunction $\psi(x)$ is $\exp(ipx/\hbar)$ up to a normalization constant. Thus,

$$\psi(x+a) = \exp(ika)\psi(x)$$

with $k \equiv p/\hbar$ being the wavevector. The crystal momentum is resulted from considering a system that is translationally invariant only for $\varepsilon = na$, where a is a fixed constant and n is an integer. That is, when the system considered is periodic, the relevant quantity is the crystal momentum. A typical example is a system with a periodic potential $V(x) = V_0 \sin(2\pi x/a)$.

(a) 5 Show that in this case even though the energy eigenfunction is no longer a plane wave, the above equation is still correct with k being some constant not proportional to the real momentum. This is know as the *Bloch theorem*, and $\hbar k$ is known as the crystal momentum.

(b) 5 From (a), prove the alternating form of the Bloch theorem, i.e., the wavefunction can rewritten as

$$\psi(x) = \exp(ikx)u(x),$$

where $u(x+a) = u(x)$ is some periodic function of period a. Wavefunctions of this kind are known as the Bloch functions.

(c) 5 Show that the conserved quantity is $\exp(ika)$, not k. This also reflects the fact that k is not uniquely defined, any combinations $k + 2\pi m/a$ (m is some integer) give the same factor $\exp(ika)$. In particular, we allow the change $k \to k + 2\pi m/a$ during the course of the time. This is known as the Umklapp process. See any textbooks on solid state for more details.

(d) 10 We now apply the above theorem to investigate the problem when a particle of mass m is in the one-dimensional potential

$$V(x) = -A \sum_{n=-\infty}^{n=\infty} \delta(x - na),$$

where A is a positive constant, n are integers and a is positive. Find the relation between the energy E and the crystal momentum k of the particle. Demonstrate that the energy spectrum consists of bands separated by forbidden gaps.

Ex.12 Discret Fourier Transformation and Phonons

(a) 10 Consider a period function $f(x)$ in $0 \le x < L$ with $f(x+L) = f(x)$. According to the Fourier expansion, one can expand

$$f(x) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} a_n e^{i\frac{2n\pi}{L}x},$$

where we introduce the factor $1/\sqrt{L}$ simply to make later forms more symmetric. By using

$$\int_0^L dx e^{i\frac{2m\pi}{L}x} e^{-i\frac{2n\pi}{L}x} = L\delta_{n,m}\ , \qquad (1)$$

one can express a_n in terms of an integration over $f(x)$. Now suppose that x only resides on discrete points, $x = ja$, $j = 0, 1, 2, ..., N-1$ with $a = \frac{L}{N}$. If we replace all integrals by summation with the understanding that $dx = a$ is small, show that one obtains the discrete Fourier transformation as follows

$$f_j = \frac{1}{\sqrt{N}} \sum_k f_k e^{ik(ja)}$$
$$f_k = \frac{1}{\sqrt{N}} \sum_j f_j e^{-ik(ja)}.$$

What are the relations between k and n & f_k and a_n? Find the discrete form of Eq.(1).
(b) Consider a system of N particles with classically equilibrium-positions being at $x = ja$, $j = 0, 1, 2, ..., N-1$. Suppose that the mass of each particle is m and nearest-neighbouring particles are connected by springs with spring constant k. Therefore, the Hamiltonian of the system is

$$H = \frac{1}{2m}\sum_n p_n^2 + \frac{1}{2}m\omega_0^2\sum_j (x_n - x_{n+1})^2,$$

where p_n and x_n are momentum and position operators of the nth particle. Using the discrete Fourier Transformation, express H in terms of the Fourier transformed momentum and position operators p_k and x_k (**10**). By appropriately defining creation and annihilation operators a_k and $a_k^\dagger$, show that H can be written in the form $H = \sum_k \hbar\omega_k(a_k^\dagger a_k + 1/2)$ and find ω_k ((**a**) **10**).
(c) Bonus (0.5) Assuming that $N \gg 1$, find the ground state wavefunction $\Psi(x_0, x_1, x_2, ..., x_{N-1})$ up to a normalization factor.
Ex.13 Tensor product and exact diagonalization
(a) 8 Consider two two-level systems. Let the associated two levels are denoted by $|1\rangle$ and $|2\rangle$, which are represented by

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In this representation, each of the system is characterized by three observables denoted by σ_x^k, σ_y^k and σ_z^k, which are given by

$$\sigma_x^k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y^k = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z^k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here k is an index for the system so that $k = 1$ or 2. Suppose that two systems are governed by the Hamiltonian $H = \sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2 + \sigma_z^1\sigma_z^2$. If we denote $|1\rangle\otimes|1\rangle$, $|1\rangle\otimes|2\rangle$, $|2\rangle\otimes|1\rangle$, and $|2\rangle\otimes|2\rangle$ by $(1,0,0,0)^T$, $(0,1,0,0)^T$, $(0,0,1,0)^T$, and $(0,0,0,1)^T$ respectively, find the matrix representation of H.
(b) 7 If we generalize (a) to three two-level systems with $H = \vec{\sigma}^1\cdot\vec{\sigma}^2 + \vec{\sigma}^2\cdot\vec{\sigma}^3$, where $\vec{\sigma}^k = (\sigma_x^k, \sigma_y^k, \sigma_z^k)$, find the matrix representation of H.
(c) Bonus (1) If we generalize (a) to N two-level systems with $H = \sum_{n=1}^{n=N-1}\vec{\sigma}^n\cdot\vec{\sigma}^{n+1}$, H describes N interacting spins. Find numerical values of three lowest eigenvalues of H for $N = 10$ (You may use any of your favorite software to write a program for solving this problem. The program must be handed in together with your solutions.)

Bonus problem (+1) How to prepare a coherent state?

In this problem, I am going to show you how to prepare a coherent state theoretically. First, consider a charge harmonic oscillator with mass m and charge q in a uniform electric field $E(t)$. The Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = (H_0 + H_1)|\psi(t)\rangle,$$

where $H_0 = \hbar\omega(a^\dagger a + 1/2)$. Verify that

$$H_1 = -q\sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)E(t).$$

(a) If we perform the transformation (i.e., use the so-called interaction picture) $|\psi(t)\rangle_I = e^{iH_0 t/\hbar}|\psi(t)\rangle$, show that the evolution operator for $|\psi(t)\rangle_I$ is

$$U_I(t,0) = \exp\left[i\phi(t)\right]\exp\left\{\int_0^t dt'\left[\alpha(t')a^\dagger - \alpha^*(t')a\right]\right\},$$

where

$$\phi(t) = \frac{q^2}{2m\hbar\omega}\int_0^t dt_1\int_0^{t_1} dt_2 E(t_1)E(t_2)\sin\hbar\omega(t_1 - t_2),$$

and $\alpha(t) = i\sqrt{1/2m\hbar\omega}E(t)\exp(i\hbar\omega t)$.

(b) Show that the above result implies that if initially the system is in the ground state (at $t = 0$), at time t

$$|\psi(t)\rangle = \exp\left(-i\omega t/2\right)\exp\left[i\phi(t)\right]|\lambda(t)\rangle,$$

where $\lambda(t) = e^{-i\omega t}\int_0^t dt'\alpha(t')$. In other words, the particle evolves into a coherent state. By choosing $E(t)$ appropriately, one can engineer the state at t into any desired coherent state.

(+1) (a) Use the coherent state to prove the Mehler summation formula

$$\frac{1}{\sqrt{1-t^2}}\exp\left\{-\frac{1}{2(1-t^2)}\left[(x^2+y^2)(1+t^2) - 4xyt\right]\right\}$$
$$= \exp(-x^2/2 - y^2/2)\sum_{n=1}^{\infty}\frac{t^n}{2^n n!}H_n(x)H_n(y)$$

(b) Exercise 21.1.21