Statistical Mechanics (II): Homework 1 (due on October 12)

Problem 1. Show that the Gibb's free energy $G = \sum_i \mu_i N_i$, where μ_i and N_i are the chemical potential and number of particles for the ith type particles. Explain why the same arguement can not be used to conclude other thermodynamic potential such as $F = \sum_i \mu_i N_i$. Use similiar arguement to show that the grand potential $\Omega = -PV$.

Problem 2. The partition function for canonical ensemble is

$$Q_N = Tre^{-\beta H} = \sum_E g(E)e^{-\beta E},$$

where $g(E)$ is the degeneracy and is related to the entropy by $S(E) = k_B \ln g(E)$. Show that $Q_N = e^{-\beta(U-TS)}$ with $U = \frac{\sum_E Eg(E)e^{-\beta E}}{Q_N}$ and $S = U = \frac{\sum_E S(E)g(E)e^{-\beta E}}{Q_N}$.

Problem 3. Given a system with hard core inter-particle potential, i.e., $v(r) = \infty$ for $r < a$. Suppose that the volume of the system is finite. Show that ρ, $\left(\frac{\partial \rho}{\partial \ln z}\right)_T$ and $\left(\frac{\partial p}{\partial \rho}\right)_T$ are all positive and finite.

Problem 4. **Yang-Lee theorem (II)** In this problem, we outline how to prove the 2nd Yang-Lee theorem: in the complex domain $\mathcal{R}$ which contains a segment of real axis and no root, the limit

$$\lim_{V\to\infty} \frac{1}{V} \frac{\partial \ln Z}{\partial \ln z}$$

exits. Furthermore, $\lim_{V\to\infty}$ and $\frac{\partial}{\partial \ln z}$ commute in $\mathcal{R}$.

(a) Consider a point $z_0 \in \mathcal{R}$ and is located on the real axis. Let the root that is closest to z_0 be z_j and consider the points inside the circle $|z - z_0|$ $(\equiv |y|) < d < |z_0 - z_j|$ $(y_j \equiv z_j - z_0)$. Show that the function $S(V, y) \equiv \frac{1}{V} \ln Z$ can be written as an infinite series of y:

$$S_n(V, y) \equiv \sum_{l=0}^{n} \alpha_l(V) y^l$$

i.e., $\lim_{n\to\infty} S_n(V, y) = S(V, y)$ and it is uniformly convergent.

(b) Verify that $\lim_{V\to\infty} \alpha_0(V)$ exits and from here show that $\lim_{V\to\infty} \alpha_l(V)$ exit for all l. Therefore, one concludes that $\lim_{V\to\infty} \lim_{n\to\infty} S_n(V, y) = \lim_{n\to\infty} \lim_{V\to\infty} S_n(V, y) = \sum_{l=0}^{\infty} \alpha_l(\infty) y^l$.

(c) Show that $\lim_{V\to\infty}$ and $\frac{\partial}{\partial \ln z}$ commute in $\mathcal{R}$. (Ref. T.D. Lee's statistical mechanics.)