Statistical Mechanics (II): Homework 5 (due on December 28)

Problem 1 Wick's theorem for classical Gaussian averages

(a) Show that if Y is a Gaussian variable (i.e., its probability density is given by $P(Y) = \frac{e^{-*^2/2\sigma_y^2}}{\sqrt{2\pi}\sigma_y}$ for some σ_y) then $\langle e^{tY} \rangle = e^{\frac{1}{2}t^2\langle Y^2\rangle}$ where t is an arbitraty (possibly complex) constant.

(b) Show that if $Y_1, Y_2, \ldots\ldots, Y_n$ are Gaussian (that is, they have a joint probability density $N^{-1}\exp(-\frac{1}{2}\sum_{i,j=1}^{n} a_{ij}Y_iY_j)$, where N is the normalization constant and a_{ij} is a symmetric positive definite matrix), then $Y = t_1Y_1 + \cdots + t_nY_n$ is also Gaussian. What is $\langle Y_iY_j \rangle$?

(c) Using part (a), expanding both sides as a Taylor series in the t's, derive Wick's theorem:

$$\begin{aligned} \langle Y_1 \cdots Y_n \rangle &= 0, \text{ n odd} \\ \langle Y_1 \cdots Y_n \rangle &= \sum_{\text{unordered pairings } (i_1,j_1)\cdots(i_p,j_p)} \langle Y_{i_1}Y_{j_1}\rangle \cdots \langle Y_{i_p}Y_{j_p}\rangle, \text{ n = 2p even} \end{aligned}$$

Problem 2 Consider the Ising spins interacting via the Hamiltonian

$$H = \frac{-J}{2N}\sum_{i,j} s_i s_j$$

i.e., we have taken $J_{ij} = J/N$ for all pairs i and j. By using the Hubbard-Stratonovich transformation, show that in the limit $N \to \infty$, mean-field theory is recovered.

Problem 3 Given a generating functional

$$Z[h] = \int D\psi e^{-\int d^d\vec{r}\left[\alpha\psi^2(\vec{r})+\beta\psi^4(\vec{r})+\frac{M}{2}(\nabla\psi)^2-h(\vec{r})\psi(\vec{r})\right]} \equiv e^{-F}$$

use the cumulant expansion to find the relation of $\frac{\delta^n F}{\delta h(\vec{r}_1)\delta h(\vec{r}_2)\cdots\delta h(\vec{r}_n)}$ to the connected Green's functions.