Quantum Mechanics (I): Homework 1
Due: Octobor 5

Ex.1
In early period of development of quantum mechanics, a few physicists (including Schrödinger himself) tried to develop so-called hidden-variable theories in which quantum fluctuations are attributed to some unknown variables in classical mechanics. In 1952, Bohm succeeded in identifying such kind of random forces in the Schrödinger equation: Suppose ψ satisfies the Schrödinger equation with the the potential $V(\mathbf{r})$. Let $\psi = |\psi| \exp(iS/\hbar)$, show that
(a) **5** S is real
(b) **5** when ψ is a plane wave, $\nabla S = \hbar\,\mathbf{k}$ is the momentum.
In general, ∇S has the meaning of momentum. Let $\mathbf{p} \equiv \nabla S$. Here $\mathbf{p}$ is a function of $\mathbf{r}$ and is the momentum of the particle that arrives at $\mathbf{r}$. This is in the Euler description. In general, we want to follow the same particle in the so-called Lagragian description. In this case, show that
(c) **10** $\frac{d\vec{p}}{dt} = -\nabla(V + V_Q)$, where $V_Q = -\frac{\hbar^2}{2m|\psi|}\nabla^2|\psi|$.
$-\nabla V_Q$ is the so-called quantum mechanical force that was interpreted as the fluctuations due to the unknown hidden-variable. It is claimed that this theory can also explain the two-slit expt. (see Philippidis et al. Nuovo Cimento, **71B**, 75-87, 1982)

Ex.2 **10** Page 139, ex 4.2.2
Ex.3 **10** Page 139, ex 4.2.3
Ex.4 **10** Show that $[p, x^n] = -ni\hbar x^{n-1}$. In general, if $f(x)$ is a differentiable function of x, this implies that $[p, f(x)] = -i\hbar\frac{df(x)}{dx}$.
Ex.5 The state space of a certain physical system is three-dimensional. Let $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ be an orthonormal basis of this space. The kets $|\psi_1\rangle$ and $|\psi_2\rangle$ are defined by :

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle$$
$$|\psi_2\rangle = \frac{1}{\sqrt{3}}|u_1\rangle + \frac{1}{\sqrt{3}}|u_2\rangle$$

(a) **5** Are these kets normalized?
(b) **5** What is $\langle\psi_1|\psi_2\rangle$?
Ex. 6 Consider the probability amplitude

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{i\omega t}$$

where A, λ, and ω are positive real constants. (a) **5** Normalize Ψ. (b) **5** Find $\langle x\rangle$ and $\langle x^2\rangle$.
Ex. 7 **10** Show that in the Schrödinger equation, the operator $H = p^2/2m + V(x)$ is hermitian.
Ex. 8 **10** Consider two *independent* and positive classical random numbers: x_1 and x_2, of which the probability densities are

$$p(x_i) = \lambda e^{-\lambda x_i}, \quad i = 1, 2,$$

where λ is positive and real. Let us form another random number x by defining $x = (x_1 + x_2)/2$. Find the probability density for x, $\mathrm{P}(x)$.
Ex. 9 **10** Suppose that at $t = 0$, a particle is described by the wavefunction

$$\Psi(x,0) = \frac{1}{\sqrt{2L}} \quad |x| < L$$
$$= 0 \qquad \text{otherwise.}$$

If, at the same instant, we measure the momentum of the particle. What are the possible values we will get and what are the corresponding probabilities?
Ex. 10 **10** Suppose that we do a measurement of the observable $\hat{\mathrm{O}}$ on some particle and get the value α. Using the concept of "collapse of state", argue that after the measurement, the state of the particle has to be an eigenstate of $\hat{\mathrm{O}}$ with eigenvalue of α.
Ex. 11 **10** Following the same arguement used in the class, show that in the momentum space, the position operator $\hat{\mathrm{x}}$ is represented by $i\hbar\frac{d}{dp}$.

Ex. 12 10 Page 46, ex.1.8.10.

Ex.13 **(a)5** Consider a particle governed by $\hat{\mathrm{H}} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$. Now, if the particle is further confined to be at $x = na$ with n be integers and a being a positive constant, one needs to discretize the system: replacing any function $f(x = na)$ by f_n and $\frac{d^2}{dx^2}\Psi(x)\Big|_{x=na}$ by $\frac{\Psi_{n+1}+\Psi_{n-1}-2\Psi_n}{a^2}$. Let us denote the state when the particle is localized at $x = na$ by $|n\rangle$, and assume that $\langle n|m\rangle = \delta_{nm}$. Express $\hat{\mathrm{H}}$ in terms of the Dirac notations (ket and bra and etc.) **(b)5** In the above discrete system, the momentum operator should be changed accordingly. However, there are many possible ways to do: For example, $\frac{\hbar}{i}\frac{d\Psi(x)}{dx} = \frac{\hbar}{i}\frac{\Psi_{n+1}-\Psi_n}{a}$ or $\frac{\hbar}{i}\frac{d\Psi(x)}{dx} = \frac{\hbar}{i}\frac{\Psi_{n+1}-\Psi_{n-1}}{2a}$, which one is correct? why? **(c)5** Suppose now that the particle is in the state $|\phi\rangle \equiv |1\rangle + i|2\rangle + 2|3\rangle$. Evaluate the averaged value of energy for $|\phi\rangle$. **(d)5** Now suppose that the particle is further confined to be at $n = 1$ and $n = 2$, i.e., $\Psi_n = 0$ when $n = 3, 4, 5, ...$ and $n = 0, -1, -2, ...$ The wavefunction $|\phi(t)\rangle$ of the particle at time t can be thus written as $|\phi(t)\rangle = \alpha(t)|1> +\beta(t)|2>$, Find the differential equations that $\alpha(t)$ and $\beta(t)$ obey (**5**). If at $t = 0$, $|\phi(0)\rangle = |1\rangle$, find $|\phi(t)\rangle$ at $t > 0$. (**5**)

Bonus(+1) Suppose that $V(x \leq 0) = \infty$ and $V(x \geq (N+1)a) = \infty$, $V(x) = 0$ otherwise. Using the above correct expression to find the eigenvalues and eigenvectors of the discrete momentum operator. Find the corresponding expressions when $a \to 0$.