Chapter 6 Free Electron Fermi Gas

Free electron model:

- The valence electrons of the constituent atoms become conduction electrons and move about freely through the volume of the metal.
- The **simplest metals are the alkali metals** lithium, sodium, potassium, Na, cesium, and rubidium.
- The **classical theory** had several conspicuous successes, notably the derivation of the form of Ohm's law and the relation between the electrical and thermal conductivity.
- The classical theory fails to explain the heat capacity and the magnetic susceptibility of the conduction electrons. $M = \chi B$
- Why the electrons in a metal can move so freely without much deflections?
- (a) A conduction electron is not deflected by ion cores arranged on a periodic lattice, because matter waves propagate freely in a periodic structure.
- (b) A conduction electron is scattered only infrequently by other conduction electrons.

Pauli exclusion principle.

Free Electron Fermi Gas: a gas of free electrons subject to the Pauli Principle

ELECTRON GAS MODEL IN METALS



Valence electrons form the electron gas

Figure 1.1

(a) Schematic picture of an isolated atom (not to scale). (b) In a metal the nucleus and ion core retain their configuration in the free atom, but the valence electrons leave the atom to form the electron gas.



Sodium Na : simple metal 3s valence electron Atomic core 1s²2s²2P⁶

Figure 1 Schematic model of a crystal of sodium metal. The atomic cores are Na⁺ ions; they are immersed in a sea of conduction electrons. The conduction electrons are derived from the 3s valence electrons of the free atoms. The atomic cores contain 10 electrons in the configuration $1s^22s^22p^6$. In an alkali metal the atomic cores occupy a relatively small part (~15 percent) of the total volume of the crystal, but in a noble metal (Cu, Ag, Au) the atomic cores are relatively larger and may be in contact with each other. The common crystal structure at room temperature is bec for the alkali metals and fee for the noble metals.

Classical Theory

Drude Model, 1900AD, after Thompson's discovery of electrons in 1897

Based on the concept of *kinetic theory* of neutral *dilute* ideal gas Apply to the *dense* electrons in metals by the free electron gas picture

Classical Statistical Mechanics: Boltzmann Maxwell Distribution The number of electrons per unit volume with velocity in the range dv about $v f_B(v) = n (m/2\pi k_B T)^{3/2} \exp(-mv^2/2k_B T)$

Success:

- (1) The Ohm's Law, the electrical conductivity $J = \sigma E$, $\sigma = n e^2 \tau / m$,
- (2) The Weidmann Frantz Law $K_e / \sigma_e T = L \sim a$ constant

Since $K = 1/3 v_F^2 C_v \tau$

 $(T_F/T) \longrightarrow 100 \text{ times}; 0.01 \text{ times} (T/T_F)$

Failure:

(1) Heat capacity $C_v \sim 3/2 NK_B$ The observed heat capacity is only 0.01 too small.

(2) The observed thermal power Q is also only ~ 0.01, as $Q = -C_v/3ne$

(3) Magnetic susceptibility χ is incorrect.

See Ashroft & Mermin, Ch. 1

Thermal Electrical Effect: (Seeback Effect)

As a temperature gradient is applied to a long thin bar, it should be accompanied by an electrical field directed opposite to the temperature gradient





*** Basic approximations:

(1) Between collisions:

- -- Neglect electron ion core interaction --- Free electron approximation
- -- Neglect electron electron interaction --- Independent electron approximation

(2) During collisions:

- -- Assuming electrons bouncing off the ion core
- -- Assuming some form of scattering

(3) Relaxation time approximation:

- -- Collision mean free time τ
- -- Independent of electron position and velocity

(4) The collisions are assumed to maintain the thermal equilibrium



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Free Electron Gas Model (Sommerfeld) : Quantum Statistical Mechanics:

The **Pauli exclusion principle** requires that the replacement of Maxwell Boltzmann distribution with **the Fermi Dirac distribution** as

(5)

$$f(\boldsymbol{\epsilon}) = \frac{1}{\exp[(\boldsymbol{\epsilon} - \boldsymbol{\mu})/k_BT] + 1} \ . \label{eq:f_eq}$$

**Can still use the dilute, neutral gas, kinetic picture as in the classical case.

** Justifications:

One can still describe the motion of an electron *classically*, If we can specify its positions and momentum as accurately as possible without violating *the Heisenberg uncertainty principle*.

> One is able to specify the position of an electron on a scale small compared with a distance λ over which the field or temperature varies.

Free Electron Gas Model (Sommerfeld) :

Success:

- Resolve the heat capacity anomaly
- Give correct C_v , thermal power, consistent with the experiments for

simple metals

• Good at low *T*, room *T*, but not at medium *T* for noble metals? transition metals?

Approximations:

- Neglect the effect of ions between collisions.
- The role of **ions** as a source of collision is unspecified.
- The contribution of **ions** to the physical phenomenon is not included.

Ashroft & Mermin: Chapter 2



Figure 2.1

(a) The Maxwell-Boltzmann and Fermi-Dirac distributions for typical metallic densities at room temperature. (Both curves are for the density given by $T = 0.01T_{0.0}$) The scale is the same for both distributions, and has been normalized so that the Fermi-Dirac distribution approaches 1 at low energies. Below room temperature the differences between the two distributions are even more marked. (b) A view of that part of (a) between x = 0and x = 10. The x-axis has been stretched by about a factor of 10, and the *f*-axis has been compressed by about 500 to get all of the Maxwell-Boltzmann distribution in the figure. On this scale the graph of the Fermi-Dirac distribution is indistinguishable from the x-axis.

Free Electron Gas in One Dimension Quantum Theory and Pauli Principle

Electron of mass M, in a 1-D line of length L confined to an infinite barrier The wavefunction $\psi_n(x)$ of the electron is a solution of the Schrödinger equation $\mathcal{H}\psi = \epsilon\psi$; with the neglect of potential energy

we have $\mathcal{H} = p^2/2m$, where p is the momentum. In quantum theory p may be represented by $-i\hbar d/dx$, so that

$$\mathcal{H}\psi_n = -\frac{\hbar^2}{2m} \frac{d^2\psi_n}{dx^2} = \epsilon_n \psi_n \quad , \tag{1}$$

the term orbital each orbital is a solution of a wave equation for one electron.

The boundary conditions are $\psi_n(0) = 0$; $\psi_n(L) = 0$, as imposed by the infinite potential energy barriers.

Fixed boundary Conditions

Standing wave solution

$$\psi_n = A \sin\left(\frac{2\pi}{\lambda_n}x\right) ; \quad \mathbf{N}\left(\lambda_n/2\right) = L \qquad L\left(\frac{2\pi}{\lambda_n}\right) = n\pi \qquad \mathbf{K} = \mathbf{n}\pi/L$$

$$\psi_n = A \sin\left(\frac{n\pi}{L}x\right) \qquad \frac{d\psi_n}{dx} = A\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi}{L}x\right) ; \qquad \frac{d^2\psi_n}{dx^2} = -A\left(\frac{n\pi}{L}\right)^2\sin\left(\frac{n\pi}{L}x\right) ,$$

$$\epsilon_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2$$

In a linear solid the <u>quantum numbers</u> of a conduction electron orbital are n and m_s , where n is any positive integer and the magnetic quantum number $m_s = \pm \frac{1}{2}$, according to spin orientation.

	Electron			Electron		
n	m_s	occupancy	n	m_s	occupancy	
				1		
1	Î	1	3	Î	1	
1	Ļ	1	3	\downarrow	1	
2	Î	1	4	î	0	
2	Ļ	1	4	Ļ	0	

degeneracy.

Let n_F denote the topmost filled energy level,

The condition $2n_F = N$ determines n_F , the value of n for the uppermost filled level.

Fermi energy
$$\epsilon_F$$
 $\epsilon_F = \frac{\hbar^2}{2m} \left(\frac{n_F \pi}{L}\right)^2 = \frac{\hbar^2}{2m} \left(\frac{N\pi}{2L}\right)^2$ Fermi wavevector k_F Fermi Temperature T_F



K

EFFECT OF TEMPERATURE ON THE FERMI-DIRAC DISTRIBUTION Ground State : at absolute zero temperature, how about for T > 0?

The Fermi-Dirac distribution the probability that an orbital at energy ϵ will be occupied in an ideal electron gas in thermal equilibrium:

$$f(\epsilon) = \frac{1}{\exp[(\epsilon - \mu)/k_BT] + 1} \ . \label{eq:f}$$

(5)

 $u = \mathcal{E}_{f} \text{ at } T = 0$

- At absolute zero $\mu = \epsilon_F$, because in the limit $T \rightarrow 0$ the function $f(\epsilon)$ changes discontinuously from the value 1 (filled) to the value 0 (empty) at $\epsilon = \epsilon_F = \mu$.
- 2 At all temperatures f(ε) is equal to ½ when ε = μ, for then the denominator of (5) has the value 2.
- 3. The quantity μ is the chemical potential (*TP*, Chapter 5), and we see that at absolute zero the chemical potential is equal to the Fermi energy
- 4. which $\epsilon \mu \ge k_B T$; so that $f(\epsilon) \cong \exp[(\mu \epsilon)/k_B T]$. This limit is called the Boltzmann or Maxwell distribution.

Chemical Potential u is a function of T, and u is such that $\int D(\varepsilon) f(\varepsilon) d\varepsilon = N$



Figure 3 Fermi-Dirac distribution function at various temperatures, for $T_F \equiv \epsilon_F/k_B = 50,000$ K. The results apply to a gas in three dimensions. The total number of particles is constant, independent of temperature. The chemical potential at each temperature may be read off the graph as the energy at which f = 0.5.

(5)



Figure 2.3

The Fermi function, $f(\delta) = 1/[e^{\beta(\delta-\mu)} + 1]$ versus δ for given μ , at (a) T = 0 and (b) $T \approx 0.01\mu$ (of order room temperature, at typical metallic densities). The two curves differ only in a region of order k_BT about μ . FREE ELECTRON GAS IN THREE DIMENSIONS

$$-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi_{\mathbf{k}}(\mathbf{r}) = \epsilon_{\mathbf{k}} \psi_{\mathbf{k}}(\mathbf{r}) \quad . \tag{6}$$

(1) For electrons confined to a cubic of edge L standing wave $\psi_n(\mathbf{r}) = A \sin(\pi n_x x/L) \sin(\pi n_y y/L) \sin(\pi n_z z/L)$, $k = n \pi/L$ (7)

(2) Periodic boundary conditions

$$\psi(x + L, y, z) = \psi(x, y, z) , \qquad (8)$$

Wavefunctions satisfying the free particle Schrödinger equation and the periodicity condition are of the form of a traveling plane wave:

and similarly for k_y and k_z . Any component of k is of the form $2n\pi/L$, $\exp[ik_x(x + L)] = \exp[i2n\pi(x + L)/L]$ $= \exp(i2n\pi x/L) \exp(i2n\pi) = \exp(i2n\pi x/L) = \exp(ik_x x)$. (11)

Fermi Sphere Fermi Surface

At the surface \mathcal{E}_f , K_f

Figure 4 In the ground state of a system of N free electrons the occupied orbitals of the system fill a sphere of radius k_F , where $\epsilon_F = \hbar^2 k_F^2 / 2m$ is the energy of an electron having a wavevector k_F .

$$\epsilon_{\mathbf{k}} = \frac{\hbar^2}{2m} k^2 = \frac{\hbar^2}{2m} (k_x^2 + k_y^2 + k_x^2) \quad . \tag{12}$$

the operator $\mathbf{p} = -i\hbar\nabla$, Linear momentum operator

Fermi surface.

at energy

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 $\mathbf{p}\psi_{\mathbf{k}}(\mathbf{r}) = -i\hbar\nabla\psi_{\mathbf{k}}(\mathbf{r}) = \hbar\mathbf{k}\psi_{\mathbf{k}}(\mathbf{r}) \ ,$ (13)

so that the plane wave ψ_k is an eigenfunction of the linear momentum with the eigenvalue $\hbar k$.

In the ground state of a system of N free electrons the occupied orbitals may be represented as points inside a sphere in k space.

$$\epsilon_F = \frac{\hbar^2}{2m} k_F^2 \cdot \text{At the Fermi surface } \ell_f \, , \, k_f \quad (14)$$

there is one allowed wavevector-that is, one dis-

tinct triplet of quantum numbers k_x , k_y , k_z —for the volume element $(2\pi/L)^3$ of k space. Thus in the sphere of volume $4\pi k_F^3/3$ the total number of orbitals is

$$2 \cdot \frac{4\pi k_F^3/3}{(2\pi/L)^3} = \frac{V}{3\pi^2} k_F^3 = N \quad , \tag{15}$$

where the factor 2 on the left comes from the two allowed values of m_s , the spin quantum number,

$$k_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3} , \qquad (16)$$

$$\epsilon_F = \frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V}\right)^{2/3} . \tag{17}$$

$$v_F = \left(\frac{\hbar k_F}{m}\right) = \left(\frac{\hbar}{m}\right) \left(\frac{3\pi^2 N}{V}\right)^{1/3} . \tag{18}$$

ε _Ρ , κ _Ε , Τ _Γ	See table 1
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the number of orbitals per unit energy range, $D(\epsilon)$, called the density of states.

From eq. 17

$$N = \frac{V}{3\pi^2} \left(\frac{2m\epsilon}{\hbar^2}\right)^{3/2} , \qquad (19)$$

$$\boxed{D(\epsilon) = \frac{dN}{d\epsilon} = \frac{V}{2\pi^2} \cdot \left(\frac{2m}{\hbar^2}\right)^{3/2} \cdot \epsilon^{1/2} .}$$

$$\ln N = \frac{3}{2} \ln \epsilon + \text{constant} ; \qquad \frac{dN}{N} = \frac{3}{2} \cdot \frac{d\epsilon}{\epsilon} ,$$

$$D(\epsilon) = \frac{dN}{d\epsilon} = \frac{3N}{2\epsilon} . \qquad (21) \text{ In 3}$$

Figure 5 Density of single-particle states as a function of energy, for a free electron gas in three dimensions. The dashed curve represents the density $f(\epsilon, T)D(\epsilon)$ of filled orbitals at a finite temperature, but such that k_BT is small in comparison with ϵ_F . The shaded area represents the filled orbitals at absolute zero. The average energy is increased when the temperature is increased from 0 to *T*, for electrons are thermally excited from region 1 to region 2. In 3-dimension



Heat Capacity of the Electron Gas

Classical theory, $C_v = 3/2 NK_B$

But the observed electronic contribu-tion at room temperature is usually less than 0.01 of this value.

The question was answered

only upon the discovery of the Pauli exclusion principle and the Fermi distribution function.



electronic heat capacity $C_{el} = \partial U/\partial T \approx Nk_B(T/T_F) \times 2$ (2) At room temperature C_{el} is smaller than the classical value $\frac{3}{2}Nk_B$ by a factor of the order of 0.01 or less, for $T_F \sim 5 \times 10^4$ K. $T/T_F \sim 0.01$

The total energy increase for heating from T=0 to T

$$\Delta U = U(T) - U(0) \qquad \Delta U = \int_{0}^{\infty} d\epsilon \ \epsilon D(\epsilon) \ f(\epsilon) - \int_{0}^{\epsilon_{r}} d\epsilon \ \epsilon D(\epsilon) \ Since \ at \ T = 0, \ f(\ell) = (24)$$
We multiply the identity
$$N = \int_{0}^{\infty} d\epsilon \ D(\epsilon) \ f(\epsilon) = \int_{0}^{\epsilon_{r}} d\epsilon \ D(\epsilon) \qquad (25)$$
by ϵ_{F} to obtain
$$\left(\int_{0}^{\epsilon_{F}} + \int_{\epsilon_{F}}^{\infty}\right) d\epsilon \ \epsilon_{F} f(\epsilon) D(\epsilon) = \int_{0}^{\epsilon_{r}} d\epsilon \ \epsilon_{F} D(\epsilon) \ . \qquad (26)$$

$$\Delta U = \int_{\epsilon_{r}}^{\infty} d\epsilon (\epsilon - \epsilon_{F}) f(\epsilon) D(\epsilon) + \int_{0}^{\epsilon_{F}} d\epsilon (\epsilon_{F} - \epsilon) [1 - f(\epsilon)] D(\epsilon) \ . \qquad (27)$$
The first integral on the right-hand side of (27) gives the energy needed to take electrons from ϵ_{F} to the orbitals of energy $\epsilon > \epsilon_{F}$

$$C_{el} = \frac{dU}{dT} = \int_{0}^{\infty} d\epsilon (\epsilon - \epsilon_{F}) \frac{df}{dT} D(\epsilon) \ . \qquad (28)$$

Since only *f*(*ε*) is temperature dependent

$$\Delta \Box = \int_{0}^{\infty} d\varepsilon \in D(\varepsilon) f(\varepsilon) - \int_{0}^{\varepsilon} d\varepsilon \in D(\varepsilon)$$

$$e_{\overline{f}}(z_{E})$$

 $\left(\int_{0}^{z_{E}} + \int_{\varepsilon_{E}}^{\infty}\right) d\varepsilon \varepsilon_{\overline{F}} f(\varepsilon) D(\varepsilon) = \int_{0}^{\varepsilon_{\overline{F}}} d\varepsilon \varepsilon_{\overline{F}} D(\varepsilon)$

(243 May be rewritten as

The hol

$$SU = \left(\int_{0}^{2\pi} + \int_{2\pi}^{\infty}\right) d\varepsilon \in D(\varepsilon) f(\varepsilon) - \int_{0}^{2\pi} d\varepsilon \in D(\varepsilon)$$

$$= \left(\int_{0}^{2\pi} + \int_{2\pi}^{\infty}\right) d\varepsilon \in \varepsilon \operatorname{D}(\varepsilon) f(\varepsilon) - t \int_{0}^{2\pi} d\varepsilon \in \varepsilon \operatorname{D}(\varepsilon)$$

$$= \int_{0}^{\infty} d\varepsilon \in D(\varepsilon) f(\varepsilon) - d\varepsilon \in \varepsilon \operatorname{D}(\varepsilon) f(\varepsilon)$$

$$= \int_{2\pi}^{\infty} d\varepsilon \in D(\varepsilon) f(\varepsilon) - d\varepsilon \in \varepsilon \operatorname{D}(\varepsilon) f(\varepsilon)$$

$$= \int_{0}^{\infty} d\varepsilon \in D(\varepsilon) f(\varepsilon) - d\varepsilon \in \varepsilon \operatorname{D}(\varepsilon) - d\varepsilon \in \varepsilon \operatorname{D}(\varepsilon) f(\varepsilon) + d\varepsilon \in \varepsilon \operatorname{D}(\varepsilon)$$

$$= \int_{\mathcal{E}_{F}} d\varepsilon \left(\varepsilon - \varepsilon_{F}\right) f(\varepsilon) D(\varepsilon) + \int_{\varepsilon}^{\varepsilon_{F}} d\varepsilon \left(\varepsilon_{F} - \varepsilon\right) \left[1 - f(\varepsilon)\right] D(\varepsilon)$$



Figure 6 Temperature dependence of the energy of a noninteracting fermion gas in three dimensions. The energy is plotted in normalized form as $\Delta U/N\epsilon_F$, where N is the number of electrons. The temperature is plotted as k_BT/ϵ_F .



Figure 7 Plot of the chemical potential μ versus temperature k_BT for a gas of noninteracting fermions in three dimensions. For convenience in plotting, the units of μ and k_BT are $0.763\epsilon_F$.



Figure 8 Variation with temperature of the chemical potential μ , for free electron Fermi gases in one and three dimensions. In common metals $\tau/\epsilon_F \approx 0.01$ at room temperature, so that μ is closely equal to ϵ_F . These curves were calculated from series expansions of the integral for the number of particles in the system.

 $(\epsilon - \epsilon_F) df/dT$ has large positive peaks at energies near ϵ_F . It is a good approximation to evaluate the density of states $D(\epsilon)$ at ϵ_F and take it outside of the integral: when $k_BT \ll \epsilon_F$ we ignore the temperature dependence of the chemical potential μ in the Fermi-Dirac distribution function and replace μ by the constant ϵ_F . $\tau \equiv k_BT$, **At very low T**, **lin** $u = E_F$

$$\frac{df}{d\tau} = \frac{\epsilon - \epsilon_F}{\tau^2} \cdot \frac{\exp[(\epsilon - \epsilon_F)/\tau]}{\{\exp[(\epsilon - \epsilon_F)/\tau] + 1\}^2} . \tag{30}$$

$$x \equiv (\epsilon - \epsilon_F)/\tau , \qquad (31)$$

$$C_{el} = k_B^2 T D(\epsilon_F) \int_{-\epsilon_F/\tau}^{\infty} dx \ x^2 \frac{e^x}{(e^x + 1)^2} \ . \tag{32}$$

We may safely replace the lower limit by $-\infty$ because $\mathcal{E}_F/\tau >> 1$

$$\int_{-\infty}^{\infty} dx \ x^2 \frac{e^x}{(e^x + 1)^2} = \frac{\pi^2}{3} \quad , \tag{33}$$



At temperatures much below both the Debye temperature and the Fermi temperature, the heat capacity of metals may be written as the sum of electron and phonon contributions: $C = \gamma T + AT^3$

 $C/T = \gamma + AT^2$,

γ, called the Sommerfeld parameter

At low **T**, the electronic term dominates

(37)

express the ratio of the observed to the free electron values of the electronic heat capacity as a ratio

Of a thermal effective mass m_{th} to the electron mass m, where m_{th} is defined

$$\frac{m_{\rm th}}{m} \equiv \frac{\gamma({\rm observed})}{\gamma({\rm free})}$$
.

See Table 2

(38)

The departure from unity involves three separate effects:

- A: The interaction of the conduction electrons with the periodic potential of the rigid crystal lattice <u>band effective mass</u>
- B: The interaction of the conduction electrons with phonons.
- C: The interaction of the conduction electrons with themselves.

Heavy Fermions The heavy fermion compounds include UBe_{13} , $CeAl_3$, and $CeCu_2Si_2$. *f* electrons in these compounds may have inertial masses as high as 1000 *m*, because of the weak over lap of wavefunctions of *f* electrons on neighboring ions (tight binding model)

ELECTRICAL CONDUCTIVITY AND OHM'S LAW

In an electrical field ${\bf E}$, magnetic field ${\bf B},$ the force ${\bf F}$ on an electron , the Newton second law of motion becomes

(CGS)
$$\mathbf{F} = m \frac{d\mathbf{v}}{dt} = \hbar \frac{d\mathbf{k}}{dt} = -e\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right) .$$
(39)

In the absence of collisions the Fermi sphere (Fig. 10) in k space is displaced at a uniform rate by a constant applied electric field.

First considering B = 0, in zero magnetic field

$$k(t) - k(0) = -eEt/\hbar$$
 (40)

If the field is applied at time t then at a later time t the sphere will be displaced to a new center at

$$\delta \mathbf{k} = -e\mathbf{E}t/\hbar \quad . \tag{41}$$



Figure 10 (a) The Fermi sphere encloses the occupied electron orbitals in k space in the ground state of the electron gas. The net momentum is zero, because for every orbital k there is an occupied orbital at -k. (b) Under the influence of a constant force F acting for a time interval t every orbital has its k vector increased by $\delta k = Ft/\hbar$. This is equivalent to a displacement of the whole Fermi sphere by δk . The total momentum is $N\hbar\delta k$, if there are N electrons present. The application of the force increases the energy of the system by $N(\hbar\delta k)^2/2m$.

If collision time is τ , the displacement of the Fermi sphere in the steady state is given by (41) with $t = \tau$. The incremental velocity is $\mathbf{v} = -e\mathbf{E}\tau/m$.

$$\mathbf{q} = -\mathbf{e}$$

$$\mathbf{j} = nq\mathbf{v} = ne^2 \tau \mathbf{E}/m \quad . \tag{42}$$

electrical conductivity σ is defined by $j = \sigma E$, so that

Conductivity



Ohm's Law (43)

electrical resistivity ρ is defined as the reciprocal of the conductivity,

See Table 3

Resistivity

$$\rho = m/ne^2\tau \quad . \tag{44}$$

 $\tau \approx 2 \times 10^{-9}$ s at 4 K. The mean free path ℓ of a conduc-

tion electron is defined as $\ell = v_F \tau$, (45)

 v_F is the velocity at the Fermi surface,

we have $v_F = 1.57 \times 10^8$ cm s⁻¹ for Cu

 $\ell(4 \text{ K}) \approx 0.3 \text{ cm}$; $\ell(300 \text{ K}) \approx 3 \times 10^{-6} \text{ cm}$.

Experimental Electrical Resistivity of Metals

The electrical resistivity of most metals is dominated at room temperature (300 K) by collisions of the conduction electrons with lattice phonons and at liquid helium temperature (4 K) by collisions with impurity atoms and mechan-

ical imperfections in the lattice (Fig. 11). Lattice phonons



To a good approximation the rates are often independent. And can be summed together

where τ_L and τ_l are the collision times for scattering by phonons and by imperfections, respectively. The net resistivity is given by

$$\rho = \rho_L + \rho_i$$
, Since $\rho \sim 1/\tau$ (47)

Often ρ_L is independent of the number of defects when their concentration is small, and often ρ_i is independent of temperature.

This empirical observation expresses Matthiessen's Rule.

The residual resistivity, $\rho_i(0)$, is the extrapolated resistivity at 0 K because ρ_L vanishes as $T \rightarrow 0$. The lattice resistivity, $\rho_L(T) = \rho - \rho_i(0)$, is the same for different specimens of a metal, even though $\rho_i(0)$ may itself vary widely.



Figure 11 Electrical resistivity in most metals arises from collisions of electrons with irregularities in the lattice, as in (a) by phonons and in (b) by impurities and vacant lattice sites.

Resistivity Ratio = ρ (300K)/ $\rho_i(0)$

resistivity ratio of a specimen is usually defined as the ratio of its resistivity at room temperature to its residual resistivity. It is a convenient approximate indicator of sample purity



The temperature-dependent part of the electrical resistivity is proportional to the rate at which an electron collides with thermal phonons One simple limit is at temperatures over the Debye temperature θ : here the phonon concentration is proportional to the temperature *T*, so that $\rho \propto T$ for $T > \theta$.

$$N_{ph} \propto T$$
 $\rho \propto T$

Umklapp Scattering

Umklapp scattering of electrons by phonons (Chapter 5) accounts for most of the electrical resistivity of metals at low temperatures. These are electronphonon scattering processes in which a reciprocal lattice vector **G** is involved,

the normal electron-phonon collision k' = k + q,Normal processThis scattering is an umklapp process,k' = k + q + GUmklapp process

At low enough temperatures the number of phonons available for umklapp scattering falls as $\exp(-\theta_U/T)$,

 ${m q}_{o}, \ {m heta}_{v}$ related to the geometry of the Fermi surface

Figure 13 Two Fermi spheres in adjacent zones: a construction to show the role of phonon umklapp processes in electrical resistivity.

Bloch obtained an analytic result for the normal scattering, with $\rho_L \propto T^5/\theta^6$

at very low temperatures.

Bloch's T⁵ Law

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G

MOTION IN MAGNETIC FIELDS

The free particle acceleration term is $(\hbar d/dt) \delta k$ and the effect of collisions (the friction) is represented by $\hbar \delta k/\tau$, where τ is the collision time.

The equation of motion is

$$\hbar \left(\frac{d}{dt} + \frac{1}{\tau}\right) \delta \mathbf{k} = \mathbf{F} \quad . \tag{48}$$

$$\mathbf{F} = -e\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right) ; \tag{49}$$

 $\mathbf{F} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \ .$

mV = hK

$$m\left(\frac{d}{dt} + \frac{1}{\tau}\right)\mathbf{v} = -e\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right) .$$
 (50)

(CGS)
$$m\left(\frac{d}{dt} + \frac{1}{\tau}\right)v_x = -e\left(E_x + \frac{B}{c}v_y\right) ;$$

$$m\left(\frac{d}{dt} + \frac{1}{\tau}\right)v_y = -e\left(E_y - \frac{B}{c}v_x\right) ;$$

$$m\left(\frac{d}{dt} + \frac{1}{\tau}\right)v_z = -eE_z .$$
(51)

In a steady state in a static electric field the time derivatives are zero,

$$v_x = -\frac{e\tau}{m}E_x - \omega_c\tau v_y \quad ; \qquad v_y = -\frac{e\tau}{m}E_y + \omega_c\tau v_x \quad ; \qquad v_z = -\frac{e\tau}{m}E_z \quad , \tag{52}$$

 $\omega_c \equiv eB/mc$ is the cyclotron frequency,



The Hall field is the electric field developed across two faces of a conductor, in the direction of $j \times B$

If current cannot flow out of the rod in the y direction we must have $\delta v_y = 0$, and $V_y = 0$ transverse electric field $E_y = -\omega_c \tau E_x = -\frac{eB\tau}{mc}E_x$; (53)B_Z ♠ $E_y = -\omega_c \tau E_x = -\frac{eB\tau}{E_x} E_x \ .$ (SI)► j_x, *E*_x E_v $R_H = \frac{E_y}{j_x B}$ When the transverse field E_y (Hall field) balances the Loro Hall coefficient. (Hall field) balances the Lorentz force $n e E_y = e j_x B$ Hall resistance, $R_H = -\frac{eB\tau E_x/mc}{ne^2\tau E_xB/m} = -\frac{1}{nec} ;$ (CGS) (55) $R_H = -$ (assume all relaxation τ are equal.) (SI)ne $\rho_H = BR_H = E_y/j_x$ (55a)



Figure 14 The standard geometry for the Hall effect: a rod-shaped specimen of rectangular crosssection is placed in a magnetic field B_z , as in (a). An electric field E_x applied across the end electrodes causes an electric current density j_x to flow down the rod. The drift velocity of the negatively-charged electrons immediately after the electric field is applied as shown in (b). The deflection in the y direction is caused by the magnetic field. Electrons accumulate on one face of the rod and a positive ion excess is established on the opposite face until, as in (c), the transverse electric field (Hall field) just cancels the Lorentz force due to the magnetic field.

See R_H listed in Table 4

The problem of an apparent positive sign for the charge carriers arises also for Be and As, as seen in the table. The anomaly of the sign was explained by Peierls (1928). The motion of carriers of apparent positive sign, which Heisenberg later called <u>"holes,"</u> cannot be explained by a free electron gas, 1 but by the energy band theory

Thermal conductivity of Metals

From eq. (36) for
$$C_V$$
 in K , and $\mathcal{E}_F = 1/2 m v_F^2$
 $K_{el} = \frac{\pi^2}{3} \cdot \frac{nk_B^2 T}{m v_F^2} \cdot v_F \cdot \ell = \frac{\pi^2 n k_B^2 T \tau}{3m}$. (56)

$$\frac{K}{\sigma} = \frac{\pi^2 k_B^2 T n \pi / 3m}{n e^2 \pi / m} = \frac{\pi^2}{3} \left(\frac{k_B}{e}\right)^2 T \quad . \tag{57}$$

The Lorenz number L is defined as $L = K/\sigma T$, (58)

Ratio of Thermal to Electrical Conductivity

The Wiedemann-Franz law states that for metals at not too low temperatures the ratio of the thermal conductivity to the electrical conductivity is directly proportional to the temperature, with the value of the constant of proportionality independent of the particular metal.

$$L = \frac{\pi^2}{3} \left(\frac{k_B}{e}\right)^2 = 2.72 \times 10^{-13} \,(\text{erg/esu} - \text{deg})^2$$
$$= 2.45 \times 10^{-8} \,\,\text{watt-ohm/deg}^2 \,\,. \tag{59}$$

This remarkable result involves neither n nor m. It does not involve τ if the relaxation times are identical for electrical and thermal processes.

NANOSTRUCTURES