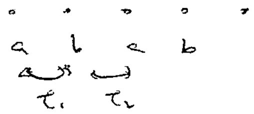
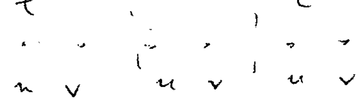
Su, Schrieffer, Heeger
PRL **42**, 1698 (1979)
PRB **22**, 2099 (1980)
Rice & Mele, PRL **49**, 1455 (1982)

- polyacetone

1D lattice





$k \pmod{2\pi}$

$$E u = -t_1 v - t_2 (e^{-ik} v)$$
$$E v = -t_2 e^{ik} u - t_1 u$$

$$H = -\begin{pmatrix} & t_1 + t_2 e^{-ik} \\ t_1 + t_2 e^{ik} & \end{pmatrix} =$$

$$= -\left[(t_1 + t_2 \cos k)\tau_x + t_2 \sin k\, \tau_y\right]$$

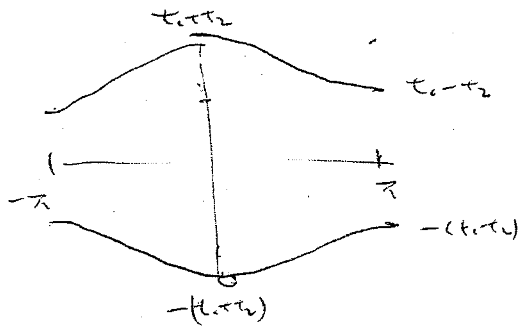
$$\boxed{\begin{aligned} E u_j &= -t_1 v_j - t_2 v_{j-1} \\ E v_j &= -t_2 u_{j+1} - t_1 u_j \end{aligned}}$$

$$E = \pm \left[(t_1 + t_2 \cos k)^2 + (t_2 \sin k)^2\right]^{1/2}$$

$$= \pm \left[t_1^2 + 2 t_1 t_2 \cos k + t_2^2\right]^{1/2}$$

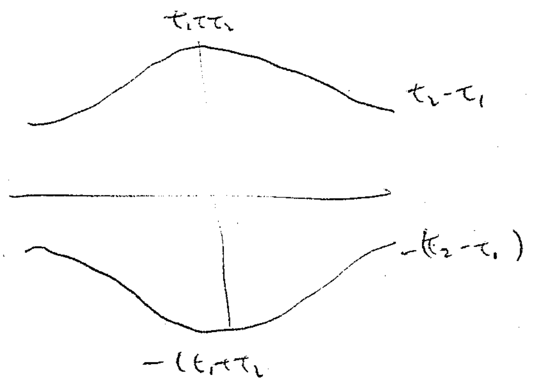
parity.
τ_x even in k
τ_y odd in k

$0 < t_2 < t_1$



insulator

$0 < t_1 < t_2$



$\{\tau_3, H\} = 0$

sublattice symmetry.

$H\psi = E\psi \qquad H\tau_3\psi = -E\psi$

$\begin{pmatrix} u_j \\ v_j \end{pmatrix}$ solution E

$\begin{pmatrix} u_j \\ -v_j \end{pmatrix}$ solution for $-E$

$$\begin{cases} Eu = (-t_1 - t_2 e^{\kappa}) v \\ Ev = (-t_1 - t_2 e^{-\kappa}) u \end{cases}$$

$Eu_j = -t_1 v_j - t_2 v_{j-1}$ 11.2

$Ev_j = -t_2 u_{j+1} - t_1 u_j$

search for decaying solution

a b a b ....

u_0 v_0 u_1 v_1

$$\begin{pmatrix} u_j \\ v_j \end{pmatrix} = e^{-\kappa j} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \qquad j \geq 1$$

$k \to i\kappa$ $\qquad \kappa_1 > 0$

$ik \to -\kappa$, $\kappa = \kappa_1 + i\kappa_2$

$$-\begin{pmatrix} 0 & t_1 + t_2 e^{\kappa} \\ t_1 + t_2 e^{-\kappa} & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = E \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

at boundary

$$Eu_0 = -t_1 v_0$$
$$Ev_0 = -(t_1 u_0 + t_2 u_1)$$

same as requiring $v_{-1} = 0$

i.e. $\underline{v_0 = 0}$

$E = 0$

$+ t_1 + t_2 e^{-\kappa} = 0$

$e^{-\kappa} = -t_1/t_2$ $\qquad$ real, < 1 if $t_1 < t_2$

$\quad \overset{\shortparallel}{e^{-(\kappa_1 + i\kappa_2)}}$

$|t_2| < |t_1|$ no decaying solution

need $|t_2| > |t_1|$

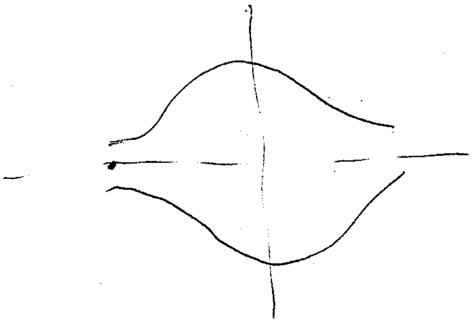
$0 < t_1 < t_2$ $\qquad$ κ real $\qquad$ (purely decaying)

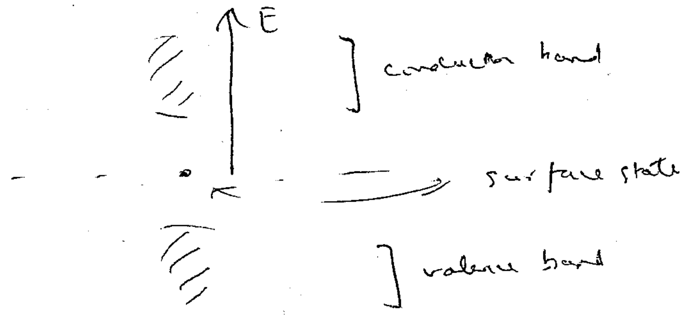
$(\kappa_2 = \pi)$ $\qquad$ if $t_1 . t_2$ opp sign. then

κ real

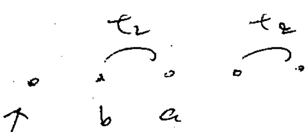
$$\begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix} e^{-xj} (-1)^j \qquad E = 0$$

only one solution



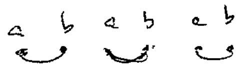


limiting case. $t_1 = 0$



$E = 0$ solution

$t_2 = 0 \quad t_1 \neq 0$



spectrum = bulk

solution exists for arbitrary $t_1 < t_2$

so long as we require $\{\tau_3, H\} = 0$

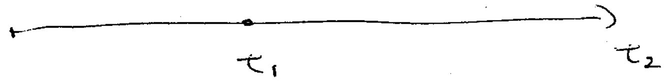
(either $\pm E$ pairs . $E \neq 0$, or $E = 0$)

$\{\tau_3 \psi = \pm \psi\}$

Solution disappears at $t_1 = t_2$ because of bulk gap

closing.

'trivial'



same symmetry, but different surface spectrum.

$t_2 < t_1$, can deform

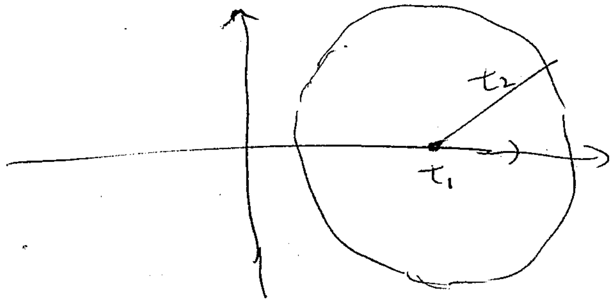
unit cell to isolated molecules

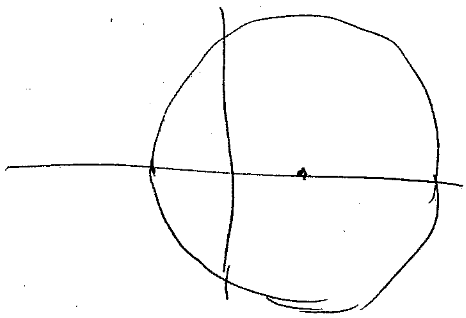
"trivial insulators"

but not for $t_1 < t_2$

$$H = -\{\underbrace{(t_1+t_2\cos k)\tau_x + t_2\sin k\,\tau_y}_{\text{effective ``magnetic'' field for `pseudospin'}}\}$$

$0 < t_2 < t_1$ $\qquad\qquad$ $0 < t_1 < t_2$





winding number: $\quad 0 \qquad\qquad 1$

$\longleftarrow\!\!\times\!\!\longrightarrow$

unless passes through origin

i.e. gapless

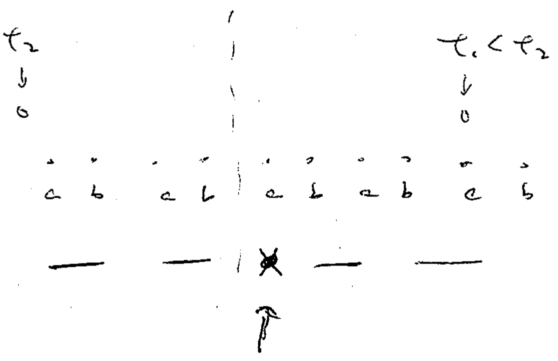
$\longleftrightarrow$

cannot go from one to the other unless close gap

and consist spectra symmetric between $E \gtrless 0$.

Ryu & Hatsugai PRL 88, 077002 (2002)

Zak PRL 62, 2747 (1989)



surface state

ground state on left and right cannot smoothly evolve from one to the other.

so resolution — <u>excitation</u>

- If system parity symmetric about $a \leftrightarrow b$ $\quad H(k) = \tau_x H(-k) \tau_x$

$k = 0$ and π special

$[H(\pi), \tau_x] = 0$

τ_x component even

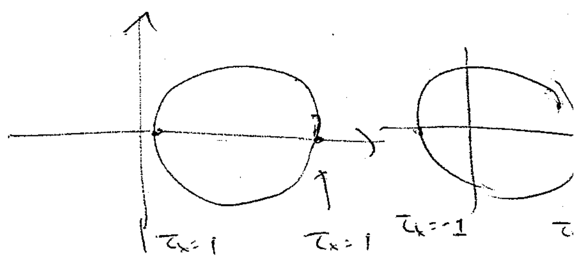
τ_y odd

eigenvectors of τ_x

If gapped and If ground state ψ

eigenvalue at $0, \pi$ different ("inverted")

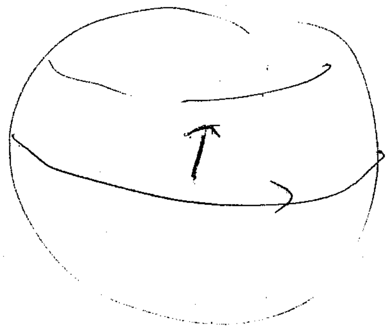
then $W \neq 0$ (odd)



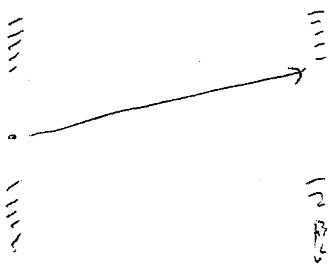
'Symmetry protected' state

('symmetry required')

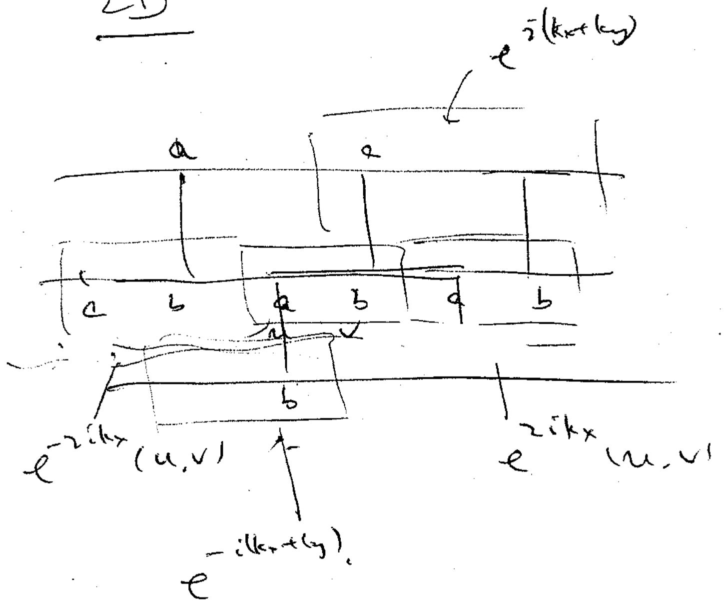
if do not insist $\{\tau_3, H\} = 0$ then



$$H = -\left[h_3 \tau_3 + h_1 \tau_1 + h_2 \tau_2\right] \qquad h_3 \neq 0$$

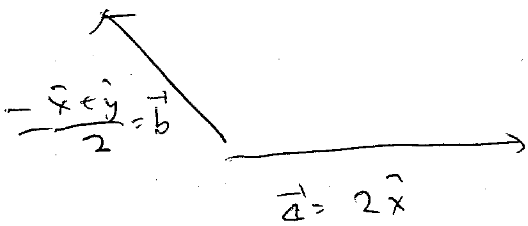


2D



$$H = -t \begin{pmatrix} 0 & 1+e^{-2ik_x}+e^{-i(k_x+k_y)} \\ 1+e^{2ik_x}+e^{i(k_x+k_y)} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$(1+\cos(2k_x)+\cos(k_x+k_y))\,\tau_x + \{\sin(2k_x)+\sin(k_x+k_y)\}\,\tau_y$$

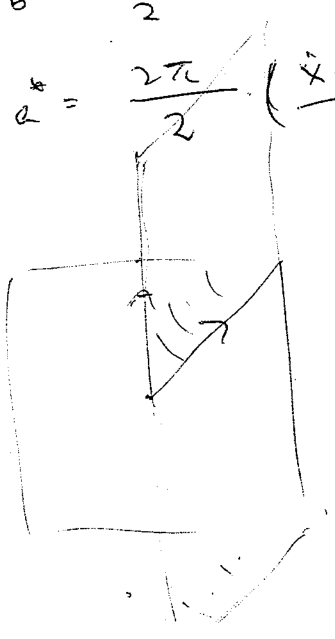


area $= 2$

$$b^* = \frac{2\pi}{2} \times (2\hat{y}) = 2\pi\,\hat{y}$$

$$a^* = \frac{2\pi}{2}\left(\frac{\hat{x}+\hat{y}}{2}\right) = \pi(\hat{x}+\hat{y})$$

BZ.



$0 < k_x < \pi$

$-\pi < k_y < \pi$

or $0 < k_y < 2\pi$

(2.2

gap vanish at

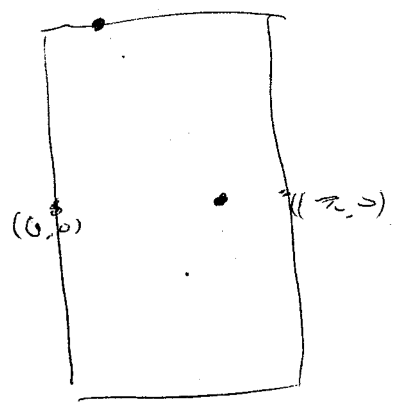
$k_x = \frac{\pi}{3}$ $\quad k_y = \pi$

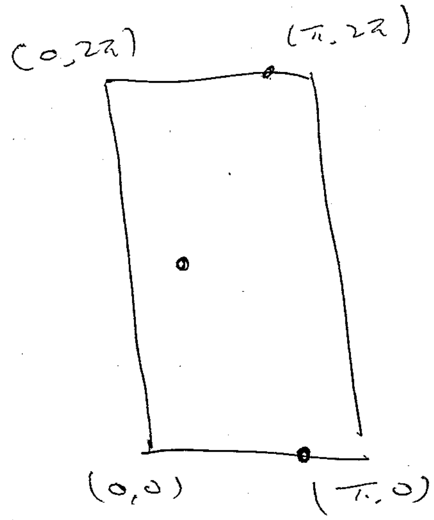
$\left(2k_x = \frac{2\pi}{3} \quad k_x + k_y = \frac{4\pi}{3} \right)$

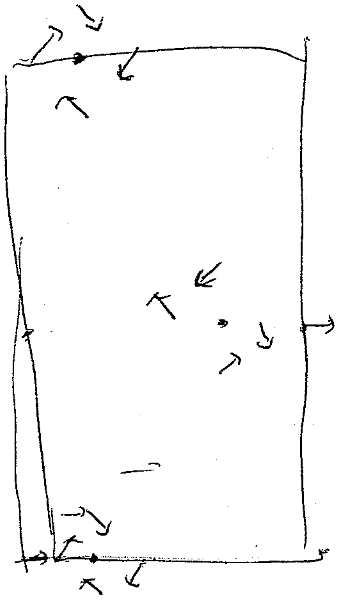
and $k_x = \frac{2\pi}{3}$ $\quad k_y = 0$ $\quad$ (equivalent to $k_x = -\frac{\pi}{3}$, $k_y = -\pi$)

(and c^*)

$\left(2k_x = \frac{4\pi}{3} \quad k_x + k_y = \frac{2\pi}{3} \right)$







special loops: (note must consider periodic paths) 2.3

$k_x = \frac{\pi}{2}$

$$\cos(\tfrac{\pi}{2}+k_y)\,\tau_x + \sin(\tfrac{\pi}{2}+k_y)\,\tau_y$$

$$= -\sin k_y\,\tau_x + \cos k_y\,\tau_y \qquad W \neq 0$$

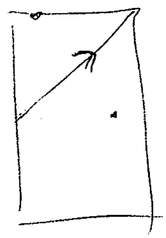
$k_x = 0$

$$(2+\cos k_y)\,\tau_x + \sin k_y\,\tau_y \qquad W = 0$$

$k_x = \pi$

$$(2-\cos k_y)\,\tau_x - \sin k_y\,\tau_y \qquad W = 0$$

$k_x = k_y = k$

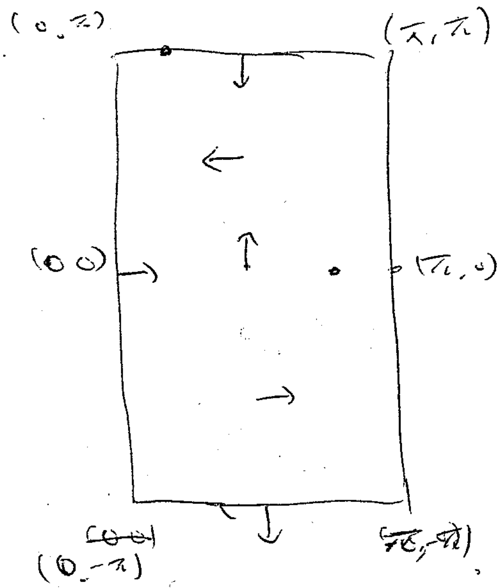


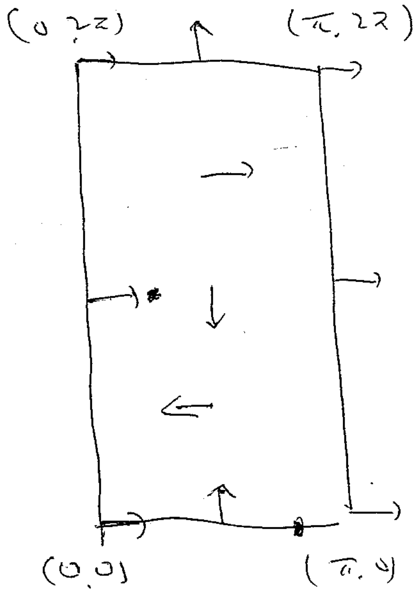
$$(1+2\cos 2k)\,\tau_x + 2(\sin 2k)\,\tau_y \qquad W \neq 0$$

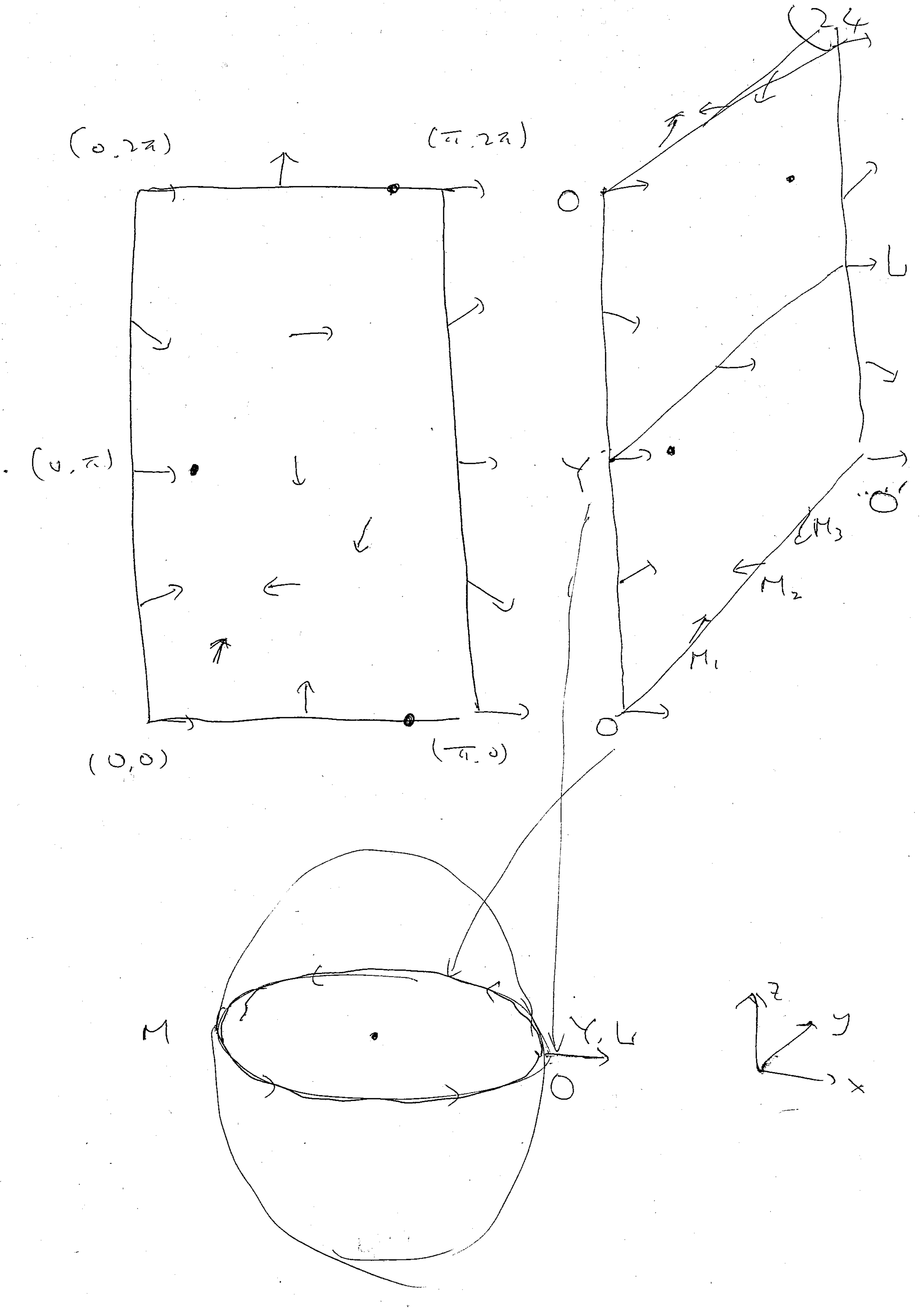
$k_x = k \quad k_y = k+\pi$

$$\left(1+\cos 2k + \cos(2k+\pi)\right]\tau_x + \left(\sin 2k_x + \sin(2k+\pi)\right)\tau_y$$

$$= \tau_x \qquad W = 0$$

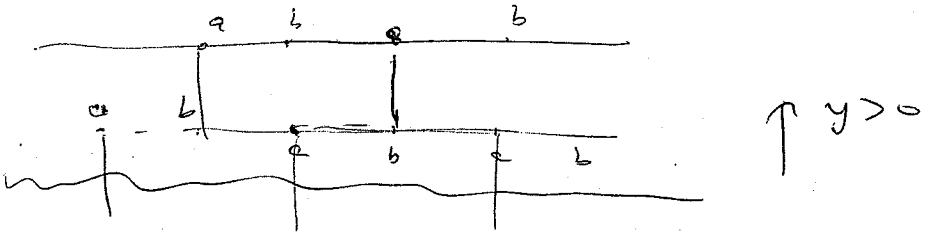






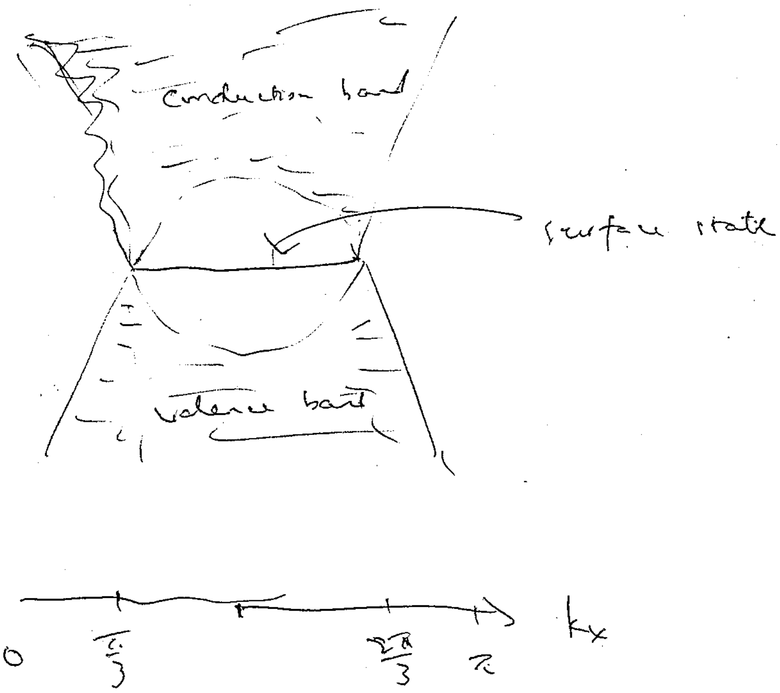
(0,2π)
(π,2π)
(0,π)
(0,0)
(π,0)
O
Y
O
L
O'
M_3
M_2
M_1
M
Y, L
O
z
y
x

surface



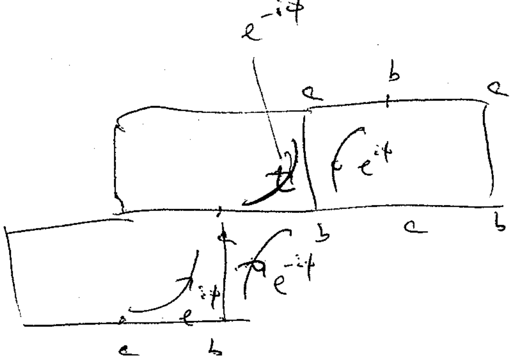
Then argument as before —

$E = 0$ state for $\frac{\pi}{3} < k_x < \frac{2\pi}{3}$



(2.6

New twist, add small extra hopping t' | Haldane PRL 61, 2015 (1988)



small $\phi' > 0$

extra terms

$a \to a$ $\quad -t'\left(e^{-i(k_x+k_y)}e^{i\phi} + e^{i(k_x+k_y)}e^{-i\phi}\right)$

$= -t' \cos[(k_x+k_y)-\phi]] \simeq -t'\left(\cos(k_x+k_y) + \sin(k_x+k_y)\phi\right)$

$b \to b$ $\quad -t'\left[e^{-i(k_x+k_y)}e^{-i\phi} + e^{i(k_x+k_y)}e^{+i\phi}\right)$

$= -t'\left[\cos(k_x+k_y+\phi)\right] = -t'\left(\cos(k_x+k_y) - \sin(k_x+k_y)\times\phi\right]$

extra terms $= -t'\cos(k_x+k_y)\,\hat{1}$

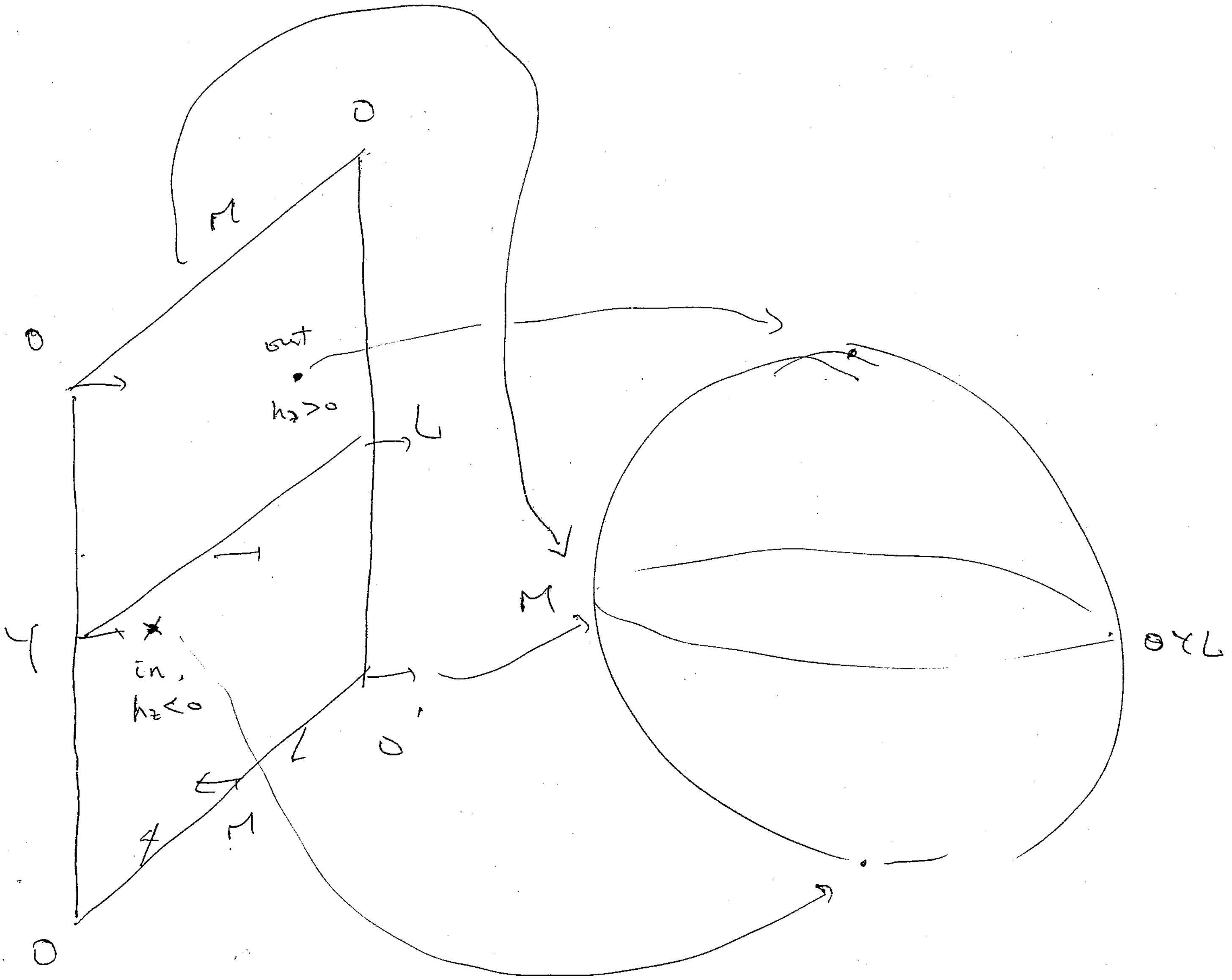
$-t' \underline{\sin(k_x+k_y) \times \phi \times \tau_z}$

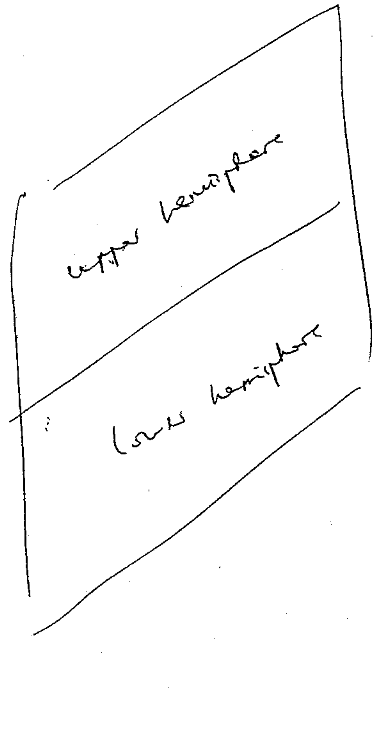
$h_z = t' \sin(k_x+k_y) \times \phi$

$k_x = \frac{2\pi}{3},\ k_y = 0$ $\quad \sin\frac{2\pi}{3} = +\frac{1}{2}$ $\quad h_z > 0$

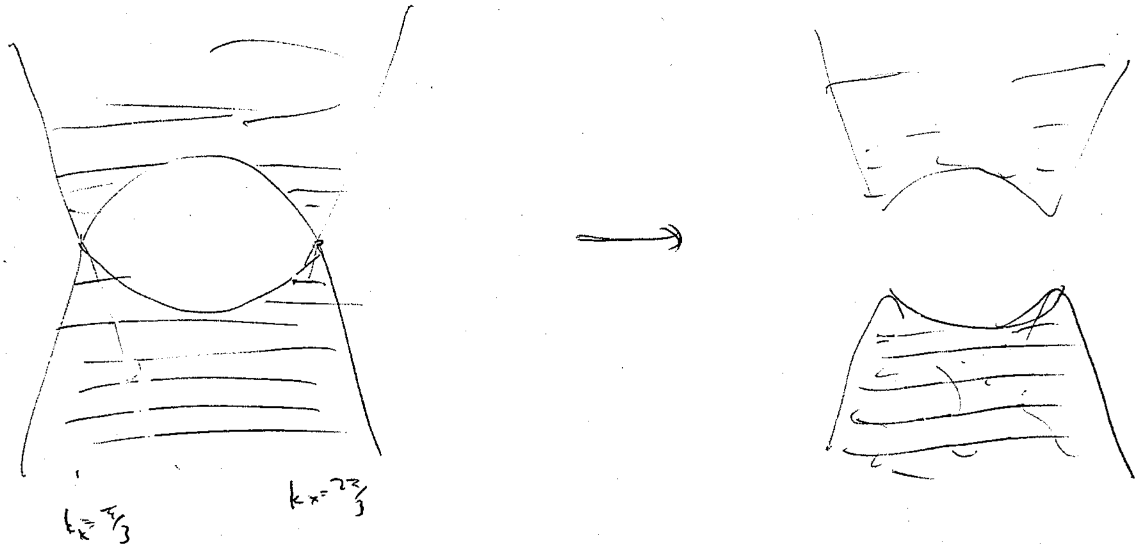
$k_x = \frac{\pi}{3},\ k_y = \pi$ $\quad \sin\frac{4\pi}{3} = -\frac{1}{2}$ $\quad h_z < 0$

then: fully gapped.



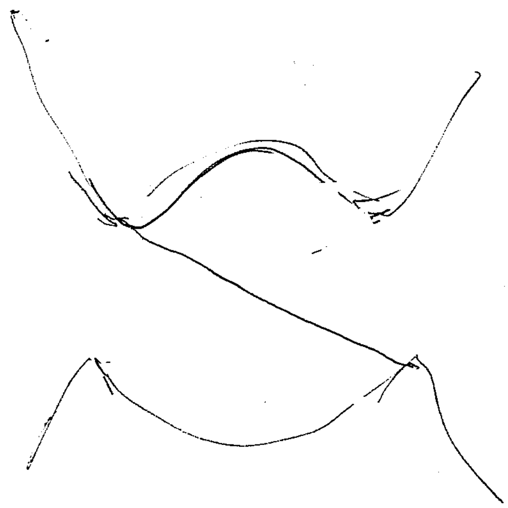


$\underline{Ch = 1}$



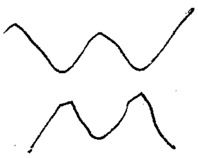
fully gapped

surface state



!

trivial insulator



smooth deformation of Hamiltonian cannot go from one to the other

Thouless et al
PRL 49 405 (1982)
PRB 31, 3372 (1985)

- gapped. $\sigma_{xx} = 0$
- $Ch \neq 0$ $\sigma_{xy} = \pm \frac{e^2}{h} \times (\text{Chern})$

$\sigma_{xy} = \pm \frac{e^2}{h} \cdot n$

'Integer' Quantum Hall

quantized Hall conductance

independent of material parameters

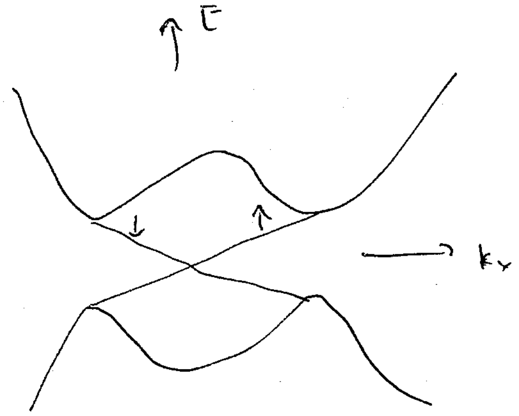
[c.f. trivial insulator $\sigma_{xx} = 0$, $\sigma_{xy} = 0$]

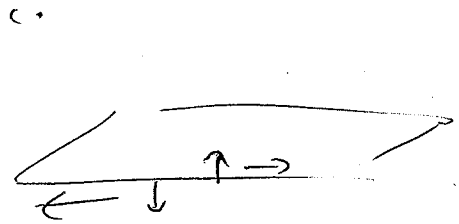
Quantum spin Hall insulator (2D) 13.1

Kane and Mele, PRL 95, 226801 (2005)

Two copies of quantum Hall,

opposite Chern numbers for ↑ & ↓





$\sigma_{xy}^{\uparrow} - \sigma_{xy}^{\downarrow}$

in other special case

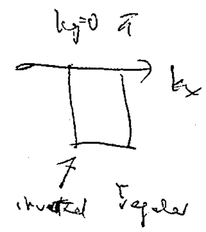
- possible model: $\phi_\uparrow = -\phi_\downarrow$ of §2, $\sigma_{xy}^{s} = \frac{2e^2}{h}$
- possible mechanism: spin-orbit coupling

dissipation less spin-current under spin-bias

- surface states remain even if spin-orbit
- crossing protected if time reversal symmetry preserved, (can deform Hamiltonian)

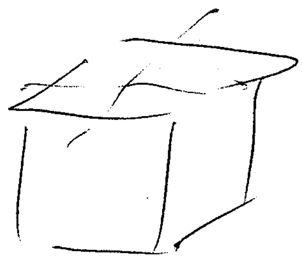
  splitting would violate time-reversal

  surface states protected though $\sigma_{xy}^{s} \neq \frac{2e^2}{h}$



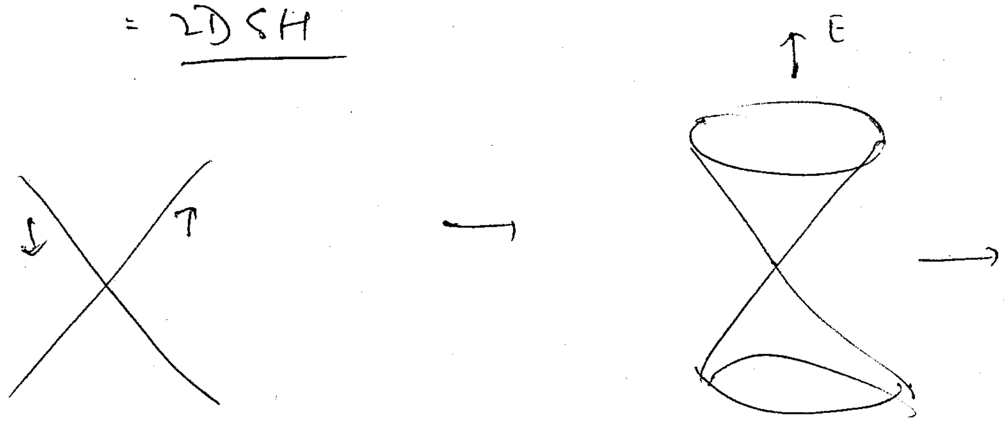
- $Z_2 = 1$: only odd number of crossings protected
- experimentally realized in quantum wells CdTe/HgTe
- König et al Science, 318, 766 (2007)
- Knez, Du, PRL 107, 136603 (2011) InAs/GaSb

3D. :



- each vertical cut

  = 2D SH



- can deform H, Dirac cone preserved

  if time reversal preserved.

  (note: spin not purely "up" or "down"

  but determined by specific system

  properties (symmetries))