Chapter 4

Ginzburg-Landau Theory

The Nobel Prize in Physics 2003
Alexei A. Abrikosov,
Vitaly L. Ginzburg, (1916-2009)
Anthony J. Leggett

Introduction

The BCS microscopic theory described in Chap. 2 gives an excellent account of the data in those cases to which it is applicable, namely, those in which the energy gap Δ is constant in space. However, there are many situations in which the entire interest derives from the existence of spatial inhomogeneity. For example, in treating the intermediate state of type I superconductors, we had to consider the interface where the superconducting state joined onto the normal state. This sort of spatial inhomogeneity becomes all-pervasive in the mixed state of type II superconductors. In such situations, the fully microscopic theory becomes very difficult, and much reliance is placed on the more macroscopic Ginzburg-Landau[1] (GL) theory.

Book of Tinkham Ch. 1

1-5 THE GINZBURG-LANDAU THEORY

Although a considerable body of work followed the appearance of the BCS theory, serving to substantiate its predictions for various processes such as nuclear relaxation and ultrasonic attenuation in which the energy gap and excitation spectrum play a key role, the most exciting developments of the ensuing decade came in another direction. This direction is epitomized by the Ginzburg-Landau (GL) theory of superconductivity, which concentrates entirely on the superconducting electrons rather than on excitations. Already in 1950, 7 years before BCS, Ginzburg and Landau[1] had introduced a complex pseudowave function ψ as an order parameter for the superconducting electrons such that the local density of superconducting electrons (as defined in the London equations) was given by

$$n_s = |\psi(x)|^2 \tag{1-15}$$

Then, using a variational principle and working from an assumed expansion of the free energy in powers of ψ and $\nabla\psi$, they derived a differential equation for ψ

$$\frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - \frac{e^*}{c}\mathbf{A}\right)^2\psi + \beta|\psi|^2\psi = -\alpha(T)\psi \tag{1-16}$$

which is very analogous to the Schrödinger equation for a free particle, but with a nonlinear term. The corresponding equation for the supercurrent

$$\mathbf{J}_s = \frac{e^*\hbar}{i2m^*}(\psi^* \nabla\psi - \psi \nabla\psi^*) - \frac{e^{*2}}{m^*c}|\psi|^2\mathbf{A} \qquad (1\text{-}17)$$

was also the same as the usual quantum-mechanical one for particles of charge e^* and mass m^*. With this formalism they were able to treat two features which were beyond the scope of the London theory, namely,

1. Nonlinear effects in fields strong enough to change n_s (or $|\psi|^2$)
2. Spatial variation of n_s

A major early triumph of the theory was in handling the so-called intermediate state of superconductors, in which superconducting and normal domains coexist in the presence of $H \approx H_c$. The interface between two such domains is shown schematically in Fig. 1-4.

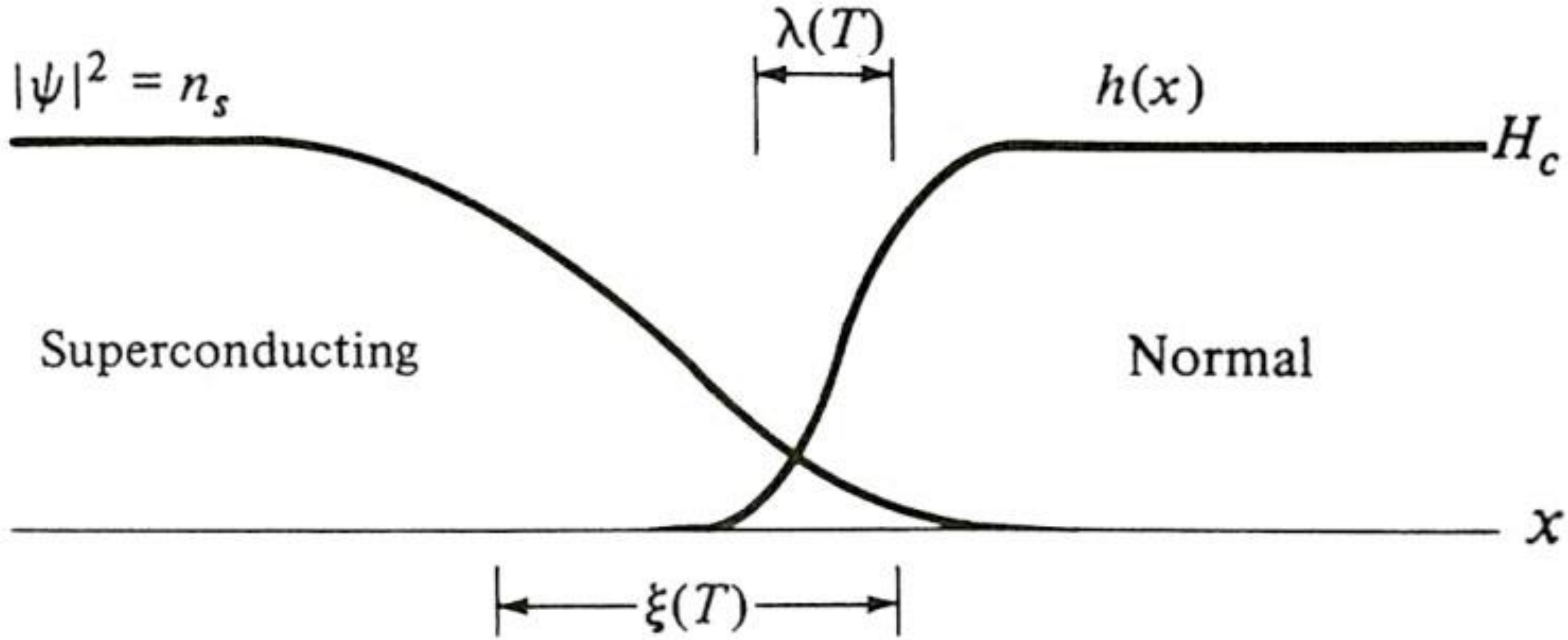

FIGURE 1-4

Interface between superconducting and normal domains in the intermediate state.

When first proposed, the theory appeared rather phenomenological, and its importance was not generally appreciated, especially in the Western literature. However, in 1959 Gor'kov[1] was able to show that the GL theory was in fact a limiting form of the microscopic theory of BCS (suitably generalized to deal with spatially varying situations), valid near T_c, in which ψ is directly proportional to the gap parameter Δ. More physically, ψ can be thought of as the wavefunction of the center-of-mass motion of the Cooper pairs. The GL theory is now universally accepted as a masterstroke of physical intuition which embodies in a most simple way the macroscopic quantum-mechanical nature of the superconducting state crucial for understanding its unique electrodynamic properties.

The Ginzburg-Landau theory introduces a characteristic length, now called the temperature-dependent coherence length,

$$\xi(T) = \frac{\hbar}{|2m^*\alpha(T)|^{1/2}} \tag{1-18}$$

which characterizes the distance over which $\psi(\mathbf{r})$ can vary without undue energy increase. In a pure superconductor far from T_c, $\xi(T) \approx \xi_0$, the Pippard coherence length; near T_c, however, $\xi(T)$ diverges as $(T_c - T)^{-1/2}$, since α vanishes as $(T - T_c)$. Thus these two "coherence lengths" are related but distinct.

The ratio of the two characteristic lengths defines the GL parameter

$$\kappa = \frac{\lambda}{\xi} \qquad (1\text{-}19)$$

Since λ and ξ diverge in the same way at T_c, this dimensionless ratio is approximately independent of temperature. For typical pure superconductors $\lambda \approx 500$ Å and $\xi \approx 3000$ Å, so $\kappa \ll 1$. In this case, one can see that there is a positive surface energy associated with a domain wall between normal and superconducting material in the intermediate state. [The qualitative argument is simply that one pays an energetic cost $\sim \xi H_c^2/8\pi$ for the variation of ψ from its superconducting value to zero, while reducing the diamagnetic energy only by $\sim \lambda H_c^2/8\pi$ (see Fig. 1-4).] This positive surface energy stabilizes a domain pattern with a scale of subdivision intermediate between the microscopic length ξ and the macroscopic sample size.

As originally proposed, this theory was a triumph of physical intuition, in which a pseudowavefunction $\psi(\mathbf{r})$ was introduced as a complex-order parameter. $|\psi(\mathbf{r})|^2$ was to represent the local density of superconducting electrons, $n_s(\mathbf{r})$. The theory was developed by applying a variational method to an assumed expansion of the free-energy density in powers of $|\psi|^2$ and $|\nabla\psi|^2$, leading to a pair of coupled differential equations for $\psi(\mathbf{r})$ and the vector potential $\mathbf{A}(\mathbf{r})$. The result was a generalization of the London theory to deal with situations in which n_s varied in space, and also to deal with the nonlinear response to fields strong enough to change n_s. The local approximation of the London electrodynamics was retained, however. Although quite successful in explaining intermediate-state phenomena, where the need for a theory capable of dealing with spatially inhomogeneous superconductivity was evident, this theory was generally given limited attention in the Western literature because of its phenomenological foundation.

This situation changed in 1959 when Gor'kov[1] showed that the GL theory was in fact derivable as a rigorous limiting case of the microscopic theory, suitably reformulated in terms of Green functions to allow treating a spatially inhomogeneous regime. The conditions for validity of the GL theory were shown to be a restriction to temperatures sufficiently near T_c and to spatial variations of ψ and $\mathbf{A}$ which were not too rapid. In this reevaluation of the GL theory, $\psi(\mathbf{r})$ turned out to be proportional to the gap parameter $\Delta(\mathbf{r})$, both being in general complex quantities. At first it was thought that $|\Delta(\mathbf{r})|$, found from solving the newly interpreted GL equations, was simply a BCS gap which might vary in space or with applied magnetic fields, or both. This led to a period in which experiments were (incorrectly) interpreted in this overly simple way. It has now become clear, however, that a solution to the GL equations for a given problem is only a useful first step toward understanding the spectral density of excitations. The key point is that fields, currents, and gradients act as "pairbreakers" which tend to blur out the sharp edge of the BCS gap as well as reducing the value of Δ.

$$\mathbf{j} = -\frac{c}{4\pi\lambda_L^2}\mathbf{A} \; ;$$

London Equation

The greatest value of the theory remains in treating the macroscopic behavior of superconductors, in which the overall free energy is important instead of the detailed spectrum of excitations. Thus, it will be quite reliable in predicting critical fields and the spatial structure of $\psi(\mathbf{r})$ in nonuniform situations. It also provides the qualitative framework for understanding the dramatic supercurrent behavior as a consequence of quantum properties on a macroscopic scale.

Although one could in principle now give a *derivation* of the GL theory following Gor'kov, this would require techniques beyond the level of our presentation. Instead, we shall follow Ginzburg and Landau in phenomenologically postulating the form of the theory on grounds of plausibility, and then simply appealing to the results of microscopic theory (or experiment) to evaluate the few parameters of the theory by considering simple special cases.

4-1 THE GINZBURG-LANDAU FREE ENERGY

The basic postulate of GL is that if ψ is small and varies slowly in space, the free-energy density f can be expanded in a series of the form

$$f = f_{n0} + \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 + \frac{1}{2m^*} \left| \left(\frac{\hbar}{i} \nabla - \frac{e^*}{c} \mathbf{A} \right) \psi \right|^2 + \frac{h^2}{8\pi} \tag{4-1}$$

Evidently, if $\psi = 0$, this reduces properly to the free energy of the normal state $f_{n0} + h^2/8\pi$, where $f_{n0}(T) = f_{n0}(0) - \frac{1}{2}\gamma T^2$. We now consider the remaining three terms describing the superconducting effects.

In the absence of fields and gradients, we have

$$f_s - f_n = \alpha |\psi|^2 + \tfrac{1}{2}\beta |\psi|^4 \tag{4-2}$$

which can be viewed as a series expansion in powers of $|\psi|^2$ or n_s, in which only the first two terms are retained.[1] These two terms should be adequate so long as one stays near the second-order phase transition at T_c, where the order parameter $|\psi|^2 \to 0$.

As is illustrated in Fig. 4-1, two cases arise, depending on whether α is positive or negative. If α is positive, the minimum free energy occurs at $|\psi|^2 = 0$, corresponding to the normal state. On the other hand, if $\alpha < 0$, the minimum occurs when

$$|\psi|^2 = |\psi_\infty|^2 \equiv -\frac{\alpha}{\beta} \qquad (4\text{-}3)$$

where the notation ψ_∞ is conventionally used because ψ approaches this value infinitely deep in the interior of the superconductor, where it is screened from any surface fields or currents. When this value of ψ is substituted back into (4-2), one finds

$$f_s - f_n = \frac{-H_c^2}{8\pi} = \frac{-\alpha^2}{2\beta} \qquad (4\text{-}4)$$

using the definition of the thermodynamic critical field H_c.

Evidently $\alpha(T)$ must change from positive to negative at T_c, since by definition T_c is the highest temperature at which $|\psi|^2 \neq 0$ gives a lower free energy than $|\psi|^2 = 0$. Making a Taylor's series expansion of $\alpha(T)$ about T_c, and keeping only the leading term, one has

$$\alpha(t) = \alpha'(t - 1) \qquad \alpha' > 0 \qquad (4\text{-}5)$$

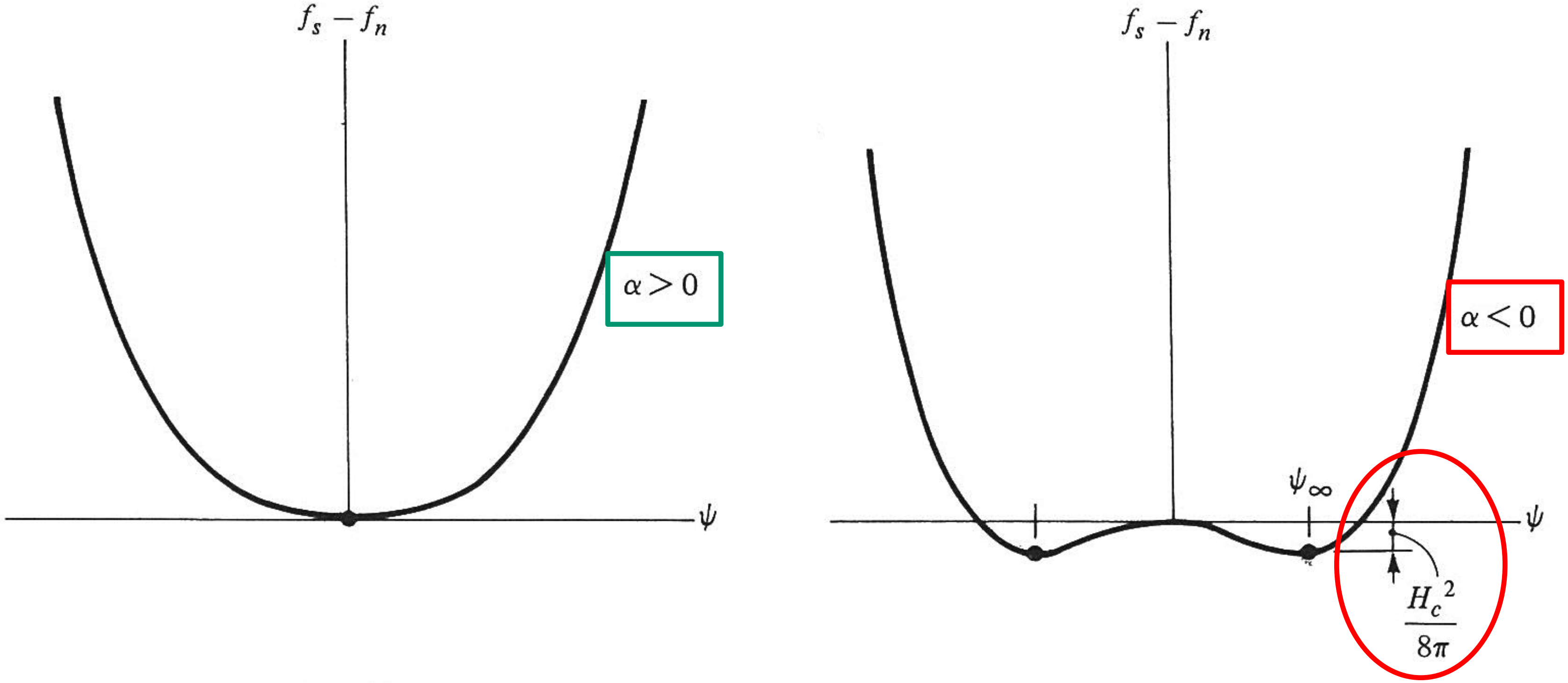

FIGURE 4-1
Ginzburg-Landau free-energy functions for $T > T_c(\alpha > 0)$ and for $T < T_c(\alpha < 0)$. Heavy dots indicate equilibrium positions. For simplicity, ψ has been taken to be real.

where $t = T/T_c$. Note that in view of (4-4), this assumption is consistent with the linear variation of H_c with $(1 - t)$, if β is regular at T_c. Putting these temperature variations of α and β into (4-3), we see that

$$|\psi|^2 \propto (1 - t) \tag{4-6}$$

near T_c. This is consistent with correlating $|\psi|^2$ with n_s, the density of superconducting electrons in the London theory, since $n_s \propto \lambda^{-2} \propto (1 - t)$ near T_c.

To make these considerations quantitative, we now consider the remaining term in the expansion (4-1), the term dealing with fields and gradients. If we write $\psi = |\psi| e^{i\varphi}$, it takes on the more transparent form

$$\frac{1}{2m^*}\left[\hbar^2(\nabla|\psi|)^2 + \left(\hbar\nabla\varphi - \frac{e^*\mathbf{A}}{c}\right)^2 |\psi|^2\right] \tag{4-7}$$

The first term gives the extra energy associated with gradients in the *magnitude* of the order parameter, as in a domain wall. The second term gives the kinetic energy associated with supercurrents in a gauge-invariant form. In the London gauge, φ is constant, and this term is simply $e^{*2}A^2|\psi|^2/2m^*c^2$. Equating this to the energy density for a London superconductor as given in (3-28), namely $A^2/8\pi\lambda_{\text{eff}}^2$, we find

$$\lambda_{\text{eff}}^2 = \frac{m^*c^2}{4\pi|\psi|^2 e^{*2}} \tag{4-8}$$

With the identification $n_s^* = |\psi|^2$, this agrees with the usual definition of the London penetration depth, except for the presence of the starred effective number, mass, and charge values. The kinetic-energy density term can then be written as $n_s^*(\frac{1}{2}m^*v_s^2)$, where the supercurrent velocity is given by

$$m^*\mathbf{v}_s = \mathbf{p}_s - \frac{e^*\mathbf{A}}{c} = \hbar\nabla\varphi - \frac{e^*\mathbf{A}}{c} \tag{4-9}$$

It should be noted that by writing the energy associated with the vector potential in the simple form (4-7), we have restricted the theory to the approximation of *local* electrodynamics. An expression of the sort found in (3-26) would be required to describe properly a *nonlocal* superconductor.

Now let us deal with the starred effective parameters. In the original formulation of the theory, it was thought that e^* and m^* would be the normal electronic values. However, experimental data turned out to be fitted better if $e^* \approx 2e$. The microscopic pairing theory of superconductivity makes it unambiguous that $e^* = 2e$ exactly, the charge of a pair of electrons. In the free-electron approximation, it would then be natural to take $m^* = 2m$ and $n_s^* = \frac{1}{2}n_s$, where n_s is the number of single electrons in the condensate. With these conventions, $n_s^* e^{*2}/m^* = n_s e^2/m$, so the London penetration depth is unchanged by the pairing.

The situation is more complicated in real metals. Band structure and phonon "dressing" effects may lead to an effective mass for a single electron in the normal state which typically differs from the free-electron mass by 50 percent. Moreover, the most important class of applications of GL theory is to dirty superconductors, in which $\lambda_{\text{eff}}^2 \approx \lambda_L^2(\xi_0/\ell) \gg \lambda_L^2$. These increased penetration depths can be attributed formally to either an increase in m^* or a decrease in n_s. It

In view of the experimental inaccessibility of m^*, we can assign it an arbitrary value, and it is probably most convenient to choose twice the mass of the free electron. (This arbitrariness was emphasized by de Gennes, who suggested that one could equally well take the mass of the sun!) With $m^* = 2m$ fixed, all variations of λ, whether due to temperature, band structure, phonons, impurities, or even nonlocal electrodynamics, are taken up by an appropriate value of $|\psi_\infty|^2 = n_s^* = \frac{1}{2}n_s$. Even at $T = 0$ this number will no longer correspond to any obvious integral number of electrons per atom. Rather, our point of view is that n_s simply measures that part of the oscillator strength in the sum rule

$$\int_0^\infty \sigma_1(\omega)\, d\omega = \frac{\pi n e^2}{2m} \tag{4-10}$$

which is located in the superfluid response at $\omega = 0$ in the form of a term $(\pi n_s e^2/2m)\,\delta(\omega)$.

Having noted that $e^* = 2e$, and taking the convention that $m^* = 2m$, we can now evaluate the parameters of the theory by solving (4-3), (4-4), and (4-8). The results are

$$|\psi_\infty|^2 \equiv n_s^* \equiv \frac{n_s}{2} = \frac{m^*c^2}{4\pi e^{*2}\lambda_{\text{eff}}^2} = \frac{mc^2}{8\pi e^2\lambda_{\text{eff}}^2} \tag{4-11a}$$

$$\alpha(T) = -\frac{e^{*2}}{m^*c^2}H_c^2(T)\lambda_{\text{eff}}^2(T) = -\frac{2e^2}{mc^2}H_c^2(T)\lambda_{\text{eff}}^2(T) \tag{4-11b}$$

$$\beta(T) = \frac{4\pi e^{*4}}{m^{*2}c^4}H_c^2(T)\lambda_{\text{eff}}^4(T) = \frac{16\pi e^4}{m^2c^4}H_c^2(T)\lambda_{\text{eff}}^4(T) \tag{4-11c}$$

where e and m are now the usual free-electron values, and λ_{eff} and H_c are measured values, or those computed from the microscopic theory.

Since the true electrodynamics of superconductors is nonlocal, it is evident that this prescription in terms of an effective London λ is straightforward only sufficiently near T_c that $\lambda_L(T) > \xi_0$, or in samples dirty enough that $\xi \approx \ell < \lambda(T)$, that is, where the nonlocality is unimportant. It is only under these conditions that the GL theory is really exact. Fortunately, the qualitative conclusions of the theory seem to have much wider validity; semiquantitative results can usually be obtained even when nonlocality is important by using a suitable λ_{eff}, such as we computed for films, (3-29) or (3-31). For pure bulk samples, as noted above, it is probably most appropriate to take $\lambda_{\text{eff}} = \lambda_{\text{exp}}$, the experimental value, if one attempts to apply the theory far enough below T_c that the nonlocality of the electrodynamics makes $\lambda_{\text{exp}} > \lambda_L$.

It is worth noting that if we insert the empirical approximations $H_c \propto (1 - t^2)$ and $\lambda^{-2} \propto (1 - t^4)$ into (4-11), we find

$$|\psi_\infty|^2 \propto 1 - t^4 \approx 4(1 - t)$$

$$\alpha \propto \frac{1 - t^2}{1 + t^2} \approx 1 - t$$

$$\beta \propto \frac{1}{(1 + t^2)^2} \approx \text{const} \qquad (4\text{-}12)$$

Since the theory is usually exactly valid only very near T_c, it is customary to carry only the leading dependence on temperature; that is, $|\psi_\infty|^2$ and α are usually taken to vary as $(1 - t)$ and β is taken to be constant, as anticipated in our preliminary discussion. Still, the more complete forms in (4-12) give some idea of how the theory can be extended over a wider range of temperature, and they have a certain amount of experimental support.

Finally, we recall that, although our discussion of (4-7) has centered on the kinetic energy of the supercurrent, this term also describes the energy associated with gradients in the magnitude of ψ. Moreover, no additional parameters are introduced, since gauge invariance requires a particular combination of ∇ and $\mathbf{A}$ in (4-1). Thus the coefficients in the theory are completely determined by the values of $\lambda_{\mathrm{eff}}(T)$ and $H_c(T)$. Since we showed earlier how the microscopic theory determines these parameters, we have effectively shown how the GL theory is set up to serve as an extension of BCS to the case of gradients and strong fields, but with a restriction to $T \approx T_c$.

4-2 THE GINZBURG-LANDAU DIFFERENTIAL EQUATIONS

In the absence of boundary conditions which impose fields, currents, or gradients, the free energy is minimized by having $\psi = \psi_\infty$ everywhere. On the other hand, when fields, currents, or gradients are imposed, $\psi(\mathbf{r}) = |\psi(\mathbf{r})|\, e^{i\varphi(\mathbf{r})}$ adjusts itself to minimize the overall free energy, given by the volume integral of (4-1). This variational problem leads, by standard methods, to the celebrated GL differential equations

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - \frac{e^*}{c}\mathbf{A}\right)^2\psi = 0 \tag{4-13}$$

and

$$\mathbf{J} = \frac{c}{4\pi}\operatorname{curl}\mathbf{h} = \frac{e^*\hbar}{2m^*i}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{e^{*2}}{m^*c}\psi^*\psi\mathbf{A} \tag{4-14}$$

or

$$\mathbf{J} = \frac{e^*}{m^*}|\psi|^2\left(\hbar\nabla\varphi - \frac{e^*}{c}\mathbf{A}\right) = e^*|\psi|^2\mathbf{v}_s \tag{4-14a}$$

where in the last step we have repeated the identification (4-9). Note that the current expression (4-14) has exactly the form of the usual quantum-mechanical expression for particles of mass m^*, charge e^*, and wavefunction $\psi(\mathbf{r})$. Similarly, apart from the nonlinear term, the first equation has the form of Schrödinger's equation for such particles, with energy eigenvalue $-\alpha$. The nonlinear term acts like a repulsive potential of ψ on itself, tending to favor wavefunctions $\psi(\mathbf{r})$ which are spread out as uniformly as possible in space.

In carrying through the variational procedure, boundary conditions must be provided. A possible choice, which assures that no current passes through the surface is

$$\left(\frac{\hbar}{i}\nabla - \frac{e^*}{c}\mathbf{A}\right)\psi\bigg|_n = 0 \qquad (4\text{-}15)$$

This is the boundary condition used by GL, and it is appropriate at an insulating surface. Using the microscopic theory, de Gennes[1] has shown that for a metal-superconductor interface with no current, (4-15) must be generalized to

$$\left(\frac{\hbar}{i}\nabla - \frac{e^*}{c}\mathbf{A}\right)\psi\bigg|_n = \frac{i}{b}\psi \qquad (4\text{-}15a)$$

where b is a real constant. It is easily seen that if $A_n = 0$, $\hbar b$ is the extrapolation length to the point outside the boundary at which ψ would go to zero if it maintained the slope it had at the surface. The value of b will depend on the nature of the material to which contact is made, approaching zero for a magnetic material and infinity for an insulator, with normal metals lying in between.

4-2.1 The Ginzburg-Landau Coherence Length

To help get a feeling for the differential equation (4-13), we first consider a simplified case in which no fields are present. Then $\mathbf{A} = 0$, and we can take ψ to be real, since the differential equation has only real coefficients. If we introduce a normalized wavefunction $f = \psi/\psi_\infty$, where $\psi_\infty^2 = -\alpha/\beta > 0$, the equation becomes (in one dimension)

$$\frac{\hbar^2}{2m^* |\alpha|} \frac{d^2 f}{dx^2} + f - f^3 = 0 \qquad (4\text{-}16)$$

This makes it natural to define the characteristic length $\xi(T)$ for variation of ψ by

$$\xi^2(T) = \frac{\hbar^2}{2m^* |\alpha(T)|} \propto \frac{1}{1-t} \qquad (4\text{-}17)$$

Note that this $\xi(T)$ is certainly not the same length as Pippard's ξ, which we used in our discussion of the nonlocal electrodynamics, since this $\xi(T)$ diverges at T_c whereas the electrodynamic ξ is essentially constant. In fact, on the face of it, it is not clear why they should even be related. We retain this traditional notation, despite its considerable power to confuse, because it is almost invariably used in the literature, and because it does turn out that $\xi(T) \approx \xi_0$ for pure materials well away from T_c. In terms of $\xi(T)$, (4-16) becomes

$$\xi^2(T)\frac{d^2f}{dx^2} + f - f^3 = 0 \tag{4-18}$$

The significance of $\xi(T)$ as a characteristic length for variation of ψ (or f) can be made even more evident by considering a linearized form of (4-18), in which we set $f(x) = 1 + g(x)$, where $g(x) \ll 1$. Then we have, to first order in g,

$$\xi^2 g''(x) + (1 + g) - (1 + 3g + \cdots) = 0$$

or

$$g'' = \left(\frac{2}{\xi^2}\right)g$$

so that

$$g(x) \sim e^{\pm\sqrt{2}\,x/\xi(T)} \tag{4-19}$$

which shows that a small disturbance of ψ from ψ_∞ will decay in a characteristic length of order $\xi(T)$.

Now that we have an idea of the significance of the length $\xi(T)$, let us see what its value is. Substituting the value of α from (4-11b) into the definition (4-17), we find

$$\xi(T) = \frac{\Phi_0}{2\sqrt{2}\,\pi H_c(T)\lambda_{\text{eff}}(T)} \qquad (4\text{-}20)$$

where

$$\Phi_0 = \frac{hc}{e^*} = \frac{hc}{2e} \qquad (4\text{-}21)$$

is the fluxoid quantum which will play an important role in our future discussions.

The fact that this $\xi(T)$ is at least related to the ξ_0 of Pippard and BCS is shown by the existence of the relation

$$\Phi_0 = \left(\frac{2}{3}\right)^{1/2} \pi^2 \xi_0 \lambda_L(0) H_c(0) \qquad (4\text{-}22)$$

which follows readily from our earlier BCS results $\xi_0 = \hbar v_F/\pi\Delta(0)$ and $H_c^2(0)/8\pi = \frac{1}{2}N(0)\Delta^2(0)$, if one assumes the free-electron relation between $N(0)$ and n. Combining (4-20) and (4-22), we can write

$$\frac{\xi(T)}{\xi_0} = \frac{\pi}{2\sqrt{3}} \frac{H_c(0)}{H_c(T)} \frac{\lambda_L(0)}{\lambda_{\text{eff}}(T)} \tag{4-23}$$

From this we can see that near T_c

Dirty limit: At very small mean free path ℓ in impure SC

$$\lambda \approx \lambda_L \ (\xi_0/\ell)^{1/2}$$

$$\xi(T) = 0.74 \frac{\xi_0}{(1-t)^{1/2}} \quad \text{pure} \tag{4-24a}$$

$$\xi(T) = 0.855 \frac{(\xi_0 \ell)^{1/2}}{(1-t)^{1/2}} \quad \text{dirty} \tag{4-24b}$$

The precise coefficients here were determined using the exact results of BCS in the limit of $T \approx T_c$, namely,

$$H_c(t) = 1.73\, H_c(0)(1-t) \tag{4-25}$$

$$\lambda_L(t) = \frac{\lambda_L(0)}{[2(1-t)]^{1/2}} \tag{4-26a}$$

$$\lambda_{\text{eff}}(t)\Big|_{\substack{\text{dirty}\\ \text{limit}}} = \lambda_L(t)\left(\frac{\xi_0}{1.33\ell}\right)^{1/2} \tag{4-26b}$$

The relation (4-24a), giving $\xi(T)$ for pure superconductors, has clear validity only in the extremely narrow temperature range near T_c in which the local electrodynamics are valid; outside this range, the appropriate effective value of ξ will be dependent on the sample configuration. Equation (4-24b) has a much broader range of validity for dirty superconductors, because there the local approximation remains good.

It is also useful to introduce the famous dimensionless Ginzburg-Landau parameter κ, which is defined as the ratio of the two characteristic lengths

$$\kappa = \frac{\lambda_{\text{eff}}(T)}{\xi(T)} = \frac{2\sqrt{2}\,\pi H_c(T)\lambda_{\text{eff}}^2(T)}{\Phi_0} \tag{4-27}$$

With the empirical approximations $H_c \propto (1 - t^2)$ and $\lambda^{-2} \propto (1 - t^4)$, we see that κ should vary as $(1 + t^2)^{-1}$. Of course this is only a rough approximation, but we can safely conclude that κ is regular at T_c, and varies only slowly with temperature. Using the numerical results above, we find the following results in the pure and dirty limits at T_c:

$$\kappa = 0.96\,\frac{\lambda_L(0)}{\xi_0} \qquad \text{pure} \tag{4-27a}$$

$$\kappa = 0.715\,\frac{\lambda_L(0)}{\ell} \qquad \text{dirty} \tag{4-27b}$$

In typical pure superconductors, $\kappa \ll 1$, but in dirty superconductors κ may be much greater than 1. As will be discussed later in more detail, the value $\kappa = 1/\sqrt{2}$ separates superconductors of types I and II.

4-4 CRITICAL CURRENT OF A THIN WIRE OR FILM

Having taken a quick look at the calculation of the interface energy, in which one immediately finds that numerical solutions are required, let us now step back and treat a number of important simpler examples in which exact analytic solutions are possible. In this way we will develop some familiarity with the GL theory before returning to more complex problems.

The very simplest applications are those in which the perturbing fields and currents are so weak that $|\psi| = \psi_\infty$ everywhere, and the GL theory reduces to the London theory.

A more interesting class of examples is that in which strong fields or currents change $|\psi|$ from ψ_∞, but in which $|\psi|$ has the same value everywhere. This will be the case if the sample is a thin wire or film so oriented with respect to any external field that any variation of $|\psi|$ would need to occur in a thickness $d \ll \xi(T)$. In that case, the term in the free energy proportional to $(\nabla|\psi|)^2$ would give an excessively large contribution if any substantial variations occurred. As a result they do not, and we can approximate $\psi(\mathbf{r})$ by $|\psi|e^{i\varphi(\mathbf{r})}$, where $|\psi|$ is constant. In this case, the expressions for the current and free-energy densities take on the simple forms

$$\mathbf{J}_s = \frac{2e}{m^*}|\psi|^2\left(\hbar\nabla\varphi - \frac{2e}{c}\mathbf{A}\right) = 2e|\psi|^2\mathbf{v}_s \tag{4-32}$$

$$f = f_{n0} + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + |\psi|^2\frac{1}{2}m^*v_s^2 + \frac{h^2}{8\pi} \tag{4-33}$$

Although it is our standard convention to set $m^* = 2m$, we retain the more general formulas here to permit other normalizations of ψ to be used, if desired, and also as a reminder of the conventional nature of the parameter m^*.

Let us now apply these equations to treat the case of a uniform current density through a thin film or wire. Since the total energy due to the field term $h^2/8\pi$ is less than the kinetic energy of the current by a factor of the order of the ratio of the cross-sectional area of the conductor to λ^2, we can always neglect it for a sufficiently thin conductor. Then, for a given v_s, we can minimize (4-33) to find the optimum value of $|\psi|^2$. The result is

$$|\psi|^2 = \psi_\infty^2\left(1 - \frac{m^*v_s^2}{2|\alpha|}\right) = \psi_\infty^2\left[1 - \left(\frac{\xi m^* v_s}{\hbar}\right)^2\right] \tag{4-34}$$

where the second form is stated in terms of ξ and $m^* v_s$, quantities invariant under changes in conventions. The corresponding current is

$$J_s = 2e\psi_\infty^2 \left(1 - \frac{m^* v_s^2}{2|\alpha|}\right) v_s \qquad (4\text{-}35)$$

As indicated in Fig. 4-3, this has a maximum value when $\partial J_s/\partial v_s = 0$, namely, when $\frac{1}{2} m^* v_s^2 = |\alpha|/3$ and $|\psi|^2/\psi_\infty^2 = 2/3$. We identify this maximum current with the critical current. Thus,

$$J_c = 2e\psi_\infty^2 \frac{2}{3}\left(\frac{2}{3}\frac{|\alpha|}{m^*}\right)^{1/2} = \frac{cH_c(T)}{3\sqrt{6}\pi\lambda(T)} \propto (1 - t)^{3/2} \qquad (4\text{-}36)$$

where, again, the second form is entirely in terms of operationally significant quantities and the indicated proportionality to $(1 - t)^{3/2}$ holds near T_c. The corresponding critical momentum is

$$p_c = m^* v_c = \frac{\hbar}{\sqrt{3}\xi(T)} \qquad (4\text{-}37)$$

The critical velocity itself is poorly defined, since it depends on the conventional choice of m^*.

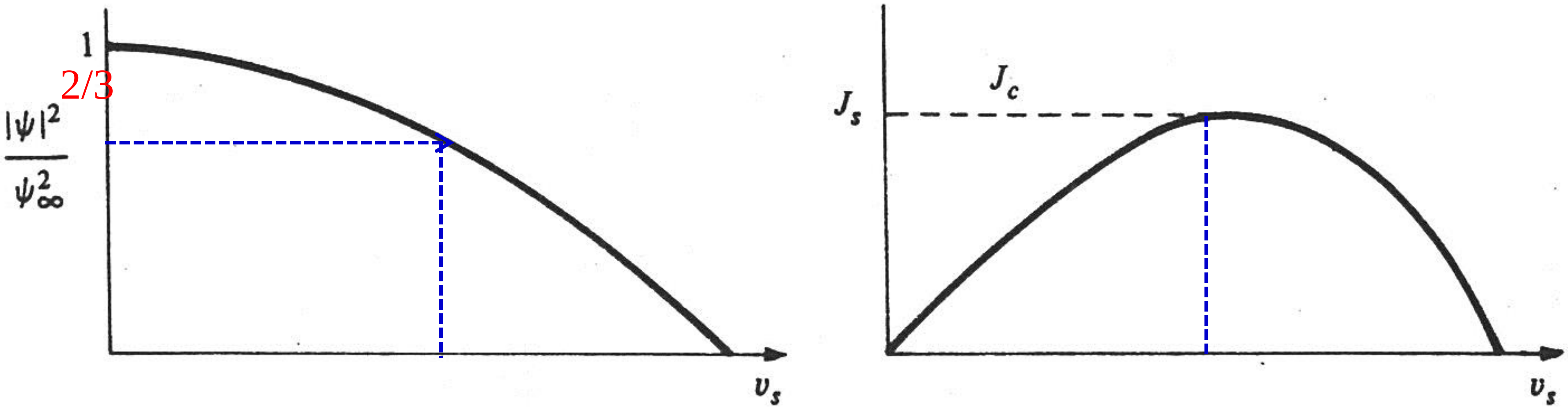

FIGURE 4-3
Variation of $|\psi|^2$ and of J_s with the superfluid velocity v_s.

It may be noted that we have taken v_s rather than $J_s \propto |\psi|^2 v_s$ as our independent variable. This was not a capricious choice; it was necessary since we are using the Helmholtz free energy, which is appropriate only if there are no induced emfs to effect energy interchanges with the source of current. This corresponds to specifying v_s, since an emf is needed to change that. If we wish to use the *current* as independent variable, then we must introduce a Legendre transformation on the free energy, as we did in (3-40) in dealing with magnetic energies. The appropriate term to subtract here to take account of work done by the generator is $m^* \mathbf{v}_s \cdot \mathbf{J}_s / 2e$, so we could consider a Gibbs free-energy density

$$g = f - \frac{m^* v_s J_s}{2e} \qquad (4\text{-}38a)$$

or

$$g = f_{n0} + \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 - \frac{m^* J_s^2}{8e^2 |\psi|^2} + \frac{h^2}{8\pi} \qquad (4\text{-}38b)$$

where we have used (4-32) to eliminate v_s. Minimizing this with respect to $|\psi|^2$ for given J_s leads to a cubic equation in $|\psi|^2$. Although algebraically more awkward, this condition is consistent with what we found above. For example, we can write it in the form

$$\frac{m^* J_s^2}{8e^2} = -\alpha |\psi|^4 - \beta |\psi|^6 \qquad (4\text{-}39)$$

whose maximum value occurs when $|\psi|^2 = -\frac{2}{3}(\alpha/\beta) = \frac{2}{3}\psi_\infty^2$, at which J_s has the critical value J_c found in (4-36).

It is of interest to compare this GL critical current with that of the London theory, where it is found by equating the density of kinetic energy to that of condensation energy

$$\tfrac{1}{2}n_s m v_s^2 = \frac{2\pi}{c^2}\lambda^2 J_s^2 = \frac{H_c^2}{8\pi}$$

so that

$$J_c = \frac{cH_c}{4\pi\lambda} \qquad (4\text{-}40)$$

This exceeds the more exact GL result (4-36) by a factor of $(3\sqrt{6}/4) = 1.84$ because it fails to take account of the decrease in $|\psi|^2$ with increasing current given by the nonlinear treatment.

It is also of interest to compare the GL result with that of the microscopic theory, where, of course, numerical computations must be made except in special cases. Bardeen has given a very useful review[1] of such calculations. Near T_c, the GL results are recovered, as expected. In the zero-temperature limit, the situation is quite different. In the presence of a uniform velocity $\mathbf{v}_s$, the quasi-particle energies are shifted by $\hbar\mathbf{k}\cdot\mathbf{v}_s$. [This may be seen from (2-108) by noting that a velocity $e\mathbf{a}(0)/mc$ is induced by a uniform vector potential $\mathbf{a}(0)$.] Thus, the gap goes to zero for some states when

$\Delta(0)$

$$v_s = \frac{\Delta(0)}{\hbar k_F} = \frac{\hbar}{\pi m \xi_0} \qquad (4\text{-}41)$$

Below this "depairing velocity," all electrons contribute to the supercurrent, and J_s is strictly proportional to v_s. Above this depairing velocity, some excitations occur at zero energy, the gap drops precipitously, and the maximum possible current is only 2 percent more than that at the velocity where depairing begins. The resulting $J_s(v_s)$ curve for $T = 0$ therefore shows a linear rise followed by a very steep drop to zero, in marked contrast with the GL result plotted in Fig. 4-3, appropriate near T_c.

Experimental confirmation of these results is most straightforward if both transverse dimensions of the conductor can be made small compared to both λ and ξ. It is then safe to take both J_s and $|\psi|^2$ to be constant over the cross section, as is assumed in the theory. Some of the first careful experiments on samples of this sort were those of Hunt,[2] who worked on very narrow strips of a thin evaporated film. More recently, several groups have been working with tin "whiskers" only about 1 μm (micrometer) in diameter, generally composed of a single crystal and having smooth surfaces, which are nearly ideal for the purpose. Since these experiments have concentrated on the fluctuation effects giving rise to resistance even at currents below the J_c computed above, we shall defer further discussion of them until a later chapter.

For reasons of experimental simplicity, many other measurements of critical currents have been made on thin-film samples which are not narrow on the scale of λ or ξ. With these, the measured J_c is usually much less than (4-36) for a number of reasons. First, it is somewhat difficult to make films of uniform thickness and structure. More seriously, the electrodynamic equations cause the supercurrent to pile up at the edges of the film because the external magnetic flux density is greatest there as the flux lines circle the film strip. This effect makes the current density nonuniform, and also emphasizes the properties of the edges of the film, which generally are thinner and less perfect. This problem can be minimized in three ways: (1) One can simply make the strip narrow enough so that the product of the thickness d and the width w is less than λ^2; in this case, J_s will be nearly uniform even if $w > \lambda$. (2) One can use a ground-plane geometry, in which the film under study is deposited on a larger thick superconductor with only a thin insulating layer in between; in this geometry, the superconducting substrate forces the field lines to be parallel to the film, which in turn requires a uniform current density in the film. (3) One can use a cylindrical film, so that there are no edges, and symmetry guarantees a uniform current density provided a concentric current return is used. It is possible to reach critical currents within about 10 percent of the theoretical values by any of these techniques if enough care is taken.

while if $\tau = \infty$, we have the Lindhard[1] result that

$$\sigma_1(\omega, q) = \frac{3\pi}{4}\frac{ne^2}{mv_F q}\left(1 - \frac{\omega^2}{v_F^2 q^2}\right) \tag{4-10b}$$

for $\omega < qv_F$, and $\sigma_1 = 0$ for $\omega > qv_F$. Naturally, both of these expressions satisfy the sum rule (4-10). Speaking qualitatively, the normal-state oscillator strength lying at frequencies below the gap frequency $\omega_g = 2\Delta/\hbar$ will be converted to n_s in the transition to the superconducting state, while that above the gap will be relatively unaffected. If we consider a superconductor with local electrodynamics, so that the approximation $q = 0$ can be used, we see from (4-10a) that almost all of the oscillator strength will appear as n_s in a pure metal, where $\omega_g \tau > 1$; in this case, $n_s \approx n$, and $\lambda \approx \lambda_L$. On the other hand, n_s/n will be reduced to something of the order of $\omega_g \tau \approx \ell/\xi_0$ if the metal is dirty enough to have $\omega_g \tau < 1$. As a result, in dirty superconductors we have $\lambda/\lambda_L = (n/n_s)^{1/2} \approx (\xi_0/\ell)^{1/2}$, a result obtained more rigorously in (2-123). If we consider instead a pure, nonlocal superconductor, (4-10b) implies that $n_s(q)/n \approx \omega_g/qv_F \approx 1/q\xi_0$. This q-dependent superfluid fraction is a reflection of the fall of $K(q)$ as $1/q$ in Fig. 3-2. Taking a typical value $q \approx 1/\lambda$ for the currents in the penetration layer, this implies that $\lambda_L^2/\lambda^2 = n_s/n \approx \lambda/\xi_0$, or $\lambda \approx (\lambda_L^2 \xi_0)^{1/3}$, as found more rigorously in (3-13). For use in the GL theory, which is local, one must take n_s to be such an average value appropriate to the actual penetration depth, not to λ_L. The survey in this paragraph reminds us of the power of the sum rule–energy gap argument in making simple physical estimates of the effective superfluid density in diverse situations.