九十八學年第一學期 PHYS5310 Electrodynamics Midterm exam (2 pages)

[Jackson Ch.1-4]　2009/11/17,　10:10am–12:00am, 教師：張存續

＊ 記得寫上學號，班別及姓名等。請依題號順序每頁答一題。

Formula: $\dfrac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \dfrac{1}{2l+1}\dfrac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)$; $\quad Y_{l0}(\theta,\phi) = \sqrt{\dfrac{2l+1}{4\pi}}P_l(\cos\theta)$

1. (10%, 10%) Using Dirac delta functions in the appropriate coordinates, express the following charge distributions as three-dimensional charge densities $\rho(\mathbf{x})$.

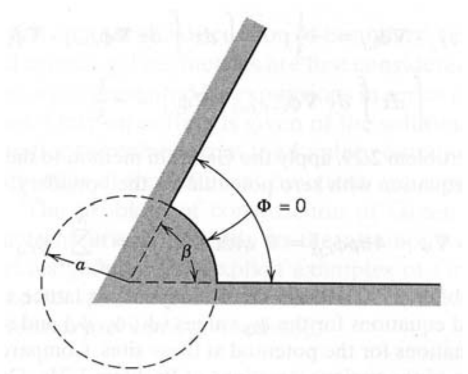
(a) In cylindrical coordinates, a charge Q per unit length uniformly distributed over a cylindrical surface of radius b.

(b) In spherical coordinates, a charge Q spreads uniformly over a ring of radius a located on the x-y plane.

2. (10%, 10%) The two-dimensional region, $\rho \geq a$, $0 \leq \phi \leq \beta$, is bounded by conducting surfaces at $\phi = 0$, $\rho \geq a$, and $\phi = \beta$ held at zero potential, as indicated in the figure below. At large ρ the potential is determined by some configuration of charges and/or conductors at fixed potentials.

(a) Write down a solution for the potential $\Phi(\rho,\phi)$ (that satisfies the boundary conditions for finite ρ).

(b) Keeping only the lowest non-vanishing terms, calculate the electric field components E_ρ and E_ϕ and also the surface-charge densities $\sigma(\rho,0)$ and $\sigma(a,\phi)$ on the two boundary surfaces.



3. (10%, 10%) Two point charges $+q$ and $-q$ are located on the z axis at $z=+a$ and $z=-a$, respectively.

(a) Find the electrostatic potential as an expansion in spherical harmonics and powers of r for both $r > a$ and $r < a$.

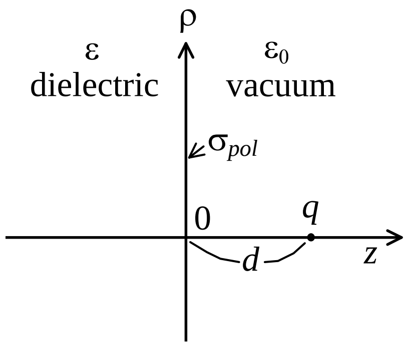
(b) Keeping the product $qa \equiv$constant, take the limit of $a \rightarrow 0$ and find the potential for $r \neq 0$ (i.e. $r > a$). This is by definition a dipole along the z axis and its potential.

4. (10%, 10%) A point charge q is placed a distant d in front of a semi-infinite dielectrics ε.

(a) Find the potential ϕ in both regions (z>0 and z<0).

(b) Find the polarization surface charge (σ_{pol}) at z=0.

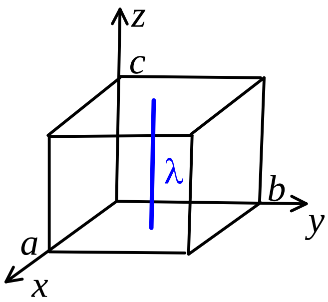
[Hint: Image charge method.]



5.(10%, 10%) Consider a grounded rectangular box with a line-change density in the middle ($\lambda = \lambda_0 \sin(z\pi / c)$).

(a) Find the Green function for the rectangular box $\nabla^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}')$ with $G(\mathbf{x}, \mathbf{x}') = 0$ at the boundaries. [Hint: Eigen function expansion.]

(b) Find the potential inside the box.



1.

(a) $\rho(r) = a\delta(r-b) \quad \lambda = \int \rho(r)\,dr d\theta = 2\pi b a \quad \Rightarrow \rho = \frac{\lambda}{2\pi b}\delta(r-b)$

(b) Lecture note p.48

The x-y plane is at $\theta = \pi/2$. The charge density $\rho(\mathbf{x})$ can be written as:

Let $\rho = c_0 \delta(r-a)\delta(\cos\theta)$

$$Q = \iiint_V \rho(\mathbf{x}) r^2 \sin\theta\, dr d\theta d\phi = c_0 2\pi a^2 \quad \Rightarrow \quad c_0 = \frac{Q}{2\pi a^2}$$

$$\rho(\mathbf{x}) = \frac{Q}{2\pi a^2}\delta(r-a)\delta(\cos\theta)$$

2. (a)

$$\Phi(\rho,\phi) = (a_0 + b_0 \ln\rho)(A_0 + B_0 \ln\phi) + \sum_\nu \left(a_\nu \rho^\nu + b_\nu \rho^{-\nu}\right)(A_\nu \cos\nu\phi + B_\nu \sin\nu\phi) \quad \text{(2.69 and 2.70)}$$

$$\Phi(\rho,\phi) = a_0 (1 + b_0' \ln\rho)(1 + B_0' \ln\phi) + \sum_\nu \left(a_\nu \rho^\nu + b_\nu \rho^{-\nu}\right)(A_\nu \cos\nu\phi + B_\nu \sin\nu\phi) \quad \text{(2.69 and 2.70)}$$

$$\Phi(\rho,0) = 0 \qquad \Rightarrow B_0' = 0,\ a_0 = 0,\ A_\nu = 0$$

$$\Phi(\rho,\beta) = 0 \qquad \Rightarrow \nu = \frac{m\pi}{\beta},\ m = 1,2,3\ldots$$

$$\Phi(\alpha,\phi) = 0 = \sum_{m=1}^{\infty}\left(A_m \alpha^{\frac{m\pi}{\beta}} + B_m \alpha^{\frac{-m\pi}{\beta}}\right)\sin\frac{m\pi\phi}{\beta} \quad \Rightarrow \ B_m = -A_m a^{\frac{2m\pi}{\beta}}$$

$$\Phi(\rho,\phi) = \sum_{m=1}^{\infty} A_m \left(\rho^{\frac{m\pi}{\beta}} - a^{\frac{2m\pi}{\beta}}\rho^{\frac{-m\pi}{\beta}}\right)\sin\frac{m\pi\phi}{\beta}$$

(b). The lowest non-vanishing term corresponding to *m*=1 is

$$\Phi(\rho,\phi) \approx A_1\left(\rho^{\pi/\beta} - a^{\frac{2\pi}{\beta}}\rho^{\frac{-\pi}{\beta}}\right)\sin\frac{\pi\phi}{\beta}$$

$$\mathrm{E}_\rho(\rho,\phi) = -\frac{\partial\Phi}{\partial\rho} = A_1\frac{\pi}{\beta}\left(\rho^{\pi/\beta - 1} - a^{\frac{2\pi}{\beta}}\rho^{\frac{-\pi}{\beta}-1}\right)\sin\frac{\pi\phi}{\beta}$$

$$\mathrm{E}_\phi(\rho,\phi) = -\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi} = A_1\frac{\pi}{\beta}\left(\rho^{\pi/\beta - 1} - a^{\frac{2\pi}{\beta}}\rho^{\frac{-\pi}{\beta}-1}\right)\cos\frac{\pi\phi}{\beta}$$

$$\Rightarrow \mathrm{E}_\rho(a,\phi) = A_1\frac{\pi}{\beta}a^{\pi/\beta - 1}\sin\frac{\pi\phi}{\beta},\ \ \mathrm{E}_\phi(a,\phi) = 0,\ \ \mathrm{E}_\rho(\rho,0) = 0$$

$$\mathrm{E}_\phi(\rho,0) = A_1\frac{\pi}{\beta}\left(\rho^{\pi/\beta - 1} - a^{\frac{2\pi}{\beta}}\rho^{\frac{-\pi}{\beta}-1}\right);\ \mathrm{E}_\phi(\rho,\beta) = A_1\frac{\pi}{\beta}\left(\rho^{\pi/\beta - 1} - a^{\frac{2\pi}{\beta}}\rho^{\frac{-\pi}{\beta}-1}\right)$$

$$\sigma = -\varepsilon_0 \frac{\partial \Phi}{\partial n} \quad \Rightarrow \begin{cases} \sigma(a,\phi) = \varepsilon_0 E_\rho\Big|_{\rho=a} = -2C_1\varepsilon_0 \dfrac{\pi}{\beta} a^{\frac{\pi}{\beta}-1} \sin\dfrac{\pi\phi}{\beta} \\ \sigma(\rho,0) = \varepsilon_0 E_\phi\Big|_{\phi=0} = -C_1\varepsilon_0 \dfrac{\pi}{\beta}\left(\rho^{\frac{\pi}{\beta}-1} - a^{\frac{2\pi}{\beta}} \rho^{-\frac{\pi}{\beta}-1}\right) \end{cases}$$

3. (a) Use the binominal expansion and expand in spherical coordinate.

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\vec{x}-\vec{x}_1|} - \frac{q}{|\vec{x}-\vec{x}_2|}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left[4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} \left(Y_{lm}^*(0,\phi_1) Y_{lm}(\theta,\phi) - Y_{lm}^*(\pi,\phi_2) Y_{lm}(\theta,\phi)\right)\right]$$

The problem is azimuthally symmetric, only *m*=0 terms survive.

$$Y_{lm}^*(0,\phi_1) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos 0) = \sqrt{\frac{2l+1}{4\pi}}$$

$$Y_{lm}^*(\pi,\phi_2) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\pi) = (-1)^l \sqrt{\frac{2l+1}{4\pi}}$$

$$\Rightarrow \Phi(\vec{x}) = \frac{q}{\varepsilon_0} \sum_{l=1} \frac{1}{2l+1} \frac{2l+1}{4\pi} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)\left[1-(-1)^l\right]$$

$$\Phi(\vec{x}) = \frac{q}{2\pi\varepsilon_0} \sum_{l=odd} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)$$

(a) $qa = \dfrac{p}{2}$, p is the dipole moment

$$\Phi(r>a,\theta,\phi) = \frac{q}{2\pi\varepsilon_0} \sum_{odd} \left(\frac{p}{2q}\right)^l \frac{1}{r^{l+1}} P_l(\cos\theta)$$

$$= \frac{1}{2\pi\varepsilon_0} \sum_{odd} \left(\frac{p}{2q}\right)^l \frac{p^l}{2^l q^{l-1}} P_l(\cos\theta)$$

$$\therefore \Phi \to \frac{p}{4\pi\varepsilon_0 r^2}\cos\theta$$

4. (a) To find Φ in the region $z \geq 0$, we put an image charge q' at $z=-d$.
To find Φ in the region $z \leq 0$, we put an image charge q'' at $z=+d$.

$$\left\{\begin{array}{c} q - q' = q'' \\ \frac{1}{\varepsilon_0}(q+q') = \frac{1}{\varepsilon} q'' \end{array}\right\} \Rightarrow q' = -\frac{\varepsilon - \varepsilon_0}{\varepsilon + \varepsilon_0} q \quad \& \quad q'' = \frac{2\varepsilon}{\varepsilon + \varepsilon_0} q$$

$$\Phi(\rho,\phi,z>0) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{(z-d)^2+\rho^2}} + \frac{q'}{\sqrt{(z+d)^2+\rho^2}}\right)$$

$$\Phi(\rho,\phi,z<0) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q''}{\sqrt{(z-d)^2+\rho^2}}\right)$$

(b) $\nabla \cdot \mathbf{P} = -\rho_{pol} \Rightarrow \sigma_{pol} = -\mathbf{P}\cdot\mathbf{n} = -\frac{q}{2\pi}\frac{\varepsilon_0(\varepsilon-\varepsilon_0)}{\varepsilon_0(\varepsilon+\varepsilon_0)}\frac{d}{(\rho^2+d^2)^{3/2}}$

where $\mathbf{P} = (\varepsilon - \varepsilon_0)\mathbf{E}(z<0) = -(\varepsilon-\varepsilon_0)\nabla\Phi(z<0)$

5. (a) Consider the corresponding eigenvalue problem with $f(\mathbf{x})=0$ and $\lambda \to k^2$

$\nabla^2\psi(\mathbf{x}) + k^2\psi(\mathbf{x}) = 0$ with the same b.c.

Let $\psi(\mathbf{x}) = X(x)Y(y)Z(z) \Rightarrow \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{-k_l^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{-k_m^2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{-k_n^2} + k^2 = 0$

$$\Rightarrow \left\{\begin{array}{l} X(x) = Ae^{ik_l x} + Be^{-ik_l x} \\ Y(x) = Be^{ik_l y} + Ce^{-ik_l y} \quad \text{with } k^2 = k_l^2 + k_m^2 + k_n^2 \\ Z(x) = De^{ik_l z} + Ee^{-ik_l z} \end{array}\right.$$

$$\text{b.c.} \left\{\begin{array}{l} X(x) = 0 \text{ at } x=0 \text{ and } a \\ Y(x) = 0 \text{ at } y=0 \text{ and } b \\ Z(x) = 0 \text{ at } z=0 \text{ and } c \end{array}\right. \Rightarrow \left\{\begin{array}{l} k_l = \frac{l\pi}{a} \\ k_m = \frac{m\pi}{b} \\ k_n = \frac{n\pi}{c} \end{array}\right. \text{ and } \left\{\begin{array}{l} X = \sin\frac{l\pi x}{a}, \\ Y = \sin\frac{m\pi y}{b}, \\ Z = \sin\frac{n\pi z}{c}, \end{array}\right.$$

$$\Rightarrow k^2 = k_{lmn}^2 = \pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)$$

$$\Rightarrow \psi(\mathbf{x}) = \sqrt{\frac{8}{abc}}\sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}$$

Sub. $\psi(\mathbf{x})$ into (3.160): $G(\mathbf{x},\mathbf{x}') = 4\pi\sum_j \frac{\psi_j^*(\mathbf{x}')\psi_j(\mathbf{x})}{\lambda_j - \lambda}$, we obtain

$$G(\mathbf{x},\mathbf{x}') = \frac{32}{\pi abc}\sum_{l,m,n=1}^{\infty}\frac{\sin\frac{l\pi x'}{a}\sin\frac{l\pi x}{a}\sin\frac{m\pi y'}{b}\sin\frac{m\pi y}{b}\sin\frac{n\pi z'}{c}\sin\frac{n\pi z}{c}}{\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}}$$

(b) The potential is given by

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \underbrace{\Phi(\mathbf{x}')}_{=0}\frac{\partial}{\partial n'}G(\mathbf{x},\mathbf{x}')da'$$

$$\lambda = \lambda_0 \sin(\frac{z\pi}{c}) \Rightarrow \ \rho(\mathbf{x}') = \lambda_0 \sin(\frac{z'\pi}{c})\delta(x' - \frac{a}{2})\delta(y' - \frac{b}{2})$$

$$G(\mathbf{x},\mathbf{x}') = \frac{32}{\pi abc}\sum_{l,m,n=1}^{\infty}\frac{\sin\frac{l\pi x'}{a}\sin\frac{l\pi x}{a}\sin\frac{m\pi y'}{b}\sin\frac{m\pi y}{b}\sin\frac{n\pi z'}{c}\sin\frac{n\pi z}{c}}{\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}}$$

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\frac{32}{\pi abc}\lambda_0\sum_{l,m,n=1}^{\infty}\frac{\sin\frac{l\pi}{2}\sin\frac{m\pi}{2}\sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}}{\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}}\int_0^c \sin\frac{n\pi z'}{c}\sin\frac{\pi z}{c}\,dz'$$

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\frac{32}{\pi abc}\lambda_0\frac{c}{2}\sum_{l,m=1}^{\infty}\frac{\sin\frac{l\pi}{2}\sin\frac{m\pi}{2}\sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{\pi z}{c}}{\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{1}{c^2}}$$