九十八學年第一學期 PHYS5310 Electrodynamics Midterm Exam (1 page)

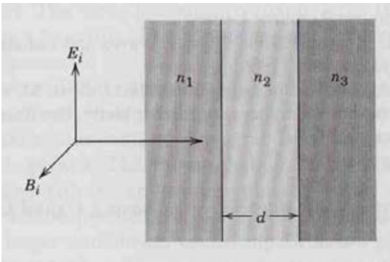
[Jackson Ch.5-7]　2010/01/14,　10:10am–12:00am, 教師：張存續

＊ 記得寫上學號，班別及姓名等。請依題號順序每頁答一題。

1. (10%, 10%) A plane wave is incident on a quartz slab of thickness d as shown in the figure. The indices of the reflection of the three media are n_1=1 (air), n_2=2 (quartz, $\varepsilon=4\varepsilon_0$), and n_3=1(vacuum).

(a) Calculate the transmission and reflection coefficients (ratios of transmitted and reflected Poynting's flux to the incident flux, i.e. power ratio.).

(b) Find the minimum thickness (d > 0) so that there is no reflected wave for a frequency of 3 GHz.



2. (10%, 10%) A plane wave is incident normally on a perfectly absorbing flat screen.

(a) Write down the law of conservation of linear momentum, and show that the radiation pressure exerted on the screen is equal to the field energy per unit volume in the wave.

(b) In the neighborhood of the Earth, the flux of electromagnetic energy from the Sun is approximately 1.5 kW/m². If an interplanetary "sailplane" had a sail of mass 1g/m² of area and negligible other weight, what would be its maximum acceleration in meters per second squared due to the solar radiation pressure?

[Hint: $T_{\alpha\beta} = \varepsilon_0\left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{E} + c^2\mathbf{B}\cdot\mathbf{B}\right)\delta_{\alpha\beta}\right]$]

3. (10%, 10%) Conservation of energy.

(a) Prove the Poynting's theorem for a system of particles and electromagnetic waves. [Hint: Starting from the rate of work done by the electromagnetic fields on the charged particles.]

(b) Show the complex Poynting's theorem using the phasor representation.

4. (10%, 10%) The generalized dielectric constant.

Assume there are N molecules per unit volume and Z electrons per molecule. The electrons could be classified into two groups, bound electrons and free electrons. Each molecule contains f_j electrons with binding frequency ω_j and damping factor γ_j. The free electrons are denoted as f_0, ω_0=0, and γ_0. $Z = f_0 + \sum f_j$.

(a) Find the general polarization **P** and the general form of the complex dielectric constant.

(b) For a uniform medium (independent of position), $\mathbf{D}(\omega) = \varepsilon(\omega)\mathbf{E}(\omega)$ is a constitutive relation in ω-space valid for all ω. How about the relation between $\mathbf{D}(t)$ and $\mathbf{E}(t)$? Do they have simple relation? When? [Hint: Use the Fourier transformation.]

5.(10%, 10%) A magnetically "hard" material is in the shape of sphere of radius a. The sphere has a uniform permanent magnetization $\mathbf{M}=M_0\hat{\mathbf{z}}$.

(a) Find the volume current density ($\mathbf{J}_M = \nabla\times\mathbf{M}$) and the surface current density ($\mathbf{K}_M = \mathbf{M}\times\hat{\mathbf{n}}$).

(b) Find the magnetic field inside the magnetized sphere. [Hint: you can use either scalar potential or vector potential.]

1.

$$\begin{cases}\mathbf{E} = E_0 e^{ikz}\mathbf{e}_x \\ \mathbf{H} = \sqrt{\frac{\varepsilon}{\mu}}\mathbf{e}_z \times \mathbf{E} = E_0\sqrt{\frac{\varepsilon}{\mu}}e^{ikz}\mathbf{e}_y\end{cases} \qquad k = \sqrt{\varepsilon\mu}\omega,\ k_2 = 2k_1$$

$$\text{Region I}\begin{cases}E_{x\mathrm{I}} = e^{\mathrm{i}k_1 z} + Ae^{-\mathrm{i}k_1 z} \\ H_{y\mathrm{I}} = \sqrt{\frac{\varepsilon_0}{\mu_0}}e^{\mathrm{i}k_1 z} - \sqrt{\frac{\varepsilon_0}{\mu_0}}Ae^{-\mathrm{i}k_1 z}\end{cases}$$

$$\text{Region II}\begin{cases}E_{x\mathrm{II}} = Be^{\mathrm{i}k_2 z} + Ce^{-\mathrm{i}k_2 z} \\ H_{y\mathrm{II}} = \sqrt{\frac{4\varepsilon_0}{\mu_0}}Be^{\mathrm{i}k_2 z} - \sqrt{\frac{4\varepsilon_0}{\mu_0}}Ce^{-\mathrm{i}k_2 z}\end{cases}$$

$$\text{Region III}\begin{cases}E_{x\mathrm{III}} = De^{\mathrm{i}k_1 z} \\ H_{y\mathrm{III}} = \sqrt{\frac{\varepsilon_0}{\mu_0}}De^{\mathrm{i}k_1 z}\end{cases}$$

Boundary conditions.   Tangential E:

$@z = 0,\ E_{x\mathrm{I}} = E_{x\mathrm{II}} \quad \Rightarrow 1 + A = B + C$ (1)

$@\, z = d,\ E_{x\mathrm{II}} = E_{x\mathrm{III}} \ \Rightarrow Be^{\mathrm{i}k_2 d} + Ce^{-\mathrm{i}k_2 d} = De^{\mathrm{i}k_1 d}$ (2)

Tangential H: (no free current)

$@z = 0,\ H_{y\mathrm{I}} = H_{y\mathrm{II}} \quad \Rightarrow 1 - A = 2B - 2C$ (3)

$@\, z = d,\ H_{y\mathrm{II}} = H_{y\mathrm{III}} \ \Rightarrow 2Be^{\mathrm{i}k_2 d} - 2Ce^{-\mathrm{i}k_2 d} = De^{\mathrm{i}k_1 d}$ (4)

What we rerally care are the coefficients of *A* and *D*.

$(4) - (2) \Rightarrow B = 3Ce^{-\mathrm{i}4k_1 d}$

$$\begin{cases}(1) \Rightarrow 1 + A = C(3e^{-\mathrm{i}4k_1 d} + 1) \\ 2(1) - (3) \Rightarrow 1 + 3A = 4C\end{cases} \quad \Rightarrow A = \frac{3(1 - e^{-\mathrm{i}4k_1 d})}{9e^{-\mathrm{i}4k_1 d} - 1}$$

$$(2) \Rightarrow 4Ce^{-\mathrm{i}2k_1 d} = De^{\mathrm{i}k_1 d} \qquad \Rightarrow D = \frac{8e^{-\mathrm{i}3k_1 d}}{9e^{-\mathrm{i}4k_1 d} - 1}$$

Reflection coefficient: $R = A \cdot A^* = \frac{3(1 - e^{-\mathrm{i}4k_1 d})}{9e^{-\mathrm{i}4k_1 d} - 1}\frac{3(1 - e^{\mathrm{i}4k_1 d})}{9e^{\mathrm{i}4k_1 d} - 1} = \frac{18(1 - \cos(4k_1 d))}{82 - 18\cos(4k_1 d)}$

Transmission coefficient: $T = D \cdot D^* = \frac{8e^{-\mathrm{i}3k_1 d}}{9e^{-\mathrm{i}4k_1 d} - 1}\frac{8e^{\mathrm{i}3k_1 d}}{9e^{\mathrm{i}4k_1 d} - 1} = \frac{64}{82 - 18\cos(4k_1 d)}$

Varification: $R + T = 1$

$R = 0$ when $1 - \cos(4k_1 d) = 0,\ d = \frac{2n\pi}{2k_2} = \frac{n}{2}\lambda_2,$ where $\lambda_2 = \frac{v}{f} = \frac{c}{2f} = 5$ cm

$\because\ d = 2.5$ cm

2. (a) $\frac{d}{dt}\left(\vec{P}_{mech}+\vec{P}_{field}\right)_\alpha = \sum_\beta \int_v \frac{\partial}{\partial x_\beta} T_{\alpha\beta} d^3x = \oint_s \sum_\beta T_{\alpha\beta} n_\beta da = \oint_s \vec{\vec{T}} \cdot \vec{n} da$

$$T_{\alpha\beta} = \varepsilon_0 \left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}\left(\vec{E}\cdot\vec{E} + c^2 \vec{B}\cdot\vec{B}\right)\delta_{\alpha\beta} \right]$$

In Cartesian coordinate system with the z-axis along the wave propagation direction

$\vec{n} = -\hat{e}_z = (0,0,-1)$ $\vec{E} = (E_x, E_y, 0)$ $\vec{B} = (B_x, B_y, 0)$

$$\vec{\vec{T}}\cdot\vec{n} = \varepsilon_0 \begin{bmatrix} E_x^2 + c^2 B_x^2 - \frac{1}{2}\left(E^2 + c^2 B^2\right) & E_x E_y + c^2 B_x B_y & 0 \\ E_y E_x + c^2 B_y B_x & E_y^2 + B_y^2 - \frac{1}{2}\left(E^2 + c^2 B^2\right) & 0 \\ 0 & 0 & -\frac{1}{2}\left(E^2 + c^2 B^2\right) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$= \frac{1}{2}\left(E^2 + c^2 B^2\right) \hat{e}_z$$

$$\frac{d}{dt}\left(P_{mech} + P_{field}\right) = \oint_s \vec{\vec{T}}\cdot\vec{n} da \quad \Rightarrow \vec{F} = \frac{d}{dt} P_{field} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) A \hat{e}_z, \text{ since } P_{mech} = 0$$

$$\frac{\vec{F}}{A} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)\hat{e}_z$$

The radiation pressure is equal to the field energy density.

(b).

$$P_z = \frac{F}{A} = \frac{\overbrace{P}^{\text{power per area}}}{c} = \frac{1.5\times10^3 w/m^2}{3\times10^8 m/s}$$

$$a = \frac{F}{m} = \frac{F/A}{m/A} = \frac{1.5\times10^3 w/m^2}{3\times10^8 m/s \times 1\times10^{-3} kg/m^2}$$

$$= 5\times10^{-3} m/s^2$$

3. (a)

The rate of work done by the **E**-field on charged particles inside a volume V is given by

$$\int_V \mathbf{f}\cdot\mathbf{v}d^3x=\int_V \rho\mathbf{v}\cdot(\mathbf{E}+\mathbf{v}\times\mathbf{B})d^3x=\int_V \mathbf{J}\cdot\mathbf{E}d^3x=\int_V (\mathbf{E}\cdot\nabla\times\mathbf{H}-\mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D})d^3x$$

$$\mathbf{J}=\nabla\times\mathbf{H}-\tfrac{\partial}{\partial t}\mathbf{D},\ \ \mathbf{E}\cdot\nabla\times\mathbf{H}=\mathbf{H}\cdot\underbrace{\nabla\times\mathbf{E}}_{-\frac{\partial}{\partial t}\mathbf{B}}-\nabla\cdot\left(\mathbf{E}\times\mathbf{H}\right)=-\mathbf{H}\cdot\tfrac{\partial}{\partial t}\mathbf{B}-\nabla\cdot\left(\mathbf{E}\times\mathbf{H}\right)$$

$$\Rightarrow \int_V \mathbf{J}\cdot\mathbf{E}d^3x=-\int_V\left[\nabla\cdot\left(\mathbf{E}\times\mathbf{H}\right)+\mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D}+\mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}\right]d^3x \qquad (6.105)$$

rate of conversion of EM energy into mechanical and thermal energies.

The terms $\mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D}$ and $\mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}$ in the integrand can be interpreted physically if we make the following assumptions:

Assumption 1: The medium is *linear* with *negligible dispersion* and *negligible losses*.

We can then write (reasons given in Ch. 7 of lecture notes)

$$\mathbf{D}(\mathbf{x},t)=\varepsilon\mathbf{E}(\mathbf{x},t),\ \mathbf{B}(\mathbf{x},t)=\mu\mathbf{H}(\mathbf{x},t)$$

$$\Rightarrow\ \mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D}=\frac{1}{2}\frac{\partial}{\partial t}\left(\mathbf{E}\cdot\mathbf{D}\right),\ \mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}=\frac{1}{2}\frac{\partial}{\partial t}\left(\mathbf{H}\cdot\mathbf{B}\right). \qquad (6)$$

Assumption 2: The field energy density for static fields

$$u=\tfrac{1}{2}\left(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H}\right) \qquad (6.106)$$

represents the field energy density even for *time-dependent* fields.

From (6) and (6.106), we have

$$\frac{\partial u}{\partial t}=\mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D}+\mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}=\begin{bmatrix}\text{rate of change of}\\ \text{field energy density}\end{bmatrix} \qquad (7)$$

Rewrite (6.105): $\int_V \mathbf{J}\cdot\mathbf{E}d^3x=-\int_V\left[\nabla\cdot\left(\mathbf{E}\times\mathbf{H}\right)+\mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D}+\mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}\right]d^3x$

Sub. $\frac{\partial u}{\partial t}$ for $\mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D}+\mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{B}$, we obtain

$$\int_V \mathbf{J}\cdot\mathbf{E}d^3x+\int_V\frac{\partial u}{\partial t}d^3x+\int_V\nabla\cdot\left(\mathbf{E}\times\mathbf{H}\right)d^3x=0 \qquad (6.107)$$

$$\Rightarrow \frac{\partial u}{\partial t}+\nabla\cdot\mathbf{S}=-\mathbf{J}\cdot\mathbf{E} \qquad (6.108)$$

where, $\mathbf{S}\equiv\mathbf{E}\times\mathbf{H}$, is called the Poynting vector.

The meaning of **S** becomes clear if we write (6.107) as

$$\underbrace{\int_V \mathbf{J}\cdot\mathbf{E}d^3x}_{\frac{d}{dt}E_{mech}}+\underbrace{\int_V\frac{\partial u}{\partial t}d^3x}_{\frac{d}{dt}E_{field}}+\underbrace{\int_V\nabla\cdot\left(\mathbf{E}\times\mathbf{H}\right)d^3x}_{\oint_S\mathbf{S}\cdot\mathbf{n}da}=0$$

$$\Rightarrow \frac{d}{dt}\left(E_{mech}+E_{field}\right)=-\oint_S\mathbf{S}\cdot\mathbf{n}da\ \text{[Poynting's theorem]} \qquad (6.111)$$

In terms of phasors, the Maxwell equations can be written as:

$$\begin{cases} \nabla\cdot\mathbf{B}(\mathbf{x},t)=0 \\ \nabla\times\mathbf{E}(\mathbf{x},t)=-\frac{\partial}{\partial t}\mathbf{B}(\mathbf{x},t) \\ \nabla\cdot\mathbf{D}(\mathbf{x},t)=\rho(\mathbf{x},t) \\ \nabla\times\mathbf{H}(\mathbf{x},t)=\mathbf{J}(\mathbf{x},t)+\frac{\partial}{\partial t}\mathbf{D}(\mathbf{x},t) \end{cases} \Rightarrow \begin{cases} \nabla\cdot\mathbf{B}(\mathbf{x})=0 \\ \nabla\times\mathbf{E}(\mathbf{x})=i\omega\mathbf{B}(\mathbf{x}) \\ \nabla\cdot\mathbf{D}(\mathbf{x})=\rho(\mathbf{x}) \\ \nabla\times\mathbf{H}(\mathbf{x})=\mathbf{J}(\mathbf{x})-i\omega\mathbf{D}(\mathbf{x}) \end{cases}$$

Complex Poynting's Theorem : Using the phasor representation of Maxwell equations, we obtain

$$\frac{1}{2}\int_V \mathbf{J}^*\cdot\mathbf{E}d^3x=\frac{1}{2}\int_V[\overbrace{\mathbf{E}\cdot\nabla\times\mathbf{H}^*}^{-\nabla\cdot(\mathbf{E}\times\mathbf{H}^*)+\mathbf{H}^*\cdot\overbrace{\nabla\times\mathbf{E}}^{i\omega\mathbf{B}}}-i\omega\mathbf{E}\cdot\mathbf{D}^*]d^3x$$

$$=\frac{1}{2}\int_V\left[-\nabla\cdot(\mathbf{E}\times\mathbf{H}^*)-i\omega(\mathbf{E}\cdot\mathbf{D}^*-\mathbf{B}\cdot\mathbf{H}^*)\right]d^3x \quad (6.131)$$

Rewrite (6.131):

$$\frac{1}{2}\int_V \mathbf{J}^*\cdot\mathbf{E}d^3x=\frac{1}{2}\int_V[-\nabla\cdot(\mathbf{E}\times\mathbf{H}^*)-i\omega(\mathbf{E}\cdot\mathbf{D}^*-\mathbf{B}\cdot\mathbf{H}^*)]d^3x \quad (6.131)$$

This equation gives the complex Poynting theorem:

$$\frac{1}{2}\int_V \mathbf{J}^*\cdot\mathbf{E}d^3x+2i\omega\int_V\left(w_e-w_m\right)d^3x+\oint_S\mathbf{S}\cdot\mathbf{n}da=0 \qquad (6.134)$$

where $\mathbf{S}\equiv\frac{1}{2}\mathbf{E}\times\mathbf{H}^*$ [called the complex Poynting vector] (6.132)

and the real part of $\mathbf{S}$ is the time-averaged power [see (10)].

In (6.134), w_e and w_m are defined as

$$\begin{cases} w_e\equiv\frac{1}{4}\mathbf{E}\cdot\mathbf{D}^*=\frac{\varepsilon}{4}|E|^2 \\ w_m\equiv\frac{1}{4}\mathbf{B}\cdot\mathbf{H}^*=\frac{\mu}{4}|H|^2 \end{cases} \left[\begin{array}{l}\text{The real part of } w_e\ (w_m) \text{ is the time} \\ \text{averaged E (B) field energy density.}\end{array}\right] \quad (6.133)$$

4. (a)

Let $\mathbf{x}(t)=\mathbf{x}_0 e^{-i\omega t}$ and substitute

$$\begin{cases}\mathbf{x}(t)=\mathbf{x}_0 e^{-i\omega t}\\ \mathbf{E}(\mathbf{x},t)=\mathbf{E}(0)e^{-i\omega t}\end{cases} \quad \text{into} \quad m(\ddot{\mathbf{x}}+\gamma\dot{\mathbf{x}}+\omega_0^2\mathbf{x})=-e\mathbf{E}(\mathbf{x},t),$$

we obtain $m(-\omega^2-i\omega\gamma+\omega_0^2)\mathbf{x}_0=-e\mathbf{E}(0)$ with the solution:

$$\mathbf{x}_0=-\frac{e}{m}\frac{\mathbf{E}(0)}{\omega_0^2-\omega^2-i\omega\gamma} \qquad \Rightarrow \qquad \mathbf{x}(t)=-\frac{e}{m}\frac{\mathbf{E}(0)e^{-i\omega t}}{\omega_0^2-\omega^2-i\omega\gamma}$$

Divide the electrons in the medium into $\begin{cases}\text{bound electrons: } \omega_j\neq 0\\ \text{free electrons: } \omega_j=0,\ f_j=f_0,\ \gamma_j=\gamma_0\end{cases}$

$$\mathbf{P}(\mathbf{x})=N\sum_j f_j\mathbf{p}_j=\left(\frac{Ne^2}{m}\sum_{\text{bound}}\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}+i\frac{Ne^2}{m}\frac{f_0}{-i\omega^2+\omega\gamma_0}\right)\mathbf{E}(\mathbf{x})=\varepsilon_0\chi_e\mathbf{E}(\mathbf{x})$$

$$\begin{cases}\mathbf{D}(\mathbf{x})\equiv\varepsilon_0\mathbf{E}(\mathbf{x})+\mathbf{P}(\mathbf{x})=\varepsilon\mathbf{E}(\mathbf{x})\\ \varepsilon=\varepsilon_0(1+\chi_e)\end{cases}$$

to fields with $\exp(-i\omega t)$ dependence, we obtain $\mathbf{D}(\mathbf{x})=\varepsilon\mathbf{E}(\mathbf{x})$

$$\text{with } \varepsilon=\underbrace{\varepsilon_0+\frac{Ne^2}{m}\sum_{j\ (\text{bound})}\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}}_{\varepsilon_b}+i\underbrace{\frac{Ne^2 f_0}{m\omega(\gamma_0-i\omega)}}_{\sigma/\omega}=\varepsilon_b+i\frac{\sigma}{\omega}$$

(b.)

$\mathbf{D}(\omega)=\varepsilon(\omega)\mathbf{E}(\omega)$ is a constitutive relation
in ω-space valid for all ω. For multi-frequency fields, we may
obtain the t-space $\mathbf{D}$ through a Fourier transformation

$$\mathbf{D}(t)=\tfrac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{D}(\omega)e^{-i\omega t}d\omega=\tfrac{1}{2\pi}\int_{-\infty}^{\infty}\varepsilon(\omega)\mathbf{E}(\omega)e^{-i\omega t}d\omega \qquad (3)$$

$$\varepsilon(\omega)=\varepsilon_0+\frac{Ne^2}{m}\sum_{j\ (\text{bound})}\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}+i\frac{Ne^2 f_0}{m\omega(\gamma_0-i\omega)}$$

$$\mathbf{E}(t)=\tfrac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{E}(\omega)e^{-i\omega t}d\omega$$

We find from (3) that, in general, $\mathbf{D}(t)\neq\varepsilon\mathbf{E}(t)$ because ε is a function of ω.

5. See Jackson, pp. 196-200. HW problems 5.13, 5.19, and 5.20
Griffith, pp. 263-264. Examples 5.11 and 6.1