清大物理系 “電動力學(一)” 任課老師:張存續
“Electrodynamics (I)” (PHYS 531000)

Fall Semester, 2010

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1. **Textbook and Contents of the Course:**

J. D. Jackson, “Classical Electrodynamics”, 3rd edition, Chapters 1-7. Other books will be referenced in the lecture notes when needed.

2. **Conduct of Class :**

Lecture notes will be projected sequentially on the screen during the class. Physical concepts will be emphasized, while algebraic details in the lecture notes will often be skipped. *Questions are encouraged*. It is assumed that students have at least gone through the algebra in the lecture notes before attending classes (*important*!).

3. **Grading Policy:**

First midterm (~30%); Second midterm (~30%); Final (~30%); Quiz (~10%); Participation (~5%, extra). The overall score will be normalized to reflect an average consistent with other courses.

4. **Lecture Notes:**

Starting from basic equations, the lecture notes follow Jackson closely with algebraic details filled in.

Equations numbered in the format of (1.1), (1.2)… refer to Jackson. Supplementary equations derived in lecture notes, which will later be referenced, are numbered (1), (2)... [restarting from (1) in each chapter.] Equations in Appendices A, B…of each chapter are numbered (A.1), (A.2)…and (B.1), (B.2)…

Page numbers cited in the text (e.g. p. 120) refer to Jackson.

Section numbers (e.g. Sec. 1.1) refer to Jackson. Main topics within each section are highlighted by **boldfaced** characters. Some words are typed in *italicized* characters for attention. Technical terms which are introduced for the first time are <u>underlined</u>.

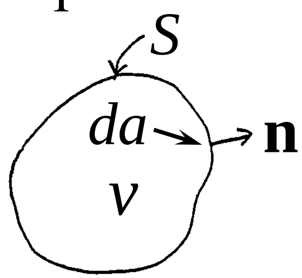
1.8 Green’s Theorem

Green's theorem, a powerful tool for treating electrostatic boundary-value problems, is a simple application of the divergence theorem:

$$\int_V \nabla\cdot\mathbf{A}\, d^3x = \oint_S \mathbf{A}\cdot\mathbf{n}\, da \qquad \theta\text{: theta;}\quad \phi\text{: phi;}\quad \psi\text{: psi}$$

Let $\mathbf{A} = \phi\nabla\psi$, where ϕ and ψ are arbitrary functions of position.

$$\Rightarrow \begin{cases} \nabla\cdot\mathbf{A} = \nabla\cdot(\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi \\ \mathbf{A}\cdot\mathbf{n} = \phi\nabla\psi\cdot\mathbf{n} = \phi\dfrac{\partial\psi}{\partial n} \end{cases}$$



Sub. these 2 expressions for $\nabla\cdot\mathbf{A}$ and $\mathbf{A}\cdot\mathbf{n}$ into the divergence theorem, we obtain Green's first identity,

$$\int_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\, d^3x = \oint_S \phi\frac{\partial\psi}{\partial n}\, da \qquad (1.34)$$

Interchange ϕ and ψ in (1.34).

$$\Rightarrow \int_V (\psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi)\, d^3x = \oint_S \psi\frac{\partial\phi}{\partial n}\, da$$

Subtract these two equations, we obtain Green's second identity,

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d^3x = \oint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) da \qquad (1.35)$$

Green's theorem relates a volume integral to a surface integral and the volume integral contains the operator ∇^2. These features are useful for the manipulateion of the Poisson equation in bounded space.

For example, applying Green's second identity:

$$\int_v (\phi\nabla^2\psi - \psi\nabla^2\phi)d^3x = \oint_s (\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n})da \qquad (1.35)$$

we may convert the Poisson equation into an integral equation. How? See next few pages.

In (1.35), letting ψ be $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$, $(\nabla^2\frac{1}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\delta(\mathbf{x}-\mathbf{x}'))$, Φ be the electrostatic potential (thus, $\nabla^2\Phi = -\frac{\rho}{\varepsilon_0}$), and $\mathbf{x}'$ be the integration variable, we obtain

$$\int_v[-4\pi\Phi\delta(\mathbf{x}-\mathbf{x}') + \frac{1}{\varepsilon_0|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}')]d^3x' = \oint_s[\Phi\frac{\partial}{\partial n'}(\frac{1}{|\mathbf{x}-\mathbf{x}'|}) - \frac{1}{|\mathbf{x}-\mathbf{x}'|}\frac{\partial\Phi}{\partial n'}]da$$

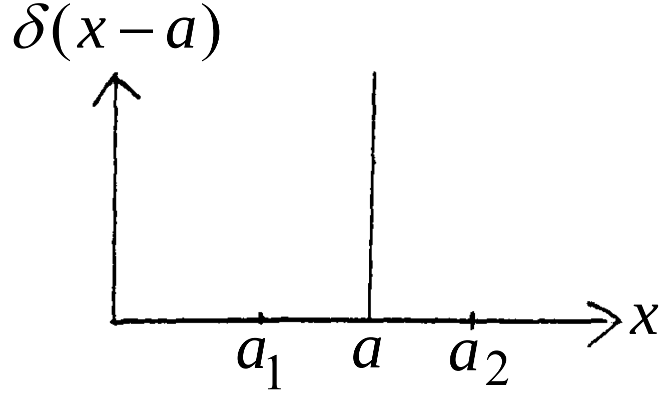
$$\overbrace{\Rightarrow}^{\mathbf{x}\text{ inside } v}\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_v\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' + \frac{1}{4\pi}\oint_s\left[\frac{1}{|\mathbf{x}-\mathbf{x}'|}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right]da' \qquad (1.36)$$

(1.36) is an integral equation (not a solution) for Φ. In infinite space, we have $\Phi \propto \frac{1}{R}$. Hence, (1.36) reduces to $\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_v\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$.

Delta Functions (pp. 26 - 27)

Definition of *delta function*:

$$\begin{cases} \delta(x-a)=0\ , & \text{if } x\neq a \\ \int_{a_1}^{a_2}\delta(x-a)dx=1\ , & \text{if } a_1<a<a_2 \end{cases}$$



Note: Since the delta function is defined in terms of an integral, it takes an integration to bring out its full meaning.

Properties of delta function:

(i) $\int_{a_1}^{a_2} f(x)\delta(x-a)dx = f(a)$ (2)

(ii) $\int_{a_1}^{a_2} f(x)\delta'(x-a)dx = \overbrace{f(x)\delta(x-a)\big|_{a_1}^{a_2}}^{0} - \int_{a_1}^{a_2} f'(x)\delta(x-a)dx$

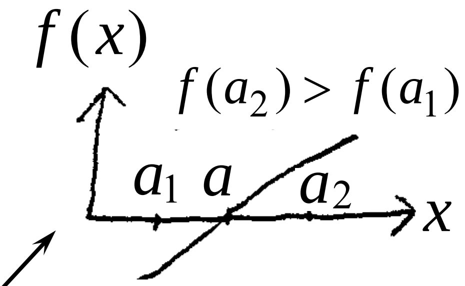
$= -f'(a)$ (3)

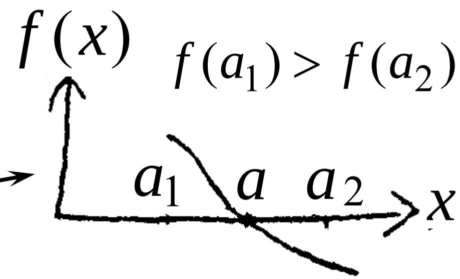
Delta Functions (*continued*)

(iii) Let $x = a$ be the root of $f(x) = 0$, then

$$\int_{a_1}^{a_2} \delta[f(x)]dx = \int_{f(a_1)}^{f(a_2)} \delta[f(x)] \frac{1}{\frac{d}{dx} f(x)} df(x)$$

$$= \begin{cases} \int_{f(a_1)}^{f(a_2)} \frac{1}{f'} \delta(f) df = \frac{1}{f'(a)} = \frac{1}{|f'(a)|}, & f'(a) > 0 \\ -\int_{f(a_2)}^{f(a_1)} \frac{1}{f'} \delta(f) df = -\frac{1}{f'(a)} = \frac{1}{|f'(a)|}, & f'(a) < 0 \end{cases}$$





Note: In both expressions above, the integration is from a samller value to a larger value, as in the definition of the delta function.

Compare with (2) $\Rightarrow \delta[f(x)] = \frac{1}{|f'(a)|} \delta(x-a) \; [= \frac{1}{|f'(x)|} \delta(x-a)]$ (4)

If $f(x)$ has multiple roots x_i $[f(x_i) = 0, \; i = 1, 2, \cdots]$, then

$$\delta[f(x)] = \sum_i \frac{1}{|f'(x_i)|} \delta(x - x_i) \; [= \sum_i \frac{1}{|f'(x)|} \delta(x - x_i)] \qquad (5)$$

Exercise: Show $\delta(a-x) = \delta(x-a)$ and $\delta(cx) = \delta(x)/|c|$

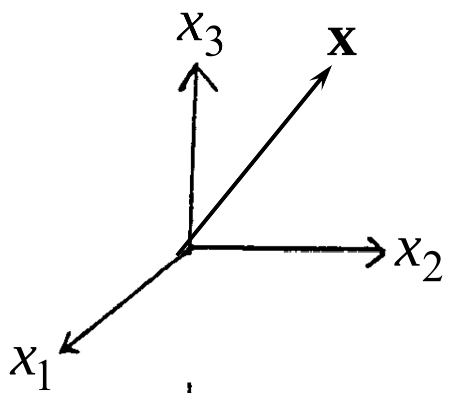
Extension to 3 dimensions:

1. Cartesian coordinates: $\mathbf{x} = (x_1, x_2, x_3)$

$$\delta(\mathbf{x}-\mathbf{x}') \equiv \delta(x_1 - x_1')\delta(x_2 - x_2')\delta(x_3 - x_3') \tag{6}$$

$$\Rightarrow \int_V \delta(\mathbf{x}-\mathbf{x}')d^3x = \int \delta(x_1 - x_1')dx_1 \int \delta(x_2 - x_2')dx_2 \int \delta(x_3 - x_3')dx_3$$

$$= \begin{cases} 0, & \text{if } \mathbf{x}' \text{ lies outside } V \\ 1, & \text{if } \mathbf{x}' \text{ lies inside } V \end{cases}$$



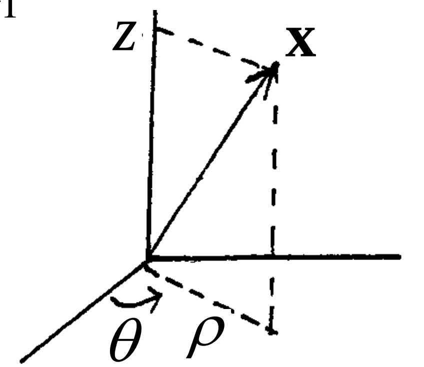
2. Cylindrical coordinates: $\mathbf{x} = (\rho, \theta, z)$

$$\delta(\mathbf{x}-\mathbf{x}') \equiv \frac{1}{\rho}\delta(\rho-\rho')\delta(\theta-\theta')\delta(z-z') \tag{7}$$

$$\Rightarrow \int_V \delta(\mathbf{x}-\mathbf{x}')d^3x = \int_V \delta(\mathbf{x}-\mathbf{x}')\rho d\rho d\theta dz$$

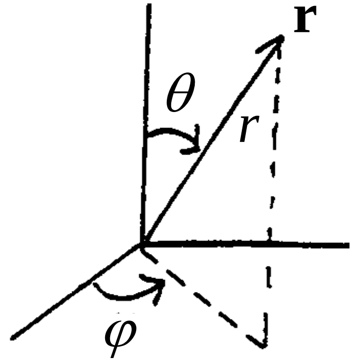
$$= \int \delta(\rho-\rho')d\rho \int \delta(\theta-\theta')d\theta \int \delta(z-z')dz$$

$$= \begin{cases} 0, & \text{if } \mathbf{x}' \text{ lies outside } V \\ 1, & \text{if } \mathbf{x}' \text{ lies inside } V \end{cases}$$



Question: If x and $\mathbf{x}$ both have the dimension of cm, what are the dimensions of $\delta(x)$ and $\delta(\mathbf{x})$? [See Appendix (A), Eq. (A.9).]

3. Spherical coordinates: $\mathbf{r} = (r,\ \theta,\ \varphi)$



$$\delta(\mathbf{r}-\mathbf{r}') \equiv \begin{cases} \dfrac{1}{r^2 \sin\theta}\delta(r-r')\delta(\theta-\theta')\delta(\varphi-\varphi'), \text{ or} \\ \dfrac{1}{r^2}\delta(r-r')\delta(\cos\theta-\cos\theta')\delta(\varphi-\varphi') \end{cases} \quad (8)$$

$$\boxed{\text{By (4), } \delta(\cos\theta-\cos\theta') = \frac{1}{|\sin\theta|}\delta(\theta-\theta') = \frac{1}{\sin\theta}\delta(\theta-\theta'),\ 0 \le \theta \le \pi}$$

$$\int_V \delta(\mathbf{r}-\mathbf{r}')d^3x = \int_V \frac{\delta(r-r')}{r^2}\delta(\cos\theta-\cos\theta')\delta(\varphi-\varphi')\underbrace{r^2 dr d(\cos\theta) d\varphi}_{\boxed{d^3x \text{ [see (9) below]}}}$$

$$= \begin{cases} 0, \text{ if } \mathbf{r}' \text{ lies outside } V \\ 1, \text{ if } \mathbf{r}' \text{ lies inside } V \end{cases}$$

Note: Volume integration in spherical coordinates

$$\int_0^\infty dr \int_0^\pi r d\theta \int_0^{2\pi} r\sin\theta d\varphi = \int_0^\infty r^2 dr \underbrace{\int_0^\pi \sin\theta d\theta}_{\boxed{-\int_1^{-1} d(\cos\theta)}} \int_0^{2\pi} d\varphi$$

$$= \int_0^\infty r^2 dr \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\varphi$$

Variables are to be integrated from smaller to larger values.

$$\Rightarrow d^3x = r^2 \sin\theta dr d\theta d\varphi \ \text{ or } \ r^2 dr d(\cos\theta) d\varphi \quad (9)$$

Approximate representations of the delta function:

The delta function, $\delta(x)$, can be represented analytically by the following functions because they satisfy the definition of the delta function in the limit $\gamma \to 0$ $(\gamma > 0)$.

$$\delta(x) = \lim_{\gamma \to 0} \frac{1}{\pi} \frac{\gamma}{x^2 + \gamma^2}$$

$$\delta(x) = \lim_{\gamma \to 0} \frac{1}{\sqrt{2\pi\gamma}} \mathrm{e}^{-\frac{x^2}{2\gamma^2}}$$

$$\delta(x) = \lim_{\gamma \to 0} \begin{cases} \frac{1}{\gamma}, & \text{for } -\frac{\gamma}{2} < x < \frac{\gamma}{2} \\ 0, & \text{otherwise} \end{cases}$$

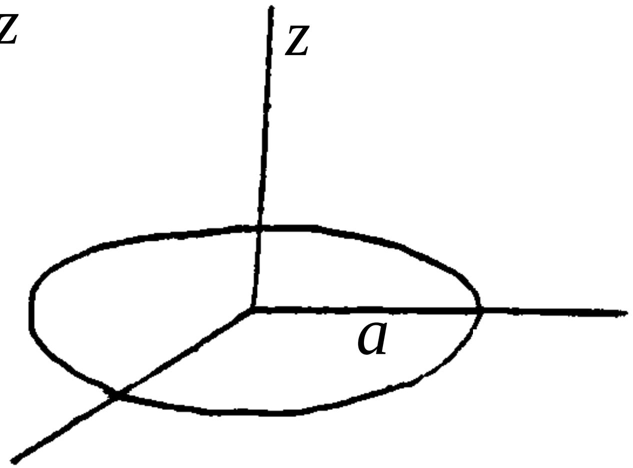
Problem 1: A total charge Q is uniformly distributed around a circular ring of radius a and infinitesimal thickness. Write the charge density $\rho(\mathbf{x})$ in cylindrical coordinates.

Solution:

Is there any θ-dependence?

Let $\rho(\mathbf{x}) = K\delta(r-a)\delta(z)$ and find K as follows.

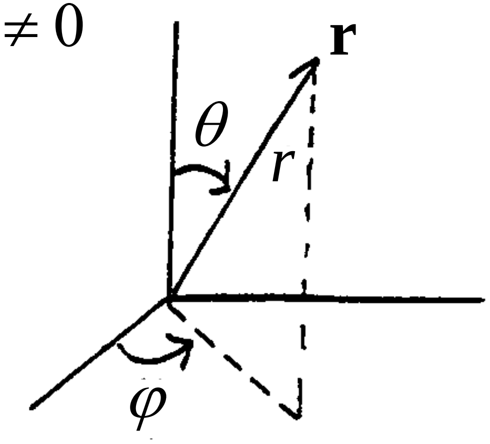
$$\int \rho(\mathbf{x})d^3x = K \int \delta(r-a)\delta(z)rdrd\theta dz$$
$$= 2\pi Ka = Q$$
$$\Rightarrow K = \frac{Q}{2\pi a}$$
$$\Rightarrow \rho(\mathbf{x}) = \frac{Q}{2\pi a}\delta(r-a)\delta(z)$$



Note: ρ has the dimension of "charge/volume" as expected.

Problem 2: Prove $\nabla^2 \frac{1}{r} = -4\pi\delta(\mathbf{r})$ $\left(\nabla^2 \frac{1}{|\mathrm{x}-\mathrm{x}'|}?\right)$

Solution: Definition of $\delta(\mathbf{r})$: $\begin{cases} \delta(\mathbf{r}) = 0, \text{ if } r \neq 0 \\ \int \delta(\mathbf{r}) d^3x = 1 \end{cases}$



Hence, we need to prove

(i) $\nabla^2 \frac{1}{r} = 0, \text{ if } r \neq 0$

(ii) $\int \nabla^2 \frac{1}{r} d^3x = -4\pi \int \delta(\mathbf{r}) d^3x = -4\pi$

It is convenient to use the spherical coordinates. To prove (i), we we write ∇^2 as (see back cover of Jackson)

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$\Rightarrow \nabla^2 \frac{1}{r} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\frac{1}{r}\right) = -\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{r^2}\right) = 0 \quad \text{if} \quad r \neq 0$$

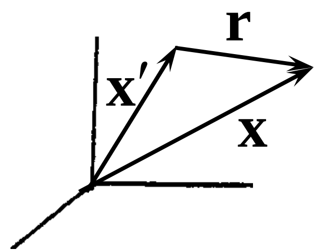
Note: $\frac{r^2}{r^2}$ is undetermined at $r = 0$. However, here we are only concerned with the region $r > 0$.

To prove (ii), we integrate $\nabla^2 \frac{1}{r}$ over a spherical volume V

$$\int_V \nabla^2 \frac{1}{r} d^3x = \int_V \nabla \cdot \nabla \frac{1}{r} d^3x \underset{\boxed{\text{divergence thm.}}}{=} \oint_S \mathbf{e}_r \cdot \underbrace{\nabla \frac{1}{r}}_{-\frac{1}{r^2}\mathbf{e}_r} \underbrace{da}_{\boxed{r^2 d\Omega}} = -\oint_S r^2 \frac{1}{r^2} d\Omega = -4\pi$$

Note: Since $r > 0$ on the spherical surface, again we do not have the problem of evaluating r^2/r^2 at $r = 0$.

Change to a coordinate system in which $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ and $r = |\mathbf{x} - \mathbf{x}'|$. We obatin from $\nabla^2 \frac{1}{r} = -4\pi\delta(\mathbf{r})$



$$\nabla^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\delta(\mathbf{x}-\mathbf{x}') \tag{1.31}$$

optional *Problem 3*: Derive $\nabla^2\Phi(\mathbf{x}) = -\frac{\rho(\mathbf{x})}{\varepsilon_0}$ from $\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$

Solution: $\nabla^2\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{x}')\nabla^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|} d^3x'$

$$= \frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{x}')\left[-4\pi\delta(\mathbf{x}-\mathbf{x}')\right] d^3x' = -\frac{\rho(\mathbf{x})}{\varepsilon_0}$$

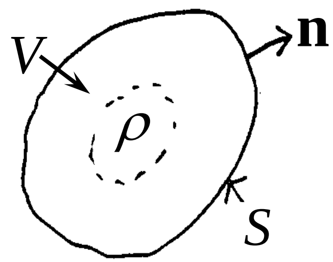
1.9 Uniqueness of Solution with Dirichlet or Neumann Boundary Conditions

$$\begin{cases} \underline{\text{Dirichlet boundary condition}}: \Phi_s \text{ specified} \\ \underline{\text{Neumann boundary condition}}: \frac{\partial}{\partial n}\Phi_s \text{ specified} \\ \underline{\text{Cauchy boundary condition}}: \Phi_s \text{ and } \frac{\partial}{\partial n}\Phi_s \text{ both specified} \end{cases}$$

As another application of Green's theorem, we use it to prove the <u>uniqueness theorem</u> for the solution of the Poisson equation.

Let there be two solutions, Φ_1 and Φ_2, which both satisfy

$$\nabla^2\Phi = -\rho/\varepsilon_0 \text{ with } \begin{cases} \Phi = \Phi_n \text{ on } S \text{ (Dirichlet b.c.), or} \\ \frac{\partial}{\partial n}\Phi = \Phi_n' \text{ on } S \text{ (Neumann b.c.)} \end{cases}$$



i.e. $\begin{cases} \nabla^2\Phi_1 = -\rho/\varepsilon_0 \\ \nabla^2\Phi_2 = -\rho/\varepsilon_0 \end{cases}$ with $\begin{cases} \Phi_{1,2} = \Phi_n \text{ on } S, \text{ or} \\ \frac{\partial}{\partial n}\Phi_{1,2} = \Phi_n' \text{ on } S \end{cases}$

Define $U \equiv \Phi_1 - \Phi_2$, then $\nabla^2 U = 0$ with $\begin{cases} U = \Phi_n - \Phi_n = 0 \text{ on } S, \text{ or} \\ \frac{\partial}{\partial n}U = \Phi_n' - \Phi_n' = 0 \text{ on } S \end{cases}$

1.9 Uniqueness of Solution… (*continued*)

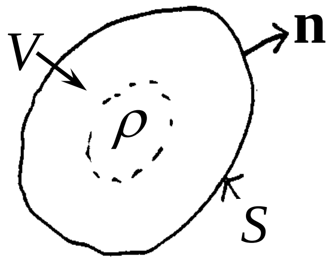
Rewrite Green's 1st identity: $\int_V \left(\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi\right)d^3x = \oint_S \phi\frac{\partial\psi}{\partial n}da$

Let $\phi = \psi = U$

$\boxed{\text{b.c. } U = 0 \text{ or } \partial U/\partial n = 0 \text{ on } S}$

$$\Rightarrow \int_V (U\underbrace{\nabla^2 U}_{0} + \nabla U\cdot\nabla U)d^3x = \oint_S U\frac{\partial U}{\partial n}da = 0 \Rightarrow \int_V |\nabla U|^2 d^3x = 0$$

$\Rightarrow \nabla U = 0$ everywhere within V



$$\Rightarrow U = \Phi_1 - \Phi_2 = \begin{cases} 0, & \text{if } U = 0 \text{ on } S \\ const, & \text{if } \partial U/\partial n = 0 \text{ on } S \end{cases}$$

$\Rightarrow \Phi_1$ and Φ_2 differ by at most a constant, hence are the same solution.

Note: Since the solution is uniquely determined by specifying either Φ or $\partial\Phi/\partial n$ on the boundary, the Cauchy boundary condition (Φ and $\partial\Phi/\partial n$ both specified on the boundary) is an over-specification, which may lead to inconsistency.

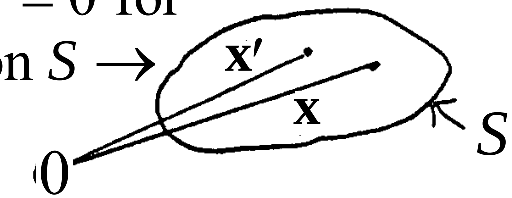
Exercise: Prove that there cannot be any static **E** inside a closed, hollow conductor if there is no charge in the hollow region.

1.10 Formal Solution of Electrostatic Boundary-Value Problem with Green Function

Green Function $G_D(\mathbf{x},\mathbf{x}')$:

In electrostatics, the Green function is the solution of the following problem:

$G_D = 0$ for $\mathbf{x}$ on $S \rightarrow$

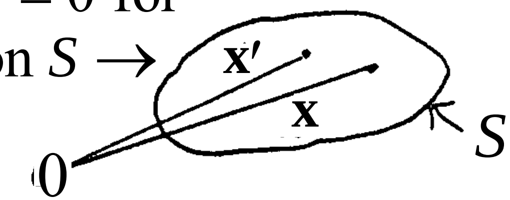


$$\nabla^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \;\; \text{with } G_D(\mathbf{x},\mathbf{x}') = 0 \text{ for } \mathbf{x} \text{ on } S,$$

where $\mathbf{x}$ is the variable of the differential equation and $\mathbf{x}'$ is treated as a constant. $G_D(\mathbf{x},\mathbf{x}')$ is the potential of a unit point source ($q \rightarrow 4\pi\varepsilon_0$) located at $\mathbf{x}'$ subject to the b. c. that $G_D(\mathbf{x},\mathbf{x}')$ vanishes for $\mathbf{x}$ on S.

Symmetry Property of $G_D(\mathbf{x},\mathbf{x}')$:

$G_D = 0$ for $\mathbf{y}$ on $S \rightarrow$



Consider two equations: one with a point source at $\mathbf{x}$, the other with a point source at $\mathbf{x}'$. The variable is $\mathbf{y}$.

$$\nabla_y^2 G_D(\mathbf{y},\mathbf{x}) = -4\pi\delta(\mathbf{y}-\mathbf{x}), \quad \text{b.c. } G_D(\mathbf{y},\mathbf{x}) = 0 \text{ for } \mathbf{y} \text{ on } S$$

$$\nabla_y^2 G_D(\mathbf{y},\mathbf{x}') = -4\pi\delta(\mathbf{y}-\mathbf{x}'), \quad \text{b.c. } G_D(\mathbf{y},\mathbf{x}') = 0 \text{ for } \mathbf{y} \text{ on } S$$

Rewrite: $\int_V \left(\phi\nabla_y^2\psi - \psi\nabla_y^2\phi\right)d^3y = \oint_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right]da$ (1.35)

Let $\phi = G_D(\mathbf{y},\mathbf{x})$ and $\psi = G_D(\mathbf{y},\mathbf{x}')$, where $\mathbf{y}$ is the variable.

$$\Rightarrow \int_V [G_D(\mathbf{y},\mathbf{x})\overbrace{\nabla_y^2 G_D(\mathbf{y},\mathbf{x}')}^{-4\pi\delta(\mathbf{y}-\mathbf{x}')} - G_D(\mathbf{y},\mathbf{x}')\overbrace{\nabla_y^2 G_D(\mathbf{y},\mathbf{x})}^{-4\pi\delta(\mathbf{y}-\mathbf{x})}]d^3y$$

$$= \oint_S [\underbrace{G_D(\mathbf{y},\mathbf{x})}_{=\,0 \text{ on } S}\frac{\partial}{\partial n}G_D(\mathbf{y},\mathbf{x}') - \underbrace{G_D(\mathbf{y},\mathbf{x}')}_{=\,0 \text{ on } S}\frac{\partial}{\partial n}G_D(\mathbf{y},\mathbf{x})]da$$

$$\Rightarrow 4\pi[G_D(\mathbf{x}',\mathbf{x}) - G_D(\mathbf{x},\mathbf{x}')] = 0$$

$$\Rightarrow G_D(\mathbf{x}',\mathbf{x}) = G_D(\mathbf{x},\mathbf{x}') \quad \text{[symmetry property of } G_D(\mathbf{x},\mathbf{x}')]$$

Questions:

1. Does $\nabla^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ imply $\nabla'^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}'-\mathbf{x})$?
2. Give two examples to show the physical meaning of the symmetry property of $G_D(\mathbf{x},\mathbf{x}')$.

Formal Solution of Electrostatic Boundary - Value Problem :

The expression $\Phi(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$ is applicable only to unbounded space. By Green's theorem, we may generalize it to an expression for bounded space with prescribed boundary conditions.

Consider a general electrostatic boundary-value problem:

$$\nabla^2\Phi(\mathbf{x})=-\rho(\mathbf{x})/\varepsilon_0 \text{ with } \Phi(\mathbf{x})=\Phi_s(\mathbf{x}) \text{ for } \mathbf{x} \text{ on } S \qquad (10)$$

Green's 2nd identity:

$$\int_V\left[\phi(\mathbf{x}')\nabla'^2\psi(\mathbf{x}')-\psi(\mathbf{x}')\nabla'^2\phi(\mathbf{x}')\right]d^3x'$$
$$=\oint_S\left[\phi(\mathbf{x}')\frac{\partial}{\partial n'}\psi(\mathbf{x}')-\psi(\mathbf{x}')\frac{\partial}{\partial n'}\phi(\mathbf{x}')\right]da' \qquad (1.35)$$

In (1.35), let $\phi(\mathbf{x}')$ be the solution of (10) with variable $\mathbf{x}'$ (i.e. $\Phi(\mathbf{x}')$). Let $\psi(\mathbf{x}')=\mathrm{G_D}(\mathbf{x},\mathbf{x}')$, where $\mathrm{G_D}(\mathbf{x},\mathbf{x}')$ is the Green function satisfying

$$\nabla'^2G_D(\mathbf{x},\mathbf{x}')=-4\pi\delta(\mathbf{x}-\mathbf{x}') \text{ with } G_D(\mathbf{x},\mathbf{x}')=0 \text{ for } \mathbf{x}' \text{ on } S \qquad (11)$$

Substitution of $\phi(\mathbf{x}')$ and $\psi(\mathbf{x}')$ into (1.35) gives

$$\int_V [\Phi(\mathbf{x}')\overbrace{\nabla'^2 G_D(\mathbf{x},\mathbf{x}')}^{-4\pi\delta(\mathbf{x}-\mathbf{x}')} - G_D(\mathbf{x},\mathbf{x}')\overbrace{\nabla'^2\Phi(\mathbf{x}')}^{-\rho(\mathbf{x}')/\varepsilon_0}]d^3x'$$
$$= \oint_S [\Phi(\mathbf{x}')\frac{\partial}{\partial n'}G_D(\mathbf{x},\mathbf{x}') - \underbrace{G_D(\mathbf{x},\mathbf{x}')}_{=\,0\text{ on }S}\frac{\partial}{\partial n'}\Phi(\mathbf{x}')]da'$$

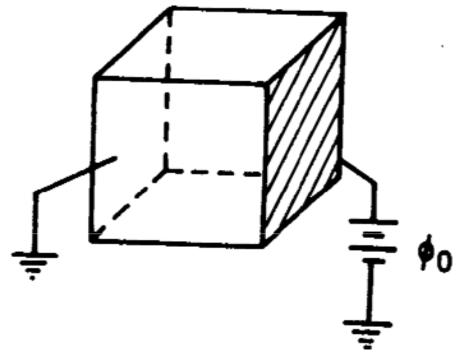
Thus, we obtain

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G_D(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n'}da' \quad (1.44)$$

(1.44) expresses the solution Φ of the general electrostatic problem in (10) in terms of the solution $G_D(\mathbf{x},\mathbf{x}')$ of the point source problem in (11) and the boundary value (Φ_s) of Φ on S. To evaluate (1.44), we first solve (11) for $G_D(\mathbf{x},\mathbf{x}')$, then substitute $G_D(\mathbf{x},\mathbf{x}')$, $\rho(\mathbf{x}')$, Φ_s into (1.44). It is often simpler to solve $G_D(\mathbf{x},\mathbf{x}')$ from (11) than solving Φ directly from (10), because (11) has the simple b.c. of $G_D(\mathbf{x},\mathbf{x}')=0$ on S. Applications of (1.44) can be found in Chs. 2 and 3. The problem below gives an application without the need to solve (11) for $G(\mathbf{x},\mathbf{x}')$.

optional **1.10 Formal Solution of Electrostatic Boundary-Value Problem…**(*continued*)

Problem: A hollow cube (see figure) has six square sides. There is no charge inside. Five sides are grounded. The sixth side, insulated from the others, is held at a constant potential Φ_0. Find the potential at the center of the cube.



Solution: Let the center of the cube be at $\mathbf{x} = 0$ and rewrite (1.44):

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \cancel{\rho(\mathbf{x}')} G_D(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial}{\partial n'}G_D(\mathbf{x},\mathbf{x}')da' \quad (1.44)$$

If all 6 sides had the potential Φ_0, then $\Phi(\mathbf{x}=0)=\Phi_0$ and by (1.44)

$$\Phi_0 = -\frac{1}{4\pi}\oint_S \Phi_0 \frac{\partial}{\partial n'}G_D(0,\mathbf{x}')da' \quad (12)$$

For the present problem, we have $\Phi = \Phi_0$ on side 1 and $\Phi = 0$ on the other 5 sides. By (1.44), the potential at the center is

$$\Phi(\mathbf{x}=0) = -\frac{1}{4\pi}\int_{side\ 1}\Phi_0\frac{\partial}{\partial n'}G_D(0,\mathbf{x}')da' = -\frac{1}{6}\underbrace{\frac{1}{4\pi}\oint_S \Phi_0\frac{\partial}{\partial n'}G_D(0,\mathbf{x}')da'}_{-\Phi_0 \text{ by } (12)}$$

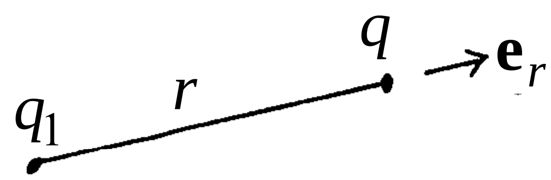
$$= \frac{1}{6}\Phi_0$$

$\because$ $G_D(0,\mathbf{x}')$ is symmetric with respect to all six sides.

1.1 Coulomb's Law

Coulomb's law, discovered experimentally, is a fundamental law governing all electrostatic phenomena. It states that the force on point charge q due to point charge q_1 obeys (see figure)

$$\mathbf{F} = \frac{qq_1}{4\pi\varepsilon_0 r^2}\mathbf{e}_r \Rightarrow \begin{cases} 1.\ F \propto q,\ q_1, \text{ and } \frac{1}{r^2}. \\ 2.\ F \text{ is along } r. \\ \quad \text{(central force)} \\ 3.\ \begin{cases} F \text{ is attractive if } q \text{ and } q_1 \text{ have opposite signs.} \\ F \text{ is repulsive if } q \text{ and } q_1 \text{ have the same sign.} \end{cases} \end{cases}$$



Furthermore, if there are multiple charges present, the total force on q is the vector sum of the two-body Coulomb forces between q and each of its surrounding charges.

Question: What is the principle of linear superposition?

1.2 Electric Field

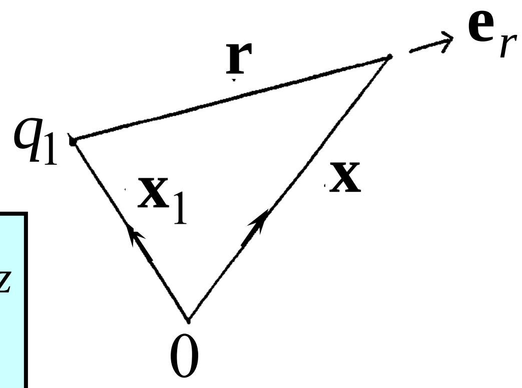
The electric field at point $\mathbf{x}$ due to one or more charges is defined as

$$\mathbf{E}(\mathbf{x}) \equiv \lim_{q \to 0} \frac{\mathbf{F}}{q}, \tag{1.1}$$

where q is a test charge and $\mathbf{F}$ is the total Coulomb force on q. We let q be infinistesimal so that it will not alter the field configuration.

Thus, $\mathbf{E}(\mathbf{x})$ due to a single point charge q_1 is

$$\mathbf{E}(\mathbf{x}) = \frac{q_1}{4\pi\varepsilon_0}\frac{1}{r^2}\mathbf{e}_r = \frac{q_1}{4\pi\varepsilon_0}\frac{\mathbf{x}-\mathbf{x_1}}{|\mathbf{x}-\mathbf{x_1}|^3}$$

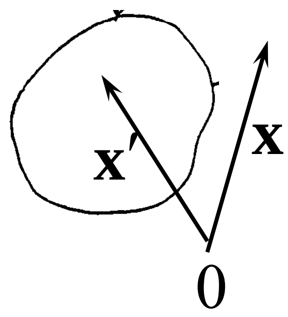


$$r\mathbf{e}_r = \mathbf{x} - \mathbf{x}_1 = (x - x_1)\mathbf{e}_x + (y - y_1)\mathbf{e}_y + (z - z_1)\mathbf{e}_z$$

$$r = |\mathbf{x} - \mathbf{x}_1| = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}$$

For distributed charges, we have by linear superposition:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d^3x' \tag{1.5}$$

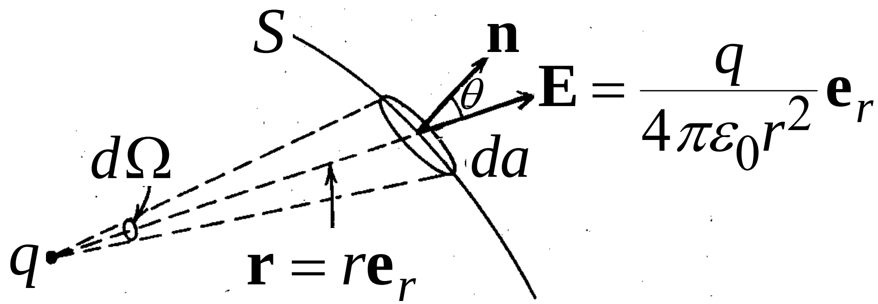


Question: Why write "$r\mathbf{e}_r$" as "$\mathbf{x} - \mathbf{x}_1$"?

1.3 Gauss's Law

Consider a point charge q and a closed surface S and adopt the following notations:

$$\begin{cases} da: & \text{infinitesimal surface area o} \\ \mathbf{n}: & \text{unit vector normal to } da \\ & \text{and pointing } \textit{outward} \\ \mathbf{e}_r: & \text{unit vector along } \mathbf{r} \\ \theta: & \text{The angle between } \mathbf{n} \text{ and } \mathbf{E} \end{cases}$$

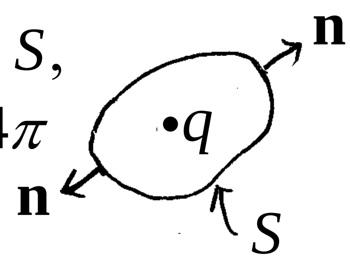


$$\mathbf{E}\cdot\mathbf{n}\ da = \frac{q}{4\pi\varepsilon_0 r^2}\mathbf{e}_r\cdot\mathbf{n}\ da = \frac{q}{4\pi\varepsilon_0 r^2}\underbrace{\cos\theta\ da}_{r^2 d\Omega} = \frac{q}{4\pi\varepsilon_0}d\Omega$$

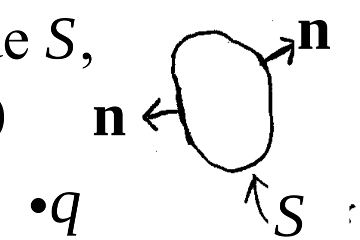
Note: $d\Omega$ carries the sign of $\cos\theta$.

$$\begin{cases} d\Omega > 0, & \text{if } \cos\theta > 0 \\ d\Omega < 0, & \text{if } \cos\theta < 0 \end{cases} \Rightarrow$$

q inside S, $\int d\Omega = 4\pi$



q outside S, $\int d\Omega = 0$



$$\Rightarrow \oint_S \mathbf{E}\cdot\mathbf{n}\ da = \frac{q}{4\pi\varepsilon_0}\int d\Omega = \begin{cases} \dfrac{q}{\varepsilon_0}, & q \text{ inside } S \\ 0, & q \text{ outside } S \end{cases} \quad \left[\begin{array}{l}\underline{\text{Gauss's law}} \text{ for} \\ \text{a single charge}\end{array}\right] \quad (1.9)$$

By the principle of linear superposition, Gauss's law for a discrete set of charges inside S is

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{1}{\varepsilon_0}\sum_i q_i \tag{1.10}$$

and Gauss's law for a distribution of charge is

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{1}{\varepsilon_0}\int_V \rho(\mathbf{x})d^3x \tag{1.11}$$

Discussion: (1.11) is the *integral* form of Gauss's law. In the next section, we will derive the *differential* form of Gauss's law. Gauss's law is a powerful mathematical representation of Coulomb's law (see example below). Furthermore, as will be shown in Ch. 6, the two forms of Gauss's law are also applicable to *time - dependent* cases where the original form of Coulomb's law (a *static* law),

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3x',$$

no longer applies.

Shell theorems - an application of Gauss's law:

Halliday, Resnick, and Walker, "Fundamentals of Physics":

"The two shell theorems that we found so useful in our study of gravitation hold equally well in electrostatics:

Theorem 1: A uniform spherical shell of charge behaves, for external points, as if all its charge were concentrated at its center.

Theorem 2: A uniform spherical shell of charge exerts no force on a charge particle placed inside the shell."

Proof:

$$[\text{Symmetry consideration}] \;\Rightarrow\; \mathbf{E} = E_r \mathbf{e}_r$$

$$[\text{Gauss' law}] \;\Rightarrow\; 4\pi r^2 E_r = \begin{cases} \dfrac{Q}{\varepsilon_0}, & r > a \\ 0, & r < a \end{cases}$$

a: radius of shell
Q: total charge on shell

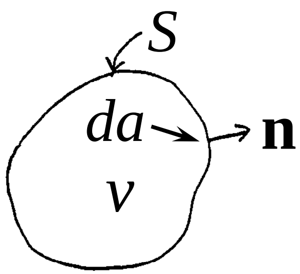
$$\Rightarrow E_r = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r^2}, & r > a \quad (\text{as if } Q \text{ were at } r = 0) \\ 0, & r < a \quad (Q \text{ produces no } \mathbf{E}) \end{cases}$$

1.4 Differential Form of Gauss's Law

Using the divergence theorem:

$$\int_V \nabla \cdot \mathbf{A} d^3x = \oint_S \mathbf{A} \cdot \mathbf{n} da,$$

[**n** is a unit vector normal to the surface element da and pointing away from the volume v enclosed by surface S.]



we obtain from $\oint_S \mathbf{E} \cdot \mathbf{n} \, da = \frac{1}{\varepsilon_0} \int_V \rho(\mathbf{x}) d^3x$ [(1.11)]

$$\oint_S \mathbf{E} \cdot \mathbf{n} \, da = \int_V \nabla \cdot \mathbf{E} d^3x = \frac{1}{\varepsilon_0} \int_V \rho(\mathbf{x}) d^3x$$

$$\Rightarrow \int_V (\nabla \cdot \mathbf{E} - \frac{\rho}{\varepsilon_0}) d^3x = 0 \qquad (1.12)$$

$$\Rightarrow \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \quad [\text{differential form of Gauss's law}] \qquad (1.13)$$

Question: If $\int_V f(x) d^3x = 0$ for an arbitrary volume v, then $f(x) = 0$ everywhere. This is the basis for obtaining (1.13) from (1.12). Does $\oint_S \mathbf{A} \cdot d\mathbf{a} = 0$ for an arbitrary closed surface S imply $\mathbf{A} = 0$ everywhere?

1.5 Another Equation of Electrostatics and the Scalar Potential

$$\nabla|\mathbf{x}-\mathbf{x}'|^n$$

$$=\frac{\partial}{\partial x}[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{n}{2}}\mathbf{e}_x$$

$$+\frac{\partial}{\partial y}[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{n}{2}}\mathbf{e}_y$$

$$+\frac{\partial}{\partial z}[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{n}{2}}\mathbf{e}_z$$

$$=\frac{n}{2}[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{n}{2}-1}2(x-x')\mathbf{e}_x$$

$$+\frac{n}{2}[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{n}{2}-1}2(y-y')\mathbf{e}_y$$

$$+\frac{n}{2}[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{n}{2}-1}2(z-z')\mathbf{e}_z$$

$$=n|\mathbf{x}-\mathbf{x}'|^{n-2}(\mathbf{x}-\mathbf{x}') \qquad (1)$$

∇ operates on $\mathbf{x}$.

∇' operates on $\mathbf{x}'$.

$$\nabla'|\mathbf{x}-\mathbf{x}'|^n=-\nabla|\mathbf{x}-\mathbf{x}'|^n$$

$$Ex:\nabla|\mathbf{x}-\mathbf{x}'|=\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|};\ \nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}=-\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3};\ \nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|^3}=-3\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^5}$$

1.5 Another Equation of Electrostatics and the Scalar Potential (*continued*)

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d^3x' = -\frac{1}{4\pi\varepsilon_0}\nabla\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$$

$$= -\nabla\Phi(\mathbf{x}),$$

$$\boxed{\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|} = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}}$$

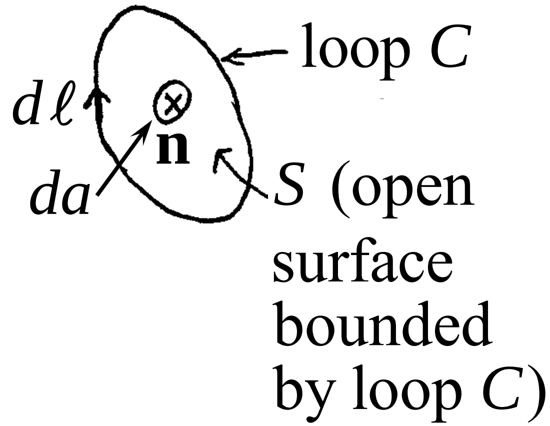
where $\Phi(\mathbf{x}) \equiv \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$ [scalar potential] (1.17)

$$\Rightarrow \nabla\times\mathbf{E} = 0 \qquad (1.14)$$

Question: $\mathbf{E} = -\nabla\Phi \Rightarrow \nabla\times\mathbf{E} = 0$. Is the reverse also true?

Below we show that $q\Phi(\mathbf{x})$ can be interpreted as the potential energy of charge q at position $\mathbf{x}$, and $\nabla\times\mathbf{E} = 0$ can be derived by an alternative method using Stokes's theorem:

$$\oint_C \mathbf{A}\cdot d\ell = \int_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\, da$$



$d\ell$: a line element on a closed loop C
S: arbitrary open surface bounded by loop C
$\mathbf{n}$: unit vector normal to surface element da in the direction given by the *right-hand rule*

1.7 Poisson and Laplace Equations

Rewrite $\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0} & (1.13) \\ \mathbf{E} = -\nabla \Phi & (1.16) \end{cases}$

Sub. (1.13) into (1.16), we obtain the Poisson equation

$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0} \tag{1.28}$$

In a charge-free region, (1.28) reduces to the Laplace euqation

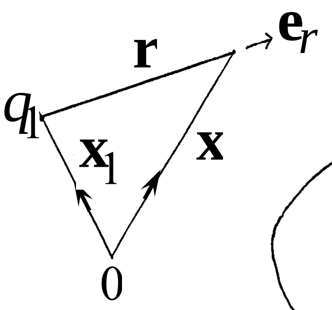
$$\nabla^2 \Phi = 0 \tag{1.29}$$

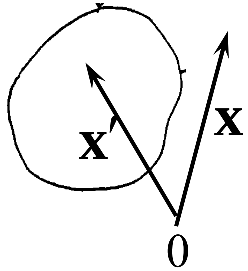
Summary of Secs. 1-5 and 7:

$$\boxed{\mathbf{F} = \frac{q q_1}{4\pi\varepsilon_0 r^2}\mathbf{e}_r} \longrightarrow$$

$$\uparrow$$

$$\mathbf{E} \equiv \lim_{q\to 0}\frac{\mathbf{F}}{q}$$





$$\mathbf{E} = \frac{q_1}{4\pi\varepsilon_0 r^2}\mathbf{e}_r \qquad \leftarrow r\mathbf{e}_r = \mathbf{x}-\mathbf{x}_1, \quad r = |\mathbf{x}-\mathbf{x}_1|$$

$$= \frac{q_1(\mathbf{x}-\mathbf{x}_1)}{4\pi\varepsilon_0|\mathbf{x}-\mathbf{x}_1|^3}$$

$\leftarrow$ principle of linear superposition

$$= \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3x'$$

$$\leftarrow \nabla|\mathbf{x}-\mathbf{x}'|^n = n|\mathbf{x}-\mathbf{x}'|^{n-2}(\mathbf{x}-\mathbf{x}')$$

$$= -\frac{1}{4\pi\varepsilon_0}\nabla\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

$$= -\nabla\Phi \qquad \leftarrow \Phi \equiv \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

$\leftarrow$ derived in Sec.1.3 using $\mathbf{E} = \dfrac{q\mathbf{e}_r}{4\pi\varepsilon_0 r^2}$

$$\nabla\times\nabla f = 0 \rightarrow$$

$$\boxed{\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{1}{\varepsilon_0}\int_V \rho(\mathbf{x})d^3x}$$

$$\boxed{\nabla\times\mathbf{E} = 0}$$

$\leftarrow$ divergence thm.

$$\boxed{\nabla\cdot\mathbf{E} = \rho/\varepsilon_0}$$

$$\boxed{\nabla^2\Phi = -\rho/\varepsilon_0}$$

Questions on Secs. 1-5 and 7:

1. Can one calculate **E** by using $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$ alone?
2. Can one calculate Φ (hence **E**) by using $\nabla^2 \Phi = -\rho / \varepsilon_0$? How?
3. Can one calculate Φ (hence **E**) by using $\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$? How?
4. Why break one equation, $\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$, into two equations: $\nabla \times \mathbf{E} = 0$ and $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$?
5. Coulomb's law gives $\nabla \times \mathbf{E} = 0$ and $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$. Can it give any other independent relation for **E**?

Helmholtz's Theorem: "A vector is uniquely specified by giving its divergence and its curl within a region and its normal component over the boundary." (Arfken, "Math. Meth. for Physicists", 3rd Ed. p.78)

6. Is the integral form of Gauss's law *mathematically* equivalent to the differential form of Gauss's law?

Answer: Yes. To prove the mathematical equivalence, we need to show that the integral form of Gauss's law is both a *sufficient* and *necessary* condition for the differential form of Gauss's law. This can be demonstrated as follows:

$$\oint \mathbf{E}\cdot\mathbf{n}da = \frac{1}{\varepsilon_0}\int_v \rho(\mathbf{x})d^3x \qquad (1.11)$$

$$\Uparrow \;\; \Downarrow \quad \leftarrow \text{divergence thm.}$$

$$\int_v (\nabla\cdot\mathbf{E} - \rho/\varepsilon_0)d^3x = 0 \;\; \text{(for arbitrary volume } v\text{)}$$

$$\Uparrow \;\; \Downarrow$$

$$\nabla\cdot\mathbf{E} = \rho/\varepsilon_0 \qquad (1.13)$$

Downward manipulation (⇓) shows that (1.11) is a sufficient condition for (1.13). Upward manipulation (⇑) shows that (1.11) is a necessary condition for (1.13). Hence, the two forms of Gauss's law are mathematically equivalent.

7. Is Gauss's law *mathematically* equivalent to Coulomb's law?

Answer: No, because Coulomb's law is a sufficient but not a necessary condition for Gauss's law. That is, we may derive Gauss's law from Coulomb's law, but not the reverse.

While Coulomb's law completely specifies the **E** field, we need more information to completely specify the **E** field from Gauss's law. This is clear when we write Gauss's law in its differential form, $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$. By Helmholtz's Theorem, we also need the curl of **E** to completely specify **E**. In electrostatics, this is given by $\nabla \times \mathbf{E} = 0$. In general, it is given by Faraday's law, $\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$ (Ch. 5).

As will be shown in later chapters, while Coulomb's law [in the form of (1.5) or (1.17)] deals only with the static **E** field, Gauss's law covers a much broader class of fields than Coulomb's law, such as the **E** field of an electromagnetic wave.

8. Is Gauss's law *physically* equivalent to Coulomb's law?

Answer: In the special case of electrostatics, the field surrounding a point charge is symmetric, implying $\mathbf{E}=E_r\mathbf{e}_r$. Choosing a spherical surface of radius r centered at the point charge, we may obtain Coulomb's law from Gauss's law,

$$\oint \mathbf{E}\cdot\mathbf{n}da = \frac{1}{\varepsilon_0}\int_V \rho(\mathbf{x})d^3x \quad \text{(Gauss's law)}$$

$$\Rightarrow \oint \mathbf{E}\cdot d\mathbf{a} = E_r 4\pi r^2 = q/\varepsilon_0$$

$$\Rightarrow E_r = \frac{q}{4\pi\varepsilon_0 r^2} \qquad \text{(Coulomb's law)}$$

In 1.3, we have also derived Gauss's law from Coulomb's law; hence, the two laws are physically equivalent in electrostatics. However, as discussed in question 7, the two laws are not mathematically equivalent, nor are they physically equivalent in electrodynamics.

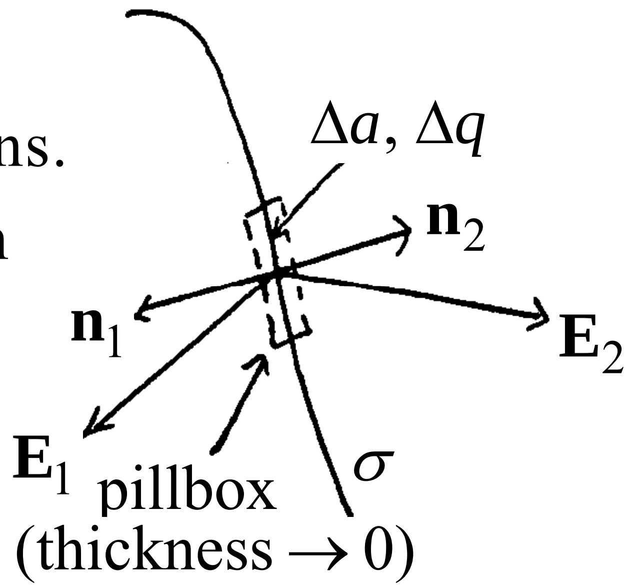
1.6 Surface Distributions of Charges and Dipoles and Discontinuities in the Electric Field and Potential

Surface Layer of Charge :

The surface charge density is defined as charge per unit area on the surface: $\sigma(\mathbf{x}) \equiv \lim_{\Delta a \to 0} \frac{\Delta q}{\Delta a}$

Note: σ and ρ have different dimensions.

Apply Gauss's law, $\oint \mathbf{E} \cdot \mathbf{n} da = \frac{q}{\varepsilon_0}$, to an infinitesimally thin pillbox, we obtain



$$(\mathbf{E}_1 \cdot \mathbf{n}_1 + \mathbf{E}_2 \cdot \mathbf{n}_2)\Delta a = \frac{\Delta q}{\varepsilon_0}$$

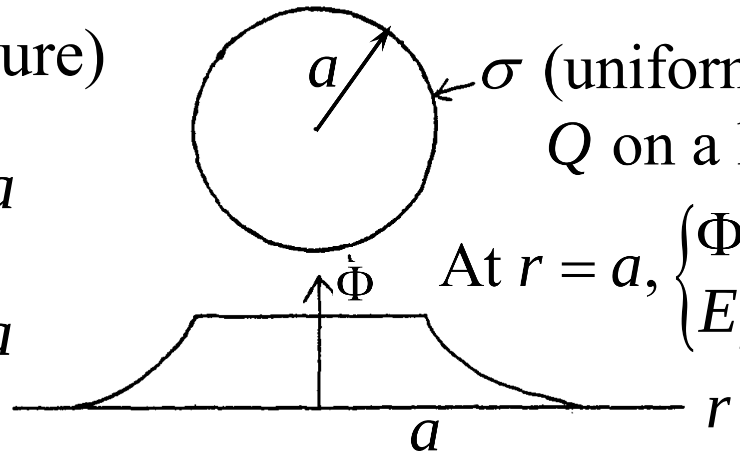
$$\mathbf{n}_1 = -\mathbf{n}_2$$

$$\Rightarrow (\mathbf{E}_2 - \mathbf{E}_1) \cdot \mathbf{n}_2 = \frac{1}{\varepsilon_0}\frac{\Delta q}{\Delta a} = \frac{\sigma}{\varepsilon_0} \tag{1.22}$$

The tangential component of $\mathbf{E}$ can be shown to be continuous across the layer by applying $\oint_C \mathbf{E} \cdot d\ell = 0$ to the loop drawn in dashed lines in the figure.

Example: (see figure)

$$\Phi = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 a}, & r \le a \\ \dfrac{Q}{4\pi\varepsilon_0 r}, & r > a \end{cases}$$



σ (uniform distribution of Q on a layer of radius a)

At $r = a$, $\begin{cases} \Phi \text{ is continuous.} \\ E_r \text{ is discontinuous.} \end{cases}$

Questions :

1. Fields (E and Φ) of a *point* charge diverge as one moves infinistesimally close to the charge. Explain why fields of the *surface* charge do not diverge as one moves infinistesimally close to the surface.

Answer: A point charge is a finite amount of charge concentrated at a *point*. However, for the surface charge, one must integrate σ over a finite surface area to obtain a finite amount of charge. Hence, there is no finite amount charge at a single point on the layer.

2. Why is Φ continuous across the layer?

1.11 Electrostatic Potential Energy and Energy Density; Capacitance

Electric Field Energy: Let $\Phi(\mathbf{x})$ be the field due to the presence of ρ. The work done to add $\delta\rho$ is

$$\delta W = \int \delta\rho(\mathbf{x})\Phi(\mathbf{x})d^3x$$

$\boxed{\delta\rho = \varepsilon_0 \nabla\cdot\delta\mathbf{E}}$ →

$$= \varepsilon_0 \int \Phi\nabla\cdot\delta\mathbf{E}(\mathbf{x})d^3x$$

Using $\nabla\cdot\psi\mathbf{a} = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$ we obtain

$$\Phi\nabla\cdot\delta\mathbf{E} = \nabla\cdot(\Phi\delta\mathbf{E}) - \delta\mathbf{E}\cdot\nabla\Phi$$
$$= \nabla\cdot(\Phi\delta\mathbf{E}) + \mathbf{E}\cdot\delta\mathbf{E}$$

$$= \varepsilon_0 \underbrace{\int \nabla\cdot(\Phi\delta\mathbf{E})d^3x} + \varepsilon_0\int \mathbf{E}\cdot\delta\mathbf{E}d^3x = \varepsilon_0\int \mathbf{E}\cdot\delta\mathbf{E}d^3x$$

$$= \oint_S \underbrace{\Phi}_{\frac{1}{r}} \underbrace{\delta\mathbf{E}}_{\frac{1}{r^2}} \cdot \underbrace{d\mathbf{a}}_{r^2} = 0, \text{ as } r\to\infty$$

[For this integral to vanish, the volume of integration must be infinite.]

By conservation of energy, this must be the total **E**-field energy.

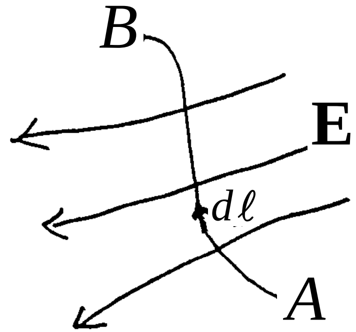
$$\Rightarrow W = \varepsilon_0\int d^3x\int_0^E \mathbf{E}\cdot d\mathbf{E} = \frac{\varepsilon_0}{2}\int|\mathbf{E}|^2 d^3x \longleftarrow \boxed{\text{infinite volume}} \qquad (1.54)$$

$$|\mathbf{E}|^2 = \mathbf{E}\cdot\mathbf{E} = -\mathbf{E}\cdot\nabla\Phi = -\nabla\cdot(\Phi\mathbf{E}) + \Phi\nabla\cdot\mathbf{E} = -\nabla\cdot(\Phi\mathbf{E}) + \frac{\rho\Phi}{\varepsilon_0}$$

$$\Rightarrow W = \frac{1}{2}\int\rho(\mathbf{x})\Phi(\mathbf{x})d^3x - \frac{\varepsilon_0}{2}\overbrace{\oint_S \Phi\mathbf{E}\cdot d\mathbf{a}}^{\to 0 \text{ as } r\to\infty} = \frac{1}{2}\int\rho(\mathbf{x})\Phi(\mathbf{x})d^3x \qquad (1.53)$$

optional **1.5 Another Equation of Electrostatics and the Scalar Potential** *(continued)*

Work done by bringing charge q from position A to position B along any path:



$$W = -\int_A^B \mathbf{F}\cdot d\ell$$
$$= -q\int_A^B \mathbf{E}\cdot d\ell$$
$$= q\int_A^B \nabla\Phi\cdot d\ell$$
$$= q\int_A^B d\Phi$$
$$= q(\Phi_B - \Phi_A)$$

$\nabla\Phi = \frac{\partial\Phi}{\partial x}\mathbf{e}_x + \frac{\partial\Phi}{\partial y}\mathbf{e}_y + \frac{\partial\Phi}{\partial z}\mathbf{e}_z;\ d\ell = dx\mathbf{e}_x + dy\mathbf{e}_y + dz\mathbf{e}_z$

$\Rightarrow d\Phi = \nabla\Phi\cdot d\ell = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz$

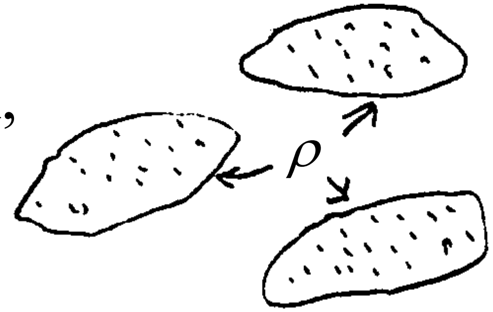
$\Rightarrow d\Phi$ is an infinitesimal change of Φ due to an infinitesimal displacement $d\ell$.

Thus, W depends only on the values of Φ at A and B, and it is independent of the charge's path from A to B. This justifies the concept of potential energy, which implies that the total work done on q in a round trip along any closed path C is 0, i.e.

$$\oint_C \mathbf{E}\cdot d\ell = 0 \text{ or, by Stokes's theorem, } \int_S (\nabla\times\mathbf{E})\cdot\mathbf{n}\,da = 0 \quad (1.21)$$

Since S is an arbitrary surface, we obtain $\nabla\times\mathbf{E} = 0$.

An alternative derivation of (1.53) and (1.54): Consider a state in which a charge density $\rho(\mathbf{x})$ has produced an electrostatic potential $\Phi(\mathbf{x})$, i.e. $\rho(\mathbf{x}) \rightarrow \Phi(\mathbf{x})$.



Then, by the principle of linear superposition, we have $\varepsilon\rho(\mathbf{x}) \rightarrow \varepsilon\Phi(\mathbf{x})$, where ε is a constant.

To find the electric field energy, we consider the energy needed to build up $\Phi(\mathbf{x})$ from $\varepsilon = 0$ (no charge and no potential) to $\varepsilon = 1$ (the present state). At any stage in the build-up process, the relative charge density (hence the relative potential) remains the same; namely, the intermediate state is characterized by the charge density $\varepsilon\rho(\mathbf{x})$ and potential $\varepsilon\Phi(\mathbf{x})$.

In such a build-up process, when the potential is $\varepsilon\Phi(\mathbf{x})$, the work done by adding an incremental charge $\rho(\mathbf{x})d\varepsilon$ is

$$dW = \int_V d^3x\,\varepsilon\Phi(\mathbf{x})\rho(\mathbf{x})d\varepsilon$$

Hence, the total work done from $\varepsilon = 0$ to $\varepsilon = 1$ is

$$W = \int_{\varepsilon=0}^{\varepsilon=1} dW = \int_V d^3x \rho(\mathbf{x})\Phi(\mathbf{x})\int_0^1 \varepsilon d\varepsilon$$

$$= \frac{1}{2}\int_V d^3x \underbrace{\rho(\mathbf{x})}_{-\varepsilon_0\nabla^2\Phi} \Phi(\mathbf{x}) \qquad (1.53)$$

For this integral to vanish, the volume of integration must be ∞.

Green's 1st identity

$$= -\frac{1}{2}\varepsilon_0\int_V \Phi\nabla^2\Phi d^3x = \frac{1}{2}\varepsilon_0[\int_V \nabla\Phi\cdot\nabla\Phi d^3x - \underbrace{\oint_S \Phi(\frac{\partial\Phi}{\partial n})da}]$$

$\sim r^2$ $\qquad$ $\sim \frac{1}{r}$ $\qquad$ $\sim \frac{1}{r^2}$

$$= \frac{1}{2}\varepsilon_0\int_V |\mathbf{E}|^2 d^3x \qquad (1.54)$$

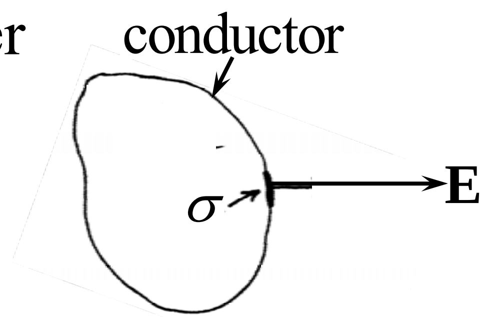
integration over infinite volume

Questions: 1. If we bring q and $-q$ toward each other, the work done is negative. Why is then $W = \frac{\varepsilon_0}{2}\int_V |\mathbf{E}|^2 d^3x$ always positive?

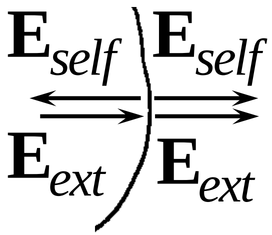
2. Give one example to show that the $\mathbf{E}$-field carries energy.

Electric Field Energy Density : $(1.54) \Rightarrow w_E = \frac{1}{2}\varepsilon_0|\mathbf{E}|^2 \qquad (1.55)$

Note: $w_E = \frac{1}{2}\varepsilon_0|\mathbf{E}|^2 = \frac{1}{2}\varepsilon_0(\sum_j \mathbf{E}_j)\cdot(\sum_j \mathbf{E}_j)\ [\neq \frac{1}{2}\varepsilon_0\sum_j(\mathbf{E}_j\cdot\mathbf{E}_j)]$



Force on the Surface of a Conductor: Consider a conductor with surface charge on it. At any point on the surface, the *total* field (**E**) outside must be normal to the surface and the total field inside must be 0. Applying Gauss's law, we find $E=\sigma/\varepsilon_0$, where σ is the local surface charge density at the observation point (upper figure). But the local σ by itself will produce equal and opposite fields on both sides of σ (call it *self* field E_{self}) and by Gauss's law



$$E_{self}\,(\text{outside}) + E_{self}\,(\text{inside}) = \frac{\sigma}{\varepsilon_0} \Rightarrow E_{self} = \frac{\sigma}{2\varepsilon_0},$$

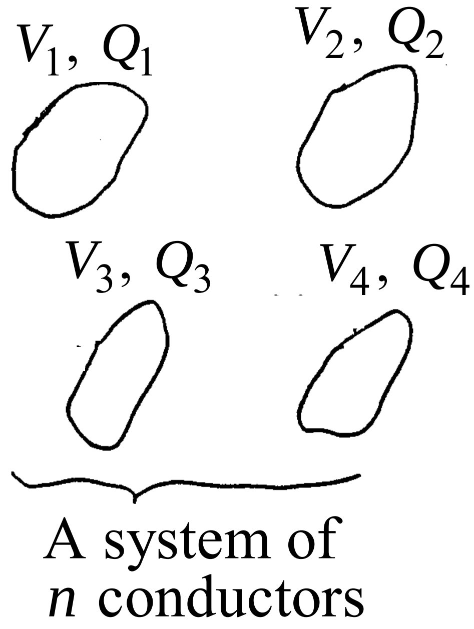
which is half of the total E outside. Since the total E inside is 0, all the external surface charge away from the local σ must have produced an *external* field with $E_{ext}=E_{self}=\sigma/2\varepsilon_0$, which cancels E_{self} (inside) and thus doubles E_{self} (outside). The local σ can only experience a force due to the field (E_{ext}) produced by the external surface charge. Thus,

$$\text{force on the surface/unit area} = \sigma E_{ext} = \frac{\sigma^2}{2\varepsilon_0}$$ (see pp. 42-43)

Capacitance: Refer to the figure

$$\begin{cases} V_1 = \sum\limits_{j=1}^{n} P_{1j} Q_j \\ V_2 = \sum\limits_{j=1}^{n} P_{2j} Q_j \\ \vdots \\ V_n = \sum\limits_{j=1}^{n} P_{nj} Q_j \end{cases} \quad \Rightarrow \quad \begin{cases} Q_1 = \sum\limits_{j=1}^{n} C_{1j} V_j \\ Q_2 = \sum\limits_{j=1}^{n} C_{2j} V_j \\ \vdots \\ Q_n = \sum\limits_{j=1}^{n} C_{nj} V_j \end{cases}$$

Invert the equations



by principle of linear superposition

C_{ii}: capacitance

C_{ij} $(i \neq j)$: coefficient of induction

P_{ij} and C_{ij} depend on the geometrical shape and position of the conductors. Potential energy of the i-th conductor is [using (1.53)]

$$W_i = \frac{1}{2}\int \rho_i(\mathbf{x})\Phi_i(\mathbf{x})d^3x = \frac{1}{2}Q_iV_i \quad \left[\Phi_i(\mathbf{x}) = V_i;\ \int \rho_i(\mathbf{x})d^3x = Q_i\right]$$

$$\Rightarrow \begin{bmatrix}\text{Potential energy} \\ \text{of the system}\end{bmatrix} = \frac{1}{2}\sum_{i=1}^{n} Q_iV_i = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}V_iV_j \qquad (1.62)$$

Homework of Chap. 1

Problems: 3, 4, 5, 6, 8,

9, 12, 14, 16, 17

Quiz: Oct. 05, 2010

Appendix A: Unit Systems and Dimensions

Unit Systems:

Two systems of electromagnetic units are in common use today: the SI and Gaussian systems. Regardless of one's personal preference, it is important to be familiar with both systems and, in particular, the conversion from one system to the other. Conversion formulae can be divided into two categories: "symbol/equation conversion [such as E and $E = q/(4\pi\varepsilon_0 r^2)$]" and "unit conversion (such as coulomb)".

Conversion formulae for symbols and equations are listed in Table 3 on p. 782 of Jackson and conversion formulae for units in Table 4 on p. 783 (both tables attached on next page). These two tables are all we need to convert between SI and Gaussian systems. Correct use of the tables requires practices.

Appendix A: Unit Systems and Dimensions (*continued*)

Table 3 Conversion Table for Symbols and Formulas

The symbols for mass, length, time, force, and other not specifically electromagnetic quantities are unchanged. To convert any equation in SI variables to the corresponding equation in Gaussian quantities, on both sides of the equation replace the relevant symbols listed below under "SI" by the corresponding "Gaussian" symbols listed on the left. The reverse transformation is also allowed. Residual powers of $\mu_0\epsilon_0$ should be eliminated in favor of the speed of light ($c^2\mu_0\epsilon_0 = 1$). Since the length and time symbols are unchanged, quantities that differ dimensionally from one another only by powers of length and/or time are grouped together where possible.

Quantity	Gaussian	SI
Velocity of light	c	$(\mu_0\epsilon_0)^{-1/2}$
Electric field (potential, voltage)	$\mathbf{E}(\Phi, V)/\sqrt{4\pi\epsilon_0}$	$\mathbf{E}(\Phi, V)$
Displacement	$\sqrt{\epsilon_0/4\pi}\,\mathbf{D}$	$\mathbf{D}$
Charge density (charge, current density, current, polarization)	$\sqrt{4\pi\epsilon_0}\,\rho(q, \mathbf{J}, I, \mathbf{P})$	$\rho(q, \mathbf{J}, I, \mathbf{P})$
Magnetic induction	$\sqrt{\mu_0/4\pi}\,\mathbf{B}$	$\mathbf{B}$
Magnetic field	$\mathbf{H}/\sqrt{4\pi\mu_0}$	$\mathbf{H}$
Magnetization	$\sqrt{4\pi/\mu_0}\,\mathbf{M}$	$\mathbf{M}$
Conductivity	$4\pi\epsilon_0\sigma$	σ
Dielectric constant	$\epsilon_0\epsilon$	ϵ
Magnetic permeability	$\mu_0\mu$	μ
Resistance (impedance)	$R(Z)/4\pi\epsilon_0$	$R(Z)$
Inductance	$L/4\pi\epsilon_0$	L
Capacitance	$4\pi\epsilon_0 C$	C

$$c = 2.997\,924\,58 \times 10^8 \text{ m/s}$$

$$\epsilon_0 = 8.854\,187\,8 \ldots \times 10^{-12} \text{ F/m}$$

$$\mu_0 = 1.256\,637\,0 \ldots \times 10^{-6} \text{ H/m}$$

$$\sqrt{\frac{\mu_0}{\epsilon_0}} = 376.730\,3 \ldots \,\Omega$$

Jackson, p. 782, Table 3

Table 4 Conversion Table for Given Amounts of a Physical Quantity

The table is arranged so that a given amount of some physical quantity, expressed as so many SI or Gaussian units of that quantity, can be expressed as an equivalent number of units in the other system. Thus the entries in each row stand for the same amount, expressed in different units. All factors of 3 (apart from exponents) should, for accurate work, be replaced by (2.997 924 58), arising from the numerical value of the velocity of light. For example, in the row for displacement (D), the entry ($12\pi \times 10^5$) is actually ($2.997\,924\,58 \times 4\pi \times 10^5$) and "9" is actually $10^{-16} c^2 = 8.987\,55 \ldots$. Where a name for a unit has been agreed on or is in common usage, that name is given. Otherwise, one merely reads so many Gaussian units, or SI units.

Physical Quantity	Symbol	SI	Gaussian	
Length	l	1 meter (m)	10^2	centimeters (cm)
Mass	m	1 kilogram (kg)	10^3	grams (g)
Time	t	1 second (s)	1	second (s)
Frequency	ν	1 hertz (Hz)	1	hertz (Hz)
Force	F	1 newton (N)	10^5	dynes
Work Energy	W U	1 joule (J)	10^7	ergs
Power	P	1 watt (W)	10^7	ergs s^{-1}
Charge	q	1 coulomb (C)	3×10^9	statcoulombs
Charge density	ρ	1 C m^{-3}	3×10^3	statcoul cm^{-3}
Current	I	1 ampere (A)	3×10^9	statamperes
Current density	J	1 A m^{-2}	3×10^5	statamp cm^{-2}
Electric field	E	1 volt m^{-1} (Vm^{-1})	$\frac{1}{3} \times 10^{-4}$	statvolt cm^{-1}
Potential	Φ, V	1 volt (V)	$\frac{1}{300}$	statvolt
Polarization	P	1 C m^{-2}	3×10^5	dipole moment cm^{-3}
Displacement	D	1 C m^{-2}	$12\pi \times 10^5$	statvolt cm^{-1} (statcoul cm^{-2})
Conductivity	σ	1 mho m^{-1}	9×10^9	s^{-1}
Resistance	R	1 ohm (Ω)	$\frac{1}{9} \times 10^{-11}$	s cm^{-1}
Capacitance	C	1 farad (F)	9×10^{11}	cm
Magnetic flux	ϕ, F	1 weber (Wb)	10^8	gauss cm^2 or maxwells
Magnetic induction	B	1 tesla (T)	10^4	gauss (G)
Magnetic field	H	1 A m^{-1}	$4\pi \times 10^{-3}$	oersted (Oe)
Magnetization	M	1 A m^{-1}	10^{-3}	magnetic moment cm^{-3}
Inductance*	L	1 henry (H)	$\frac{1}{9} \times 10^{-11}$	

Jackson, p. 783, Table 4

Conversion of symbols and equations:

Consider, for example, the conversion of the SI equation

$$E = \frac{q}{4\pi\varepsilon_0 r^2} \qquad \text{(A.1)}$$

into the Gaussian system.

This involves the conversion of symbols and equations. So we use Table 3. First, we note from Table 3 (top) that mechanical symbols (e.g. time, length, mass, force, energy, and frequency) are unchanged in the conversion. Thus, we only need to deal with electromagnetic symbols on *both* sides of (A.1).

From Table 3, we find $E^{SI} \to \dfrac{E^G}{\sqrt{4\pi\varepsilon_0}}$ and $q^{SI} \to \sqrt{4\pi\varepsilon_0}\, q^G$ (A.2)

Sub. $E^G/\sqrt{4\pi\varepsilon_0}$ and $\sqrt{4\pi\varepsilon_0}\, q^G$, respectively, for E and q in (A.1), we obtain the corresponding equation in the Gaussian system:

$$\frac{E^G}{\sqrt{4\pi\varepsilon_0}} = \frac{\sqrt{4\pi\varepsilon_0}\, q^G}{4\pi\varepsilon_0 r^2} \Rightarrow E^G = \frac{q^G}{r^2} \qquad \text{(A.3)}$$

Conversion of units and evaluation of physical quantities:

Consider again the SI equation : $E = \dfrac{q}{4\pi\varepsilon_0 r^2}$ (A.1)

Given $r = 0.01$ m, $q = 1$ statcoulomb, we may evaluate E in 3 steps:

Step 1: Express r, q, and ε_0 in SI units. From Table 3 (bottom) and Table 4, we find

$$\begin{cases} \varepsilon_0 = 8.854\times 10^{-12}\ \text{Farad/m} = \dfrac{1}{36\pi\times 10^9}\ \text{Farad/m} \\ r = 0.01\ \text{m (same as given)} \\ q(=1\ \text{statcoulomb}) = \dfrac{1}{3\times 10^9}\ \text{coulomb} \end{cases} \quad (\text{A.4})$$

Step 2: Sub. the numbers (*but not the units*) from (A.4) into (A.1).

This gives $E = \dfrac{q}{4\pi\varepsilon_0 r^2} = \dfrac{\frac{1}{3\times 10^9}}{4\pi\times\frac{1}{36\pi\times 10^9}\times(0.01)^2} = 3\times 10^4$

Step 3: Look up Table 4 for the SI unit of E. As shown in Table 4, the SI unit of E is V/m. Thus, $E = 3\times 10^4$ V/m (A.5)

As another exercise, we write (A.1) in the Gaussian system:

$$E = \frac{q}{r^2} \qquad \text{(A.3)}$$

and evaluate E for the same r ($= 0.01$ m) and q ($= 1$ statcoulomb).

Step 1: Express r and q in Gaussian units. From Table 4, we find

$$\begin{cases} r(= 0.01 \text{ m}) = 1 \text{ cm} \\ q = 1 \text{ statcoulomb (same as given)} \end{cases} \qquad \text{(A.6)}$$

Step 2: Sub. the numbers (*but not the units*) from (A.6) into (A.3).

This gives $E = \dfrac{q}{r^2} = \dfrac{1}{1} = 1$

Step 3: Look up Table 4 for the Gaussian unit of E. We find the unit to be statvolt/cm. Thus, $E = 1$ statvolt/cm (A.7)

Table 4 shows 1 statvolt/cm $= 3 \times 10^4$ V/m. Hence, the 2 results

in (A.5) and (A.7): $\left\{ \begin{array}{l} E = 3 \times 10^4 \text{ V/m} \\ E = 1 \text{ statvolt/cm} \end{array} \right\}$ are identical as expected.

Units and Dimensions :

In the Gaussian system, the basic units are length (ℓ), mass (m), and time (t). In the SI system, they are the above plus the current (I). [See Table 1 (top) on p. 779 of Jackson.] All other units are derived units.

If a physical quantity is expressed in term of the basic units, we have the dimension of this quantity.

A mechanical quantity has the same dimension in both systems. For example, the acceleration a $(= d^2x/dt^2)$ has the dimension of ℓt^{-2}. From $f = ma$, we obtain the dimension of force : $m\ell t^{-2}$, which in turn gives the dimension of work ($f \cdot \ell$) or energy: $m\ell^2 t^{-2}$.

An electromagnetic quantity has different dimensions in different systems. For example, the charge q has the SI dimension of It. From the Gaussian equation $f = q_1 q_2 / r^2$ and the dimensions of force and length, we find the Gaussian dimension of q to be $m^{1/2}\ell^{3/2}t^{-1}$. Since $q\phi$ has the dimension of energy ($m\ell^2 t^{-2}$), the potential ϕ has the SI dimension of $m\ell^2 t^{-3} I^{-1}$ and the Gaussian dimension of $m^{1/2}\ell^{1/2}t^{-1}$.

All physical quantities in an equation must be expressed in the same unit system and all terms must have the same dimension. For example, by Stokes's theorem, we have

$$\oint_C \mathbf{E} \cdot d\ell = \int_S (\nabla \times \mathbf{E}) \cdot \mathbf{n}\, da \qquad \text{(A.8)}$$

where both terms have the dimension of $\ell \cdot$(the dimension of E).

In the definition of the delta function:

$$\int_{a_1}^{a_2} \delta(x-a)dx = 1, \qquad \text{(A.9)}$$

the RHS is dimensionless. Thus, if x has the dimension of ℓ, $\delta(x-a)$ must have the dimension of ℓ^{-1}. However, "0" is *not* to be regarded as a dimensionless quantity. This is clear if we write (A.8) as

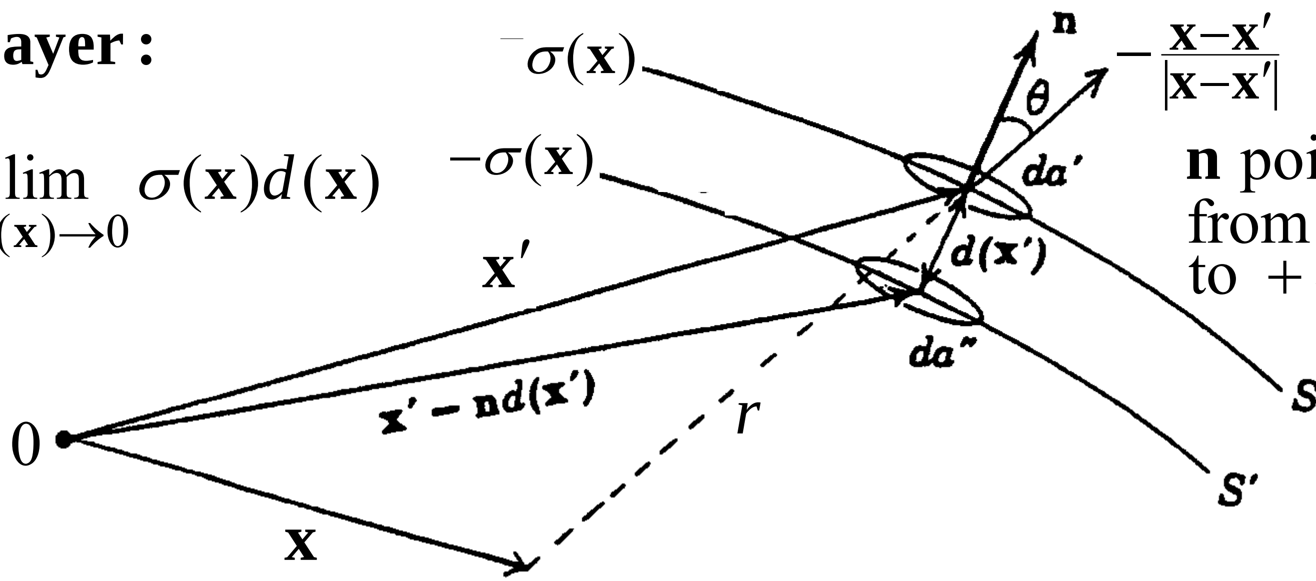
$$\oint_C \mathbf{E} \cdot d\ell - \int_S (\nabla \times \mathbf{E}) \cdot \mathbf{n}\, da = 0.$$

Well known equations need not be checked for dimensional consistency. However, for newly derived equations, a dimensional check can be a convenient way to find mistakes.

optional

Dipole Layer :

$$D(\mathbf{x}) \equiv \lim_{d(\mathbf{x})\to 0} \sigma(\mathbf{x}) d(\mathbf{x})$$



$-\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}$

n points from $-\sigma$ to $+\sigma$

Assume that, at any given point, the two layers have equal and opposite surface charge densities (see figure).

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' = \frac{1}{4\pi\varepsilon_0}\left[\int_S \frac{\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} da' - \int_{S'} \frac{\sigma(\mathbf{x}')}{|\mathbf{x}-(\mathbf{x}'-\mathbf{n}d)|} da''\right]$$

$\boxed{da' = da''}$

$$= \frac{1}{4\pi\varepsilon_0}\int_S \sigma(\mathbf{x}')\left[\frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{|\mathbf{x}-(\mathbf{x}'-\mathbf{n}d)|}\right] da'$$

optional

Using the binomial expansion: What is the Taylor expansion?

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots,$$

we obtain

$$\frac{1}{|\mathbf{b}+\mathbf{a}|} = \frac{1}{\left(b^2+a^2+2\mathbf{a}\cdot\mathbf{b}\right)^{1/2}} = \frac{1}{b}\left(1+\frac{a^2}{b^2}+2\frac{\mathbf{a}\cdot\mathbf{b}}{b^2}\right)^{-1/2}$$

$$= \frac{1}{b}\left(1-\frac{a^2}{2b^2}-\frac{\mathbf{a}\cdot\mathbf{b}}{b^2}+\cdots\right) \approx \frac{1}{b}-\frac{\mathbf{a}\cdot\mathbf{b}}{b^3}$$

$\boxed{a/b \to 0}$

$$\Rightarrow \frac{1}{|\mathbf{x}-(\mathbf{x}'-\mathbf{n}d)|} \approx \frac{1}{|\mathbf{x}-\mathbf{x}'|} - d\mathbf{n}\cdot\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$$

[valid for d $\ll |\mathbf{x}-\mathbf{x}'|$]

$\boxed{\begin{array}{l}\mathbf{b}\to\mathbf{x}-\mathbf{x}'\\ \mathbf{a}\to\mathbf{n}d\end{array}}$

optional

Sub. $$\frac{1}{|\mathbf{x}-(\mathbf{x}'-\mathbf{n}d)|} \approx \frac{1}{|\mathbf{x}-\mathbf{x}'|} - d\mathbf{n}\cdot\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$$

into $\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_S \sigma(\mathbf{x}')\left[\frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{|\mathbf{x}-(\mathbf{x}'-\mathbf{n}d)|}\right]da'$, we obtain

$$\boxed{= -\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}}$$

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_S \underbrace{\sigma(\mathbf{x}')d(\mathbf{x}')}_{D(\mathbf{x}')}\mathbf{n}\cdot\overbrace{\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}}\,da' = \frac{1}{4\pi\varepsilon_0}\int_S D(\mathbf{x}')\mathbf{n}\cdot\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}da' \qquad (1.24)$$

σ and *d* appear as a product here, so it's meaningful to define the product as the dipole layer strength.

$$\text{or } \Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_S D(\mathbf{x}')\underbrace{\underbrace{\mathbf{n}\cdot\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}}_{-\cos\theta}\underbrace{\frac{1}{|\mathbf{x}-\mathbf{x}'|^2}}_{1/r^2}da'}_{-d\Omega} = -\frac{1}{4\pi\varepsilon_0}\int_S D(\mathbf{x}')d\Omega \qquad (1.26)$$

$$\begin{cases} d\Omega \geq 0, \text{ if } \cos\theta > 0 \\ d\Omega < 0, \text{ if } \cos\theta < 0 \end{cases}$$

See figure two pages back.

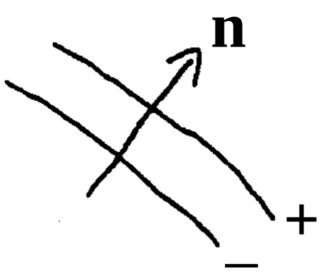
optional

1.6 Surface Distributions of Charges and Dipoles… (*continued*)

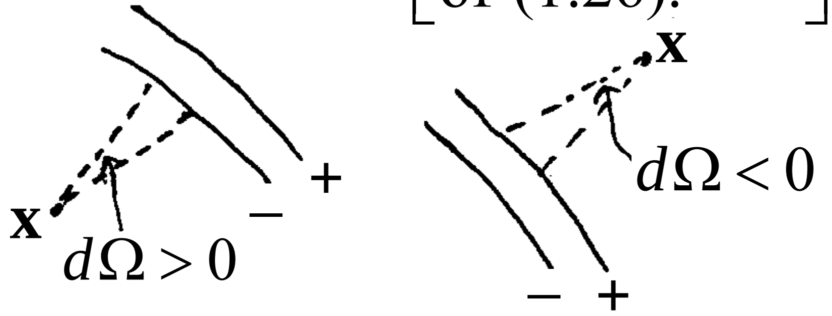
$$\text{Rewrite}: \Phi(\mathbf{x}) = \begin{cases} -\dfrac{1}{4\pi\varepsilon_0}\int_S D(\mathbf{x}')d\Omega & (1.26) \\ \dfrac{1}{4\pi\varepsilon_0}\int_S D(\mathbf{x}')\mathbf{n}\cdot\nabla'\dfrac{1}{|\mathbf{x}-\mathbf{x}'|}da' & (1.24) \end{cases}$$

Note: (1) The direction of $\mathbf{n}$ and sign of $d\Omega$ are shown below with respect to the polarity of the dipole layer:

direction of $\mathbf{n}$:



sign of $d\Omega$: $\left[\begin{array}{l}\text{See derivation} \\ \text{of (1.26).}\end{array}\right]$



(2) The RHS of (1.24) is an explicit function of $\mathbf{x}$ (the position of observation). The RHS of (1.26) is an implicit function of $\mathbf{x}$, because the total solid angle depends on $\mathbf{x}$.

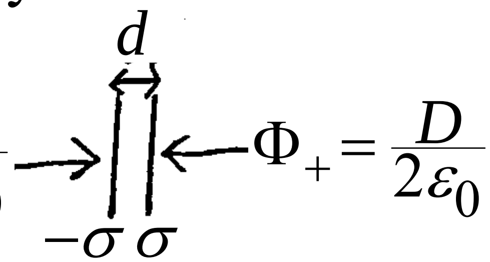
Question: Under what condition will (1.24) and (1.26) be invalid?

optional

Special case 1: A flat-disc shaped double layer with $D = const$.

$$\boxed{\Phi = -\frac{1}{4\pi\varepsilon_0}\int_S D(\mathbf{x}')d\Omega \quad (1.26)}$$

$$\Phi_+ - \Phi_- = \frac{D}{2\varepsilon_0} - \left(-\frac{D}{2\varepsilon_0}\right) = \frac{D}{\varepsilon_0}$$



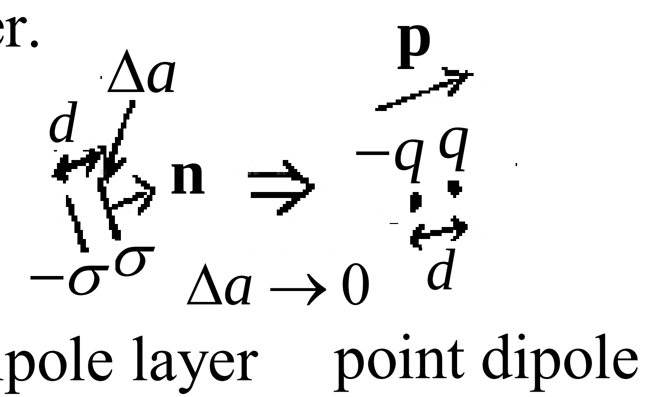
$$\Rightarrow \begin{cases} \text{electric field between layers: } E_\perp = \dfrac{D}{\varepsilon_0 d}. \\ \Phi \text{ is discontinuous across the dipole layer.} \end{cases}$$

Special case 2: Point dipole

$$\underbrace{\mathbf{p}}_{\substack{\text{point}\\\text{dipole}}} = \lim_{\Delta a \to 0}\int_{\Delta a} \underbrace{\mathbf{n}D}_{\substack{\text{dipole}\\\text{layer}}}\, da'$$

$$\boxed{\sigma\Delta a = q}$$

$$= \lim_{\Delta a \to 0}\int_{\Delta a} \mathbf{n}(\sigma d)da' = \mathbf{n}\sigma d\Delta a = \mathbf{n}qd$$



$$\Rightarrow \Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\lim_{\Delta a \to 0}\int_{\Delta a} D(\mathbf{x}')\mathbf{n}\cdot \underbrace{\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}}_{=\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}} da' = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \quad (1.25)$$

$\boxed{(1.24)}$