CHAPTER 2: Boundary-Value Problems in Electrostatics: I

Applications of Green’s theorem

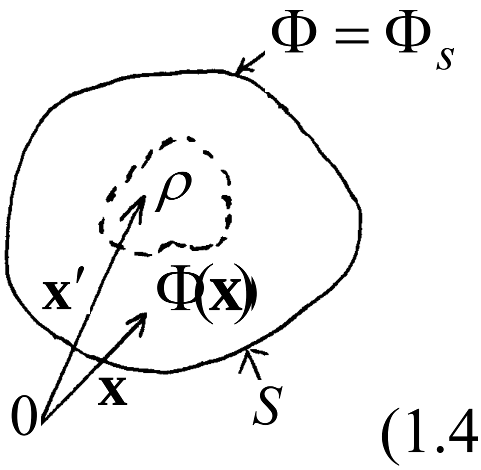
2.6 Green Function for the Sphere; General Solution for the Potential

The general electrostatic problem (upper figure):

$$\nabla^2\Phi(\mathbf{x}) = -\frac{1}{\varepsilon_0}\rho(\mathbf{x}) \text{ with b.c. } \Phi = \Phi_s$$

has the formal solution: (see Sec. 1.10)

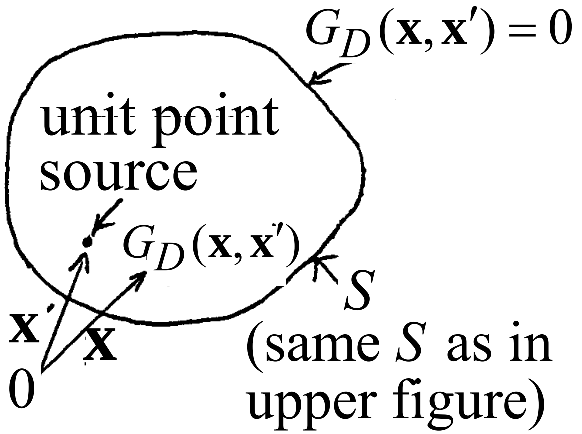
$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G_D(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial}{\partial n'}G_D(\mathbf{x},\mathbf{x}')da', \qquad (1.44)$$



where the Green function $G_D(\mathbf{x},\mathbf{x}')$ is the solution of (lower figure)

$$\nabla^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \text{ with b.c. } G_D(\mathbf{x},\mathbf{x}') = 0 \text{ for } \mathbf{x} \text{ on } S$$

$G_D(\mathbf{x},\mathbf{x}')$ can be regarded as the potential due to a unit point source ($q \to 4\pi\varepsilon_0$, p. 64) at an arbitrary position $\mathbf{x}'$ inside the same surface S, but with the homogeneous b.c. $G_D(\mathbf{x},\mathbf{x}') = 0$ for $\mathbf{x}$ on S.



Example 1: Use (1.44) to find Φ due to a point charge q at $\mathbf{x}=\mathbf{b}$ in infinite space.

$$\nabla^2\Phi(\mathbf{x})=-\frac{q}{\varepsilon_0}\delta(\mathbf{x}-\mathbf{b}) \text{ with b.c. } \Phi(\infty)=0$$

In order to use (1.44), we first obtain the Green function from

$$\nabla^2 G_D(\mathbf{x},\mathbf{x}')=-4\pi\delta(\mathbf{x}-\mathbf{x}') \text{ with } G_D(\mathbf{x},\mathbf{x}')=0 \text{ for } \mathbf{x} \text{ on S} \qquad (2)$$

The solution of (2) is $G_D(\mathbf{x},\mathbf{x}')=\dfrac{1}{|\mathbf{x}-\mathbf{x}'|}$

Sub. $q\delta(\mathbf{x}'-\mathbf{b})$ for $\rho(\mathbf{x}')$ and $1/|\mathbf{x}-\mathbf{x}'|$ for $G_D(\mathbf{x},\mathbf{x}')$ into (1.44)

$$\Rightarrow \Phi(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int_V \overbrace{\rho(\mathbf{x}')}^{q\delta(\mathbf{x}'-\mathbf{b})}\ \overbrace{G_D(\mathbf{x},\mathbf{x}')}^{\frac{1}{|\mathbf{x}-\mathbf{x}'|}} d^3x' - \frac{1}{4\pi}\oint_S \overbrace{\Phi(\mathbf{x}')}^{0}\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n}da'$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{q}{|\mathbf{x}-\mathbf{b}|}$$

2.6 Green Function for the Sphere… (*continued*)

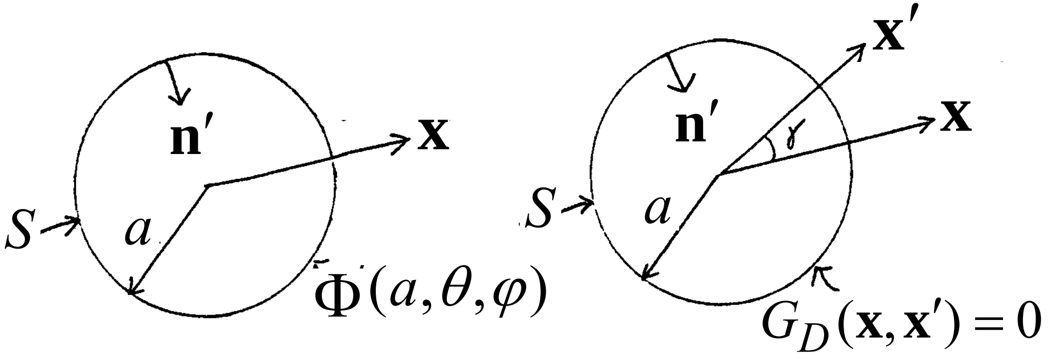
Example 2: $\nabla^2\Phi(\mathbf{x}) = 0$ with b.c. $\Phi(r=a) = \Phi(a,\theta,\varphi)$

Find $\Phi(\mathbf{x})$ in the region $r \geq a$ (see left figure).

First, find $G_D(\mathbf{x},\mathbf{x}')$ from the equation (see right figure

$$\nabla^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$$

with $G_D(\mathbf{x},\mathbf{x}') = 0$ on S.



Note: $\mathbf{n}$ points outward from the volume of interest.

This equation has the solution (see Sec. 2.2):

$$G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{a}{x'\left|\mathbf{x} - \frac{a^2}{x'^2}\mathbf{x}'\right|}$$

$\nabla^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ in region of interest $(r \geq a)$

$$= \frac{1}{\left(x^2 + x'^2 - 2xx'\cos\gamma\right)^{1/2}} - \frac{1}{\left(\frac{x^2x'^2}{a^2} + a^2 - 2xx'\cos\gamma\right)^{1/2}} \quad (3)$$

angle between **x** and **x'**

Note: $G_D(\mathbf{x},\mathbf{x}') = G_D(\mathbf{x}',\mathbf{x})$

2.6 Green Function for the Sphere… *(continued)*

$$\boxed{\left.\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n'}\right|_{x'=a} = -\left.\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial x'}\right|_{x'=a} = -\frac{(x^2-a^2)}{a(x^2+a^2-2ax\cos\gamma)^{3/2}}}$$

Sub. (3) into (1.44)

$$\Rightarrow \Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \overbrace{\rho(\mathbf{x}')}^{=0} G_D(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\overbrace{\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n'}} da'$$

$$= \frac{1}{4\pi}\oint_S \Phi(a,\theta',\varphi')\frac{a(x^2-a^2)}{(x^2+a^2-2ax\cos\gamma)^{3/2}}d\Omega' \qquad (2.19)$$

Questions:

1. In (3), we have $G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{a}{x'\left|\mathbf{x}-a^2\mathbf{x}'/x'^2\right|}$ as a solution of $\nabla^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$. But $G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|}$ apparently also satisfies the same equation. Does this violate the uniqueness thm.?

2. Can the solution of $\nabla^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ be written in the form $G_D(\mathbf{x},\mathbf{x}') = G_D(\mathbf{x}-\mathbf{x}')$? why?

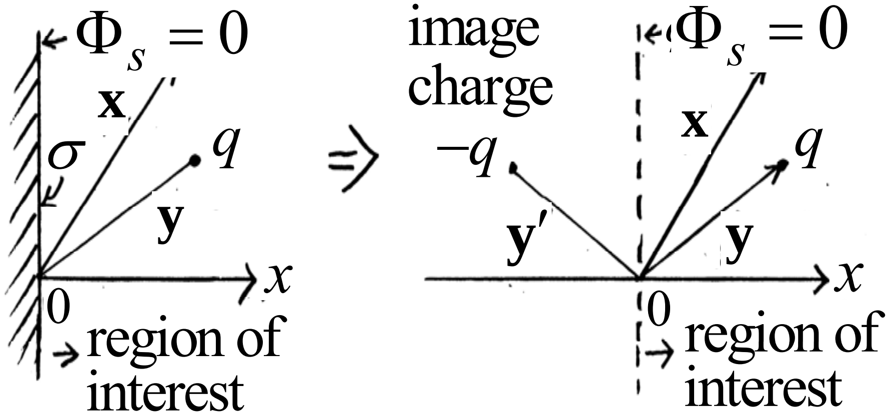
2.1 Method of Images

The method of images is not a general method. It works for some problems with a simple geometry. Consider a point charge q located in front of an infinite and grounded plane conductor (see figure). The region of interest is $x \geq 0$ and Φ is governed by the Poisson equation:

$$\nabla^2 \Phi(\mathbf{x}) = -\frac{q}{\varepsilon_0} \delta(\mathbf{x} - \mathbf{y})$$

subject to the boundary condition

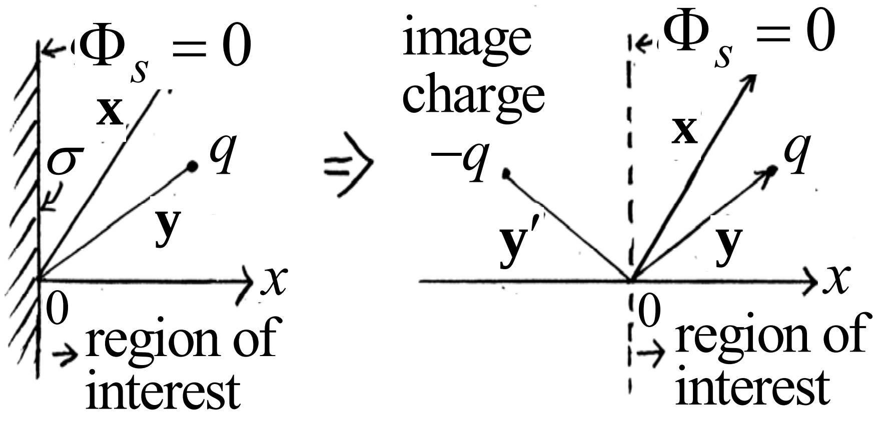
$$\Phi(x = 0) = 0.$$



In order to maintain a zero potential on the conductor, surface charge will be induced (by q) on the conductor. We may simulate the effects of the surface charge with a hypothetical "image charge", $-q$, located symmetrically behind the conductor. Then,

2.1 Method of Images (*continued*)

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{|\mathbf{x}-\mathbf{y}'|}\right]$$



and, by symmetry, $\Phi(\mathbf{x})$ satisfies the boundary condition

$$\Phi(x=0)=0.$$

Operate $\Phi(\mathbf{x})$ with ∇^2

$$\Rightarrow \nabla^2\Phi(\mathbf{x}) = -\frac{q}{\varepsilon_0}[\delta(\mathbf{x}-\mathbf{y}) - \delta(\mathbf{x}-\mathbf{y}')] \qquad (1)$$

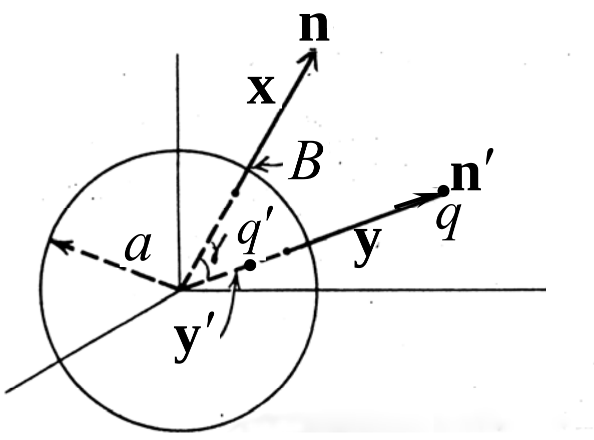
In the region of interest ($x \geq 0$), we have $\delta(\mathbf{x}-\mathbf{y}')=0$. Thus, $\Phi(\mathbf{x})$ obeys the original Poisson equation

$$\nabla^2\Phi(\mathbf{x}) = -\frac{q}{\varepsilon_0}\delta(\mathbf{x}-\mathbf{y})$$

[This shows that we must put the image charge outside the region of interest]

Since $\Phi(\mathbf{x})$ satisfies both the Poisson equation and the boundary condition in the region of interest, it is a solution. By the uniqueness theorem, it is the only solution. Note that the Poisson equation (1) and the solution $\Phi(\mathbf{x})$ are irrelevant outside the region of interest.

2.2 Point Charge in the Presence of a Grounded Conducting Sphere



Refer to the conducting sphere of radius a shown in the figure. Assume a point charge q is at $r = y$ $(> a)$. To find Φ for $r \geq a$, we put an image charge q' at $r = y'$ $(< a)$. Then,

$$\Phi(\mathbf{x}) = \frac{q/4\pi\varepsilon_0}{|\mathbf{x}-\mathbf{y}|} + \frac{q'/4\pi\varepsilon_0}{|\mathbf{x}-\mathbf{y}'|}$$

$$= \frac{q/4\pi\varepsilon_0}{|x\mathbf{n}-y\mathbf{n}'|} + \frac{q'/4\pi\varepsilon_0}{|x\mathbf{n}-y'\mathbf{n}'|}$$

Bounday condition requires

$$\Phi(a) = \frac{q/4\pi\varepsilon_0}{a\left|\mathbf{n}-\frac{y}{a}\mathbf{n}'\right|} + \frac{q'/4\pi\varepsilon_0}{y'\left|\frac{a}{y'}\mathbf{n}-\mathbf{n}'\right|} = 0$$

$$\Rightarrow \Phi(\mathbf{x}) = \frac{q/4\pi\varepsilon_0}{|\mathbf{x}-\mathbf{y}|} - \frac{aq/4\pi\varepsilon_0}{y\left|\mathbf{x}-\frac{a^2}{y^2}\mathbf{y}\right|}$$

First, set $\dfrac{y}{a} = \dfrac{a}{y'}$, or $y' = \dfrac{a^2}{y}$,

so that $\left|\mathbf{n} - \frac{y}{a}\mathbf{n}'\right| = \left|\frac{a}{y'}\mathbf{n} - \mathbf{n}'\right|$

[Note: $y' < a$; hence, q' lies outside the region of interest.]

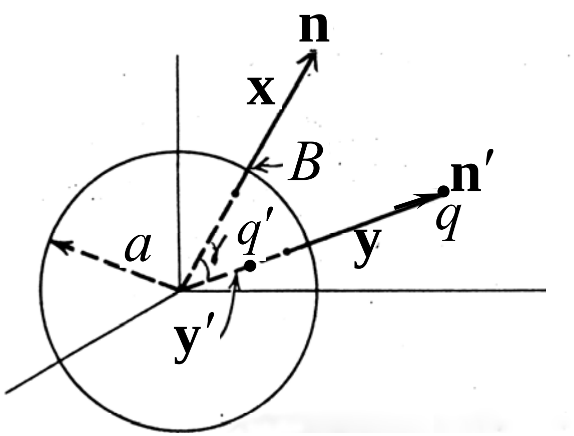
Next, set $\dfrac{q}{a} = -\dfrac{q'}{y'}$ so that RHS $= 0$.

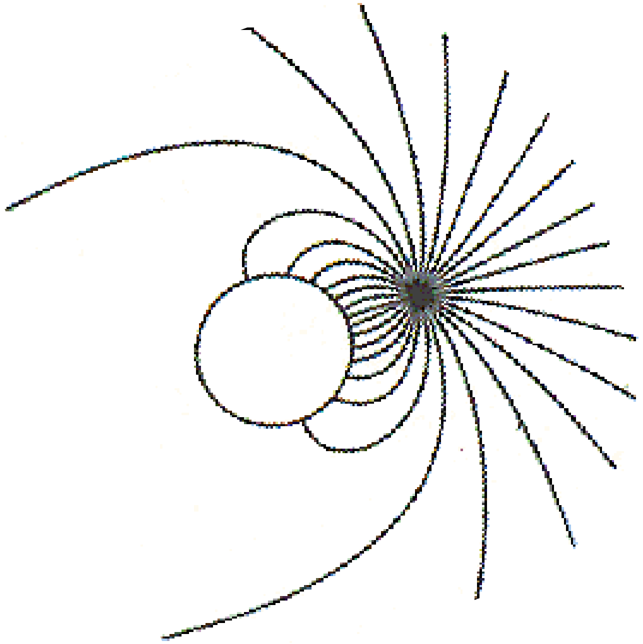
This gives $q' = -\dfrac{y'}{a}q = -\dfrac{a}{y}q.$

2.2 Point Charge in the Presence of a Grounded Conducting Sphere (*continued*)

Rewrite $\Phi(\mathbf{x}) = \frac{q/4\pi\varepsilon_0}{|\mathbf{x}-\mathbf{y}|} - \frac{aq/4\pi\varepsilon_0}{y\left|\mathbf{x}-\frac{a^2}{y^2}\mathbf{y}\right|}$ $\left[\text{This is equivalent to (2.1) and (2.4) of Jackson.}\right]$

In the region of interest ($r \geq a$), we have $\nabla^2\Phi(\mathbf{x}) = -\frac{q}{\varepsilon_0}\delta(\mathbf{x}-\mathbf{y})$. Thus, as in the case of the plane conductor, Φ satisfies the Poisson equation and the b.c. It is hence the only solution. The **E**-field lines are shown in the figure below.

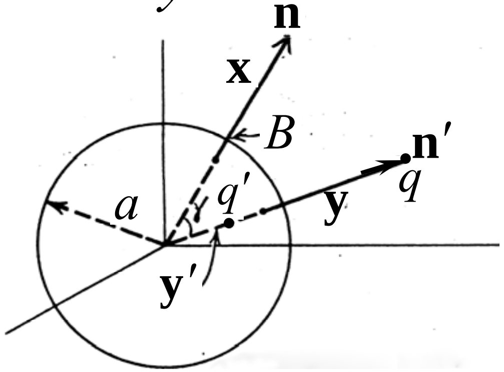




Surface charge density on the sphere: The solution for $\Phi(\mathbf{x})$ can be expressed in terms of scalars as

$$\Phi(\mathbf{x})=\frac{q}{4\pi\varepsilon_0}\Big[\frac{1}{(x^2+y^2-2xy\cos\gamma)^{1/2}}-\frac{a}{y(x^2+\frac{a^4}{y^2}-2\frac{xa^2}{y}\cos\gamma)^{1/2}}\Big]$$

where γ is the angle between $\mathbf{x}$ and $\mathbf{y}$.



By Gauss's law, the surface charge density at point B is

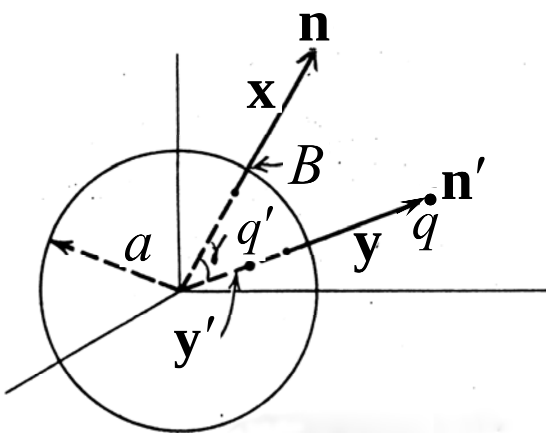
$$\sigma=\varepsilon_0 E_r(x=a)=-\varepsilon_0\left.\frac{\partial\Phi}{\partial x}\right|_{x=a}$$

$$=\frac{q}{8\pi}\Big[\frac{2a-2y\cos\gamma}{(a^2+y^2-2ay\cos\gamma)^{3/2}}-\frac{a(2a-2\frac{a^2}{y}\cos\gamma)}{y(a^2+\frac{a^4}{y^2}-2\frac{a^3}{y}\cos\gamma)^{3/2}}\Big]$$

$$=\frac{-q}{4\pi a^2}\left(\frac{a}{y}\right)\frac{1-\frac{a^2}{y^2}}{(1+\frac{a^2}{y^2}-2\frac{a}{y}\cos\gamma)^{3/2}} \qquad (2.5)$$

2.2 Point Charge in the Presence of a Grounded Conducting Sphere (*continued*)

Total charge on the sphere:



The total surface charge can be obtained by integrating σ over the spherical surface. However, it can be deduced from a simple argument: In the region $r \geq a$, the electric field due to the surface charge is exactly the same as that due to the image charge q'. Hence, by Gauss's law, the total surface charge must be $q'(= -\frac{a}{y}q)$.

Force on q:

Since, at the position of charge q, the field produced by the image charge q' is the same as that produced by the surface charge, the force on q is the Coulomb force between q' and q.

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{(y-y')^2}\mathbf{n}' = \frac{-1}{4\pi\varepsilon_0}\frac{q(\frac{a}{y}q)}{(y-\frac{a^2}{y})^2}\mathbf{n}' = \frac{-1}{4\pi\varepsilon_0}\frac{q^2}{a^2}\frac{\left(\frac{a}{y}\right)^3}{(1-\frac{a^2}{y^2})^2}\mathbf{n}' \qquad (2.6)$$

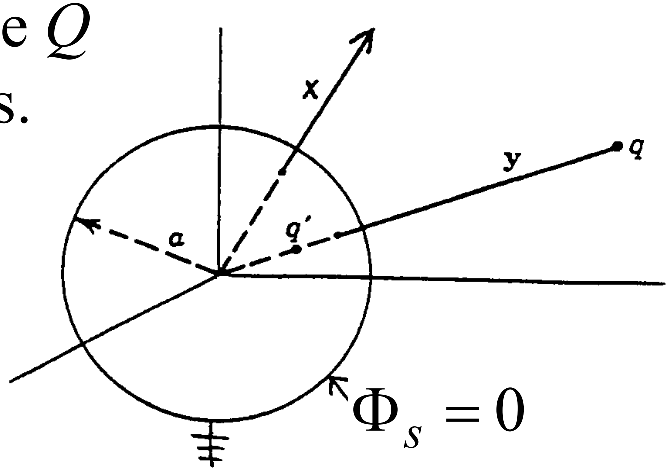
2.3 Point Charge in the Presence of a Charged, Insulated, Conducting Sphere (with Total Charge *Q*)

If the sphere is insulated with total charge Q on its surface, we may obtain Φ in two steps.

Step 1: Ground the sphere

$\Rightarrow$ same problem as in Sec. 2.2

$$\Rightarrow \Phi(\mathbf{x}) = \frac{q/4\pi\varepsilon_0}{|\mathbf{x}-\mathbf{y}|} - \frac{aq/4\pi\varepsilon_0}{y\left|\mathbf{x}-a^2\mathbf{y}/y^2\right|}$$

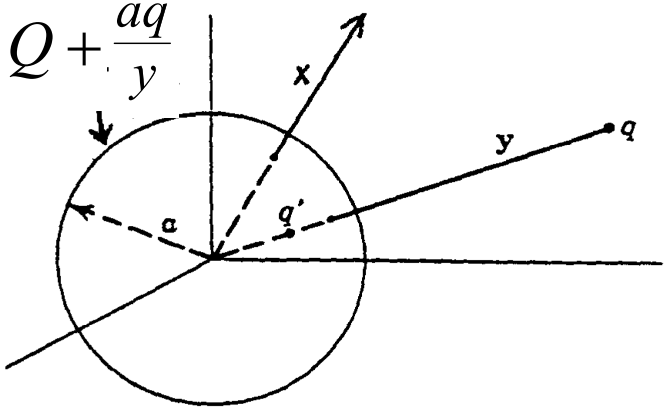


$\Phi_s = 0$

with total surface charge $q' = -aq/y$.

Step 2: Disconnect the ground wire. Add $Q + aq/y$ to the sphere so that the total charge on the sphere is Q. Then, $Q + aq/y$ will be distributed uniformly on the surface because the charges were already in static equilibrium.

$Q + \frac{aq}{y}$



$$\Rightarrow \Phi \text{ due to } Q + aq/y \text{ is } \Phi(\mathbf{x}) = \frac{Q + aq/y}{4\pi\varepsilon_0 |\mathbf{x}|}$$

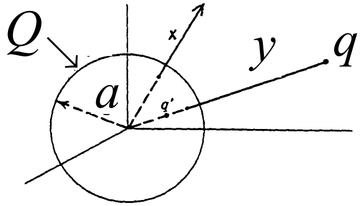
2.3 Point Charge in the Presence of a Charged, Insulated, Conducting Sphere (*continued*)

Hence, the total Φ is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\mathbf{x}-\mathbf{y}|} - \frac{aq}{y\left|\mathbf{x}-a^2\mathbf{y}/y^2\right|} + \frac{Q+aq/y}{|\mathbf{x}|}\right] \qquad (2.8)$$

The force on q is the force in (2.6) plus $\dfrac{q(Q+aq/y)}{4\pi\varepsilon_0}\dfrac{\mathbf{y}}{y^3}$ $\begin{bmatrix}\text{force due to}\\ \text{added charge}\end{bmatrix}$

$$\Rightarrow \mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{q}{y^2}\left[Q - \frac{qa^3(2y^2-a^2)}{y(y^2-a^2)^2}\right]\frac{\mathbf{y}}{y} \qquad (2.9)$$



$$\Rightarrow \begin{cases} \text{As } y \to \infty,\ F \to \dfrac{qQ}{4\pi\varepsilon_0 y^2} \ \text{(Coulomb force between point charges)} \\ \text{As } y \to a,\ F \text{ is always attractive even if } q \text{ and } Q \text{ have the same} \\ \qquad \text{sign.} \end{cases}$$

Question: If there is an excess of electrons on the surface, why don't they leave the surface due to mutual repulsion?

(See p. 61 for a discussion on the work function of a metal.)

2.7 Conducting Spheres with Hemisphere…

(to be discussed in Sec. 3.3)

2.8 Orthogonal Functions and Expansions

Definition of Orthogonal Functions :

Consider a set of real or complex functions $U_n(\xi)$ $(n=1,2,\cdots)$ which are square integrable on the interval $a\le\xi\le b$.

$$U_n(\xi)\text{'s are }\begin{cases}\underline{\text{orthogonal}},\text{ if }\overbrace{\int_a^b U_n^*(\xi)U_m(\xi)d\xi}^{\text{inner product}}\begin{cases}=0, & m\neq n\\ \neq 0, & m=n\end{cases}\\ \underline{\text{orthonormal}},\text{ if }\int_a^b U_n^*(\xi)U_m(\xi)d\xi=\delta_{mn}=\begin{cases}0, & m\neq n\\ 1, & m=n\end{cases}\end{cases}$$

Geometrical analogue: $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are an orthonormal set of unit vectors, i.e. $\mathbf{e}_m\cdot\mathbf{e}_n=\delta_{mn}$. By comparison, the dot product $\mathbf{e}_m\cdot\mathbf{e}_n$ is similar to the inner product . But the algebraic set $U_n(\xi)$ can be infinite in number.

Linearly Independent Functions :

The set of $U_n(\xi)$'s are said to be linearly independent if the only solution of $\sum_n a_n U_n(\xi) = 0$ (for every ξ in the range of $a \le \xi \le b$) is $a_n = 0$ for any n.

If a set of functions are orthogonal, they are also linearly independent.

Proof :

$$\sum_n a_n U_n(\xi) = 0$$

$$\Rightarrow \int_a^b \sum_n a_n U_n(\xi) U_m^*(\xi) d\xi = \sum_n a_n \overbrace{\int_a^b U_n(\xi) U_m^*(\xi) d\xi}^{=0,\text{ unless } m=n}$$

$$= a_n \int_a^b \left| U_n(\xi) \right|^2 d\xi = 0$$

$$\Rightarrow a_n = 0 \text{ for any } n$$

Gram-Schmidt Orthogonalization Procedure:

Orthogonality is a sufficient, but not necessary, condition for linear independence, i.e. linearly independent functions do not have to be orthogonal. However, they can be reconstructed into an orthogonal set by the Gram-Schmidt orthogonalization procedure.

Consider two vectors, $\mathbf{e}_x$ and $(\mathbf{e}_x + \mathbf{e}_y)$, as a simple example. These two vectors are not orthogonal, because $\mathbf{e}_x \cdot (\mathbf{e}_x + \mathbf{e}_y) \neq 0,$ but are linearly independent because $a\mathbf{e}_x + b(\mathbf{e}_x + \mathbf{e}_y) = 0 \Rightarrow a = b = 0.$

We may form two new vectors as linear combinations of the old vectors, $\mathbf{e}_1 = \mathbf{e}_x$ and $\mathbf{e}_2 = \mathbf{e}_x + \mathbf{e}_y + \alpha \mathbf{e}_x$, and demand $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0.$

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = 0 \Rightarrow 1 + \alpha = 0 \Rightarrow \alpha = -1 \Rightarrow \mathbf{e}_2 = \mathbf{e}_y$$

The new set, $\mathbf{e}_1 (= \mathbf{e}_x)$ and $\mathbf{e}_2 (= \mathbf{e}_y)$, are thus orthogonal (as well as linearly independent).

The same procedure can be applied to algebraic functions.

Completeness of a Set of Functions:

Expand an arbitrary, square integrable function $f(\xi)$ in terms of a finite number (N) of functions in the orthonormal set $U_n(\xi)$,

$$f(\xi) \leftrightarrow \sum_{n=1}^{N} a_n U_n(\xi) \tag{2.30}$$

and define the mean square error (M_N) as

$$M_N \equiv \int_a^b \left| f(\xi) - \sum_{n=1}^{N} a_n U_n(\xi) \right|^2 d\xi.$$

If there exists a finite number N_0 such that for $N > N_0$ the mean square error M_N can be made smaller than any arbitrarily small positive quantity by proper choice of a_n's, then the set $U_n(\xi)$ is said to be complete and the series representation

$$\sum_{n=1}^{\infty} a_n U_n(\xi) = f(\xi) \tag{2.33}$$

is said to converge in the mean to $f(\xi)$.

2.8 Orthogonal Functions and Expansions (*continued*)

Rewrite (2.33): $f(\xi) = \sum_{n=1}^{\infty} a_n U_n(\xi)$ (2.33)

Using the orthonormal property of $U_n(\xi)$'s, we get

$$a_n = \int_a^b U_n^*(\xi) f(\xi) d\xi \tag{2.32}$$

Change ξ in (2.32) to ξ' and substitute (2.32) into (2.33)

$$f(\xi) = \int_a^b \left[\sum_{n=1}^{\infty} U_n^*(\xi') U_n(\xi) \right] f(\xi') d\xi' \tag{2.34}$$

$$f(\xi) \text{ is arbitrary} \Rightarrow \underbrace{\sum_{n=1}^{\infty} U_n^*(\xi') U_n(\xi) = \delta(\xi - \xi')}_{\text{(completeness or closure relation)}} \tag{2.35}$$

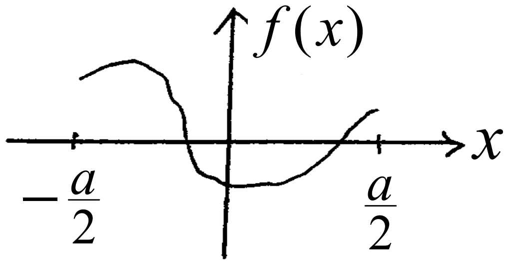
Jackson, p. 68: "All orthonormal sets of functions normally occurring in mathematical physics have been proved to be complete." (This statement will be illustrated in Sec. 2.9.)

2.8 Orthogonal Functions and Expansions (*continued*)

Fourier Series : example of complete set of orthogonal functions

Exponential representation of $f(x)$ *on the interval* $-\frac{a}{2} \leq x \leq \frac{a}{2}$:

$$\begin{cases} f(x) = \sum\limits_{n=-\infty}^{\infty} a_n e^{ik_n x} \\ k_n = \dfrac{2\pi n}{a};\ a_n = \dfrac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) e^{-ik_n x} dx \end{cases} \qquad (4)$$



In (4), $f(x)$ is in general a complex function and, even when $f(x)$ is real, a_n is in general a complex constant.

In the case $f(x)$ is *real*, we have the realty condition: $a_n = a_{-n}^*$

Proof: $f(x) = real \Rightarrow f(x) = f^*(x)$

$\boxed{n \rightarrow -n}$

$$\Rightarrow \sum_{n=-\infty}^{\infty} a_n e^{ik_n x} = \sum_{n=-\infty}^{\infty} a_n^* e^{-ik_n x} = \sum_{n=-\infty}^{\infty} a_{-n}^* e^{ik_n x}$$

$\Rightarrow a_n = a_{-n}^*$ (since $e^{ik_n x}$ is linearly independent)

Questions: 1. Why "$n = -\infty$ to ∞" instead of "$n = 0$ to ∞" ?

2. Why $k_n = 2\pi n/a$ instead of $k_n = \pi n/a$?

2.8 Orthogonal Functions and Expansions (*continued*)

Sinusoidal representation of $f(x)$ *on the interval* $-\frac{a}{2}\le x\le\frac{a}{2}$:

$$f(x)=\sum_{n=-\infty}^{\infty} a_n e^{ik_nx} = a_0+\sum_{n=1}^{\infty}\left(a_n e^{ik_nx}+a_{-n}e^{-ik_nx}\right)$$

$$= a_0+\sum_{n=1}^{\infty}\left[\left(a_n\cos k_nx+a_{-n}\cos k_nx\right)+i\left(a_n\sin k_nx-a_{-n}\sin k_nx\right)\right]$$

$$= a_0+\sum_{n=1}^{\infty}\left(a_n+a_{-n}\right)\cos k_nx+\sum_{n=1}^{\infty}i\left(a_n-a_{-n}\right)\sin k_nx$$

$$\Rightarrow f(x)=\frac{A_0}{2}+\sum_{n=1}^{\infty}\left[A_n\cos k_nx+B_n\sin k_nx\right],\qquad k_n=\frac{2\pi n}{a} \qquad (5)$$

where

Same as (2.36) and (2.37)

$$\left\{\begin{array}{l} A_n=a_n+a_{-n}=\dfrac{1}{a}\displaystyle\int_{\frac{-a}{2}}^{\frac{a}{2}} f(x)\underbrace{\left(e^{-ik_nx}+e^{ik_nx}\right)}_{2\cos k_nx}dx=\dfrac{2}{a}\displaystyle\int_{\frac{-a}{2}}^{\frac{a}{2}} f(x)\cos k_nx\,dx \\ (n=0\to\infty) \\ B_n=i\left(a_n-a_{-n}\right)=\dfrac{i}{a}\displaystyle\int_{\frac{-a}{2}}^{\frac{a}{2}} f(x)\underbrace{\left(e^{-ik_nx}-e^{ik_nx}\right)}_{-2i\sin k_nx}dx=\dfrac{2}{a}\displaystyle\int_{\frac{-a}{2}}^{\frac{a}{2}} f(x)\sin k_nx\,dx \\ (n=1\to\infty) \end{array}\right.$$

Discussion: It is often more convenient to express a physical quantity (a real number) in the exponential representation than in the sinusoidal representation, because the complex coefficient (a_n) of an exponential term carries twice the information of the real coefficient (A_n or B_n) of a sinusoidal term. For example, if

$$x(t) = \mathrm{Re}[a\mathrm{e}^{i\omega t}]$$

is the displacement of a simple harmonic oscillator, the complex a contains both the magnitude and phase of the displacement. In the sinusoidal representation, the same quantity will be written

$$x(t) = A\cos(\omega t) + B\sin(\omega t).$$

Exponential terms are also easier to manipulate (such as multiplication and differentiation). This point will be further discussed in Ch. 7.

Fourier Transform :

If the interval becomes infinite ($a \to \infty$), we obtain the Fourier transform (see Jackson p.68).

$$\begin{cases} f(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} A(k) e^{ikx} dk & (2.44) \\ A(k) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{-ikx} dx & (2.45) \end{cases}$$

Change x to x' in (2.45) and sub. (2.45) into (2.44)

$$f(x) = \int_{-\infty}^{\infty} dx' f(x') \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk}_{\delta(x-x')}$$

$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk = \delta(x - x') \quad \text{[completeness relation]} \qquad (2.47)$$

Question 1: Does $A(k)$ contain any more or any less information than $f(x)$?

Question 2 : Does a_n in $f(x)=\sum_n a_n e^{ik_n x}$ have the same dimension

as $A(k)$ in $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}A(k)e^{ikx}dk$?

[assuming x is a dimensional quantity.]

Rewrite (2.47): $\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ik(x-x')}dk=\delta(x-x')$

Interchange x and k

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(k-k')x}dx=\delta(k-k'),\quad \text{[orthogonality condition]} \qquad (2.46)$$

Let $y=k-k'$ and sub. it into (2.46)

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ixy}dx=\delta(y)$$

Since $\delta(y)=\delta(-y)$, we may write more generally,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\pm ixy}dx=\delta(y) \qquad (6)$$

There are two useful theorems concerning the Fourier integral:

(1) Parseval's theorem :

The Parseval's theorem states $\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |A(k)|^2 \, dk \quad (7)$

Proof:

Rewrite the Fourier transform: $\begin{cases} f(x) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx} dk & (2.44) \\ A(k) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx & (2.45) \end{cases}$

$$\Rightarrow \int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} f(x) f^*(x) dx$$

$$= \int_{-\infty}^{\infty} dx \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx} dk \right] \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A^*(k') e^{-ik'x} dk' \right]$$

$$= \int_{-\infty}^{\infty} dk A(k) \int_{-\infty}^{\infty} dk' A^*(k') \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{i(k-k')x}}_{\delta(k-k')} = \int_{-\infty}^{\infty} |A(k)|^2 \, dk$$

(2) Convolution theorem :

The convolution theorem states

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}[\underbrace{\int_{-\infty}^{\infty} f_1(x-\xi)f_2(\xi)d\xi}]e^{-ikx}dx = A_1(k)A_2(k) \qquad (8)$$

This is called the convolution of $f_1(x)$ and $f_2(x)$

where the factor $\frac{1}{2\pi}$ follows the convention in (2.44) and (2.45).

Proof: LHS of (8) $= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_2(\xi)d\xi \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} f_1(x-\xi)e^{-ikx}dx}_{\text{Let } \eta=x-\xi \ (\Rightarrow dx=d\eta)}$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_2(\xi)d\xi \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(\eta)e^{-ik(\xi+\eta)}d\eta$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_2(\xi)^{-ik\xi}d\xi \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(\eta)e^{-ik\eta}d\eta$$

$$= A_1(k)A_2(k)$$

2.9 Separation of Variables, Laplace Equation in Rectangular Coordinates

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \quad \begin{bmatrix}\text{Laplace equation in} \\ \text{Cartesian coordinates}\end{bmatrix} \tag{2.48}$$

$$\text{Let } \Phi(x, y, z) = X(x)Y(y)Z(z) \tag{2.49}$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \tag{2.50}$$

Since this equation holds for arbitrary values of x, y, and z, each of the three terms must be separately constant.

$$\Rightarrow \frac{d^2X}{dx^2} = -\alpha^2 X;\ \frac{d^2Y}{dy^2} = -\beta^2 Y;\ \frac{d^2Z}{dz^2} = \gamma^2 Z \text{ subject to } \gamma^2 = \alpha^2 + \beta^2$$

$$\Rightarrow X(x) = \begin{cases} e^{i\alpha x} \\ e^{-i\alpha x} \end{cases};\ Y(y) = \begin{cases} e^{i\beta y} \\ e^{-i\beta y} \end{cases};\ Z(z) = \begin{cases} e^{\gamma z} \\ e^{-\gamma z} \end{cases} \text{ with } \gamma = \sqrt{\alpha^2 + \beta^2}$$

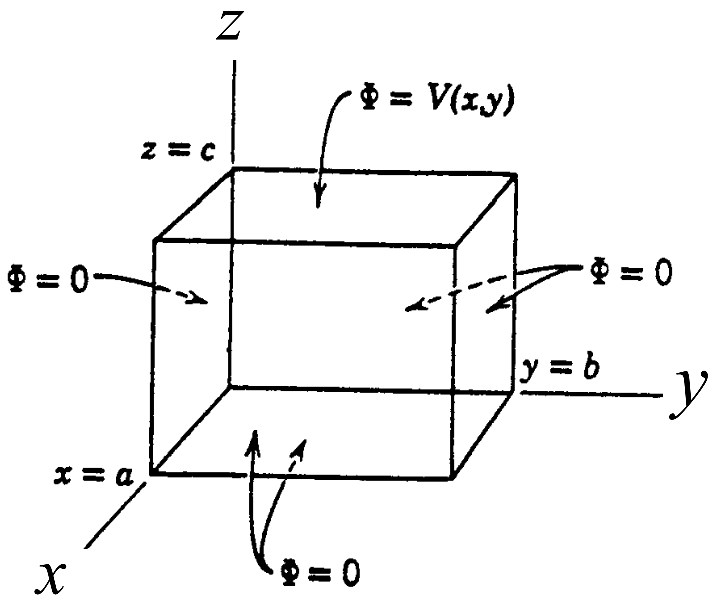
2.9 Separation of Variables, Laplace Equation in Rectangular Coordinates (*continued*)

Example: Find Φ inside a charge-free rectangular box (see figure) with the b.c. $\Phi\ (x,\ y,\ z=c)=V(x,\ y)$ and $\Phi=0$ on other sides.

$$X(x)=Ae^{i\alpha x}+Be^{-i\alpha x}$$

$$\begin{cases} X(0)=0 \Rightarrow B=-A \Rightarrow X=A(e^{i\alpha x}-e^{-i\alpha x})=A'\sin\alpha x \\ X(a)=0 \Rightarrow \alpha=\alpha_n=\dfrac{\pi n}{a},\quad n=1,2,\ldots \end{cases}$$

$$\Rightarrow X(x)=\sum_{n=1}^{\infty}A_n\sin\alpha_n x$$



Similarly, $Y(y)=Ae^{i\beta y}+Be^{-i\beta y}$.

$Y(0)=0$ and $Y(b)=0$ give

$$Y(y)=\sum_{m=1}^{\infty}A_m\sin\beta_m y,\quad \beta_m=\frac{\pi m}{b}$$

Solution for Z: $Z(z)=Ae^{\gamma z}+Be^{-\gamma z}$

$$Z(0)=0 \Rightarrow B=-A \Rightarrow Z(z)=A(e^{\gamma z}-e^{-\gamma z})=A''\sinh\gamma z$$

$$\Rightarrow \Phi=\sum_{n,m=1}^{\infty}A_{nm}\sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm}z),\ \gamma_{nm}=\sqrt{\alpha_n^2+\beta_m^2}\quad (2.56)$$

To find A_{nm}, we apply the b.c. on the $z = c$ plane:

$$\Phi(x, y, z = c) = V(x, y)$$

$$\Rightarrow V(x, y) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c) \qquad (2.57)$$

$$\Rightarrow A_{nm} = \frac{4}{ab \sinh(\gamma_{nm} c)} \int_0^a dx \int_0^b dy V(x, y) \sin(\alpha_n x) \sin(\beta_m y) \qquad (2.58)$$

Questions:

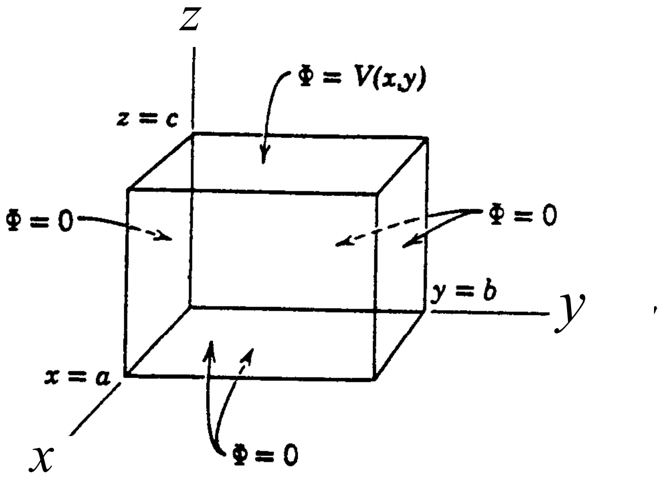
1. The method of images is not a general method, but the method of expansion in orthogonal functions is. Why?
2. In electrostatics, only charges can produce Φ. In this problem, $\rho = 0$, how can there be Φ ?
3. Can we find the surface charge distribution (σ) on the walls from the knowledge of Φ inside the box? If so, under what condition?

2.9 Separation of Variables, Laplace Equation in Rectangular Coordinates (*continued*)

Discussion:

Rewrite (2.57): $V(x,y)=\sum_{n,m=1}^{\infty}A_{nm}\sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm}c)$,

where $\alpha_n=\frac{\pi n}{a}$ and $\beta_m=\frac{\pi m}{b}$.



This is a good example to substantiate the following statement on physics ground: "All orthonormal sets of functions normally occurring in mathematical physics have been proved to be complete." (p. 68)

In (2.57), $\sin(\alpha_n x)$ and $\sin(\beta_m y)$ are orthogonal functions generated in a physics problem. Physically, we expect the problem to have a solution for any boundary condition on the surface $z=c$, i.e. for any function $V(x,y)$ specified in (2.57). Thus, $\sin(\alpha_n x)$ and and $\sin(\beta_m y)$ must each form a complete set in order to represent an arbitrary $V(x,y)$.

Homework of Chap. 2

Problems: 1, 2, 3, 4, 5,

9, 23, 26