# Chapter 3: Boundary-Value Problems in Electrostatics: II

We begin this chapter with 3 sections (Secs. 3.2, 3.5, & 3.6) on mathematics.

## 3.2 Legendre Equation and Legendre Polynomials

**Legendre Equation:**

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{du}{dx}\right]+\nu\left(\nu+1\right)u=0, \quad -1\leq x\leq 1 \qquad (3.9)$$

The solutions are: $u(x)=AP_\nu(x)+BQ_\nu(x)$

$$\begin{cases} P_\nu(x): & \underline{\text{Legendre function of the first kind}} \\ Q_\nu(x): & \underline{\text{Legendre function of the second kind}} \end{cases}$$

*Ref. 1*: Gradshteyn & Ryzhik, "Table of Integrals, Series, and Products," Chs. 7 & 8.

*Ref. 2*: Abramowitz & Stegun, "Handbook of Mathematical Functions," Ch. 8.

Rewrite the solution: $u(x) = AP_\nu(x) + BQ_\nu(x)$

$Q_\nu(x)$ diverges as $x \to \pm 1$. Hence, $Q_\nu(x)$ is not commonly used in physics.

$P_\nu(x)$ is finite for $|x| < 1$ and $x = 1$, but $P_\nu(-1)$ diverges unless $\nu$ is an integer (see p.105.)

In many physics problems, boundary conditions require $\nu$ to be an integer. Since the form of the Legendre equation is unchanged if $\nu \to -\nu - 1$, we have $P_{-\nu-1}(x) = P_\nu(x)$. Hence, when $\nu$ is an integer (denoted by $l$), negative $l$ is redundant. Thus, $l = 0,\ 1,\ 2\cdots$ and $P_l(x)$ becomes a polynomial (properties on following pages).

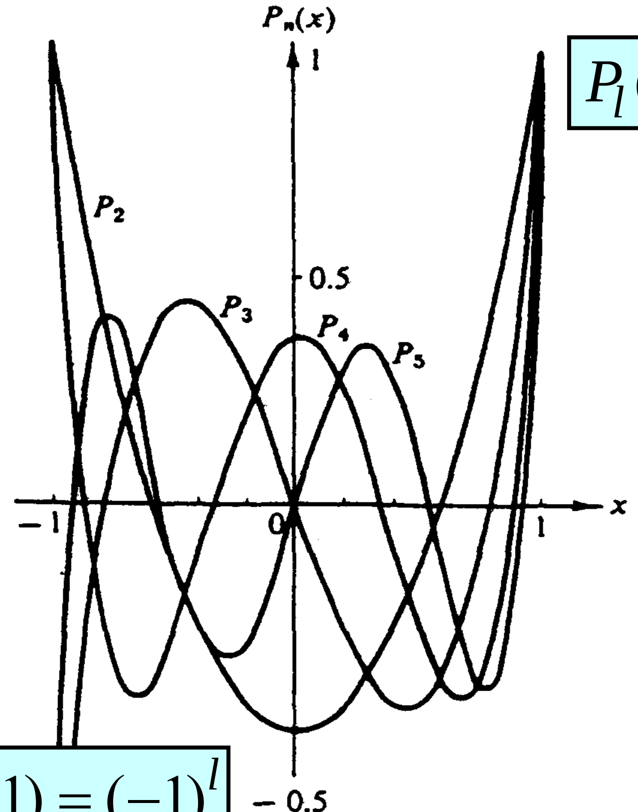
*Note*: The range $(-1 \le x \le 1)$ considered here is often encountered in physics problems. Mathmatically, the range of $P_\nu(x)$ and $Q_\nu(x)$ can be extended to the entire complex $x + iy$ plane. Furthermore, $\nu$ can also be a complex number (See Gradshteyn & Ryzhik).

**Legendre Polynomial :** $P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l,\ \ l = 0,\ 1,\ 2...$ (3.16)

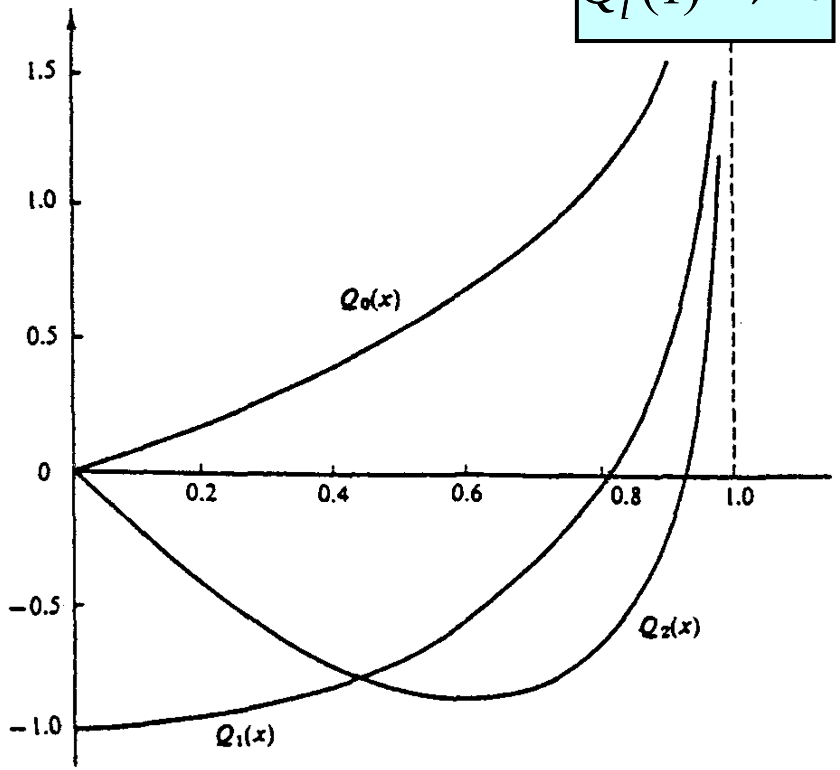
$\boxed{P_l(-x) = (-1)^l P_l(x)}$

$\boxed{P_l(1) = 1}$

$\boxed{Q_l(1) \to \infty}$




$\boxed{P_l(-1) = (-1)^l}$




Lengendre polynomials $P_2(x)$ - $P_5(x)$
$[P_0(x) = 1,\ P_1(x) = x]$

Second Lengendre functions
$Q_0(x)$, $Q_1(x)$, and $Q_2(x)$

The set $P_l(x)$ is orthogonal: $\int_{-1}^{1} P_{l'}(x)P_l(x)dx = \frac{2}{2l+1}\delta_{l'l}$ (3.21)

It is also complete in index $l$. Hence, any function $f(x)$ can be expanded as

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x) \tag{3.23}$$

## 3.5 Associated Legendre Functions and the Spherical Harmonics

**Associated Legendre Equation :**

$$\frac{d}{dx}\left(1-x^2\right)\frac{du}{dx} + \left[\nu\left(\nu+1\right) - \frac{m^2}{1-x^2}\right]u = 0 \text{ , for } -1 \le x \le 1$$

The solutions are: $u(x) = AP_\nu^m(x) + BQ_\nu^m(x)$

$$\begin{cases} P_\nu^m : \text{ \underline{associated Legendre function of the first kind}} \\ Q_\nu^m : \text{ \underline{associated Legendre function of the second kind}} \end{cases}$$

(Refs.: Gradshteyn & Ryzhik; Abramowitz & Stegun)

Rewrite the solution: $u(x) = AP_\nu^m(x) + BQ_\nu^m(x)$

$Q_\nu^m(x)$ diverges as $x \to \pm 1$, hence is not commonly used in physics.

$P_\nu^m(x)$ is finite on the interval $-1 \le x \le 1$ only when

$$\begin{cases} \nu \text{ is zero or a positive integer } (\nu = l = 0,\ 1,\ 2...) \text{ and} \\ m = -l,\ -(l-1), \ldots,\ -1,\ 0,\ 1, \ldots,\ (l-1),\ l \end{cases}$$ [p. 107.]

Under these conditions, we have (for positive or negative $m$)

$$P_l^m(x) = \frac{(-1)^m}{2^l\, l!}(1-x^2)^{\frac{m}{2}}\left(\frac{d}{dx}\right)^{l+m}(x^2-1)^l \qquad (3.50)$$

with the properties:

$$\begin{cases} P_l^0(x) = P_l(x) \\ P_l^m(-x) = (-1)^{l+m} P_l^m(x) \\ P_l^{-m}(x) = (-1)^m \dfrac{(l-m)!}{(l+m)!} P_l^m(x) & (3.51) \\ \int_{-1}^{1} P_{l'}^m(x) P_l^m(x)\, dx = \dfrac{2}{2l+1}\dfrac{(l+m)!}{(l-m)!}\delta_{ll'} & (3.52) \end{cases}$$

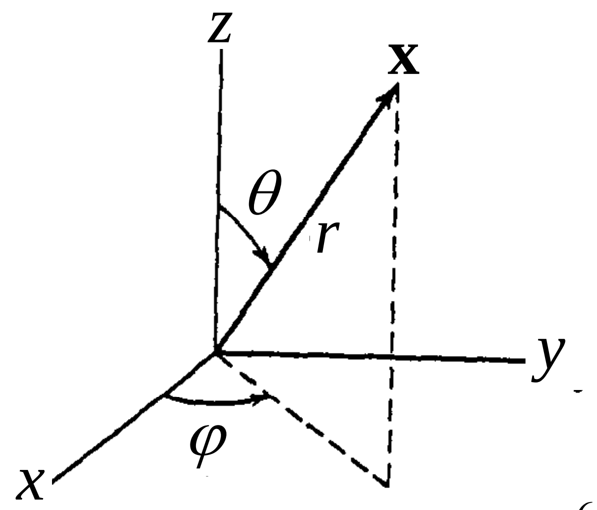
The set $P_l^m(x)$ is complete in index $l$ in the sense any function $f(x)$ can be expanded as $f(x)=\sum_{l=|m|}^{\infty} C_l P_l^m(x)$ $\left[\begin{array}{l} m: \text{ a fixed integer} \\ \text{See (A.3) in Appendix A.} \end{array}\right]$

**Spherical Harmonics** $Y_{lm}(\theta,\varphi)$:

$$Y_{lm}(\theta,\varphi) \equiv \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\varphi}, \qquad (3.53)$$

where $l=0$ or a positive integer; $m=-l,\ -(l-1),\ldots,\ 0,\ldots,\ (l-1),\ l$

$$\textit{Examples}: \begin{cases} Y_{0,0}(\theta,\varphi)=\sqrt{\frac{1}{4\pi}} \\ Y_{1,-1}(\theta,\varphi)=\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\varphi} \\ Y_{1,0}(\theta,\varphi)=\sqrt{\frac{3}{4\pi}}\cos\theta \\ Y_{1,1}(\theta,\varphi)=-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi} \end{cases}$$



*Properties of spherical harmonics*:

(i) Using the orthogonality relation,

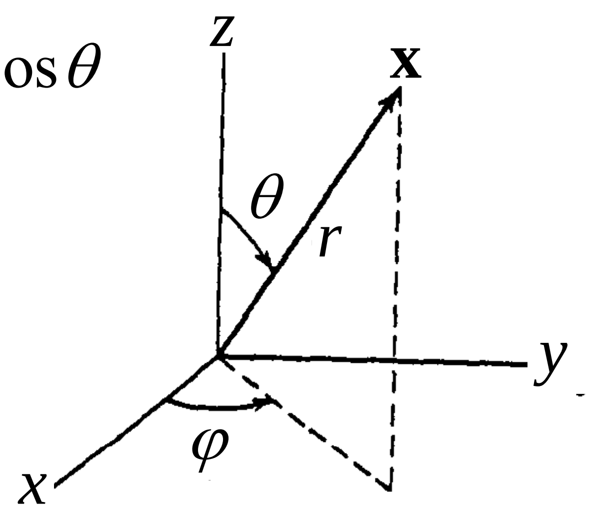
$$\int_{-1}^{1} P_{l'}^{m}(x) P_{l}^{m}(x) dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ll'} \tag{3.52}$$

we can show that the spherical harmonics are orthonormal, i.e

$$\int d\Omega Y_{l'm'}^{*}(\theta,\varphi) Y_{lm}(\theta,\varphi) = \delta_{ll'} \delta_{mm'}, \tag{3.55}$$

where

$$\int d\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta = \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta$$



(ii) The set $Y_{lm}(\theta,\varphi)$ is complete, i.e. an arbitrary function $g(\theta,\varphi)$ can be expanded as

$$g(\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}Y_{lm}(\theta,\varphi) \tag{3.58}$$

Multiplying both sides by $Y_{lm}^{*}(\theta,\varphi)$, integrating over $\theta,\varphi$, and making use of (3.55), we obtain

$$A_{lm}=\int d\Omega Y_{lm}^{*}(\theta,\varphi)g(\theta,\varphi)$$

Substitution of $A_{lm}$ into (3.58) gives the following expression for $g(\theta,\varphi)$,

$$g(\theta,\varphi)=\int d\Omega'\left[\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y_{lm}^{*}(\theta',\varphi')Y_{lm}(\theta,\varphi)\right]g(\theta',\varphi')$$

$$\Rightarrow \sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y_{lm}^{*}(\theta',\varphi')Y_{lm}(\theta,\varphi)=\delta(\varphi-\varphi')\delta(\cos\theta-\cos\theta') \tag{3.56}$$

This is the completeness relation of $Y_{lm}(\theta,\varphi)$ [cf. (2.34) & (2.35).]

(iii) Other properties of $Y_{lm}(\theta,\varphi)$:

$$\begin{cases} Y_{l,-m}(\theta,\varphi) = (-1)^m Y_{lm}^*(\theta,\varphi) \\ Y_{l,0}(\theta,\varphi) = \sqrt{\dfrac{2l+1}{4\pi}} P_l(\cos\theta) \end{cases}$$

This can be seen from the definition of $Y_{lm}(\theta,\varphi)$:

$$Y_{lm}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi}$$

and the relations:

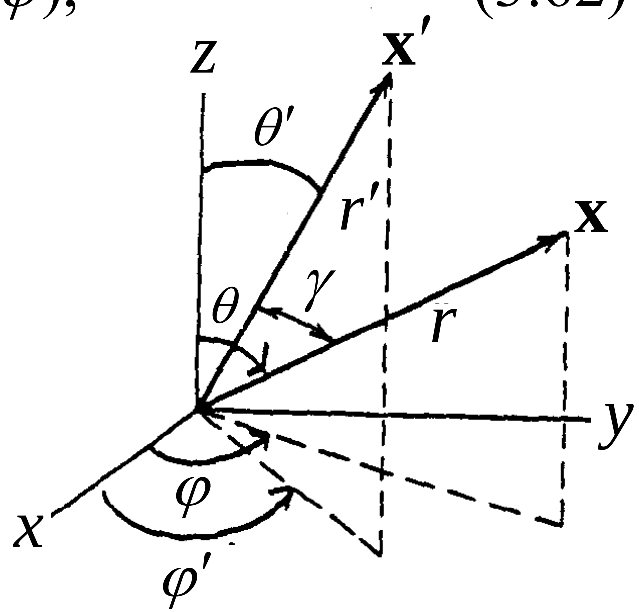
$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) \qquad (3.51)$$

$$P_l^0(x) = P_l(x)$$

## 3.6 Addition Theorem for Spherical Harmonics

The addition theorem for spherical harmonics is derived on pp. 110-111. Here we write the theorem without derivation:

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi), \qquad (3.62)$$




where $\gamma$ is the angle between $\mathbf{x}$ and $\mathbf{x}'$.

Setting $l=1$ in (3.62) gives

$$P_1(\cos\gamma) = \frac{4\pi}{3}[Y_{1,-1}^*(\theta',\varphi')Y_{1,-1}(\theta,\varphi) + Y_{1,0}^*(\theta',\varphi')Y_{1,0}(\theta,\varphi) + Y_{1,1}^*(\theta',\varphi')Y_{1,1}(\theta,\varphi)]$$

Using $P_1(\cos\gamma)=\cos\gamma$, $Y_{1,-1}=\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\varphi}$, $Y_{1,0}=\sqrt{\frac{3}{4\pi}}\cos\theta$,

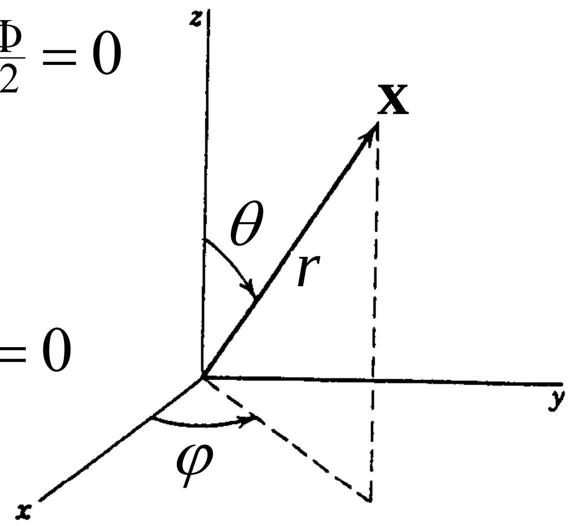
and $Y_{1,1}=-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi}$, we obtain a useful expression:

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi'). \qquad (1)$$

# 3.1 Laplace Equation in Spherical Coordinates

$$\nabla^2\Phi(\mathbf{x})=0$$

$$\Rightarrow \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}=0$$




Let $\Phi(\mathrm{x})=\dfrac{U(r)}{r}P(\theta)Q(\varphi)$

$$\Rightarrow PQ\frac{d^2U}{dr^2}+\frac{UQ}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\frac{UP}{r^2\sin^2\theta}\frac{d^2Q}{d\varphi^2}=0$$

Multiply by $\dfrac{r^2\sin^2\theta}{UPQ}$

Dividing all terms by $\sin^2\theta$, we see that the $r$-dependence is isolated within this term. So this term must be a constant. Let it be $\nu(\nu+1)$.

$$\Rightarrow \sin^2\theta\,[\overbrace{\tfrac{1}{U}r^2\frac{d^2U}{dr^2}}^{=\nu(\nu+1)}+\frac{1}{P\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dP}{d\theta})]+\underbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}_{=-m^2}=0 \qquad (3.3)$$

The $\varphi$-dependence is isolated within this term, so this term must be a constant. Let it be $-m^2$.

Rewrite (3.3): $\sin^2\theta\,[\overbrace{\frac{1}{U}r^2\frac{d^2U}{dr^2}+\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)}^{=\nu(\nu+1)}]+\overbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}^{=-m^2}=0$

The equation for $Q(\varphi)$ is: $\frac{d^2Q}{d\varphi^2}+m^2Q=0$ (3.4)

$\Rightarrow Q=e^{im\varphi},e^{-im\varphi}$

$m$ is to be determined from the b.c.

The equation for $P(\theta)$ is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[\nu(\nu+1)-\frac{m^2}{\sin^2\theta}\right]P=0. \qquad (3.6)$$

Let $x=\cos\theta$, then the equation takes the form of the associated Legendre equation:

$$\frac{d}{dx}\left(1-x^2\right)\frac{dP}{dx}+\left[\nu\left(\nu+1\right)-\frac{m^2}{1-x^2}\right]P=0$$

$$\Rightarrow P=\begin{Bmatrix}P_\nu^m(x)\\ Q_\nu^m(x)\end{Bmatrix}=\begin{Bmatrix}P_\nu^m(\cos\theta)\\ Q_\nu^m(\cos\theta)\end{Bmatrix} \qquad (2)$$

$\nu$ is to be determined from the b.c.

Rewrite (3.3): $\sin^2\theta\,[\overbrace{\frac{1}{U}r^2\frac{d^2U}{dr^2}}^{=\nu(\nu+1)}+\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)]+\overbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}^{=-m^2}=0$

The equation for $U(r)$ is: $\frac{d^2U}{dr^2}-\frac{\nu(\nu+1)}{r^2}U=0$ (3.7)

Since $\nu$ is determined from the b.c. for (3.6), this is not an eigenvalue problem.

$$\Rightarrow U=r^{\nu+1},r^{-\nu}\Rightarrow\frac{U}{r}=r^{\nu},r^{-\nu-1}$$

Thus,

$$\Phi=\begin{Bmatrix}r^{\nu}\\ r^{-\nu-1}\end{Bmatrix}\begin{Bmatrix}P_\nu^m(\cos\theta)\\ Q_\nu^m(\cos\theta)\end{Bmatrix}\begin{Bmatrix}e^{im\varphi}\\ e^{-im\varphi}\end{Bmatrix},$$

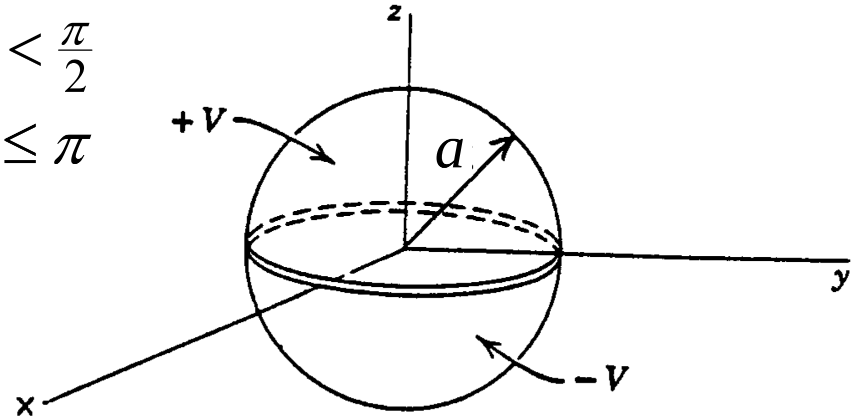
where each bracket represents a linear combination of the two functions inside. Because the differential equation is linear, the linear combination of any number of solutions is also a solution.

Note that $\nu$ and $m$ are arbitrary constants until we apply boundary conditions.

# 3.3 Boundary-Value Problems with Azimuthal Symmetry

*Problem 1*: Find $\Phi$ inside 2 hemispheres held at opposite potentials as shown in the figure.

$$\nabla^2\Phi = 0, \quad \Phi(a,\theta) = \begin{cases} V, & 0 \le \theta < \frac{\pi}{2} \\ -V, & \frac{\pi}{2} \le \theta \le \pi \end{cases}$$




$$\Phi = \begin{Bmatrix} r^{\nu} \\ r^{-\nu-1} \end{Bmatrix} \begin{Bmatrix} P_\nu^m(\cos\theta) \\ Q_\nu^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$$

(i) $\Phi$ is independent of $\varphi$. $\Rightarrow m = 0$

(ii) $\Phi$ is finite at $\theta = 0$ and $\pi$ (i.e. at $\cos\theta = 1$ and $-1$).

$\Rightarrow \nu = l = 0,\ 1,\ 2, \ldots$ and drop $Q_\nu^m$

(iii) $\Phi$ is finite at $r = 0$. $\Rightarrow$ drop $r^{-\nu-1}$

$$\Rightarrow \Phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

*Note*:

1. $P_\nu(-1)$ diverges unless $\nu$ is an integer (p.105.)
2. We have set $l = 0,\ 1,\ 2, \cdots$ because $P_{-l-1}(x) = P_l(x)$.
3. $Q_\nu(x) \to \infty$ as $x \to \pm 1$.

The b.c. at $r=a$ is: $\Phi(a,\theta)=\sum_l A_l a^l P_l(\cos\theta)=\begin{cases} V, & 0\le\theta<\frac{\pi}{2} \\ -V, & \frac{\pi}{2}\le\theta\le\pi \end{cases}$

$\boxed{\text{Use } \int_{-1}^{1} P_l(x)P_{l'}(x)dx=\frac{2}{2l+1}\delta_{ll'} \quad (3.21)}$

$$\Rightarrow \int_{-1}^{1} P_l(\cos\theta)\phi(a,\theta)d\cos\theta = A_l a^l \overbrace{\int_{-1}^{1} P_l^2(\cos\theta)d\cos\theta} = A_l a^l \frac{2}{2l+1}$$

$$\Rightarrow A_l = \frac{V}{a^l}\frac{2l+1}{2}\left[\int_0^1 P_l(\cos\theta)d\cos\theta - \int_{-1}^{0} P_l(\cos\theta)d\cos\theta\right]$$

$$=\begin{cases} \dfrac{V}{a^l}\dfrac{\left(-\frac{1}{2}\right)^{\frac{l-1}{2}}(2l+1)(l-2)!!}{2\left(\frac{l+1}{2}\right)!}, & \text{for odd } l \\ 0, & \text{for even } l \end{cases}$$

$\boxed{(2n+1)!!=(2n+1)(2n-1)(2n-3)\ldots5\times3\times1}$

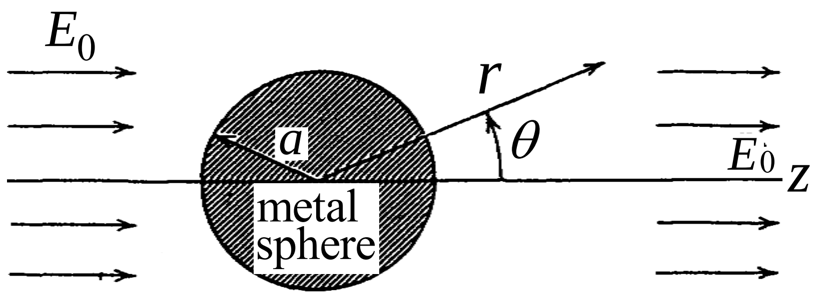
$\boxed{\text{pp. 99-100}}$

$$\Rightarrow \Phi(r,\theta)=V\left[\frac{3}{2}\frac{r}{a}P_1(\cos\theta)-\frac{7}{8}\left(\frac{r}{a}\right)^3 P_3(\cos\theta)+\cdots\right],\quad r\le a \qquad (3.36)$$

To find $\Phi$ for $r>a$, replace $\left(\frac{r}{a}\right)^l$ in (3.36) by $\left(\frac{a}{r}\right)^{l+1}$ [see (2.27)]

*Problem 2*: A conducting sphere of radius $a$ with net charge $Q$ on its surface is placed in a uniform electric field $E_0\mathbf{e}_z$. Use the method of expansion to find $\Phi$ outside the sphere and $\sigma$ on the sphere.

$$\Phi = \begin{Bmatrix} r^{\nu} \\ r^{-\nu-1} \end{Bmatrix} \begin{Bmatrix} P_\nu^m(\cos\theta) \\ Q_\nu^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$$




(i) $\Phi$ is independent of $\varphi$. $\Rightarrow m=0$

(ii) $\Phi$ is finite at $\theta=0$ and $\pi$ (i.e. at $\cos\theta=1$ and $-1$).

$\Rightarrow \quad \nu = l = 0,\ 1,\ 2, \ldots$ and drop $Q_\nu^m$

Hence, $\Phi(r,\theta) = \sum_{l=0}^{\infty} [A_l r^l + B_l r^{-(l+1)}] P_l(\cos\theta)$

b.c. at $r\to\infty$: $\Phi = \underbrace{-E_0 r\cos\theta}_{\text{external field}} + \underbrace{\frac{Q}{r}}_{\text{due to net chrage } Q}$

$\Rightarrow A_1 = -E_0,\ A_l = 0$ (for $l \neq 1$), and $B_0 = Q$

Question:
$r^l \to \infty$ ($l \geq 1$) as $r\to\infty$. Why keep the $A_l r^l$ terms?

$$\Rightarrow \Phi(r,\theta) = -E_0 r\cos\theta + \frac{Q}{r} + \sum_{l=1}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)$$

b.c. at $r = a$: $\Phi(r = a) = const.$

$$\Rightarrow \Phi(r=a) = \underbrace{(-E_0 a + \frac{B_1}{a^2})}_{0}\underbrace{\cos\theta}_{\text{not a const.}} + \frac{Q}{a} + \sum_{l=2}^{\infty} \underbrace{B_l}_{0}\, a^{-(l+1)} \underbrace{P_l(\cos\theta)}_{\text{not a const.}}$$

$$\Rightarrow B_1 = E_0 a^3 \text{ and } B_l = 0 \text{ for } l \geq 2$$

$$\Rightarrow \Phi(r,\theta) = -E_0 r\cos\theta + \frac{Q}{r} + \underbrace{E_0 \frac{a^3}{r^2}\cos\theta}_{\substack{\text{due to induced surface charge} \\ \text{density } \sigma \text{ on the sphere}}}$$

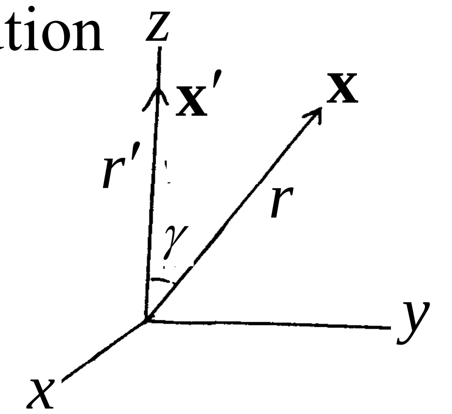
As will become clear in Ch. 4 [Eq. (4.56)], the $E_0 \frac{a^3}{r^2}\cos\theta$ term in $\Phi$ is due to an electric dipole of dipole moment $p = 4\pi\varepsilon_0 a^3 E_0$. (see p.64)

The induced surface charge density $\sigma$ is

$$\sigma = -\varepsilon_0 \left.\frac{\partial\phi}{\partial r}\right|_{r=a} = 3\varepsilon_0 E_0\cos\theta + \frac{\varepsilon_0 Q}{a^2}$$

*Problem 3*: $\Phi$ due to a unit point source at $\mathbf{x}'$ in infinite space

First, let's assume the point source is on the $z$-axis (at a distance $r'$ from the origin) and divide the space into two regions: $r < r'$ and $r > r'$. In each region, we have $\nabla^2\Phi = 0$ with the solution




$$\Phi = \begin{Bmatrix} r^{\nu} \\ r^{-\nu-1} \end{Bmatrix} \begin{Bmatrix} P_\nu^m(\cos\gamma) \\ Q_\nu^m(\cos\gamma) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$$

(i) $\Phi$ is indep. of $\varphi$. $\Rightarrow$ $m = 0$

(ii) $\Phi$ is finite at $\gamma = 0$ and $\pi$. $\Rightarrow \nu = l = 0, 1, 2, \ldots$ and drop $Q_\nu^m$

(iii) $\Phi$ is finite $\begin{cases} \text{at } r = 0. & \Rightarrow \text{drop } r^{-l-1} \text{ in region } r < r' \\ \text{as } r \to \infty. & \Rightarrow \text{drop } r^{l} \text{ in region } r > r' \end{cases}$

$$\Rightarrow \Phi = \begin{cases} \sum\limits_{l=0}^{\infty} A_l r^l P_l(\cos\gamma), & r < r' \\ \sum\limits_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos\gamma), & r > r' \end{cases}$$

The formal method to solve for $A_l$ and $B_l$ is to match the b.c. at $r = r'$ (as will be done in Sec. 3.9). Here we obtain $A_l$ and $B_l$ by exploiting the fact that we already know $\Phi = 1/|\mathbf{x}-\mathbf{x}'|$ (for a unit point source, $q \leftrightarrow 4\pi\varepsilon_0$). So, by the uniqueness theorem, we have

$$\Phi = \frac{1}{|\mathbf{x}-\mathbf{x}'|} = \begin{cases} \sum\limits_{l=0}^{\infty} A_l r^l P_l(\cos\gamma) & , \quad r < r' \\ \sum\limits_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos\gamma) & , \quad r > r' \end{cases}$$

For $\gamma = 0$, we have $P_l(1) = 1$ and $|\mathbf{x}-\mathbf{x}'| = |r - r'|$. Hence,

$$\frac{1}{|r-r'|} = \begin{cases} \sum\limits_{l=0}^{\infty} A_l r^l & , \quad r < r' \\ \sum\limits_{l=0}^{\infty} B_l r^{-l-1} & , \quad r > r' \end{cases}$$

$$\boxed{(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots}$$

$$\text{But } \frac{1}{|r-r'|} = \begin{cases} \dfrac{1}{r'-r} = \dfrac{1}{r'}\dfrac{1}{1-\frac{r}{r'}} = \dfrac{1}{r'}\sum\limits_{l=0}^{\infty}\left(\frac{r}{r'}\right)^l = \sum\limits_{l=0}^{\infty}\dfrac{r^l}{r'^{l+1}}, & r<r' \\ \dfrac{1}{r-r'} = \dfrac{1}{r}\dfrac{1}{1-\frac{r'}{r}} = \dfrac{1}{r}\sum\limits_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l = \sum\limits_{l=0}^{\infty}\dfrac{r'^l}{r^{l+1}}, & r>r' \end{cases}$$

Equating the RHS of this equation to the RHS of the equation on the previous page, we obtain

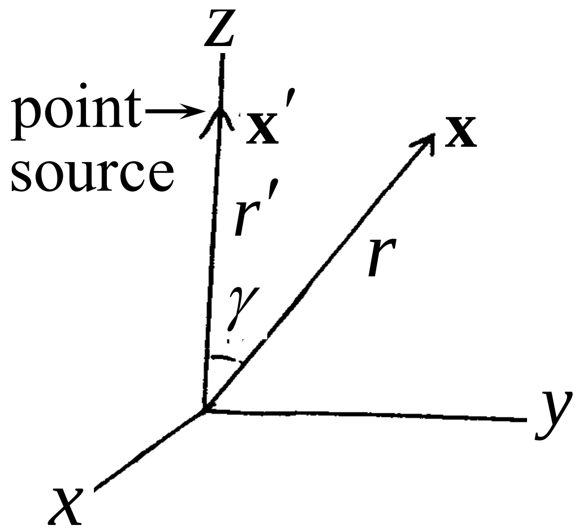
$$A_l = \frac{1}{r'^{l+1}}, \; B_l = r'^l \;\Rightarrow\; \frac{1}{|\mathbf{x}-\mathbf{x}'|} = \begin{cases} \sum\limits_{l=0}^{\infty}\dfrac{r^l}{r'^{l+1}}P_l(\cos\gamma), & r<r' \\ \sum\limits_{l=0}^{\infty}\dfrac{r'^l}{r^{l+1}}P_l(\cos\gamma), & r>r' \end{cases}$$

$$\text{or } \frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma), \qquad \text{[two equations in one]} \qquad (3.38)$$

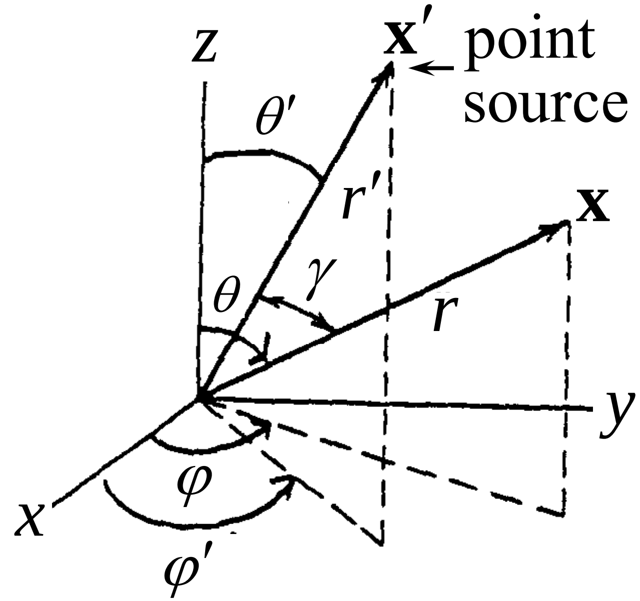
where $r_<$ ($r_>$) is the smaller (larger) of $r$ and $r'$.

Rewrite (3.38):

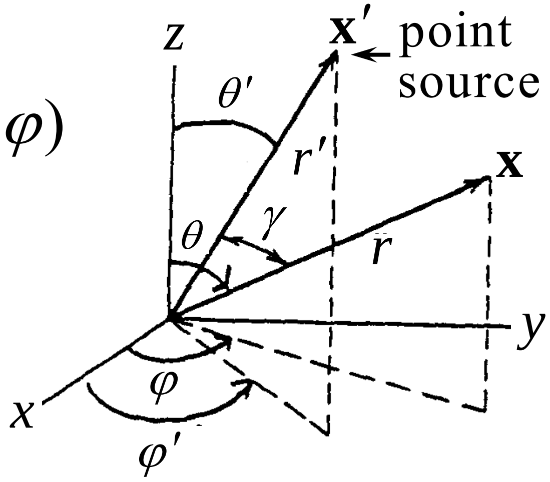
$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma)$$




This equation was derived with the unit point source located on the $z$-axis (upper figure). However, it depends only on the magnitudes ($r$, $r'$) of $\mathbf{x}$ and $\mathbf{x}'$ and the angle ($\gamma$) between $\mathbf{x}$ and $\mathbf{x}'$. So we expect the expression in (3.38) can be cast into a general form which holds true when the unit point charge is at an arbitrary point (lower figure). We may obtain the general form by way of the addition theorem.



Sub. the RHS of the addition theorem




$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \qquad (3.62)$$

for $P_l(\cos\gamma)$ in $\dfrac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\dfrac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma)$,

we get $$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \qquad (3.70)$$

So, we started with a physics problem (the potential of a point charge in infinite space), but end up with a mathematical relation in (3.70).

*Question*: Why write a simple function $\Phi = 1/|\mathbf{x}-\mathbf{x}'|$ in such a complicated form? (See next problem.)

*Problem 4*: Potential due to charge $q$ uniformly distributed on a circular ring of radius $a$.

Let $\rho(\mathbf{x}) = K\delta(\theta-\alpha)\delta(r-c)$ in spherical coordinates

$$q = \int \rho(\mathbf{x})d^3x$$

$$= K\int \delta(\theta-\alpha)\delta(r-c)\overbrace{r^2\sin\theta dr d\theta d\varphi}^{d^3x}$$

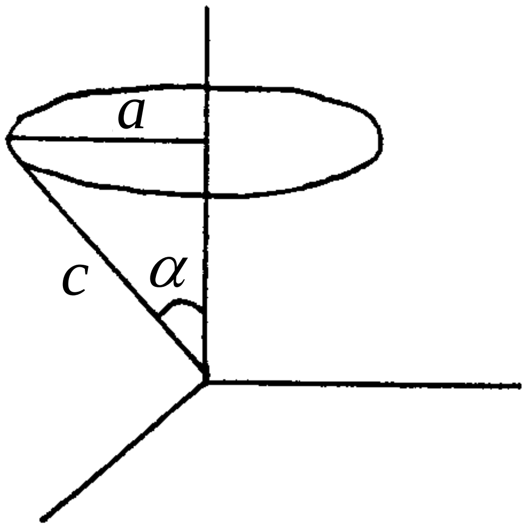
$$= 2\pi K c^2 \sin\alpha$$

$$\Rightarrow \quad K = \frac{q}{2\pi c^2 \sin\alpha}$$

$$\Rightarrow \rho(\mathbf{x}) = \frac{q}{2\pi c^2 \sin\alpha}\delta(\theta-\alpha)\delta(r-c)$$

$$= \frac{q}{2\pi c^2}\delta(\cos\theta-\cos\alpha)\delta(r-c)$$

$$\boxed{\delta[f(x)] = \frac{\delta(x-a)}{|f'(a)|}}$$



$$\rho(\mathbf{x}') = \frac{q}{2\pi c^2}\delta(\cos\theta' - \cos\alpha)\delta(r' - c)$$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

$$= \frac{q}{2\pi\varepsilon_0 c^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\int_V r'^2 dr' d\cos\theta' d\varphi' \begin{bmatrix} \dfrac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \\ \cdot\delta(\cos\theta' - \cos\alpha)\delta(r'-c) \end{bmatrix}$$

$$Y_{lm}(\theta',\varphi') = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta')e^{im\varphi'}$$

Apparently, only the $m=0$ terms survive the $\varphi'$ integration.

$$\Rightarrow \Phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0 c^2}\sum_{l=0}^{\infty}\int_0^{\infty} r'^2 dr' \int_{-1}^{1} d\cos\theta' \begin{bmatrix} \dfrac{r_<^l}{r_>^{l+1}}P_l(\cos\theta')P_l(\cos\theta) \\ \cdot\delta(\cos\theta' - \cos\alpha)\delta(r'-c) \end{bmatrix}$$

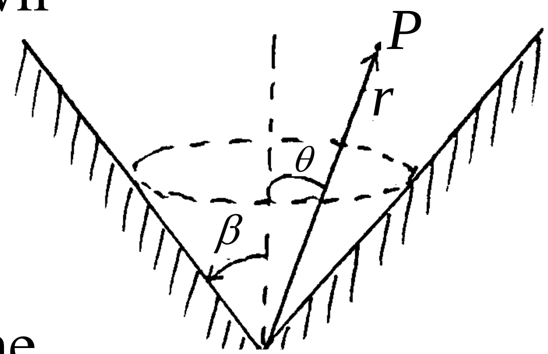
$$= \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\alpha)P_l(\cos\theta)$$

Jackson uses a slightly different method to derive this. See p.103.

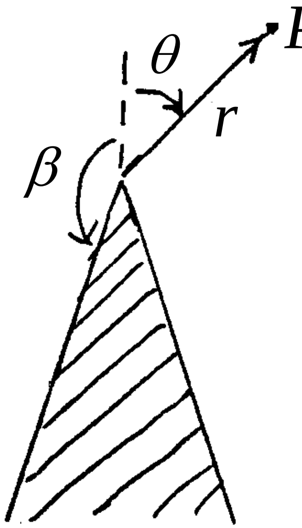
# 3.4 Behavior of Fields in a Conical Hole or Near a Sharp Point

Consider the source-free configurations shown in the figures.

$$\nabla^2\Phi = 0 \Rightarrow \Phi = \begin{Bmatrix} r^{\nu} \\ r^{-\nu-1} \end{Bmatrix} \begin{Bmatrix} P_\nu^m(\cos\theta) \\ Q_\nu^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$$




(i) The geometry is indep. of $\varphi$ (We also assume that the b.c. is indep. of $\varphi$.) $\Rightarrow$ $m = 0$




(ii) $Q_\nu^m(\cos\theta)$ diverges at $\theta = 0$ or $\cos\theta = 1$.

$\Rightarrow$ drop $Q_\nu^m(\cos\theta)$

Hence, $\Phi = \begin{Bmatrix} r^{\nu} \\ r^{-\nu-1} \end{Bmatrix} P_\nu(\cos\theta)$

*Note*: $P_\nu(x)$ diverges at $x = -1$ unless $\nu =$ integer. However, in this problem, we have $\theta \le \beta < \pi \;\Rightarrow\; \cos\theta \ne -1$ in the region of interest. Hence, $\nu$ is not required to be an integer.

Rewrite: $\Phi = \begin{Bmatrix} r^{\nu} \\ r^{-\nu-1} \end{Bmatrix} P_\nu(\cos\theta)$

(iii) $\Phi$ is finite at $r = 0$.

$$\Rightarrow \begin{cases} \text{(a) demand } \nu > 0 \text{ and drop } r^{-\nu-1} \Rightarrow \Phi = r^{\nu} P_\nu(\cos\theta) \\ \text{(b) demand } -\nu-1 > 0 \text{ and drop } r^{\nu} \Rightarrow \Phi = r^{-\nu-1} P_\nu(\cos\theta) \end{cases}$$

But $P_\nu(\cos\theta) = P_{-\nu-1}(\cos\theta)$, hence $\Phi = r^{-\nu-1} P_{-\nu-1}(\cos\theta)$

$\Rightarrow$ Either option (a) or option (b) gives $\Phi = r^{\nu} P_\nu(\cos\theta)$, $\nu > 0$

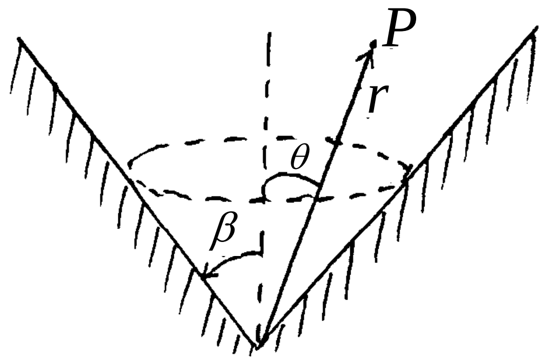
(iv) $\Phi = 0$ at $\theta = \beta \Rightarrow P_\nu(\cos\beta) = 0 \Rightarrow \nu = \nu_1, \nu_2, \nu_3, \ldots$ $(\nu > 0)$

*Note*: In the boundary condition: $P_\nu(\cos\beta) = 0$, $\beta$ is fixed and $\nu$ is the eigenvalue to be solved.

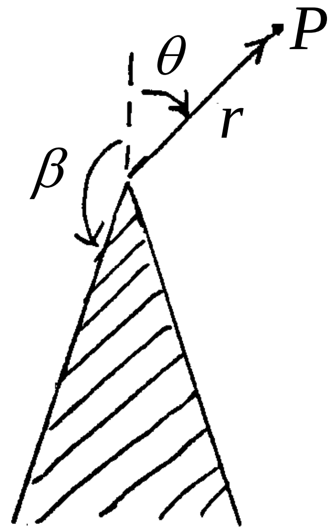
$$\Rightarrow \Phi = \sum_{k=1}^{\infty} A_k r^{\nu_k} P_{\nu_k}(\cos\theta) \overset{r \to 0}{\approx} A_1 r^{\nu_1} P_{\nu_1}(\cos\theta), \tag{3.44}$$

where $\nu_1$ is the smallest eigenvalue [the first root of $P_\nu(\cos\beta) = 0$].

## 3.4 Behavior of Fields in a Conical Hole or Near a Sharp Point (*continued*)

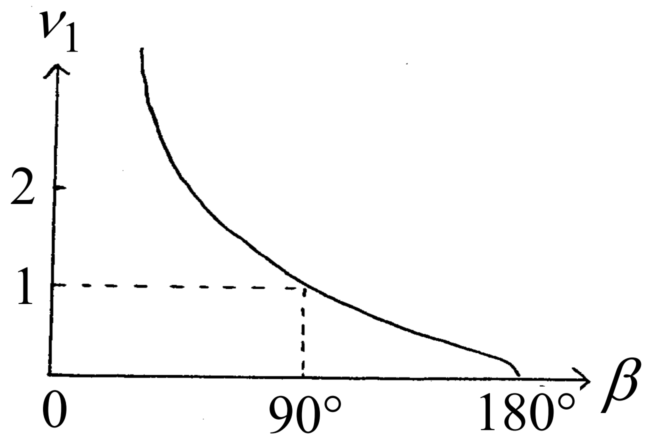



$$\Rightarrow \begin{cases} E_r = -\dfrac{\partial \phi}{\boxed{\partial r}} \approx -\nu_1 A_1 r^{\nu_1 - 1} P_{\nu_1}(\cos\theta) \propto r^{\nu_1 - 1} \quad (r \to 0) \\ E_\theta = -\dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} \approx A_1 r^{\nu_1 - 1} \sin\theta P'_{\nu_1}(\cos\theta) \propto r^{\nu_1 - 1} \\ \sigma = -\varepsilon_0 E_\theta(\theta = \beta) \approx -A_1 \varepsilon_0 r^{\nu_1 - 1} \sin\beta P'_{\nu_1}(\cos\beta) \propto r^{\nu_1 - 1} \end{cases}$$




Behavior of $\nu_1$ as a function of $\beta$ is shown in the figure below. Note that

$$\begin{cases} \nu_1 > 1, \text{ if } \beta < 90^\circ \\ \nu_1 = 1, \text{ if } \beta = 90^\circ \\ \nu_1 < 1, \text{ if } \beta > 90^\circ \end{cases}$$




When $\beta < 90^\circ$ (conical hole), both $E$ and $\sigma \to 0$ as $r \to 0$.

However, when $\beta > 90^{\mathrm{o}}$ (sharp point), both $E$ and $\sigma \to \infty$ as $r \to 0$. Large electric field ($E > 2.5\times10^4$ V/cm) can cause the air to breakdown and form a conducting path in the air for the sharp point to discharge. This is the principle of the lightning rod (pp. 77-78.)

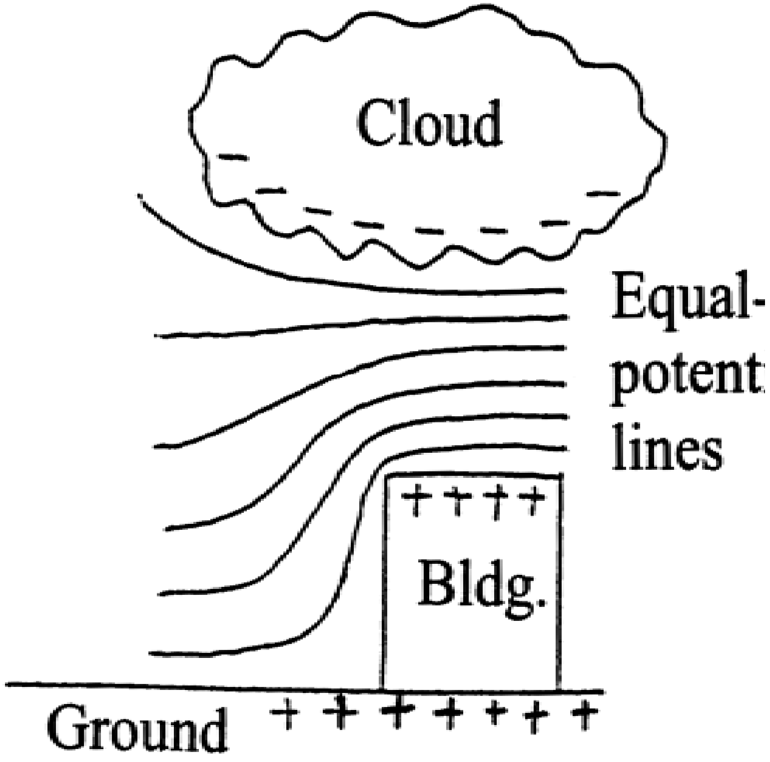
If the region of interest is bounded by the surface at $r = a$, the coefficients $A_k$ in (3.44) can be determined by the b.c. $\Phi(r = a) = \Phi(\theta)$ through

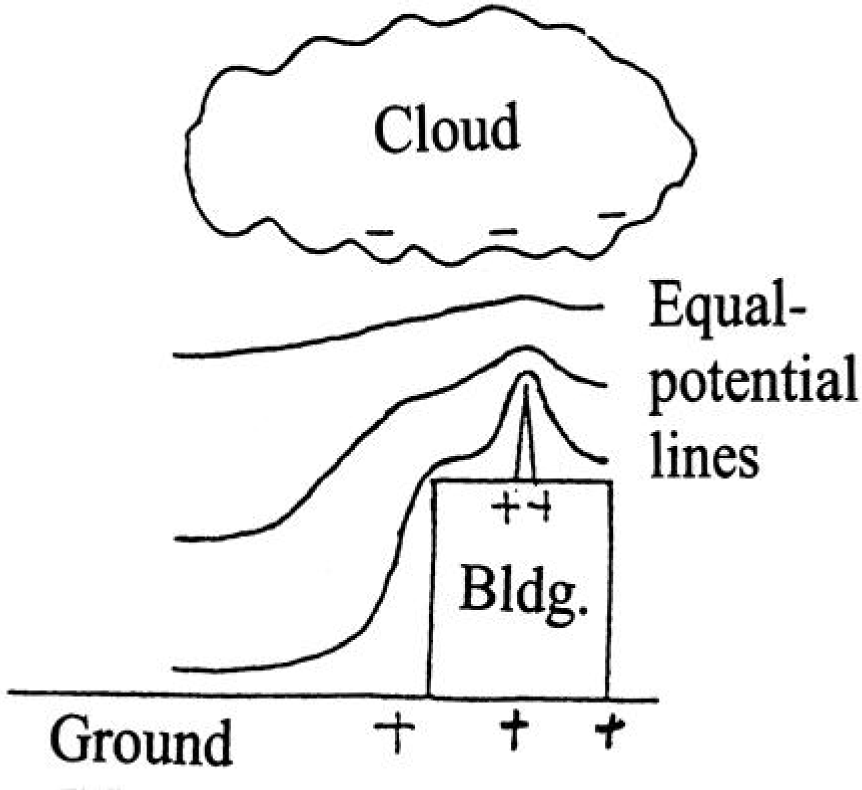
$$\Phi(\theta) = \sum_{k=1}^{\infty} A_k a^{\nu_k} P_{\nu_k}(\cos\theta)$$

If $\Phi(r = a) = \Phi(\theta) = 0,$ then all $A_k = 0. \Rightarrow \Phi = 0$ everywhere

In reality, the lightning rod is not perfectly sharp. Hence, $\Phi$ is finite at the tip, and on a clear day when there is a small potential difference between the ground the clouds, the lightning rod will not discharge.

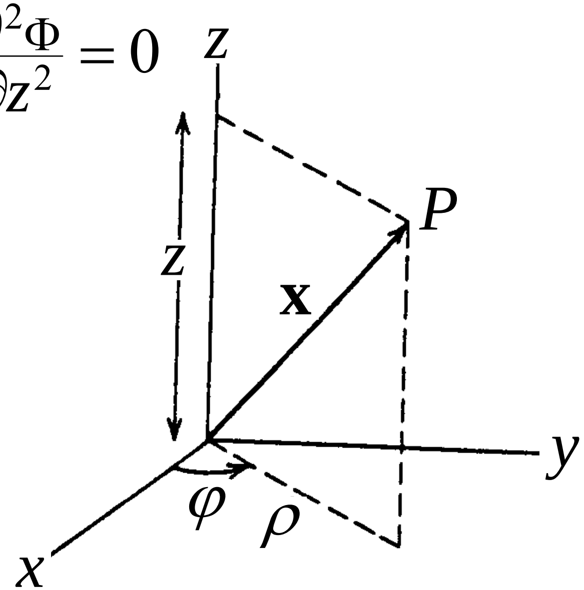
# A physical picture of the lightning rod



# 3.7 Laplace Equation in Cylindrical Coordinates; Bessel Functions

$$\nabla^2\Phi(\mathbf{x})=0 \Rightarrow \frac{\partial^2\Phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\Phi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$




Let $\Phi(\mathrm{x})=R(\rho)Q(\varphi)Z(z)$

$$\Rightarrow \begin{cases} \dfrac{\partial^2 Z}{\partial z^2}-k^2 Z=0 \Rightarrow Z=e^{\pm kz} \\ \dfrac{\partial^2 Q}{\partial\varphi^2}+\nu^2 Q=0 \Rightarrow Q=e^{\pm i\nu\varphi} \\ \dfrac{\partial^2 R}{\partial\rho^2}+\dfrac{1}{\rho}\dfrac{\partial R}{\partial\rho}+\left(k^2-\dfrac{\nu^2}{\rho^2}\right)R=0 \Rightarrow R=J_\nu(k\rho),\ N_\nu(k\rho) \end{cases}$$

where $J_\nu$ and $N_\nu$ are Bessel functions of the first and second kind, respectively (see following pages).

$$\Rightarrow \Phi=\begin{Bmatrix} J_\nu(k\rho) \\ N_\nu(k\rho) \end{Bmatrix}\begin{Bmatrix} e^{i\nu\varphi} \\ e^{-i\nu\varphi} \end{Bmatrix}\begin{Bmatrix} e^{kz} \\ e^{-kz} \end{Bmatrix} \qquad (3)$$

**Bessel Functions :** If we let $x = k\rho$, the equation for $R$ takes the standard form of the Bessel equation,

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0 \tag{3.77}$$

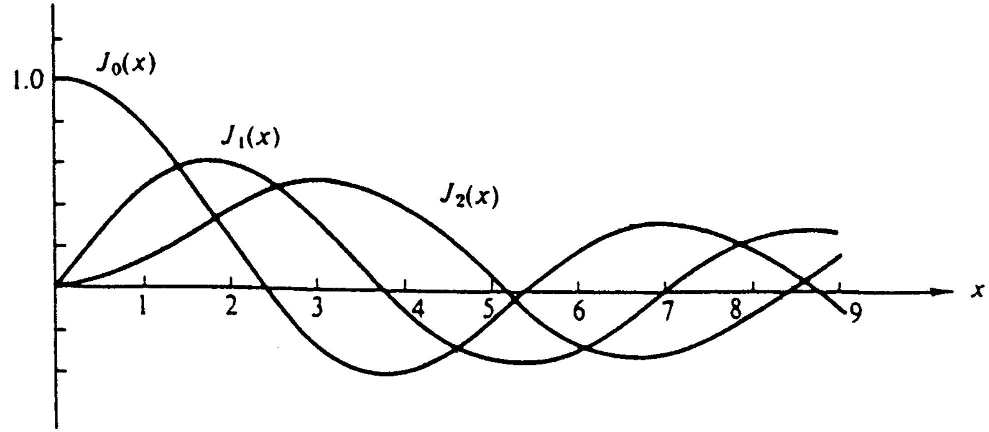
with solutions $J_\nu(x)$ and $N_\nu(x)$, from which we define the Hankel functions:

$$\begin{cases} H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x) \\ H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x) \end{cases} \tag{3.86}$$

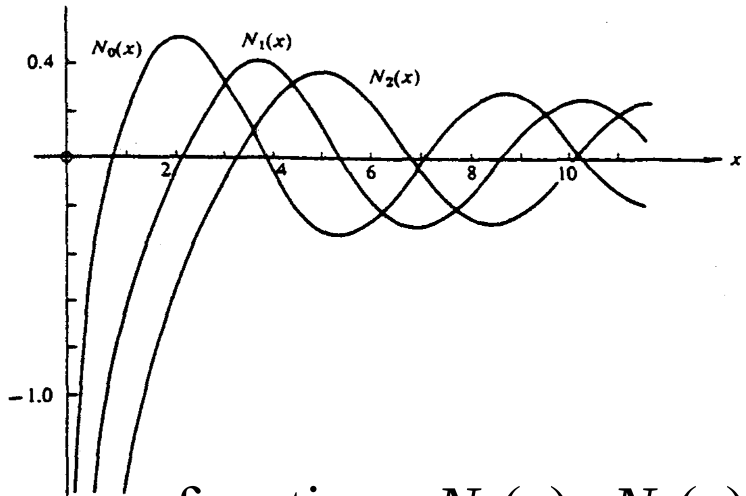
and the modified Bessel functions (Bessel functions of imaginary argument)

$$\begin{cases} I_\nu(x) = i^{-\nu} J_\nu(ix) & (3.100) \\ K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) & (3.101) \end{cases}$$

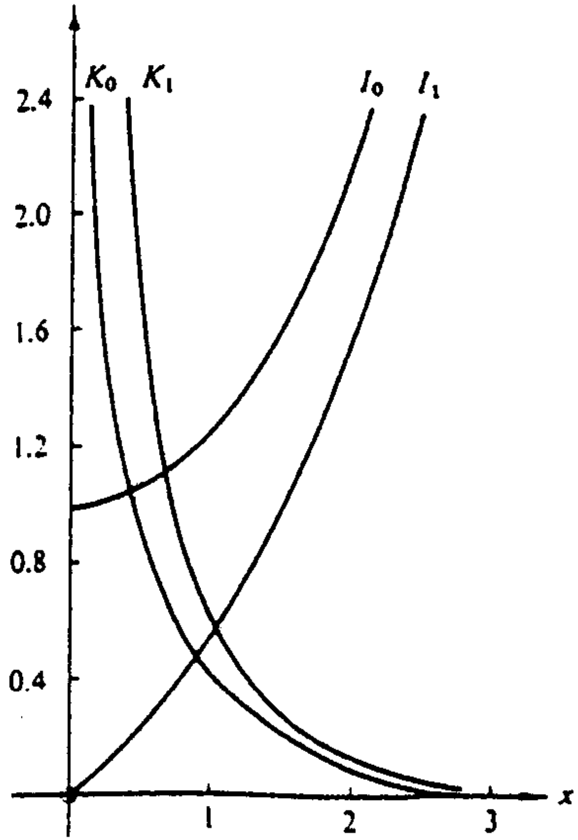
See Jackson pp. 112-116, Gradshteyn & Ryzhik, and Abramowitz & Stegun for properties of these special functions.




Bessel functions $J_0(x)$, $J_1(x)$, and $J_2(x)$




Neumann functions $N_0(x)$, $N_1(x)$, and $N_2(x)$

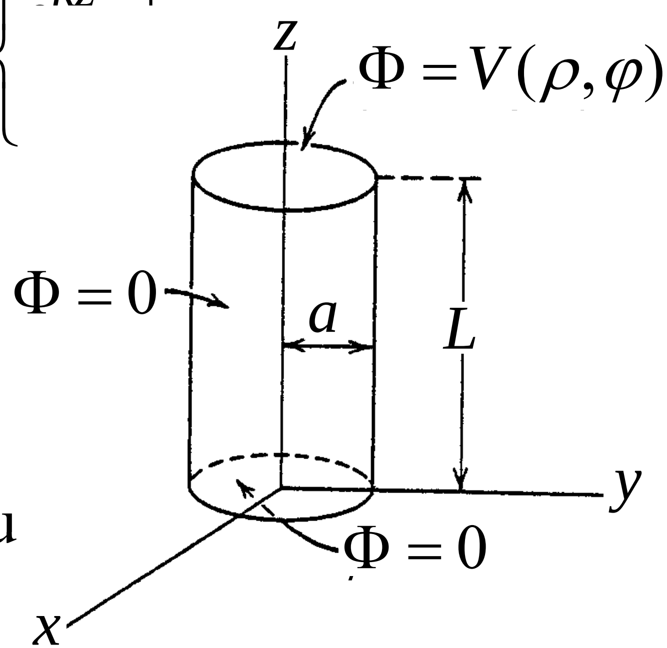



Modified Bessel functions $I_0(x)$, $I_1(x)$, $K_0(x)$, and $K_1(x)$

# 3.8 Bounday-Value Problems in Cylindrical Coordinates

*Example 1*: Potential inside a charge-free cylinder (see figure) with the b.c. $\Phi(z=L)=V(\rho,\varphi)$ and $\Phi=0$ on other surfaces.

$$\nabla^2\Phi(\mathbf{x})=0 \Rightarrow \Phi=\begin{Bmatrix} J_\nu(k\rho) \\ N_\nu(k\rho) \end{Bmatrix}\begin{Bmatrix} e^{i\nu\varphi} \\ e^{-i\nu\varphi} \end{Bmatrix}\begin{Bmatrix} kz \\ \end{Bmatrix}$$




(i) $Z(z)=Ae^{kz}+Be^{-kz}$

$\Phi=0$ at $z=0 \Rightarrow Z(0)=0 \Rightarrow B=-A$

$\Rightarrow Z(z)=A\left(e^{kz}-e^{-kz}\right)=A'\sinh kz$

(ii) $\Phi(\varphi)=\Phi(\varphi+2\pi)$, i. e. $\Phi$ is single-valu

$\Rightarrow \nu=m=$ integer

$$\Rightarrow Q(\varphi)=\sum_{m=-\infty}^{\infty} C_m e^{im\varphi}=\sum_{m=0}^{\infty}\left(A_m\sin m\varphi+B_m\cos m\varphi\right)$$

(iii) $\Phi$ is finite at $\rho=0$. $\Rightarrow$ drop $N_m(k\rho) \Rightarrow R=J_m(k\rho)$

Rewrite: $R = J_m(k\rho)$

(iv) $\Phi = 0$ at $\rho = a \Rightarrow J_m(ka) = 0 \Rightarrow k = k_{mn} = \frac{x_{mn}}{a}$, $n = 1, 2, 3\ldots$

where $x_{mn}$ is the $n$-th root of $J_m(x) = 0$. Thus,

$$\Phi(\rho, \varphi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho) \sinh(k_{mn} z)(A_{mn} \sin m\varphi + B_{mn} \cos m\varphi)$$

With $k$ fixed by the boundary condition to a set of dicrete values ($k_{mn}$), we may introduce two properties of $J_m(k_{mn}\rho)$:

The set $J_m(k_{mn}\rho)$ is orthogonal in index $n$: [$m$: a fixed number.]

$$\int_0^a J_m(k_{mn'}\rho) J_m(k_{mn}\rho)\rho d\rho = \frac{a^2}{2}[J_{m+1}(k_{mn}a)]^2 \delta_{n'n} \qquad (3.95)$$

The set $J_m(k_{mn}x)$ is complete in index $n$. Hence, any function $f(x)$ can be expanded as $f(x) = \sum_{n=1}^{\infty} C_n J_m(k_{mn}x)$

*Questions*: (See last page of Appendix A.)

1. Why is $J_m(k_{mn}x)$ orhtogonal and complete in index $n$ instead of $m$?
2. Why is there a factor $\rho$ in the integrand of (3.95), but not in (3.52)?

Rewrite:

$$\Phi(\rho,\varphi,z)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m(k_{mn}\rho)\sinh(k_{mn}z)(A_{mn}\sin m\varphi+B_{mn}\cos m\varphi)$$

(v) $\Phi(\rho,\varphi,z=L)=V(\rho,\varphi)$

$$\Rightarrow V(\rho,\varphi)=\sum_{m,n}\sinh(k_{mn}L)\,J_m(k_{mn}\rho)(A_{mn}\sin m\varphi+B_{mn}\cos m\varphi)$$

Operating both sides with $\int_0^{2\pi}d\varphi\int_0^{a}\rho d\rho J_m(k_{mn}\rho)\begin{Bmatrix}\sin m\varphi\\ \cos m\varphi\end{Bmatrix}$ and

making use of the orthogonal properties of $\sin m\varphi$ and $\cos m\varphi$, and

the relation: $\int_0^{a}J_m(k_{mn'}\rho)J_m(k_{mn}\rho)\rho d\rho=\frac{a^2}{2}[J_{m+1}(k_{mn}a)]^2\delta_{n'n}$ (3.95)

$$\Rightarrow\begin{Bmatrix}A_{mn}\\ B_{mn}\end{Bmatrix}=\frac{2\operatorname{cosech}(k_{mn}L)}{\pi a^2J_{m+1}^2(k_{mn}a)}\int_0^{2\pi}d\varphi\int_0^{a}\rho d\rho V(\rho,\varphi)J_m(k_{mn}\rho)\begin{Bmatrix}\sin m\varphi\\ \cos m\varphi\end{Bmatrix}$$

(for $m=0$, use $\frac{1}{2}B_{0n}$)

*Example 2*: Potential in the charge-free semi-infinite space $z \geq 0$

subject to the b.c. $\begin{cases} \Phi(\rho, \varphi, z=0) = V(\rho, \varphi) \\ \Phi(\rho \to \infty, \varphi, z) = 0 \end{cases}$

$$\nabla^2 \Phi(\mathbf{x}) = 0 \Rightarrow \phi = \begin{Bmatrix} J_\nu(k\rho) \\ N_\nu(k\rho) \end{Bmatrix} \begin{Bmatrix} e^{i\nu\varphi} \\ e^{-i\nu\varphi} \end{Bmatrix} \begin{Bmatrix} e^{kz} \\ e^{-kz} \end{Bmatrix}$$

(i) $\Phi$ remains fini te as $z \to \infty$. $\Rightarrow$ drop $e^{kz} \Rightarrow Z(z) = Ae^{-kz}$

(ii) $\Phi(\varphi) = \Phi(\varphi + 2\pi) \Rightarrow \nu = m =$ integer

$$\Rightarrow Q(\varphi) = \sum_{m=0}^{\infty} \left( A_m \sin m\varphi + B_m \cos m\varphi \right)$$

(iii) $\Phi$ is finite at $\rho = 0$. $\Rightarrow$ drop $N_m(k\rho) \Rightarrow R = J_m(k\rho)$

(iv) $\Phi = 0$ at $\rho \to \infty \Rightarrow J_m(k \cdot \infty) = 0 \Rightarrow$ continuous eigenvalue $k$

$$\Rightarrow \Phi(\rho, \varphi, z) = \sum_{m=0}^{\infty} \int_0^\infty dk e^{-kz} J_m(k\rho) \left[ A_m(k) \sin m\varphi + B_m(k) \cos m\varphi \right] \tag{3.106}$$

Rewrite (3.106) with variable $k$ changed to $k'$:

$$\Phi(\rho,\varphi,z)=\sum_{m=0}^{\infty}\int_0^\infty dk' e^{-k'z} J_m(k'\rho)\left[A_m(k')\sin m\varphi+B_m(k')\cos m\varphi\right]$$

(v) $\Phi(\rho,\varphi,z=0)=V(\rho,\varphi)$

$$\Rightarrow\ V(\rho,\varphi)=\sum_{m=0}^{\infty}\int_0^\infty dk' J_m(k'\rho)\left[A_m(k')\sin m\varphi+B_m(k')\cos m\varphi\right]$$

Operating both sides with $\int_0^{2\pi} d\varphi \int_0^\infty \rho d\rho J_m(k\rho)\begin{Bmatrix}\sin m\varphi\\ \cos m\varphi\end{Bmatrix}$ and making use of the orthogonal properties of $\sin m\varphi$ and $\cos m\varphi$, and the relation: $\int_0^\infty x J_m(kx) J_m(k'x) dx=\frac{1}{k}\delta(k-k')$ (3.108)

$$\Rightarrow\ \begin{Bmatrix}A_m(k)\\ B_m(k)\end{Bmatrix}=\frac{k}{\pi}\int_0^{2\pi} d\varphi\int_0^\infty \rho d\rho V(\rho,\varphi) J_m(k\rho)\begin{Bmatrix}\sin m\varphi\\ \cos m\varphi\end{Bmatrix} \qquad (3.109)$$

For $m=0$, use $\frac{1}{2}B_0(k)$ in series (3.106).

# 3.9 Expansion of Green Functions in Spherical Coordinates

The Green function for an electrostatic potential problem satisfies

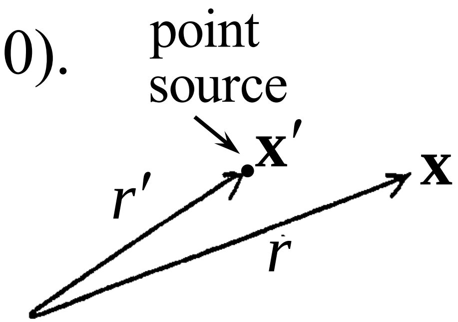
$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}')$$

with $G(\mathbf{x}, \mathbf{x}') = 0$ for $\mathbf{x}$ on the boundary surface.

*Question*: Jackson p.120 states the b.c. as "$G(\mathbf{x}, \mathbf{x}') = 0$ for either $\mathbf{x}$ or $\mathbf{x}'$ on the boundary surface." Why?

*Case 1*: Green function in infinite space

The simplest form is $G(\mathbf{x}, \mathbf{x}') = \dfrac{1}{|\mathbf{x} - \mathbf{x}'|}$ (Sec. 1.10).

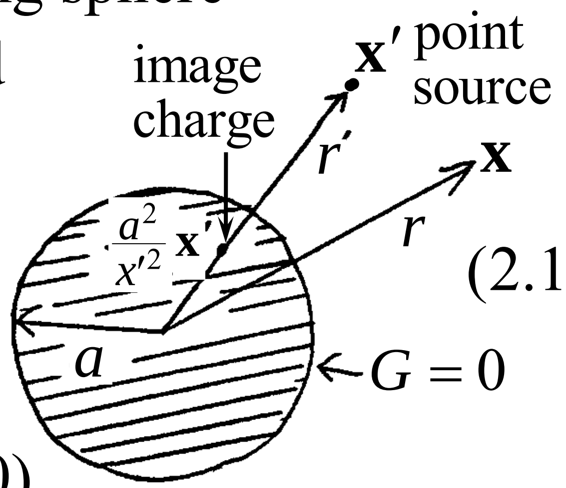



It can be expressed as an expansion in spherical coordinates as (Sec. 3.7)

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) \qquad (3.70)$$

*Case 2*: Green function outside a conducting sphere

By the method of images, we have obtained the Green function in Sec. 2.6,




$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{a}{x'\left|\mathbf{x}-\frac{a^2}{x'^2}\mathbf{x}'\right|} \qquad (2.16)$$

The first term in (2.16) is expanded in (3.70).

The second term can be expanded using (3.70). Since $|\mathbf{x}| > \left|\frac{a^2}{x'^2}\mathbf{x}'\right|$,
we substitute $r_> = |\mathbf{x}| = r$ and $r_< = \left|\frac{a^2}{x'^2}\mathbf{x}'\right| = \frac{a^2}{r'}$ into (3.70) to obtain

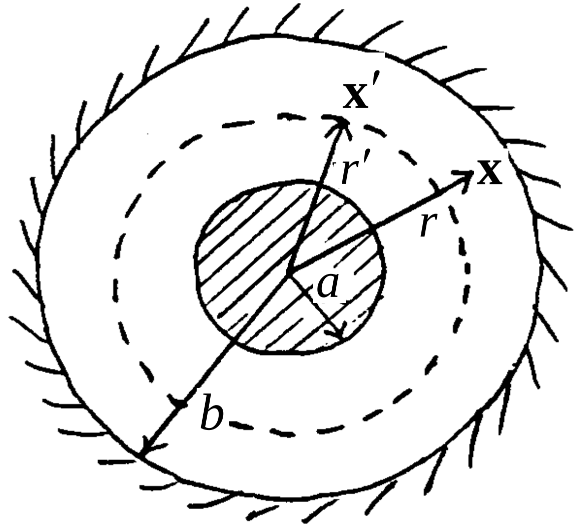
$$\frac{a}{x'\left|\mathbf{x}-\frac{a^2}{x'^2}\mathbf{x}'\right|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{a\left(\frac{a^2}{r'}\right)^l}{r'r^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

$\boxed{r'}$

$$\Rightarrow G(\mathbf{x},\mathbf{x}') = 4\pi\sum_{l,m}\frac{1}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right]Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi), \quad (3.114)$$

*Case 3*: Green function inside a spherical shell bounded by grounded conductors (see figure)

$$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$$




with the b.c. $G(r=a) = G(r=b) = 0$

This problem is difficult to solve by the method of images. We will solve it by a systematic method: **method of expansion**.

Write $\delta(\mathbf{x}-\mathbf{x}')$ in spherical coordinates,

$$\delta(\mathbf{x}-\mathbf{x}') = \tfrac{1}{r^2}\delta(r-r')\delta(\varphi-\varphi')\delta(\cos\theta-\cos\theta')$$

Use the completeness relation (3.56) for $\delta(\varphi-\varphi')\delta(\cos\theta-\cos\theta')$

$$\Rightarrow \quad \delta(\mathbf{x}-\mathbf{x}') = \tfrac{1}{r^2}\delta(r-r')\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \qquad (3.117)$$

Note that, in (3.117), we have decomposed a point charge into an infinite number of spherical "charge layers", all of radius $r'$.

$$\Rightarrow \nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi \frac{1}{r^2}\delta(\underset{\text{variable}}{r} - \underset{\text{constant}}{r'}) \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\underbrace{\theta',\varphi'}_{\text{constants}}) Y_{lm}(\underbrace{\theta,\varphi}_{\text{variables}}) \qquad (4)$$

The RHS of this equation is an expansion in spherical harmonics, which suggests that we expand $G(\mathbf{x},\mathbf{x}')$ similarly. This is possible since $Y_{lm}(\theta,\varphi)$ form a complete set.

$$G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}(\underset{\text{variable}}{r} \,|\, \underbrace{r',\theta',\varphi'}_{\text{constants}}) Y_{lm}(\underbrace{\theta,\varphi}_{\text{variables}}), \qquad (3.118)$$

where $A_{lm}$ is a function of $r$ to be solved from (4).

Expressing $A_{lm}$ as

$$A_{lm}(r \,|\, r',\theta',\varphi') = g_l(r,r') Y^*_{lm}(\theta',\varphi') \qquad (5)$$

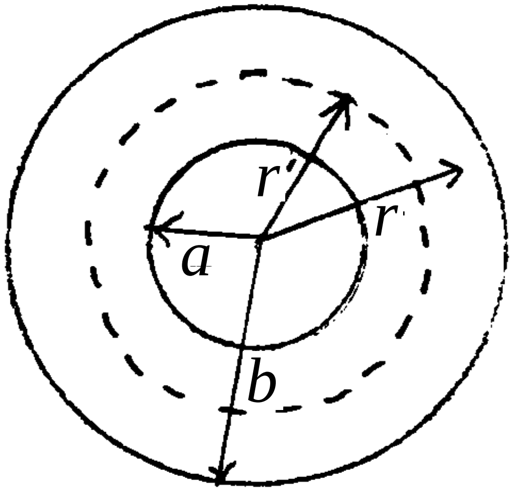
and sub. (5) into (4), we get the equation for $g_l(r,r')$ (see Sec. 3.1),

$$\frac{1}{r}\frac{d^2}{dr^2}[rg_l(r,r')]-\frac{l(l+1)}{r^2}g_l(r,r')=-\frac{4\pi}{r^2}\delta(r-r') \qquad (3.120)$$

Divide the space into $r<r'$ and $r>r'$. In each region, (3.120) reduces to

$$\frac{1}{r}\frac{d^2}{dr^2}[rg_l(r,r')]-\frac{l(l+1)}{r^2}g_l(r,r')=0$$

$$\Rightarrow \quad g_l(r,r')=\begin{cases} Ar^l+Br^{-l-1}, & r<r' \\ A'r^l+B'r^{-l-1}, & r>r' \end{cases} \qquad (6)$$

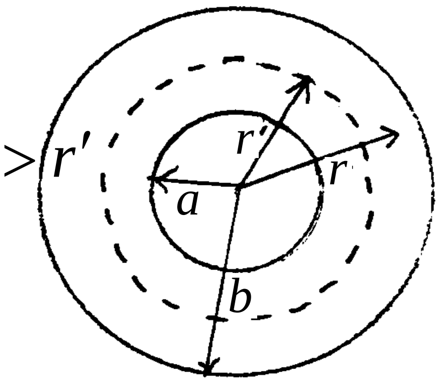



The remaining job is to find 4 boundary conditions to solve for the 4 constants $A,\ B,\ A',$ and $B'$ in (6).

(i) $g_l(r=a,r')=0 \;\Rightarrow g_l(r,r')=A\left(r^l-\frac{a^{2l+1}}{r^{l+1}}\right), \quad r<r'$

(ii) $g_l(r=b,r')=0 \;\Rightarrow g_l(r,r')=B'\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right), \quad r>r'$

(iii) $g_l(r,r')$ is continuous at $r=r'$.




Physical reason: $\phi$ is continuous across the charge layer at $r=r'$. ($E$ is finite at $r=r'$. $\Rightarrow \Delta\phi = \lim\limits_{\Delta r\to 0} E\Delta r = 0$). Thus,

$$A\left(r'^l-\frac{a^{2l+1}}{r'^{l+1}}\right)=B'\left(\frac{1}{r'^{l+1}}-\frac{r'^l}{b^{2l+1}}\right) \Rightarrow \frac{A}{B'}=\frac{\frac{1}{r'^{l+1}}-\frac{r'^l}{b^{2l+1}}}{r'^l-\frac{a^{2l+1}}{r'^{l+1}}} \Rightarrow \begin{cases} A=C\left(\frac{1}{r'^{l+1}}-\frac{r'^l}{b^{2l+1}}\right) \\ B'=C\left(r'^l-\frac{a^{2l+1}}{r'^{l+1}}\right) \end{cases}$$

$$\Rightarrow g_l(r,r')=\begin{cases} C\left(\frac{1}{r'^{l+1}}-\frac{r'^l}{b^{2l+1}}\right)\left(r^l-\frac{a^{2l+1}}{r^{l+1}}\right), & r<r' \\ C\left(r'^l-\frac{a^{2l+1}}{r'^{l+1}}\right)\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right), & r>r' \end{cases}$$

$$=C\left(r_<^l-\frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right) \qquad (3.122)$$

Rewrite (3.120): $\dfrac{1}{r}\dfrac{d^2}{dr^2}[rg_l(r,r')]-\dfrac{l(l+1)}{r^2}g_l(r,r')=-\dfrac{4\pi}{r^2}\delta(r-r')$

(iv) We need one more condition to get the remaining constant $C$ in (3.122). Physically, this condition is related to the discontinuity of $E_r(\propto \frac{d}{dr}g_l)$ across the charge layer at $r=r'$. Mathematicaly, we integrate the delta funtion in (3.120) to bring out the discontinuity. Multiply (3.120) by $r$ and integrate from $r'-\varepsilon$ to $r'+\varepsilon$ $(\varepsilon\to 0)$

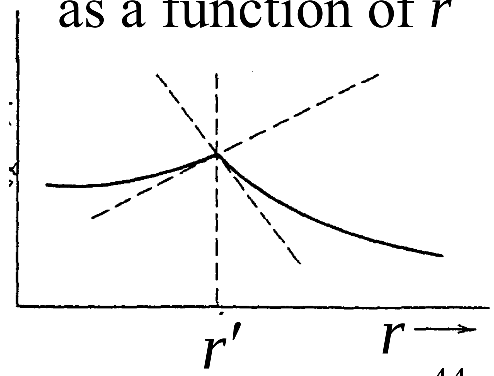
$$\Rightarrow \frac{d}{dr}[rg_l(r,r')]_{r'+\varepsilon}-\frac{d}{dr}[rg_l(r,r')]_{r'-\varepsilon}=-\frac{4\pi}{r'}$$

$$\Rightarrow -\tfrac{C}{r'}\left[1-(\tfrac{a}{r'})^{2l+1}\right]\left[l+(l+1)(\tfrac{r'}{b})^{2l+1}\right]$$

$$-\tfrac{C}{r'}\left[(l+1)+l(\tfrac{a}{r'})^{2l+1}\right]\left[1-(\tfrac{r'}{b})^{2l+1}\right]=-\frac{4\pi}{r'}$$

$$\Rightarrow C=\frac{4\pi}{(2l+1)\left[1-(\frac{a}{b})^{2l+1}\right]}$$

$rg_l$ (solid line)
as a function of $r$



Sub. $C$ into (3.122), we get

$$G(\mathbf{x},\mathbf{x}')$$

$$=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}^{*}(\theta',\varphi')Y_{lm}(\theta,\varphi)}{(2l+1)\left[1-(\frac{a}{b})^{2l+1}\right]}\left(r_<^l-\frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right) \qquad (3.125)$$

*Limiting case 1*: $a\rightarrow 0$ & $b\rightarrow\infty$, (3.125) $\Rightarrow$ (3.70)

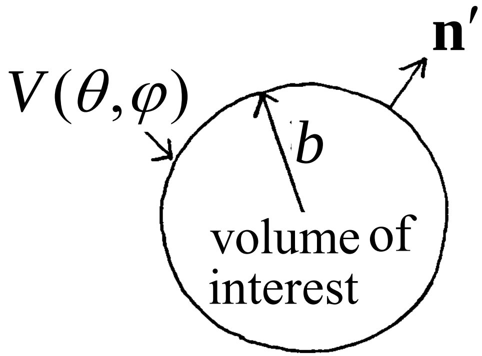
$$G(\mathbf{x},\mathbf{x}')=\frac{1}{|\mathbf{x}-\mathbf{x}'|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^{*}(\theta',\varphi')Y_{lm}(\theta,\varphi) \qquad (3.70)$$

*Limiting case 2*: $b\rightarrow\infty$, (3.125) $\Rightarrow$ (3.114)

$$G(\mathbf{x},\mathbf{x}')=4\pi\sum_{l,m}\frac{1}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}}-\frac{1}{a}(\frac{a^2}{rr'})^{l+1}\right]Y_{lm}^{*}(\theta',\varphi')Y_{lm}(\theta,\varphi), \quad (3.114)$$

# 3.10 Solution of Potential Problems with the Spherical Green Function Expansion

*Example 1*: Potential inside a charge-free sphere of radius $b$ subject to the b.c. $\Phi(r=b)=V(\theta,\varphi)$




Since we already have the Green function for this problem, it is convenient to use the formal solution derived in Sec. 1.10:

$$\Phi(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int_V \underbrace{\rho(\mathbf{x}')}_{=0}G(\mathbf{x},\mathbf{x}')d^3x'-\frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial}{\partial n'}G(\mathbf{x},\mathbf{x}')da' \quad (1.44)$$

There is no charge inside. $\Rightarrow \Phi(\mathbf{x})=-\frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial}{\partial n'}G(\mathbf{x},\mathbf{x}')da'$

*Note*: The unit vector $\mathbf{n}'$ is normal to the surface and pointing outward from volume of interest. $\frac{\partial}{\partial n'}$ is a differentiation along $\mathbf{n}'$ ($\frac{\partial}{\partial n'}=\frac{\partial}{\partial r'}$ for this example).

Rewrite (3.125):

$$G(\mathbf{x},\mathbf{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)}{(2l+1)\left[1-(\frac{a}{b})^{2l+1}\right]} \left( r_<^l - \frac{a^{2l+1}}{r_<^{l+1}} \right) \left( \frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right)$$

For this example, $a = 0$, $r_> = r'$, and $r_< = r$, hence

$$G(\mathbf{x},\mathbf{x}') = 4\pi \sum_{l,m} \frac{1}{2l+1} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) r^l \left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right)$$

$$\Rightarrow \frac{\partial G}{\partial r'} = 4\pi \sum_{l,m} \frac{1}{2l+1} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) r^l \left(-\frac{l+1}{r'^{l+2}} - \frac{l r'^{l-1}}{b^{2l+1}}\right)$$

$$\Rightarrow \left.\frac{\partial G}{\partial n'}\right|_{r'=b} = \left.\frac{\partial G}{\partial r'}\right|_{r'=b} = -\frac{4\pi}{b^2} \sum_{l,m} \left(\frac{r}{b}\right)^l Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \qquad (7)$$

$$da' = r'^2 \sin\theta' d\theta' d\varphi' = b^2 d\Omega' \qquad (8)$$

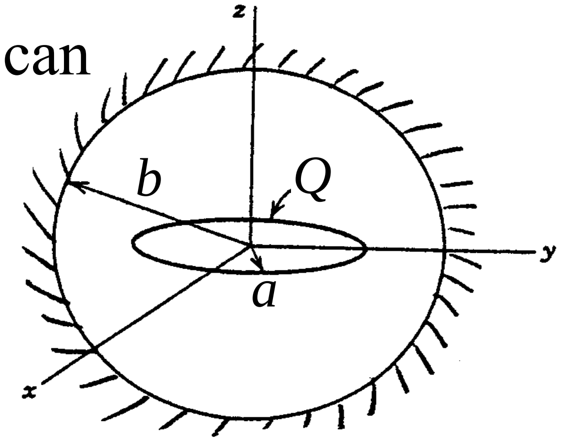
$$\Phi(\mathbf{x}')\big|_S = \Phi(r'=b) = V(\theta',\varphi') \qquad (9)$$

Sub. (7) - (9) into $\Phi(\mathbf{x}) = -\frac{1}{4\pi}\oint_S \Phi(\mathbf{x}') \frac{\partial}{\partial n'} G(\mathbf{x},\mathbf{x}') da'$, we get

$$\Phi(\mathbf{x}) = \sum_{l,m} \left[\int V(\theta',\varphi') Y_{lm}^*(\theta',\varphi') d\Omega'\right] \left(\frac{r}{b}\right)^l Y_{lm}(\theta,\varphi) \qquad (3.128)$$

*Example 2*: Potential due to a uniformly charged ring of radius $a$ and total charge $Q$ located on the $x$-$y$ plane inside a grounded conducting sphere of radius $b$

In spherical coordinates, the $x$-$y$ plane is at $\theta = \pi/2$. The charge density $\rho(\mathbf{x})$ can be written as




$$\rho(\mathbf{x}) = \frac{Q}{2\pi a^2}\delta(r-a)\delta(\cos\theta) \quad (3.129)$$

The potental is given by

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \underbrace{\Phi(\mathbf{x}')}_{=0}\frac{\partial}{\partial n'}G(\mathbf{x},\mathbf{x}')da' \quad (1.44)$$

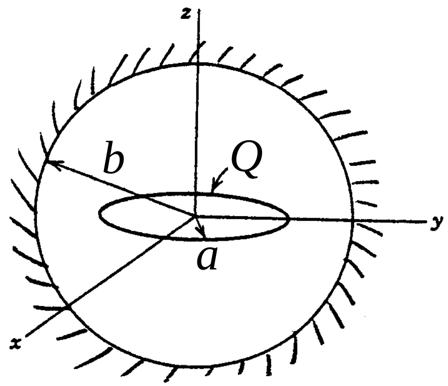
There is no inner conductor in this problem. $\Rightarrow$ (3.125) reduces to

$$G\left(\mathbf{x},\mathbf{x}'\right) = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}Y_{lm}^{*}(\theta',\varphi')Y_{lm}(\theta,\varphi)r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \quad (10)$$

Symmetry in $\varphi \Rightarrow m=0$. Hence,

$$Y_{lm}(\theta,\varphi) \to Y_{l0}(\theta,\varphi) = \sqrt{\tfrac{2l+1}{4\pi}} P_l(\cos\theta)$$

$$\Rightarrow \quad G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty} P_l(\cos\theta')P_l(\cos\theta) r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \qquad (11)$$




Sub. (11) and $\rho(\mathbf{x}) = \dfrac{Q}{2\pi a^2}\delta(r-a)\delta(\cos\theta)$ into (1.44), we obtain

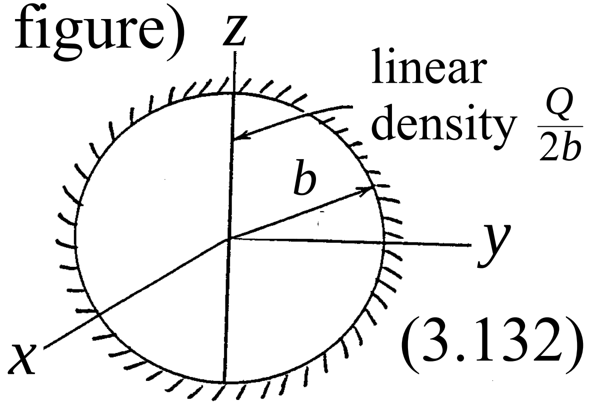
$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')$$

$$= \frac{Q}{8\pi^2\varepsilon_0 a^2}\int r'^2 dr' d\cos\theta' d\varphi' \left[\begin{array}{l} \delta(r'-a)\delta(\cos\theta') \\ \cdot \sum_{l=0}^{\infty} P_l(\cos\theta')P_l(\cos\theta) r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \end{array}\right]$$

$$= \frac{Q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty} P_l(0)P_l(\cos\theta) r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \qquad (3.130)$$

where $r_<$ $(r_>)$ is the smaller (larger) of $r$ and $a$.

*Example* 3: Potential due to a uniformly charged line of length $2b$ and total charge $Q$ located on the $z$-axis inside a grounded conducting sphere of radius $b$ (see figure)




The charge density $\rho(\mathbf{x})$ can be written in spherical coordinates as (see problem below.)

$$\rho(\mathbf{x}) = \frac{Q}{2b}\frac{1}{2\pi r^2}\left[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\right] \qquad (3.132)$$

The potential is given by

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \underbrace{\Phi(\mathbf{x}')}_{=0}\frac{\partial}{\partial n'}G(\mathbf{x},\mathbf{x}')da' \qquad (1.44)$$

Rewrite (11), which is applicable to this problem:

$$G\left(\mathbf{x},\mathbf{x}'\right) = \sum_{l=0}^{\infty} P_l(\cos\theta')P_l(\cos\theta)r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \qquad (11)$$

Sub. (11) into (1.44), we obtain

$$\Phi(\mathbf{x}) = \frac{Q}{8\pi\varepsilon_0 b}\int r'^2 dr' d\cos\theta' d\varphi' \left[ \frac{\delta(\cos\theta'-1)+\delta(\cos\theta'+1)}{2\pi r'^2} \cdot \sum_{l=0}^{\infty} P_l(\cos\theta')P_l(\cos\theta) r_<^l \left( \frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right) \right]$$

$$= \frac{Q}{8\pi\varepsilon_0 b}\sum_{l=0}^{\infty}\left[P_l(1)+P_l(-1)\right]P_l(\cos\theta)\underbrace{\int_0^b r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) dr'} \quad (3.133)$$

$$\boxed{\begin{aligned} &= \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\int_0^r r'^l dr' + r^l \int_r^b \left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right) dr' \\ &= \frac{2l+1}{l(l+1)}\left[1-\left(\tfrac{r}{b}\right)^l\right] \end{aligned}}$$

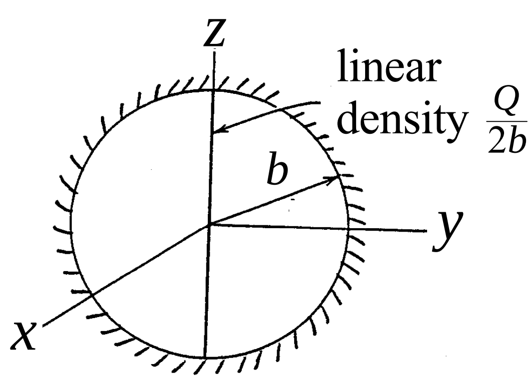
$P_l(-1) = (-1)^l$ and $P_l(1) = 1 \Rightarrow$ Odd $l$ terms cancel.

$$\Rightarrow \Phi(\mathbf{x}) = \frac{Q}{4\pi\varepsilon_0 b}\left\{ \ln\left(\tfrac{b}{r}\right) + \sum_{j=1}^{\infty} \frac{4j+1}{2j(2j+1)}\left[1-\left(\tfrac{r}{b}\right)^{2j}\right]P_{2j}(\cos\theta)\right\} \quad (3.136)$$

Note that the $l=0$ term in (3.133) is given by $\ln(\frac{b}{r})$. See p.124.

*Problem*: Show the charge density in (3.132):

$$\rho(\mathbf{x}) = \frac{Q}{2b}\frac{1}{2\pi r^2}[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)]$$




represents a unifom charge distribution along z.

*Solution*: The total charge is

$$\int \rho(\mathbf{x}) d^3x = \frac{Q}{2b}\int_0^b r^2 dr \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\varphi \frac{\delta(\cos\theta - 1) + \delta(\cos\theta + 1)}{2\pi r^2}$$

$$= \frac{Q}{2b}\int_0^b dr \int_{-1}^1 d\cos\theta [\underbrace{\delta(\cos\theta - 1)}_{\theta=0,\ +z\text{-axis}} + \underbrace{\delta(\cos\theta + 1)}_{\theta=\pi,\ -z\text{-axis}}]$$

$$= \frac{Q}{2b}\int_{-b}^b dz \Rightarrow \text{uniform distribution from } z = -b \text{ to } z = b.$$

*Note*: The above integration over $\cos\theta$ starts from $\cos\theta = -1$ and and ends at $\cos\theta = 1$. It does not cross 1 or $-1$. This issue can be resolved by a limiting procedure; namely, we write

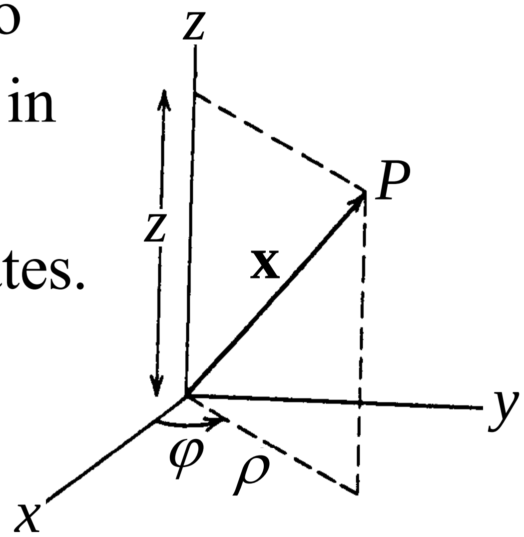
$$\delta(\cos\theta - 1) + \delta(\cos\theta + 1) = \lim_{\varepsilon \to 0}[\delta(\cos\theta - 1 + \varepsilon) + \delta(\cos\theta + 1 - \varepsilon)]$$

optional

# 3.11 Expansion of Green Functions in Cylindrical Coordinates

Consider the Green equation:

$$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}'), \text{ with } G(\mathbf{x},\mathbf{x}') = 0 \text{ as } |\mathbf{x}| \to \infty$$




An obvious solution is $1/|\mathbf{x}-\mathbf{x}'|$. We have also solved this equation by the method of expansion in spherical coordinates [(3.70)]. Here, by the same method, we solve it again in cylindrical coordinates.

Write $\delta(\mathbf{x}-\mathbf{x}')$ as

$$\delta(\mathbf{x}-\mathbf{x}') = \tfrac{1}{\rho}\delta(\rho-\rho')\delta(\varphi-\varphi')\delta(z-z')$$

with

$$\begin{cases} \delta(\varphi-\varphi') = \dfrac{1}{2\pi}\sum\limits_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')} \\ \delta(z-z') = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(z-z')} = \tfrac{1}{\pi}\int_0^{\infty} dk \cos[k(z-z')] \end{cases}$$

$$\Rightarrow \nabla^2 G(\mathbf{x},\mathbf{x}') = -\frac{2}{\pi}\frac{\delta(\rho-\rho')}{\rho}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk e^{im(\varphi-\varphi')}\cos[k(z-z')] \qquad (12)$$

**optional** 

Since $e^{im\varphi}$ and $e^{ikz}$ are complete sets, we may expand $G(\mathbf{x},\mathbf{x}')$ in variables $\varphi$ and $z$

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{2\pi}\frac{1}{\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk g_m(k,\rho,\rho')e^{im(\varphi-\varphi')}\cos[k(z-z')] \quad (3.140)$$

where the coefficient $g_m(k,\rho,\rho')$ is a function of $m$, $k$, $\rho$ and $\rho'$.

Sub. (3.140) into (12) we get

$$\frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}\right)$$

$$\cdot g_m(k,\rho,\rho')e^{im(\varphi-\varphi')}\cos[k(z-z')]$$

$$= -\frac{2}{\pi}\frac{\delta(\rho-\rho')}{\rho}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk e^{im(\varphi-\varphi')}\cos[k(z-z')] \quad (13)$$

In (13), $\frac{\partial^2}{\partial\varphi^2}\to -m^2$, $\frac{\partial^2}{\partial z^2}\to -k^2$, $\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}=\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}$. Hence,

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}-\left(k^2+\frac{m^2}{\rho^2}\right)\right]g_m(k,\rho,\rho') = -\frac{4\pi}{\rho}\delta(\rho-\rho') \quad (3.141)$$

optional



See (3.98)-(3.101) in Jackson.

$$\Rightarrow g_m(k,\rho,\rho') = \begin{cases} AI_m(k\rho) + BK_m(k\rho), & \rho < \rho' \\ A'I_m(k\rho) + B'K_m(k\rho), & \rho > \rho' \end{cases}$$

(i) $g_m$ is finite at $\rho = 0. \Rightarrow B = 0$

(ii) $g_m$ remains finite as $\rho \to \infty. \Rightarrow A' = 0$

(iii) $g_m$ is continuous at $\rho = \rho'$.

$$\Rightarrow \ AI_m(k\rho') = B'K_m(k\rho')$$

$$\Rightarrow \frac{A}{B'} = \frac{K_m(k\rho')}{I_m(k\rho')} \Rightarrow \begin{cases} A = CK_m(k\rho') \\ B' = CI_m(k\rho') \end{cases}$$

$$\Rightarrow g_m(k,\rho,\rho') = \begin{cases} CK_m(k\rho')I_m(k\rho), & \rho < \rho' \\ CI_m(k\rho')K_m(k\rho), & \rho > \rho' \end{cases}$$

$$= CI_m(k\rho_<)K_m(k\rho_>) \qquad (14)$$

(iv) To obtain the coefficient $C$ in $g_m(k,\rho,\rho') = CI_m(k\rho_<)K_m(k\rho_>)$, mutiply (3.141) by $\rho$ and integrate form $\rho' - \varepsilon$ to $\rho' + \varepsilon$ $(\varepsilon \to 0)$

$$\left.\frac{dg_m}{d\rho}\right|_{\rho'+\varepsilon} - \left.\frac{dg_m}{d\rho}\right|_{\rho'-\varepsilon} = -\frac{4\pi}{\rho'}$$

$$\Rightarrow Ck[I_m(k\rho')K'_m(k\rho') - K_m(k\rho')I'_m(k\rho')] = -\frac{4\pi}{\rho'}$$

Use the relation: $I_m(x)K'_m(x) - I'_m(x)K_m(x) = -1/x$ (3.147)

$$\Rightarrow Ck\left(\frac{-1}{k\rho'}\right) = -\frac{4\pi}{\rho'} \Rightarrow C = 4\pi \Rightarrow g_m(k,\rho,\rho') = 4\pi I_m(k\rho_<)K_m(k\rho_>)$$

Sub. the above expression for $g_m(k,\rho,\rho')$ into (3.140)

$$\Rightarrow G(\mathbf{x},\mathbf{x}') = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk e^{im(\varphi-\varphi')}\cos[k(z-z')]I_m(k\rho_<)K_m(k\rho_>)$$

Since $G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|}$, we have by the uniqueness theorem

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk e^{im(\varphi-\varphi')}\cos[k(z-z')]I_m(k\rho_<)K_m(k\rho_>) \quad (3.148)$$

## 3.12 Eigenfunction Expansion for Green Functions

**Eigenfunction Expansion of Green Function in 3 Dimensions :**

We have obtained the Green function for the Poisson equation by the method of eigenfunction expansion in 2 dimensions [e.g. (3.118), in $\theta$, $\varphi$]. Here, we develop a general technique to obtain the Green function by eigenfunction expansion in 3 dimensions. Consider the Green function for a more general equation (with homogeneous b.c.):

$$\nabla^2 G(\mathbf{x},\mathbf{x}') + [f(\mathbf{x}) + \lambda] G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \qquad (3.156)$$

a given real function

a given constant

We shall solve (3.156) by expanding $G(\mathbf{x},\mathbf{x}')$ and $\delta(\mathbf{x}-\mathbf{x}')$ in eigenfunctions of a related problem formulated as follows.

same $f(\mathbf{x})$ as in (3.156)

an eigenvalue to be determined by the b.c., not the same λ as in (3.156)

$$\nabla^2 \psi(\mathbf{x}) + [f(\mathbf{x}) + \lambda] \psi(\mathbf{x}) = 0 \qquad (3.153)$$

with the same boundary surface and homogeneous b.c. as for (3.156), Assume the (3-dimensional) eigenfunctions for

$$\nabla^2\psi(\mathbf{x})+[f(\mathbf{x})+\lambda]\psi(\mathbf{x})=0$$

are $\psi_n(\mathbf{x})$. Since the operator $[\nabla^2+f(\mathbf{x})]$ is Hermitian, we have

$$\int_V \psi_m^*(\mathbf{x})\psi_n(\mathbf{x})d^3x=\delta_{mn}$$

and $\psi_n$ form a complete set with *real* eigenvalue $\lambda_n$ [see Apendix A].

Write $\quad G(\mathbf{x},\mathbf{x}')=\sum_n a_n(\mathbf{x}')\psi_n(\mathbf{x}) \qquad (3.157)$

Sub. (3.157) and $\delta(\mathbf{x}-\mathbf{x}')=\sum_n \psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})$ [see (2.35)] into

$\nabla^2 G(\mathbf{x},\mathbf{x}')+[f(\mathbf{x})+\lambda]G(\mathbf{x},\mathbf{x}')=-4\pi\delta(\mathbf{x}-\mathbf{x}')$, we obtain

$$\sum_n a_n(\mathbf{x}')\{\nabla^2\psi_n(\mathbf{x})+[f(\mathbf{x})+\lambda]\psi_n(\mathbf{x})\}=-4\pi\sum_n \psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})$$

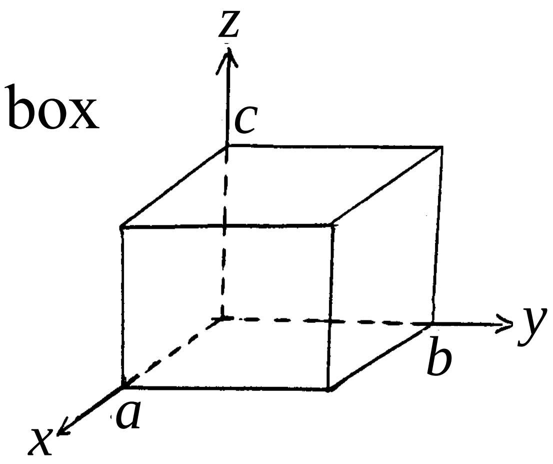
Since $\psi_n$ satisfies $\nabla^2\psi_n(\mathbf{x})+[f(\mathbf{x})+\lambda_n]\psi_n(\mathbf{x})=0$, we have

$$\sum_n a_n(\mathbf{x}')(\lambda-\lambda_n)\psi_n(\mathbf{x})=-4\pi\sum_n \psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})$$

$$\Rightarrow \quad a_n(\mathbf{x}')=4\pi\frac{\psi_n^*(\mathbf{x}')}{\lambda_n-\lambda} \Rightarrow G(\mathbf{x},\mathbf{x}')=4\pi\sum_n \frac{\psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})}{\lambda_n-\lambda} \qquad (3.160)$$

We now specialize to the Green function for the Poisson equation i.e. (3.156) with $f(\mathbf{x}) = \lambda = 0$.

*Example 1*: Green function for a rectangular box




$$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$$

$$\text{with } G(\mathbf{x},\mathbf{x}') = 0 \text{ at } \begin{cases} x = 0 \text{ and } a \\ y = 0 \text{ and } b \\ z = 0 \text{ and } c \end{cases}$$

Consider the corresponding eigenvalue problem [(3.153) with $f(\mathbf{x}) = 0$ and $\lambda \to k^2$]: $\nabla^2\psi(\mathbf{x}) + k^2\psi(\mathbf{x}) = 0$ with the same b.c.

$$\text{Let } \psi(\mathbf{x}) = X(x)Y(y)Z(z) \Rightarrow \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{-k_l^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{-k_m^2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{-k_n^2} + k^2 = 0$$

$$\Rightarrow \begin{cases} X(x) = Ae^{ik_l x} + Be^{-ik_l x} \\ Y(x) = Be^{ik_l y} + Ce^{-ik_l y} \\ Z(x) = De^{ik_l z} + Ee^{-ik_l z} \end{cases} \quad \text{with } k^2 = k_l^2 + k_m^2 + k_n^2$$

$$\text{b.c.}\ \begin{cases} X(x)=0 \text{ at } x=0 \text{ and } a \\ Y(x)=0 \text{ at } y=0 \text{ and } b \\ Z(x)=0 \text{ at } z=0 \text{ and } c \end{cases} \Rightarrow \begin{cases} k_l = \dfrac{l\pi}{a} \\ k_m = \dfrac{m\pi}{b} \\ k_n = \dfrac{n\pi}{c} \end{cases} \text{ and } \begin{cases} X = \sin\dfrac{l\pi x}{a}, \\ Y = \sin\dfrac{m\pi y}{b}, \\ Z = \sin\dfrac{n\pi z}{c}, \end{cases}$$

$$\Rightarrow k^2 = k_{lmn}^2 = \pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)$$

$$\Rightarrow \psi(\mathbf{x}) = \sqrt{\frac{8}{abc}}\sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}$$

Sub. $\psi(\mathbf{x})$ into (3.160): $G(\mathbf{x},\mathbf{x}') = 4\pi\sum_j \dfrac{\psi_j^*(\mathbf{x}')\psi_j(\mathbf{x})}{\lambda_j-\lambda}$, we obtain

$$\boxed{\sum_j \to \sum_{l,m,n}\ ;\ \lambda_j \to k_{lmn}^2;\ \lambda = 0}$$

$$G(\mathbf{x},\mathbf{x}') = \frac{32}{\pi abc}\sum_{l,m,n=1}^{\infty}\frac{\sin\frac{l\pi x'}{a}\sin\frac{l\pi x}{a}\sin\frac{m\pi y'}{b}\sin\frac{m\pi y}{b}\sin\frac{n\pi z'}{c}\sin\frac{n\pi z}{c}}{\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}} \quad (3.167)$$

*Example 2*: Green function for infinite space

$$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \text{ with } G(\mathbf{x},\mathbf{x}') = 0 \text{ as } |\mathbf{x}| \to \infty$$

Consider the corresponding eigenvalue problem

$$\nabla^2 \psi(\mathbf{x}) + k^2 \psi(\mathbf{x}) = 0 \quad \text{in infinite space}$$

The solution is

$$\psi_{\mathbf{k}}(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} = \frac{1}{(2\pi)^{3/2}} e^{ik_x x + ik_y y + ik_z z} \tag{3.162}$$

where $\mathbf{k} = k_x \mathbf{e}_x + k_y \mathbf{e}_y + k_z \mathbf{e}_z$

Since the region of interest is infinite space, we have continuous engenvalue $k^2$ and the factor $1/(2\pi)^{3/2}$ gives the normalization condition for $\psi_{\mathbf{k}}(\mathbf{x})$:

$$\int \psi^*_{\mathbf{k}'}(\mathbf{x}) \psi_{\mathbf{k}}(\mathbf{x}) d^3x = \delta(\mathbf{k}-\mathbf{k}') \tag{3.163}$$

[see p.69 for a one dimensional example.]

So the series expansion: $G(\mathbf{x},\mathbf{x}') = 4\pi\sum_n \frac{\psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})}{\lambda_n - \lambda}$ [(3.160)]
becomes

$$G(\mathbf{x},\mathbf{x}') = 4\pi\int \frac{\psi_{\mathbf{k}}^*(\mathbf{x}')\psi_{\mathbf{k}}(\mathbf{x})}{\lambda_{\mathbf{k}} - \lambda} d^3k$$

With $\lambda_{\mathbf{k}} = k^2$, $\lambda = 0$, and $\psi_{\mathbf{k}} = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}}$, we have

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{2\pi^2}\int d^3k \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k^2}$$

Since $G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|}$, we get another mathematical expression for $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ by the uniqueness theorem

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{2\pi^2}\int d^3k \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{k^2} \qquad (3.164)$$

**Solution of Inhomogeneous Differential Equation by the Green Function Method :**

To show the usefulness of the 3-dimensional Green function just obtained, we consider an inhomogeneous differential equation:

$$\nabla^2 u(\mathbf{x}) + [f(\mathbf{x}) + \lambda] u(\mathbf{x}) = -4\pi S(\mathbf{x}) \qquad (15)$$

wth homogeneous b.c. In (15), $S(\mathbf{x})$ is a distributed source. We have shown that the solution for the same equation with a point source:

$$\nabla^2 G(\mathbf{x},\mathbf{x}') + [f(\mathbf{x}) + \lambda] G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \qquad (3.156)$$

is $$G(\mathbf{x},\mathbf{x}') = 4\pi \sum_n \psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})/(\lambda_n - \lambda), \qquad (3.160)$$

where $\psi_n(\mathbf{x})$ is the eigenfunction of $\nabla^2\psi_n(\mathbf{x}) + [f(\mathbf{x}) + \lambda_n]\psi_n(\mathbf{x}) = 0.$

Then, the solution of (15) is $u(\mathbf{x}) = \int_V G(\mathbf{x},\mathbf{x}')S(\mathbf{x}')d^3x',$ (16) which can be verified if we operate both sides with $\nabla^2 + f(\mathbf{x}) + \lambda$ and apply (3.156) to the RHS.

*Note*: If $\lambda = \lambda_n$, there is no solution unless $\int_V u_n^*(\mathbf{x})S(\mathbf{x})d^3x = 0.$

# Homework of Chap. 3

**Problems: 1, 2, 3, 6, 7,**

**9, 17, 20, 22**