4.1 Multipole Expansion

\[ \Phi(x) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(x')}{|x - x'|} d^3x' \tag{1.17} \]

In Ch. 3, we developed various methods of expansion for the solution of the Poisson equation. In this chapter, we continue the subject of electrostatics by taking a closer look at the source \( \rho(x) \). By the method of expansion, we first decompose \( \Phi(x) \) in (1.17) into multipole fields and thereby express the source in multipole moments, then show that the atomic/molecular dipole moments account for the macroscopic properties of a dielectric medium and allow a concise characterization of the medium by a single number called the dielectric constant.
4.1 Multipole Expansion (continued)

Multipole Expansion in Spherical Coordinates:

\[
\frac{1}{|\mathbf{x} - \mathbf{x}'|} = 4\pi \sum_{l=0}^{\infty} \frac{1}{2l + 1} \sum_{m=-l}^{l} \frac{1}{r_{\leq l+1}} Y_{lm}^*(\theta', \varphi')Y_{lm}(\theta, \varphi) \tag{3.70}
\]

For \( \mathbf{x} \) outside the sphere enclosing \( \rho, r_\leq = r', r_\geq = r \).

\[
\Rightarrow \Phi(\mathbf{x}) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \\
= \frac{1}{\varepsilon_0} \sum_{lm} \frac{1}{2l + 1} \left[ \int_{V} Y_{lm}^*(\theta', \varphi') r'^l \rho(\mathbf{x}') d^3x' \right] Y_{lm}(\theta, \varphi) r^{l+1} \equiv q_{lm} \text{ (multipole moments)} \tag{4.2}
\]

monopole \( (l = 0) \) \[+\] \( \Rightarrow \Phi \propto \frac{1}{r} \)

dipole \( (l = 1) \) \[+ -\] \( \Rightarrow \Phi \propto \frac{1}{r^2} \) \(\text{partial cancellation of monopoles}\)

quadrupole \( (l = 2) \) \[+ -\] \[- +\] \( \Rightarrow \Phi \propto \frac{1}{r^3} \) \(\text{partial cancellation of dipoles}\)
4.1 Multipole Expansion (continued)

Multipole Expansion in Cartesian Coordinates:

Expansion in Cartesian coordinates is more useful for our purposes. We first summarize the formulae needed for the expansion.

*Taylor expansion:* [see Appendix A]

\[ f(x + a) = f(x) + (a \cdot \nabla) f(x) + \frac{1}{2} (a \cdot \nabla)(a \cdot \nabla) f(x) + \cdots, \quad (1) \]

where

\[ a \cdot \nabla = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3} = \sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i} \]

\[ (a \cdot \nabla)(a \cdot \nabla) = \sum_{i} a_i \frac{\partial}{\partial x_i} \sum_{j} a_j \frac{\partial}{\partial x_j} = \sum_{ij} a_i a_j \frac{\partial^2}{\partial x_i \partial x_j} \quad (2) \]

*Other useful relations:*

\[ \nabla |x - x'|^n = n |x - x'|^{n-2} (x - x') \quad [\text{derived in Sec. 1.5}] \quad (3) \]

\[ \frac{\partial}{\partial x_i} |x - x'|^n = n |x - x'|^{n-2} (x_i - x_i') \quad (4) \]
4.1 Multipole Expansion (continued)

Now apply (1)-(4) to expand $1/|\mathbf{x} - \mathbf{x}'|$.

Use (1); Write $r = |\mathbf{x}|$

$$
\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{r} - \mathbf{x}' \cdot \nabla \frac{1}{r} + \frac{1}{2} \left( \mathbf{x}' \cdot \nabla \right) \left( \mathbf{x}' \cdot \nabla \right) \frac{1}{r} + \cdots
$$

Use (2)-(4)

$$
= \frac{1}{r} + \frac{\mathbf{x}' \cdot \mathbf{x}}{r^3} + \frac{1}{2} \sum_{ij} x'_i x'_j \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r} + \cdots
$$

$$
= - \frac{\partial}{\partial x_i} \frac{x_j}{r^3} = - \mathbf{x} \cdot \frac{\partial}{\partial x_i} \mathbf{x} \frac{1}{r^3} - \frac{1}{r^3} \frac{\partial x_j}{\partial x_i} \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r} = \frac{3 x_i x_j}{r^5} - \frac{\delta_{ij}}{r^3}
$$

$$
= \sum_{ij} x_i x_j \delta_{ij}
$$

$$
= \frac{1}{r} + \frac{\mathbf{x}' \cdot \mathbf{x}}{r^3} + \frac{1}{2} \sum_{ij} 3 x'_i x'_j x_i x_j - \frac{1}{2} \frac{r'}{r^5} \sum_{ij} x'_i x'_j \delta_{ij} + \cdots
$$

$$
= \frac{1}{r} + \frac{\mathbf{x}' \cdot \mathbf{x}}{r^3} + \frac{1}{2} \sum_{ij} x_i x_j \left( 3 x'_i x'_j - r'^2 \delta_{ij} \right) + \cdots (5)
$$
4.1 Multipole Expansion (continued)

Multipole moments with respect to $\mathbf{x} = 0$:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}'$$

Use (5)

$$= \frac{1}{4\pi\varepsilon_0} \left[ \int \frac{\rho(\mathbf{x}')d^3\mathbf{x}'}{r} + \mathbf{x} \cdot \int \frac{\mathbf{x}'\rho(\mathbf{x}')d^3\mathbf{x}'}{r^3} \right] + \frac{1}{2r^5} \sum_{ij} x_i x_j \left[ (3x_i' x_j' - r'^2 \delta_{ij}) \rho(\mathbf{x}')d^3\mathbf{x}' + \cdots \right]$$

$$= \frac{1}{4\pi\varepsilon_0} \left[ \frac{q}{r} + \frac{\mathbf{p} \cdot \mathbf{x}}{r^3} + \frac{1}{2r^5} \sum_{ij} Q_{ij} x_i x_j + \cdots \right]$$

$$\Rightarrow \Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \left[ \frac{q}{r} + \frac{\mathbf{p} \cdot \mathbf{x}}{r^3} + \frac{1}{2r^5} \mathbf{Q} \cdot \mathbf{xx} + \cdots \right] \quad (4.10)$$

Note: Multipole moments are defined with respect to a point of reference. In (4.10), it is the origin of coordinates ($\mathbf{x} = 0$).
4.1 Multipole Expansion (continued)

Multipole moments with respect to \( \mathbf{x} = \mathbf{x}_0 \):

In general, values of the multiple moments depend upon the choice of the point of reference, although the sum of all multipole fields has the same value. Consider the general case in which the point of reference \( \mathbf{x} = \mathbf{x}_0 \) is separate from the origin of coordinates \( \mathbf{x} = 0 \).

\[
\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{|(\mathbf{x} - \mathbf{x}_0) - (\mathbf{x}' - \mathbf{x}_0)|} \\
= \frac{1}{|\mathbf{x} - \mathbf{x}_0|} + \frac{(\mathbf{x}' - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|^3} \\
+ \frac{1}{2} \sum_{ij} \frac{(x_i - x_{0i})(x_j - x_{0j})}{|\mathbf{x} - \mathbf{x}_0|^5} \left[ 3(x'_i - x_{0i})(x'_j - x_{0j}) - |\mathbf{x}' - \mathbf{x}_0|^2 \delta_{ij} \right] + \cdots
\]
4.1 Multipole Expansion (continued)

\[ \Phi(x) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(x')}{|x - x'|} d^3x' \]

\[ = \frac{1}{4\pi\varepsilon_0} \left[ \frac{q}{|x - x_0|} \right] + \frac{p \cdot (x - x_0)}{|x - x_0|^3} + \frac{1}{2} \sum_{ij} \frac{Q_{ij} (x_i - x_{0i}) (x_j - x_{0j})}{|x - x_0|^5} + \ldots \]

\[ = \frac{1}{4\pi\varepsilon_0} \left[ \frac{q}{|x - x_0|} \right] + \frac{p \cdot (x - x_0)}{|x - x_0|^3} + \frac{1}{2} \sum_{ij} \frac{Q_{ij} (x_i - x_{0i}) (x_j - x_{0j})}{|x - x_0|^5} + \ldots \]  \hspace{1cm} (6)

\text{due to monopole}

\text{due to dipole}

\text{due to quadrupole}

\text{p and } Q_{ij} \text{ above are defined with respect to the point of reference at } x_0. \text{ We may regard } x_0 \text{ as the position of these multipoles.}
4.1 Multipole Expansion (continued)

**Dipole field:**

\[ \mathbf{E}_{\text{dipole}} = -\nabla \phi_{\text{dipole}} \]

\[ = -\nabla \frac{\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}_0)}{4\pi \varepsilon_0 |\mathbf{x} - \mathbf{x}_0|^3} \]

\[ = -\frac{\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}_0)}{4\pi \varepsilon_0} \left( \frac{1}{|\mathbf{x} - \mathbf{x}_0|^3} - \frac{1}{4\pi \varepsilon_0 |\mathbf{x} - \mathbf{x}_0|^3} \nabla [\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}_0)] \right) \]

\[ = \frac{3\mathbf{n}(\mathbf{p} \cdot \mathbf{n}) - \mathbf{p}}{4\pi \varepsilon_0 |\mathbf{x} - \mathbf{x}_0|^3} \]

where \( \mathbf{n} \equiv \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|} \)

\[ = (\mathbf{p} \cdot \nabla)(\mathbf{x} - \mathbf{x}_0) + [0]p \]

\[ + p \times [\nabla \times (\mathbf{x} - \mathbf{x}_0)] + (\mathbf{x} - \mathbf{x}_0) \times (\nabla \times \mathbf{p}) \]

\[ = (\mathbf{p} \cdot \nabla)(\mathbf{x} - \mathbf{x}_0) = \mathbf{p} \]
4.1 Multipole Expansion (continued)

Relation between spherical and Cartesian mutipole moments:

\[
Y_{0,0}(\theta, \varphi) = \sqrt{\frac{1}{4\pi}} \\
Y_{1,1}(\theta, \varphi) = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\varphi} \right \} \text{ p.109}
\]

\[
q_{lm} = \int Y_{lm}^*(\theta', \varphi')r'^l \rho(x')d^3x' \\
q_{00} = \int Y_{00}^*(\theta', \varphi')\rho(x')d^3x' \\
= \frac{1}{\sqrt{4\pi}} \int \rho(x')d^3x' = \frac{q}{\sqrt{4\pi}} \\
q_{11} = \int Y_{11}^*(\theta', \varphi')r' \rho(x')d^3x' \\
= -\sqrt{\frac{3}{8\pi}} \int r' \sin \theta'e^{-i\varphi'} \rho(x')d^3x' \\
= -\sqrt{\frac{3}{8\pi}}(p_x - ip_y)
\]
Problem: Prove that the lowest non-vanishing multipole moment is independent of the point of reference (see pp. 147-8).

Solution: Each component of the $\ell$-th multipole moment with respect to reference points $\mathbf{a}$ and $\mathbf{b}$ consists, respectively, of integrals of the form $I_{ijk}^{(a)} = \int \rho(x)(x-a_x)^i(y-a_y)^j(z-a_z)^k \, d^3 x$ and 

$$I_{ijk}^{(b)} = \int \rho(x)(x-b_x)^i(y-b_y)^j(z-b_z)^k \, d^3 x$$

$$= \int \rho(x)(x-a_x-c_x)^i(y-a_y-c_y)^j(z-a_z-c_z)^k \, d^3 x,$$

where $i, j, k$ are zero or positive integers $(i + j + k = \ell)$, $\mathbf{a} = (a_x, a_y, a_z)$, $\mathbf{b} = (b_x, b_y, b_z)$, and $\mathbf{b} = \mathbf{a} + \mathbf{c}$ with $\mathbf{c}$ given by $\mathbf{c} = (c_x, c_y, c_z)$.

For example, the monopole moment has only one term ($i = j = k = 0$), each component of the dipole moment consists of one term ($i$ or $j$ or $k = 1$), and each component of the quadrupole moment consists of multiple terms (all having $i + j + k = 2$).
optional

4.1 Multipole Expansion (continued)

The monopole moment \( q(= \int \rho(x')d^3x') \) is clearly independent of the reference point. If \( q = 0 \) and the lowest nonvanishing multipole moment with respect to reference point \( a \) is the \( \ell \)-th moment, i.e.

\[
I_{ijk}^{(a)} = \begin{cases} \\
= 0 \ , \ i + j + k < \ell \\
\neq 0 \ , \ i + j + k = \ell \end{cases}
\]

then, with respect to reference point \( b \), we have

\[
I_{ijk}^{(b)} = \int \rho(x) \left( x-a_x-c_x \right)^i \left( y-a_y-c_y \right)^j \left( z-a_z-c_z \right)^k d^3x
\]

\[
= \int_{i+j+k=\ell} \rho(x) (x-a_x)^i (y-a_y)^j (z-a_z)^k d^3x
\]

\[
+ \sum_{\alpha\beta\gamma} C_{\alpha\beta\gamma} \int \rho(x) (x-a_x)^\alpha (y-a_y)^\beta (z-a_z)^\gamma d^3x
\]

\[
= I_{ijk}^{(a)} = 0 \text{ multiplications of } c_x, c_y, \text{ & } c_z \ 	ext{ of } \alpha+\beta+\gamma < \ell \quad \text{Q.E.D.}
\]
4.2 Multipole Expansion of the Energy of a Charge Distribution in an External Field

In (1.53), we have \( W = \frac{1}{2} \int \rho(x) \Phi(x) d^3x \) [self energy]

potential due to \( \rho(x) \) in the integrand, \( \nabla^2 \Phi(x) = -\rho(x)/\varepsilon_0 \)

Here, we consider the relative energy between \( \rho(x) \) and external charges:

\[
W = \int \rho(x) \Phi(x) d^3x
\]

potential due to external charges, \( \nabla^2 \Phi(x) = 0 \) in region of \( \rho(x) \)

Expand the external field \( \Phi(x) \) [Use (A.3) in appendix A]:

\[
\Phi(x) = \Phi(0) + x \cdot \nabla \Phi(0) + \frac{1}{2} \sum_{ij} x_i x_j \frac{\partial^2 \Phi(0)}{\partial x_i \partial x_j} + \ldots
\]

\[
= \Phi(0) - x \cdot E(0) - \frac{1}{2} \sum_{ij} x_i x_j \frac{\partial E_j(0)}{\partial x_i} + \ldots
\]

\[
= \Phi(0) - x \cdot E(0) - \frac{1}{6} \sum_{ij} \left( 3x_i x_j - r^2 \delta_{ij} \right) \frac{\partial E_j(0)}{\partial x_i} + \ldots
\]

\[
= \Phi(0) - x \cdot E(0) - \frac{1}{6} \sum_{ij} \left( 3x_i x_j - r^2 \delta_{ij} \right) \frac{\partial E_j(0)}{\partial x_i} + \ldots
\]

\[
= \Phi(0) - x \cdot E(0) - \frac{1}{6} \sum_{ij} \left( 3x_i x_j - r^2 \delta_{ij} \right) \frac{\partial E_j(0)}{\partial x_i} + \ldots
\]
4.2 Multipole Expansion of the Energy of a Charge Distribution in an External Field (continued)

\[ \Phi(x) = \Phi(0) - x \cdot E(0) - \frac{1}{6} \sum_{ij} \left( 3x_i x_j - r^2 \delta_{ij} \right) \frac{\partial E_j(0)}{\partial x_i} + \ldots \]

Thus,

\[ W = \int \rho(x) \Phi(x) d^3 x \]

\[ = q\Phi(0) - p \cdot E(0) - \frac{1}{6} \sum_{ij} Q_{ij} \frac{\partial E_j(0)}{\partial x_i} + \ldots \]  

(4.24)

\[ \begin{cases} 
q [+] \text{ interacts with } \Phi \\
p [+] [+] \text{ interacts with } E \text{ (non-uniform } \Phi) \\
Q_{ij} [+] [-] \text{ interacts with non-uniform } E 
\end{cases} \]

Note: The multipole moments here are not induced by \( E \). See Sec. 4.6 for induced moments.

Questions:
1. Higher order moments can “see” finer structure of \( \Phi(x) \). Why?
2. How does a charged rod attract a piece of paper?
3. How does a microwave oven heat food?
4.6 Models for the Molecular Polarizability

Induced Dipole Moment:

In the presence of an external electric field, the electrons and ions in a molecule (or atom) will be very slightly displaced, in opposite directions, from their equilibrium positions. The molecule is thus polarized. The resulting induced dipole moment is calculated below.

The molecular electrons and ions are bound charges. When a charge is displaced from its equilibrium position \( x = 0 \), it will be subject to a restoring force \( F(x) \), which we expand as

\[
F(x) = F(0) + F'(0) x + \frac{1}{2} F''(0) x^2 + \cdots
\]

because \( x = 0 \) is the equilibrium position.

\( \omega_0 \) is the natural frequency of the charge if it oscillates as a simple harmonic oscillator.

\( \omega_0 \) is negligible if \( x \to 0 \)
4.6 Models for the Molecular Polarizability (continued)

For small displacements \((x \to 0)\) and assuming \(F\) and \(x\) are along the same line (property of an isotropic medium) but in opposite directions, we have \(F(x) \approx -m\omega_0^2 x\) \((4.71)\)

Under the action of a static \(E\), a charge will be displaced to a position \(x\), at which the restoring force equals the electric force,

\[
m\omega_0^2 x = eE\quad \text{Note: } e \text{ carries a sign.}
\]

This induces a dipole moment given by

\[
p = ex = \frac{e^2}{m\omega_0^2}E = \varepsilon_0\gamma E,
\]

where \(\gamma \equiv e^2/(\varepsilon_0 m\omega_0^2)\) is the polarizability of a single charge. For all the charges in the molecule, we have

\[
p_{mol} = \sum_j e_j x_j = \sum_j \frac{e_j^2}{m_j\omega_j^2}E = \varepsilon_0\gamma_{mol}E, \quad \text{[induced molecular dipole moment]}
\]

where \(\gamma_{mol} \equiv \frac{1}{\varepsilon_0} \sum_j \frac{e_j^2}{m_j\omega_j^2}\) (molecular polarizability). \((4.73)\)
4.6 Models for the Molecular Polarizability (continued)

Discussion:

(1) The dipole moment for a single charge as calculated above \( p = ex \) is with respect to the equilibrium position of the charge. Since different charges have different equilibrium positions, the dipole moments \( e_jx_j \) of individual charges in the expression \( p_{mol} = \sum_j e_jx_j \) are with respect to different reference points.

This will not cause any inconsistency for an equal amount of \(+/-\) charges in the sum, in which case the monopole moment vanishes and hence \( p_{mol} \) is independent of the reference point (proved in Sec. 4.1). For this reason, we will assume \( p_{mol} \) to be contributed by an equal amount of \(+/-\) charges in the molecule. If there is a net charge in the molecule, the net charge will be treated separately [see (4.29)].

(ii) The approximation made in (4.71) \( \mathbf{F}(\mathbf{x}) \approx -m\omega_0^2\mathbf{x} \) has led to a linear relation between \( p_{mol} \) and \( \mathbf{E} : p_{mol} = \varepsilon_0\gamma_{mol}\mathbf{E} \).
4.6 Models for the Molecular Polarizability (continued)

Electric Polarization, Polarization Charge, and Free Charge:

The electric polarization is defined as the total dipole moment per unit volume and is given by

\[ P(x) = \sum_i N_i \langle p_i \rangle \]

(4.28)

We now divide the charge density in a medium into two categories: polarization charge density \( \rho_{pol} \) and free charge density \( \rho_{free} \).

\( \rho_{pol} \) results from the polarization of (equal amount of) \( +/- \) charges in each molecule. \( \rho_{free} \) consists of the net molecular charge (usually 0) and the excess charge (such as free electrons) in the medium:

\[ \rho_{free}(x) = \sum_i N_i \langle e_i \rangle + \rho_{excess} \]

(4.29)

Note: We have used the notation \( \rho_{free} \) to distinguish it from \( \rho_{pol} \). \( \rho_{free} \) here is denoted by \( \rho \) in Jackson [e.g. in (4.29), (4.35), etc.]
4.3 Elementary Treatment of Electrostatics with Ponderable Media

Macroscopic Poisson Equation: Consider a general medium and divide its charge into \( \rho_{free} \) and \( \rho_{pol} \). By linear superposition, we may write \( \Phi = \Phi_{free} + \Phi_{pol} \), where \( \Phi_{free} \) and \( \Phi_{pol} \) are due to \( \rho_{free} \) and \( \rho_{pol} \), respectively. Obviously, \( \Phi_{free}(x) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \frac{\rho_{free}(x')}{|x-x'|} \)

For \( \Phi_{pol} \), we have the expression for \( \mathbf{p} \), but not yet for \( \rho_{pol} \). So we approximate \( \Phi_{pol} \) by the dipole term in (6) (with \( x_0 \) replaced by \( x' \)).

\[
\Phi_{pol}(x) = \frac{1}{4\pi\varepsilon_0} \left[ \frac{q}{|x-x'|} + \frac{\mathbf{p} \cdot (x-x')}{|x-x'|^3} + \cdots \right], \\
q = \int_\nu \rho_{pol} d^3x = 0 \quad (\rho_{pol} \text{ contains equal amount of } +/\ - \text{ charges}); \\
\text{hence, } \mathbf{p} \text{ (in the volume } \nu \text{) is independent of the point of reference.}
\]

To represent \( \Phi_{pol} \) by the dipole term in (6), we must have \(|x| \gg \text{ the dimension of } \mathbf{p} \). So we divide \( \rho_{pol} \) into infinitesimal volumes.
4.3 Elementary Treatment of Electrostatics with Ponderable Media (continued)

Let $\Delta \Phi_{pol}(x)$ be the potential due to $\rho_{pol}$ in an infinitesimal volume $\Delta v$ at $x'$. Then, in this volume, we have $p = P(x')\Delta v$ and (6) gives

$$\Delta \Phi_{pol}(x) = \frac{1}{4\pi \varepsilon_0} \frac{P(x')(x-x')\Delta v}{|x-x'|^3}$$

Volume of integration includes all the charge.

$$\Rightarrow \Phi_{pol}(x) = \frac{1}{4\pi \varepsilon_0} \int d^3x' \frac{P(x') \cdot \nabla'(\frac{1}{x-x'})}{|x-x'|}$$

$$= \frac{1}{4\pi \varepsilon_0} \left[ -\int d^3x' \frac{\nabla'.P(x')}{|x-x'|} + \oint_S \frac{P(x')}{|x-x'|} da' \right] = -\frac{1}{4\pi \varepsilon_0} \int d^3x' \frac{\nabla'.P(x')}{|x-x'|}$$

$$= 0 \ (P = 0 \text{ on } S)$$

Thus, $\Phi = \Phi_{free} + \Phi_{pol} = \frac{1}{4\pi \varepsilon_0} \int d^3x' \frac{\rho_{free}(x') - \nabla'.P(x')}{|x-x'|}$

**Question:** If $x - x' \to 0$, higher multipole terms are important. Can we still write $\Delta \Phi_{pol}(x)$ and $\Phi_{pol}(x)$ as above?
4.3 Elementary Treatment of Electrostatics with Ponderable Media (continued)

Rewrite: $\Phi = \Phi_{\text{free}} + \Phi_{\text{pol}} = \frac{1}{4\pi \varepsilon_0} \int d^3 x' \frac{\rho_{\text{free}}(x') - \nabla' \cdot P(x')}{|x-x'|}$

$$\Rightarrow \nabla^2 \Phi(x) = \frac{1}{4\pi \varepsilon_0} \int d^3 x' \left[ \rho_{\text{free}}(x') - \nabla' \cdot P(x') \right] \nabla^2 \frac{1}{|x-x'|} - 4\pi \delta(x-x')$$

$$\Rightarrow \nabla \cdot E(x) = \frac{1}{\varepsilon_0} \left[ \rho_{\text{free}}(x) - \nabla \cdot P(x) \right] \quad (4.33)$$

In electrostatics, only the electric charge can produce $E$. The equal footing of $\rho_{\text{free}}$ and $-\nabla \cdot P$ in (4.33) suggests that $-\nabla \cdot P$ (due to the electric polarization $P$) must be the polarization charge density $\rho_{\text{pol}}$ (see p. 153 and p. 156). Thus, (4.33) can be written

$$\nabla \cdot E = \frac{1}{\varepsilon_0} (\rho_{\text{free}} + \rho_{\text{pol}})$$

where

$$\rho_{\text{pol}} = -\nabla \cdot P \quad (7)$$

(7) is obtained here by inference. A direct derivation can be found in Appendix B [see Eq. (B.2)].
4.3 Elementary Treatment of Electrostatics with Ponderable Media (continued)

We may also put
\[ \nabla \cdot \mathbf{E}(x) = \frac{1}{\varepsilon_0} [\rho_{\text{free}}(x) - \nabla \cdot \mathbf{P}(x)] \]

[(4.33)] in the form:
\[ \nabla \cdot \mathbf{D} = \rho_{\text{free}} \] [macroscopic Poisson equation] \hspace{1cm} (4.35)
by defining an electric displacement: \( \mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P} \) \hspace{1cm} (4.34)

In Sec. 4.6, by assuming an isotropic medium and approximating the restoring force by \( \mathbf{F}(x) \approx -m\omega_0^2 x \) [(4.71)], we have obtained the linear relation \( \mathbf{p}_{\text{mol}} = \varepsilon_0 \chi_{\text{mol}} \mathbf{E} \) for a single molecule. Then, \( \mathbf{P} \) (the sum of \( \mathbf{p}_{\text{mol}} \) per unit volume) must also be a linear function of \( \mathbf{E} \):

\[ \mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}, \]

where the proportionality constant \( \chi_e \) is the electric susceptibility (see Jackson Sec. 4.5 for further discussion on \( \chi_e \)).

Sub. (4.36) into \( \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \), we obtain

\[ \mathbf{D} = \varepsilon \mathbf{E} \hspace{1cm} \varepsilon: \text{electric permittivity} \]
and

\[ \mathbf{P} = (\varepsilon - \varepsilon_0) \mathbf{E}, \hspace{1cm} \varepsilon/\varepsilon_0: \text{dielectric constant or relative electric permittivity} \]

where \( \varepsilon = \varepsilon_0(1+\chi_e) \)

\[ \hspace{1cm} (4.37) \]
\[ \hspace{1cm} (4.38) \]

**Question:** Is \( \mathbf{D} \) a physical quantity? If so, what is its physical meaning?
4.3 Elementary Treatment of Electrostatics with Ponderable Media (continued)

*Special case:* For a uniform medium, $\varepsilon$ is independent of $x$. Hence, (4.35) gives $\nabla \cdot \mathbf{D} = \nabla \cdot \varepsilon \mathbf{E} = \varepsilon \nabla \cdot \mathbf{E} = \rho_{\text{free}}$

$$\Rightarrow \quad \nabla \cdot \mathbf{E} = \frac{\rho_{\text{free}}}{\varepsilon} \quad \text{[for uniform media]} \quad (4.39)$$

**Conversion of $\varepsilon$ to the Gaussian System:**

$\varepsilon$ in the SI system is called the electric permittivity ($\varepsilon_0$ is its value in vacuum). It has no counterpart in the Gaussian system. However, $\varepsilon/\varepsilon_0$ in the SI system (called dielectric constant or relative permittivity, see p.154) has a counterpart denoted by $\varepsilon$ in the Gaussian system, According to the table on p.782, we have the following conversion formula:

$$\begin{bmatrix} \varepsilon_{\text{SI}} \\ \varepsilon \end{bmatrix} \leftrightarrow \begin{bmatrix} \varepsilon/\varepsilon_0 \\ \varepsilon \end{bmatrix}$$

Although $\varepsilon$ in the Gaussian system has the same notation as the electric permittivity of the SI system, it is really the dielectric constant, which correspond to $\varepsilon/\varepsilon_0$ of the SI system. Thus, $\varepsilon$ in these two systems are not quite the same physical quantity.
4.4 Boundary-Value Problems with Dielectrics

**Boundary conditions:** [\( \sigma_{\text{free}} \) here is denoted by \( \sigma \) in (4.40)]

(i) \( \nabla \cdot \mathbf{D} = \rho_{\text{free}} \)

\[ \nabla \cdot \mathbf{D} = \rho_{\text{free}} \quad \Rightarrow \quad D_{\perp 2} - D_{\perp 1} = \sigma_{\text{free}} \]

(ii) \( \nabla \times \mathbf{E} = 0 \)

\[ \nabla \times \mathbf{E} = 0 \quad \Rightarrow \quad E_{t2} = E_{t1} \]

**Example 1:** Two semi-infinite dielectrics have an interface plane at \( z = 0 \) (lower figure). A point charge \( q \) is at \( z = d \). Find \( \Phi \) everywhere.

To find \( \Phi \) in the region \( z \geq 0 \), we put an image charge \( q' \) at \( z = -d \).

To find \( \Phi \) in the region \( z \leq 0 \), we put an image charge \( q'' \) at \( z = d \).
4.4 Boundary-Value Problems with Dielectrics (continued)

\[
\Phi_2 = \frac{1}{4\pi\varepsilon_2} \frac{q''}{R_1}, \quad \Phi_1 = \frac{1}{4\pi\varepsilon_1} \left( \frac{q}{R_1} + \frac{q'}{R_2} \right)
\]

\[R_2 = \sqrt{\rho^2 + (d+z)^2}\]

\[R_1 = \sqrt{\rho^2 + (d-z)^2}\]

Now apply boundary conditions at \(z = 0\).

b.c. 1: \(\varepsilon_1 E_{\perp 1} - \varepsilon_2 E_{\perp 2} = \sigma_{\text{free}} = 0 \Rightarrow \varepsilon_1 \frac{\partial \Phi_1}{\partial z} \bigg|_{z=0} = \varepsilon_2 \frac{\partial \Phi_2}{\partial z} \bigg|_{z=0}
\]

\[
\Rightarrow \left[ q \frac{\partial}{\partial z} \frac{1}{R_1} + q' \frac{\partial}{\partial z} \frac{1}{R_2} \right]_{z=0} = q'' \frac{\partial}{\partial z} \frac{1}{R_1} \bigg|_{z=0} \Rightarrow q - q' = q''
\]

b.c. 2: \(E_{t1} = E_{t2} \Rightarrow \frac{\partial \Phi_1}{\partial \rho} \bigg|_{z=0} = \frac{\partial \Phi_2}{\partial \rho} \bigg|_{z=0}
\]

\[
\Rightarrow \frac{1}{\varepsilon_1} \left[ q \frac{\partial}{\partial \rho} \frac{1}{R_1} + q' \frac{\partial}{\partial \rho} \frac{1}{R_2} \right]_{z=0} = \frac{1}{\varepsilon_2} q'' \frac{\partial}{\partial \rho} \frac{1}{R_1} \bigg|_{z=0} \Rightarrow \frac{1}{\varepsilon_1} (q + q') = \frac{1}{\varepsilon_2} q''
\]
4.4 Boundary-Value Problems with Dielectrics (continued)

\[ \Phi_2 = \frac{1}{4\pi\varepsilon_2} \frac{q''}{R_1} \]

\[ R_2 = \sqrt{\rho^2 + (d+z)^2} \]

\[ R_1 = \sqrt{\rho^2 + (d-z)^2} \]

\[ \Phi_1 = \frac{1}{4\pi\varepsilon_1} \left( \frac{q}{R_1} + \frac{q'}{R_2} \right) \]

\[ q' \rightarrow \text{actual charge} \]

\[ q'' \rightarrow \text{image charge} \]

(for \( z > 0 \))

\[ (\text{for } z < 0) \]

\[ \begin{align*}
q - q' &= q'' \\
\frac{1}{\varepsilon_1} (q + q') &= \frac{1}{\varepsilon_2} q''
\end{align*} \]

\[ \Rightarrow q' = -\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} q \quad \& \quad q'' = \frac{2\varepsilon_2}{\varepsilon_2 + \varepsilon_1} q \quad (4.45) \]

\[ \nabla \cdot \mathbf{P} = -\rho_{\text{pol}} \]

\[ \Rightarrow \sigma_{\text{pol}} = -(\mathbf{P}_2 - \mathbf{P}_1) \cdot \mathbf{n} = -\frac{q}{2\pi} \frac{\varepsilon_0 (\varepsilon_2 - \varepsilon_1)}{\varepsilon_1 (\varepsilon_2 + \varepsilon_1)} \frac{d}{(\rho^2 + d^2)^{3/2}} \quad (4.47) \]
Example 2: A dielectric sphere is placed in a uniform electric field. Find $\Phi$ everywhere.

We choose the spherical coordinates and divide the space into two regions: $r < a$ and $r > a$. In both regions, we have $\nabla^2 \Phi = 0$ with the solution:

$$
\Phi = \begin{cases}
  r^l \\
  r^{-l-1}
\end{cases} \left\{ \begin{array}{c}
P_l^m (\cos \theta) \\
Q_l^m (\cos \theta)
\end{array} \right\} \left\{ \begin{array}{c}
e^{im\phi} \\
e^{-im\phi}
\end{array} \right\} \quad [\text{Sec. 3.1 of lecture notes}]
$$

\begin{align*}
\begin{cases}
\Phi \text{ is independent of } \phi. \\
\Phi \text{ is finite at } \cos \theta = \pm 1.
\end{cases} & \Rightarrow \\
\Phi_{\text{in}} & = \sum_{l=0}^{\infty} A_l r^l P_l (\cos \theta) \\
\Phi_{\text{out}} & = \sum_{l=0}^{\infty} \left[ B_l r^l + C_l r^{-l-1} \right] P_l (\cos \theta)
\end{align*}

**Question:** If $l > 0$, $\Phi_{\text{out}} \to \infty$ as $r \to \infty$. Why then keep the $l > 0$ terms in $\Phi_{\text{out}}$?
Rewrite: \( \Phi_{\text{in}} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta), \quad \Phi_{\text{out}} = \sum_{l=0}^{\infty} \left[ B_l r^l + C_l r^{-l-1} \right] P_l(\cos \theta) \)

b.c. (i): \( \Phi_{\text{out}}(\infty) = -E_0 z + \text{const.} = -E_0 r \cos \theta + \text{const.} \)
\[ \Rightarrow B_0 = \text{const.}; \quad B_1 = -E_0; \quad B_l(l > 1) = 0 \]

b.c. (ii): \( \Phi_{\text{in}}(a) = \Phi_{\text{out}}(a) \) \[ \Rightarrow E_t^{\text{in}}(a) = E_t^{\text{out}}(a) \]
\[ \Rightarrow A_l a^l = B_l a^l + \frac{C_l}{a^{l+1}} \Rightarrow \left\{ \begin{array}{l} A_0 = B_0 + C_0/a \\ A_1 = -E_0 + C_1/a^3 \\ A_l = C_l/a^{2l+1}, \ l > 1 \end{array} \right. \] (8) (9) (10)

b.c. (iii): \( \varepsilon E_r^{\text{in}}(a) = \varepsilon E_r^{\text{out}}(a) \Rightarrow -\varepsilon \frac{\partial}{\partial r} \Phi_{\text{in}} \bigg|_{r=a} = -\varepsilon_0 \frac{\partial}{\partial r} \Phi_{\text{out}} \bigg|_{r=a} \)
\[ \Rightarrow \varepsilon l A_l a^{l-1} = \varepsilon_0 \left[ l B_l a^{l-1} - (l+1) C_l / a^{l+2} \right] \]
\[ \Rightarrow \left\{ \begin{array}{l} 0 = -\varepsilon_0 C_0 / a^2, \quad l = 0 \\ \varepsilon A_1 = -\varepsilon_0 [E_0 + 2C_1 / a^3], \ l = 1 \\ \varepsilon l A_l = -\varepsilon_0 (l+1) C_l / a^{2l+1}, \ l > 1 \end{array} \right. \] (11) (12) (13)
(7), (11) ⇒ \( A_0 = B_0 = \text{const.} \) (let it be 0.)

(9), (12) ⇒ \( A_1 = -\frac{3E_0}{2+\varepsilon/\varepsilon_0} \); \( C_1 = \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right)a^3E_0 \)

(10), (13) ⇒ \( A_l = C_l = 0 \) for \( l > 1 \)

\[
\begin{aligned}
\Phi_{in} &= -\frac{3}{2 + \varepsilon/\varepsilon_0} E_0 r \cos \theta \\
\Phi_{out} &= -E_0 r \cos \theta + \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \frac{E_0 a^3}{r^2} \cos \theta 
\end{aligned}
\]

(4.54)

This is the only way (3) & (6) can both be satisfied.

[Diagram showing polarization charge and total electric field]
4.7 Electrostatic Energy in Dielectric Media

Let \( \Phi(x) \) be the field due to charge density \( \rho_{\text{free}} \) already present in a dielectric medium. The work done to add \( \delta \rho_{\text{free}} \) is

\[
\delta W = \int \delta \rho_{\text{free}}(x) \Phi(x) d^3 x
\]

\[
\delta \rho_{\text{free}} = \nabla \cdot \delta D
\]

\[
= \int \Phi \nabla \cdot \delta D(x) d^3 x
\]

\[
= \int \nabla \cdot (\Phi \delta D) d^3 x + \int E \cdot \delta D d^3 x
\]

\[
= \oint_s \frac{\Phi \delta D \cdot da}{r} = 0, \quad \text{as} \quad r \to \infty
\]

\[
= \int E \cdot \delta D d^3 x
\]

(4.86)

**Note:**

1. \( \rho_{\text{free}}(x) \) here is denoted by \( \rho(x) \) in Jackson (4.84).
2. In a dielectric medium, the addition of \( \delta \rho_{\text{free}}(x) \) will induce \( \delta \rho_{\text{pol}}(x) \). Hence, \( \Phi(x) \) in the above equation is due to both \( \rho_{\text{free}} \) and \( \rho_{\text{pol}} \). The effect of \( \rho_{\text{pol}} \) is implicit in \( D (=\varepsilon E) \).
4.7 Electrostatic Energy in Dielectric Media (continued)

\[ \delta W = \int \mathbf{E} \cdot \delta \mathbf{D} d^3x \quad [(4.86)] \quad \Rightarrow \quad W = \int d^3x \int_0^D \mathbf{E} \cdot \delta \mathbf{D} \quad (4.87) \]

For linear and isotropic media (\( \mathbf{D} = \varepsilon \mathbf{E}; \varepsilon \) indep. of \( \mathbf{E} \)):  
\[
\mathbf{E} \cdot \delta \mathbf{D} = \mathbf{E} \cdot \delta (\varepsilon \mathbf{E}) = \varepsilon \mathbf{E} \cdot \delta \mathbf{E} = \frac{1}{2} \varepsilon \delta (\mathbf{E} \cdot \mathbf{E}) = \frac{1}{2} \delta (\mathbf{E} \cdot \mathbf{D})
\]

For linear and anisotropic media (\( \mathbf{D} = \tilde{\varepsilon} \cdot \mathbf{E}; \tilde{\varepsilon} \) indep. of \( \mathbf{E} \)):  
\[
\mathbf{E} \cdot \delta \mathbf{D} = \mathbf{E} \cdot \delta (\tilde{\varepsilon} \cdot \mathbf{E}) = \mathbf{E} \cdot \tilde{\varepsilon} \cdot \delta \mathbf{E} = \frac{1}{2} \delta (\mathbf{E} \cdot \tilde{\varepsilon} \cdot \mathbf{E}) = \frac{1}{2} \delta (\mathbf{E} \cdot \mathbf{D})
\]

\[ \Rightarrow W = \frac{1}{2} \int d^3x \int_0^D \delta (\mathbf{E} \cdot \mathbf{D}) = \frac{1}{2} \int \mathbf{E} \cdot \mathbf{D} d^3x \quad [\text{for linear media}] \quad (4.89) \]

\[ \Rightarrow W = \frac{1}{2} \int \rho_{\text{free}}(\mathbf{x}) \Phi(\mathbf{x}) d^3x - \frac{1}{2} \oint_S \frac{\Phi \mathbf{D}}{r^2} \cdot \frac{d\mathbf{a}}{r^2} \quad \rightarrow 0 \text{ as } r \rightarrow \infty \]

\[ \mathbf{E} \cdot \mathbf{D} = -\mathbf{D} \cdot \nabla \Phi \]

\[ = -\nabla \cdot (\Phi \mathbf{D}) + \Phi \nabla \cdot \mathbf{D} \]

\[ = -\nabla \cdot (\Phi \mathbf{D}) + \rho_{\text{free}} \Phi \]

\[ = \frac{1}{2} \int \rho_{\text{free}}(\mathbf{x}) \Phi(\mathbf{x}) d^3x \quad [\text{for linear media}] \]

Here, \( \Phi \) is due to \( \rho_{\text{free}} \) and \( \rho_{\text{pol}} \). In (1.53), \( W = \frac{1}{2} \int \rho(\mathbf{x}) \Phi(\mathbf{x}) d^3x \)

(valid for a vacuum medium), \( \Phi \) is due entirely to \( \rho \) in the integrand.
4.7 Electrostatic Energy in Dielectric Media (continued)

**Problem 1:** Refer to the mechanism of dipole formation discussed in Sec. 4.6. Find the energy required to induce a dipole on an atomic or molecular charge $e$ by an electric field $E$.

**Solution:** Under the restoring force:

$$F = -m\omega_0^2 x,$$

the energy required to displace $e$ by a distance $x$ is

$$w_{dipole} = -\int_0^x F(x')dx' = \frac{1}{2} m\omega_0^2 x^2$$

Using the relations:

\[
\begin{align*}
\text{Force balance: } & \quad m\omega_0^2 x = eE \Rightarrow E = \frac{m\omega_0^2 x}{e} \\
\text{Induced dipole moment: } & \quad \mathbf{p} = e\mathbf{x}
\end{align*}
\]

we obtain

$$w_{dipole} = \frac{1}{2} m\omega_0^2 x^2 = \frac{1}{2} \frac{m\omega_0^2 x}{e} \cdot e\mathbf{x} = \frac{1}{2} \mathbf{p} \cdot \mathbf{E}$$

internal energy of a single dipole
4.7 Electrostatic Energy in Dielectric Media (continued)

Problem 2: From $W = \frac{1}{2} \int \mathbf{E} \cdot \mathbf{D} d^3 x$ [(4.89)], we deduce that, in a dielectric, the energy density due to the presence of $\mathbf{E}$ is $w = \frac{1}{2} \mathbf{E} \cdot \mathbf{D}$. Derive this relation using the result in problem 1.

Solution: Problem 1 gives the internal energy of a single dipole:

$$w_{\text{dipole}} = \frac{1}{2} \mathbf{p} \cdot \mathbf{E} \quad (14)$$

Hence, the internal energy of all dipoles per unit volume is

$$w_{\text{int}} = \frac{1}{2} \sum_i N_i \mathbf{p}_i \cdot \mathbf{E} = \frac{1}{2} \mathbf{P} \cdot \mathbf{E} = \frac{1}{2} (\varepsilon - \varepsilon_0) |\mathbf{E}|^2$$

From Ch. 1, we have the electric field energy per unit volume:

$$w_E = \frac{1}{2} \varepsilon_0 |\mathbf{E}|^2 \quad (1.55)$$

Hence, the total energy per unit volume is

$$w = w_{\text{int}} + w_E = \frac{1}{2} (\varepsilon - \varepsilon_0) |\mathbf{E}|^2 + \frac{1}{2} \varepsilon_0 |\mathbf{E}|^2 = \frac{1}{2} \varepsilon |\mathbf{E}|^2 = \frac{1}{2} \mathbf{E} \cdot \mathbf{D}$$
We now apply (4.89) to find the energy change due to a dielectric object with linear $\varepsilon_1(x)$ in the field $\mathbf{E}_0$ of a fixed external source.

Without the object:

$$ W_0 = \frac{1}{2} \int \mathbf{E}_0 \cdot \mathbf{D}_0 d^3x $$

With the object:

$$ W_1 = \frac{1}{2} \int \mathbf{E} \cdot \mathbf{D} d^3x $$

$$ \Rightarrow \Delta W = W_1 - W_0 = \frac{1}{2} \int (\mathbf{E} \cdot \mathbf{D} - \mathbf{E}_0 \cdot \mathbf{D}_0) d^3x $$

$$ = \frac{1}{2} \int (\mathbf{E} \cdot \mathbf{D}_0 - \mathbf{D} \cdot \mathbf{E}_0) d^3x + \frac{1}{2} \int (\mathbf{E} + \mathbf{E}_0) \cdot (\mathbf{D} - \mathbf{D}_0) d^3x $$

$$ = \frac{1}{2} \int (\mathbf{E} \cdot \mathbf{D}_0 - \mathbf{D} \cdot \mathbf{E}_0) d^3x $$

$$ - \int \nabla(\Phi + \Phi_0) \cdot (\mathbf{D} - \mathbf{D}_0) d^3x = \int (\Phi + \Phi_0) \nabla \cdot (\mathbf{D} - \mathbf{D}_0) d^3x = 0 $$

integration by parts

Reason for $\nabla \cdot \mathbf{D}_0 = \nabla \cdot \mathbf{D} = \rho_{\text{free}}$: A dielectric object contains no $\rho_{\text{free}}$ and the external source is fixed. $\Rightarrow \rho_{\text{free}}$ is unchanged before and after the introduction of the object.
4.7 Electrostatic Energy in Dielectric Media (continued)

\[ \mathbf{D}_0 = \varepsilon_0 \mathbf{E}_0 \]

\[ \Delta W = \frac{1}{2} \int \left( \mathbf{E} \cdot \mathbf{D}_0 - \mathbf{D} \cdot \mathbf{E}_0 \right) d^3x \]

Outside the object: \( \mathbf{D} = \varepsilon_0 \mathbf{E} \)

Inside the object: \( \mathbf{D} = \varepsilon_1 \mathbf{E} \)

\[ \Rightarrow \Delta W \text{ (outside the object)} = 0 \]

\[ \Rightarrow \Delta W = -\frac{1}{2} \int_{v_1} (\varepsilon_1 - \varepsilon_0) \mathbf{E} \cdot \mathbf{E}_0 d^3x \]

\( v_1 \) is the volume of the object.

(4.92)

\[ \Rightarrow \] The dielectric object tends to move toward (away from) the region of increasing \( \mathbf{E}_0 \) if \( \varepsilon_1 > \varepsilon_0 \) (\( \varepsilon_1 < \varepsilon_0 \)).

\[ \mathbf{D} = \varepsilon_1 \mathbf{E} \quad \& \quad \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad \Rightarrow \mathbf{P} = (\varepsilon_1 - \varepsilon_0) \mathbf{E} \]

\[ \Rightarrow \Delta W = -\frac{1}{2} \int_{v_1} \mathbf{P} \cdot \mathbf{E}_0 d^3x \]

induced polarization of the object

(4.93)

\[ \Rightarrow \] The energy density of a dielectric object placed in the field \( \mathbf{E}_0 \) of a fixed external source is

\[ w = -\frac{1}{2} \mathbf{P} \cdot \mathbf{E}_0 \]

(4.94)

**Question:** Explain the factor \( \frac{1}{2} \) which is in (4.94) but not in the 2nd term of:

\[ W = q \Phi(0) - \mathbf{p} \cdot \mathbf{E}(0) - \frac{1}{6} \sum_{ij} Q_{ij} \frac{\partial E_j(0)}{\partial x_i} + \cdots \] (4.24)

[34]
Homework of Chap. 4

Problems:  1,  2,  7,  8,  10

Quiz:  Nov. 9, 2010
Appendix A. Taylor Expansion

Define \( e^{a \cdot \nabla} \equiv \sum_{n=0}^{\infty} \frac{1}{n!} (a \cdot \nabla)^n \) [a translational operator, which translates the argument of the function it operates on to a distance \( a \) away from the argument.]

Taylor expansion of \( f(x + a) \) and \( A(x + a) \) about point \( x \):

\[
\begin{align*}
  f(x + a) &= e^{a \cdot \nabla} f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} (a \cdot \nabla)^n f(x) \\
  &= f(x) + (a \cdot \nabla) f(x) + \frac{1}{2} (a \cdot \nabla)(a \cdot \nabla) f(x) + \cdots \quad (A.1) \\
  A(x + a) &= e^{a \cdot \nabla} A(x) = \sum_{n=0}^{\infty} \frac{1}{n!} (a \cdot \nabla)^n A(x) \\
  &= A(x) + (a \cdot \nabla) A(x) + \frac{1}{2} (a \cdot \nabla)(a \cdot \nabla) A(x) + \cdots \quad (A.2)
\end{align*}
\]

Similarly, operating \( f(x)|_{x=a} \) and \( A(x)|_{x=a} \) with \( e^{(x-a) \cdot \nabla} \), we obtain the Taylor expansion of \( f(x) \) and \( A(x) \) about point \( a \):

\[
\begin{align*}
  f(x) &= f(a) + [(x-a) \cdot \nabla] f(a) + \frac{1}{2} [(x-a) \cdot \nabla][(x-a) \cdot \nabla] f(a) + \cdots \quad (A.3) \\
  A(x) &= A(a) + [(x-a) \cdot \nabla] A(a) + \frac{1}{2} [(x-a) \cdot \nabla][(x-a) \cdot \nabla] A(a) + \cdots (A.4)
\end{align*}
\]
Appendix A. Taylor Expansion (continued)

In (A.1) and (A.2), we have [in Cartesian coordinates]

\[ \mathbf{a} \cdot \nabla = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3} = \sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i} \]  

(A.5)

\[ (\mathbf{a} \cdot \nabla)(\mathbf{a} \cdot \nabla) = \sum_{i} a_i \frac{\partial}{\partial x_i} \sum_{j} a_j \frac{\partial}{\partial x_j} = \sum_{ij} a_i a_j \frac{\partial^2}{\partial x_i \partial x_j} \]  

(A.6)

\[ (\mathbf{a} \cdot \nabla) f(\mathbf{x}) = \sum_{i} a_i \frac{\partial}{\partial x_i} f(\mathbf{x}) = \mathbf{a} \cdot \nabla f(\mathbf{x}) \]  

(A.7)

\[ (\mathbf{a} \cdot \nabla) \mathbf{A}(\mathbf{x}) = \sum_{i} a_i \frac{\partial}{\partial x_i} \left( \sum_{j} A_j \mathbf{e}_j \right) = \sum_{j} \left( \sum_{i} a_i \frac{\partial}{\partial x_i} A_j \right) \mathbf{e}_j \]  

(A.8)

Example: \( (\mathbf{a} \cdot \nabla)(\mathbf{x} - \mathbf{x}') = \sum_{j} \left[ \sum_{i} a_i \frac{\partial}{\partial x_i} (x_j - x'_j) \right] \mathbf{e}_j = \sum_{i} a_i \mathbf{e}_j = \mathbf{a} \)  

For scalar functions with a scalar argument, (A.1) \& (A.3) reduce to

\[ f(x + a) = f(x) + af'(x) + \frac{1}{2} a^2 f''(x) + \cdots \]  

(A.9)

\[ f(x) = f(a) + (x - a)f'(a) + \frac{1}{2} (x - a)^2 f''(a) + \cdots \]  

(A.10)
Appendix B. Polarization Current Density and Polarization Charge Density in Dielectric Media

We divide the bound charges (electrons and ions) in a dielectric into different groups. The $i$-th group has $N_i$ identical charged particles per unit volume. Each particle in the group carries a charge $e_i$ and has a dipole moment given by $p_i = e_i x_i$, where $x_i$ is the particle’s displacement from its equilibrium position under the influence of a static or time-dependent electric field. We assume that all particles in the group have the same $x_i$ at all times and that the variation of $x_i$ is so small that it will not change $N_i$. Then, the electric polarization $P$ as a function of position and time can be written as

$$P(x,t) = \sum_i N_i(x) p_i(t) = \sum_i N_i(x) e_i x_i(t) = \sum_i \rho_i(x) x_i(t)$$

and the polarization current density is the time derivative of $P(x,t)$

$$\frac{\partial}{\partial t} P(x,t) = \sum_i \rho_i(x) \frac{d}{dt} x_i(t) = \sum_i \rho_i(x) v_i(t) = J_{pol}(x,t) \quad (B.1)$$
Appendix B. Polarization Current Density and Polarization Charge Density… (continued)

Let $\rho_{pol}$ be the polarization charge density of the medium, then

$$\frac{\partial}{\partial t} \rho_{pol} + \nabla \cdot \mathbf{J}_{pol} = 0 \quad \text{(conservation of charge)}$$

$$\Rightarrow \frac{\partial}{\partial t} \rho_{pol} + \nabla \cdot \frac{\partial}{\partial t} \mathbf{P} = 0 \Rightarrow \frac{\partial}{\partial t} \left( \rho_{pol} + \nabla \cdot \mathbf{P} \right) = 0$$

$$\Rightarrow \rho_{pol} + \nabla \cdot \mathbf{P} = K$$

If $\mathbf{P} = 0$, we have $\rho_{pol} = 0$. Hence, $K = 0$.

$$\Rightarrow \rho_{pol} = -\nabla \cdot \mathbf{P} \quad \text{(B.2)}$$

$\mathbf{J}_{pol}$ is due to the motion of bound charges, whereas $\rho_{pol}$ is due to the displacement of bound charges. The presence of $\mathbf{J}_{pol}$ does not necessarily imply the presence of $\rho_{pol}$, and vice versa. For example, in a static electric field $\mathbf{E}$, we have $\mathbf{J}_{pol} = 0$ because bound charges are stationary. But the stationary charges will be displaced by $\mathbf{E}$; hence $\rho_{pol} \neq 0$ if $\nabla \cdot \mathbf{P} \neq 0$. In time-dependent cases, there must be a $\mathbf{J}_{pol}$ if $\mathbf{P} \neq 0$ [hence $\frac{\partial}{\partial t} \mathbf{P}(\mathbf{x}, t) \neq 0$] but not necessarily a $\rho_{pol}$ unless $\nabla \cdot \mathbf{P} \neq 0$. 