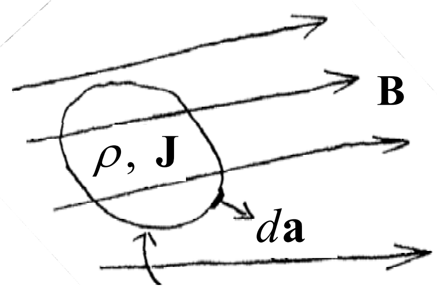
Chapter 5: Magnetostatics, Faraday's Law, Quasi-Static Fields

5.1 Introduction and Definitions

We begin with the law of conservation of charge:

$$\int_V \nabla \cdot \mathbf{J} d^3x = \oint \mathbf{J} \cdot d\mathbf{a} = -\frac{\partial Q}{\partial t} = -\frac{\partial}{\partial t} \int_V \rho d^3x$$



$$\Rightarrow \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad \begin{bmatrix} \text{conservation} \\ \text{of charge} \end{bmatrix} \qquad (5.2)$$

Magnetostatics is applicable under the static condition. Hence, $\frac{\partial \rho}{\partial t} = 0$ and (5.2) gives $\quad \nabla \cdot \mathbf{J} = 0$ [for magnetoststics] (5.3)

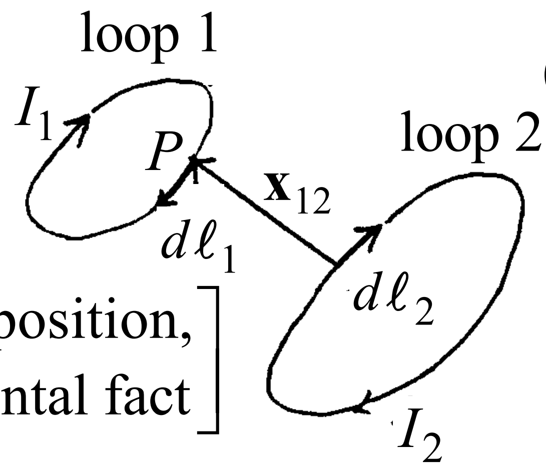
Assuming a magnetic force $\mathbf{F}_B$ is experienced by charge q moving at velocity $\mathbf{v}$, we define the magnetic induction $\mathbf{B}$ by the relation:

$$\mathbf{F}_B = q\mathbf{v} \times \mathbf{B},$$

which is consistent with the definition in (5.1).

5.2 Biot and Savart Law

The Biot-Savart law states that the differential magnetic field $d\mathbf{B}$ at point P (see figure) due to a differential current element $d\ell_2$ in loop 2 is given by $d\mathbf{B} = \frac{\mu_0}{4\pi} I_2 \frac{d\ell_2 \times \mathbf{x}_{12}}{|\mathbf{x}_{12}|^3}$ (5.4)



Thus, the total field at P due to I_2 in loop 2 is: $\mathbf{B} = \frac{\mu_0}{4\pi} I_2 \oint \frac{d\ell_2 \times \mathbf{x}_{12}}{|\mathbf{x}_{12}|^3}$ [linear superposition, an experimental fact] (1)

Integrating the force on I_1 in loop 1 due to I_2 in loop 2, we obtain

$$\mathbf{F}_{12} = I_1 \oint d\ell_1 \times \mathbf{B} \quad (5.7)$$

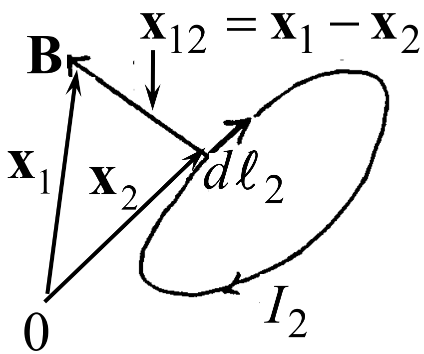
$$= \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \underbrace{\frac{d\ell_1 \times (d\ell_2 \times \mathbf{x}_{12})}{|\mathbf{x}_{12}|^3}}$$

$$= \oint d\ell_2 \underbrace{\oint \frac{d\ell_1 \cdot \mathbf{x}_{12}}{|\mathbf{x}_{12}|^3}} - \oint\oint \frac{(d\ell_1 \cdot d\ell_2)\mathbf{x}_{12}}{|\mathbf{x}_{12}|^3}$$

$$= \oint \frac{d|\mathbf{x}_{12}|}{|\mathbf{x}_{12}|^2} = -\oint d\left(\frac{1}{|\mathbf{x}_{12}|}\right) = 0$$

$$= -\frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{(d\ell_1 \cdot d\ell_2)\mathbf{x}_{12}}{|\mathbf{x}_{12}|^3}$$

5.3 Differential Equations of Magnetostatics and Ampere's Law



Gauss Law of Magnetism :

Rewrite (1): $\mathbf{B} = \frac{\mu_0}{4\pi} I_2 \oint \frac{d\ell_2 \times \mathbf{x}_{12}}{|\mathbf{x}_{12}|^3}$

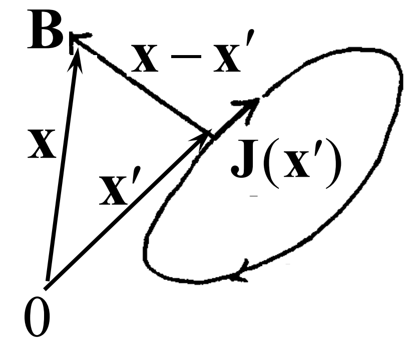
cross section of wire

Change $\mathbf{x}_1$ to $\mathbf{x}$, $\mathbf{x}_2$ to $\mathbf{x}'$, and let $I_2 d\ell_2 = \mathbf{J}\, da\, d\ell_2 = \mathbf{J} d^3x$, we obtain

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{x}') \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} d^3x' = \frac{\mu_0}{4\pi}\int \nabla \frac{1}{|\mathbf{x}-\mathbf{x}'|} \times \mathbf{J}(\mathbf{x}') d^3x'$$

$\nabla \times \psi \mathbf{a} = \nabla \psi \times \mathbf{a} + \psi \nabla \times \mathbf{a}$

$\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\nabla \frac{1}{|\mathbf{x}-\mathbf{x}'|}$



$$= \frac{\mu_0}{4\pi}\int \Big[\nabla \times \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{|\mathbf{x}-\mathbf{x}'|} \underbrace{\nabla \times \mathbf{J}(\mathbf{x}')}_{=0}\Big] d^3x'$$

$$= \frac{\mu_0}{4\pi} \nabla \times \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \qquad (5.16)$$

∇ operates on $\mathbf{x}$

$$\Rightarrow \qquad \nabla \cdot \mathbf{B} = 0 \quad \text{[Gauss law of magnetism]} \qquad (5.17)$$

Ampere's Law : Rewrite (5.16):

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \nabla \times \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$$

$$\Rightarrow \nabla \times \mathbf{B}(\mathbf{x})$$

$$= \frac{\mu_0}{4\pi} \nabla \times \nabla \times \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$$

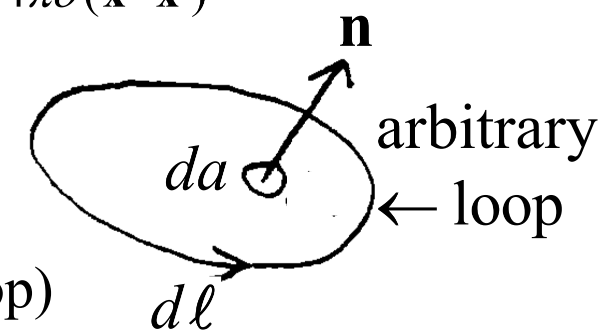
$$= \frac{\mu_0}{4\pi} [\nabla \overbrace{\int \nabla \cdot \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'}^{0} - \int \mathbf{J}(\mathbf{x}') \underbrace{\nabla^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|}}_{-4\pi\delta(\mathbf{x}-\mathbf{x}')} d^3x']$$

$$\int \nabla \cdot \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$$
$$= \int [\mathbf{J}(\mathbf{x}') \cdot \nabla \frac{1}{|\mathbf{x}-\mathbf{x}'|} + \frac{1}{|\mathbf{x}-\mathbf{x}'|} \overbrace{\nabla \cdot \mathbf{J}(\mathbf{x}')}^{0}] d^3x'$$
$$= -\int \mathbf{J}(\mathbf{x}') \cdot \nabla' \frac{1}{|\mathbf{x}-\mathbf{x}'|} d^3x'$$
$$= \int \frac{1}{|\mathbf{x}-\mathbf{x}'|} \underbrace{\nabla' \cdot \mathbf{J}(\mathbf{x}')}_{0} d^3x' - \underbrace{\int \nabla' \cdot \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'}_{0}$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$

$$\Rightarrow \nabla \times \mathbf{B}(\mathbf{x}) = \mu_0 \mathbf{J}(\mathbf{x}) \qquad (5.22)$$

$$\Rightarrow \underbrace{\int \nabla \times \mathbf{B} \cdot \mathbf{n}\, da}_{\oint \mathbf{B} \cdot d\ell} = \mu_0 \underbrace{\int \mathbf{J} \cdot \mathbf{n}\, da}_{I \text{ (through the loop)}}$$



$$\Rightarrow \oint \mathbf{B} \cdot d\ell = \mu_0 I \qquad (5.25)$$

[Ampere's law, a much more elaborate representation of the Biot-Savart law]

5.4 Vector Potential

Vector Potential : Rewrite (5.16): $\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \nabla \times \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$

$$\Rightarrow \mathbf{B} = \nabla \times \mathbf{A}, \tag{5.27}$$

where the vector potential **A** is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' + \nabla \psi, \tag{5.28}$$

which shows that **A** may be freely transformed (without changing **B**)

according to $\quad \mathbf{A} \to \mathbf{A} + \nabla \psi \quad$ (gauge transformation) $\quad$ (5.29)

We may exploit this freedom by choosing a ψ so that

$$\nabla \cdot \mathbf{A} = 0 \qquad \text{(Coulomb gauge)} \tag{5.31}$$

See proof on previous page.

$$\nabla \cdot (5.28) \Rightarrow \nabla \cdot \mathbf{A} = \frac{\mu_0}{4\pi} \overbrace{\int \nabla \cdot \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'}^{0} + \nabla^2 \psi = \nabla^2 \psi,$$

$\Rightarrow$ Coulomb gauge requires $\nabla^2 \psi = 0$ everywhere and hence $\psi = \text{const}.$

Rewrite: $\begin{cases} \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \\ \mathbf{B} = \nabla \times \mathbf{A} \end{cases}$

$$\Rightarrow \nabla \times \nabla \times \mathbf{A} = \mu_0 \mathbf{J}$$

$$\Rightarrow \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

Choose the Coulomb gauge $(\nabla \cdot \mathbf{A} = 0)$

$$\Rightarrow \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \qquad (5.31)$$

$$\Rightarrow \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \qquad (5.32)$$

Note:

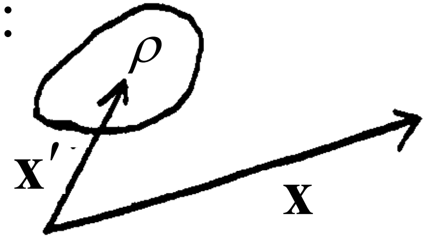
(5.32) is valid in unbounded (infinite) space, i.e. the volume of integration must include all currents. If there is a boundary surface, the currents on the boundary must be accounted for by application of boundary conditions (See example in Sec. 5.12.)

A Comparison of Electrostatics and Magnetostatics:

Electrostatics

Definition of **E**:

$$\mathbf{F}_E = q\mathbf{E}$$



Coulomb's law:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d^3x'$$

$$\Downarrow$$

$$\nabla\cdot\mathbf{E} = \rho/\varepsilon_0 \qquad \nabla\times\mathbf{E} = 0$$

$$\Downarrow \qquad \Downarrow$$

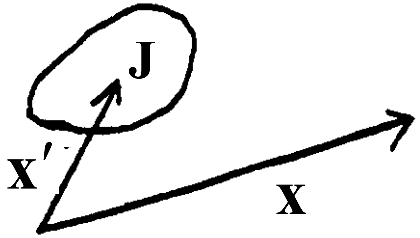
$$\oint \mathbf{E}\cdot d\mathbf{a} = q/\varepsilon_0 \qquad \oint \mathbf{E}\cdot d\ell = 0$$

Gauss's law of electrostatics

Magnetostatics

Definition of **B**:

$$\mathbf{F}_B = q\mathbf{v}\times\mathbf{B}$$



Biot-Savart law:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{x}')\times\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} d^3x'$$

$$\Downarrow$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J}$$

$$\Downarrow \qquad \Downarrow$$

$$\oint \mathbf{B}\cdot d\mathbf{a} = 0 \qquad \oint \mathbf{B}\cdot d\ell = \mu_0 I$$

Gauss's Law of magnetism — Ampere's law

5.6 Magnetic Field of Localized Current Distribution, Magnetic Moment

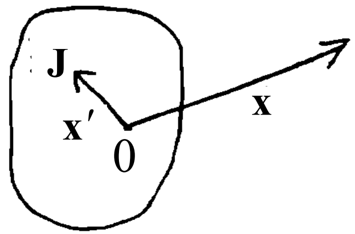
Magnetic (Dipole) Moment :

$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{1}{|\mathbf{x}|}+\frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^3}+\cdots$$ [Eq. (5), Ch. 4]

$$=\frac{\mu_0}{4\pi}\Big[\frac{1}{|\mathbf{x}|}\underbrace{\int\mathbf{J}(\mathbf{x}')d^3x'}_{=0}+\frac{1}{|\mathbf{x}|^3}\underbrace{\mathbf{x}\cdot\int\mathbf{x}'\mathbf{J}(\mathbf{x}')d^3x'}_{=-\frac{1}{2}\int\mathbf{x}\times[\mathbf{x}'\times\mathbf{J}(\mathbf{x}')]d^3x'}+\cdots\Big]$$



Proved on next page.

Proved on p.185 under the conditions:
1. $\mathbf{J}$ is localized within volume of integration
2. $\nabla\cdot\mathbf{J}=0$

$$=-\frac{\mu_0}{8\pi}\frac{\int\mathbf{x}\times[\mathbf{x}'\times\mathbf{J}(\mathbf{x}')]d^3x'}{|\mathbf{x}|^3}+\cdots$$

$$\approx\frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3}\quad\left[\text{If } \mathbf{x} \text{ is far from source.}\right] \qquad (5.55)$$

where $\mathbf{m}\equiv\frac{1}{2}\int\mathbf{x}'\times\mathbf{J}(\mathbf{x}')d^3x'$ [magnetic (dipole) moment] (5.54)

Note: In (5.54), $\mathbf{m}$ is defined with respect to a *point of reference.* Here, it coincides with the origin of the coordinates ($\mathbf{x}=0$).

Problem: Prove the relation $\int \mathbf{J}(\mathbf{x})d^3x = 0$ under the conditions:
$\nabla \cdot \mathbf{J} = 0$ and $\mathbf{J}$ is localized within volume of integration.

Proof: Since $\mathbf{J} = 0$ outside the volume of integration, we may extend the volume of integration to ∞ without changing the integral value.

$$\int \mathbf{J}(\mathbf{x})d^3x = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz (J_x\mathbf{e}_x + J_y\mathbf{e}_y + J_z\mathbf{e}_z)$$

Consider the x-component first:

$$\mathbf{e}_x \cdot \int \mathbf{J}(\mathbf{x})d^3x = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \underbrace{\int_{-\infty}^{\infty} J_x dx}$$

$$= -\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} x \frac{\partial J_x}{\partial x} dx$$

$$= \cancel{xJ_x\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} x \frac{\partial J_x}{\partial x} dx$$

$$= -\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} x\left(\frac{\partial J_x}{\partial x} + \underbrace{\frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}}\right)dx$$

$$= -\int x\nabla \cdot \mathbf{J} d^3x = 0$$

The insertion of these 2 terms will not change the value of the integral because $\int_{-\infty}^{\infty} (\frac{\partial J_y}{\partial y})dy = J_y\Big|_{-\infty}^{\infty} = 0$ & $\int_{-\infty}^{\infty} (\frac{\partial J_z}{\partial z})dz = J_z\Big|_{-\infty}^{\infty} = 0$

Similarly, the y- and z-components also vanish.

Thus, $\int \mathbf{J}(\mathbf{x})d^3x = 0$

Anti-symmetric unit tensor (ε_{ijk}): (used on p.185 and p.188)

$$\varepsilon_{ijk} \equiv \begin{cases} 0 & , \text{ if two or more indices are equal} \\ 1 & , \text{ if } i, j, k \text{ is an even permutation of } 1,2,3 \\ -1 & , \text{ if } i, j, k \text{ is an odd permutation of } 1,2,3 \end{cases} \quad (2)$$

Levi-Civita symbol

Examples: $\varepsilon_{112} = 0,\ \varepsilon_{123} = 1,\ \varepsilon_{132} = -1,\ \varepsilon_{312} = 1$

$$(\mathbf{A} \times \mathbf{B})_i = \sum_{jk} \varepsilon_{ijk} A_j B_k \ , \quad (\nabla \times \mathbf{A})_i = \sum_{jk} \varepsilon_{ijk} \frac{\partial}{\partial x_j} A_k$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \sum_{ijk} \varepsilon_{ijk} \frac{\partial}{\partial x_i} \left(A_j B_k \right)$$

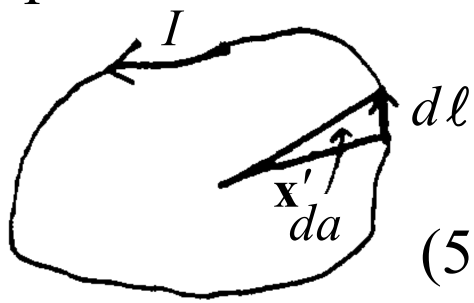
$$= \sum_{ijk} \left[\varepsilon_{ijk} \frac{\partial A_j}{\partial x_i} B_k + \varepsilon_{ijk} A_j \frac{\partial B_k}{\partial x_i} \right]$$

$$= \sum_{ijk} \left[\varepsilon_{kij} B_k \frac{\partial A_j}{\partial x_i} - \varepsilon_{jik} A_j \frac{\partial B_k}{\partial x_i} \right]$$

$$= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

Example 1 of magnetic moment: plane loop

$$\mathbf{m} = \frac{1}{2}\int \mathbf{x}' \times \mathbf{J}(\mathbf{x}') d^3x' = \frac{I}{2}\underbrace{\oint \mathbf{x}' \times d\boldsymbol{\ell}}_{2\cdot(area)}$$



$$\Rightarrow \begin{cases} |\mathbf{m}| = I \cdot (area) \\ \mathbf{m} \text{ is normal (by right hand rule) to the plane of the loop.} \end{cases} \quad (5.57)$$

Example 2 of magnetic moment: a number of charged particles in motion

$$\mathbf{J} = \sum_i q_i \mathbf{v}_i \delta(\mathbf{x} - \mathbf{x}_i)$$

angular momentum $\mathbf{L}_i = M_i \mathbf{x}_i \times \mathbf{v}_i$

$$\Rightarrow \mathbf{m} = \frac{1}{2}\int \mathbf{x}' \times \mathbf{J}(\mathbf{x}') d^3x' = \frac{1}{2}\sum_i q_i \mathbf{x}_i \times \mathbf{v}_i = \sum_i \frac{q_i}{2M_i}\mathbf{L}_i \quad (5.58)$$

$$= \frac{e}{2M}\mathbf{L} \quad (5.59)$$

if $q_i / M_i = e / M$ for all particles.

L: total angular momentum

Dipole Field : (valid far from the source)

Rewrite (5.55) : $\mathbf{A} = \dfrac{\mu_0}{4\pi}\dfrac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3}$ (5.55)

$$\Rightarrow \mathbf{B} = \nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\nabla\times\left(\mathbf{m}\times\frac{\mathbf{x}}{|\mathbf{x}|^3}\right)$$

$$\nabla\cdot\frac{\mathbf{x}}{|\mathbf{x}|^3} = \frac{1}{|\mathbf{x}|^3}\nabla\cdot\mathbf{x} + \mathbf{x}\cdot\nabla\frac{1}{|\mathbf{x}|^3} = \frac{3}{|\mathbf{x}|^3} - \mathbf{x}\cdot\frac{3\mathbf{x}}{|\mathbf{x}|^5} = 0$$

$$= \frac{\mu_0}{4\pi}\Big[\mathbf{m}\underbrace{\nabla\cdot\frac{\mathbf{x}}{|\mathbf{x}|^3}}_{=0} - \frac{\mathbf{x}}{|\mathbf{x}|^3}\underbrace{\nabla\cdot\mathbf{m}}_{=0} + \underbrace{\left(\frac{\mathbf{x}}{|\mathbf{x}|^3}\cdot\nabla\right)\mathbf{m}}_{=0\ (\because\ \mathbf{m}\text{ is a constant.})} - (\mathbf{m}\cdot\nabla)\frac{\mathbf{x}}{|\mathbf{x}|^3}\Big]$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$$

$$= \frac{\mu_0}{4\pi}\left[-m_x\frac{\partial}{\partial x}\frac{\mathbf{x}}{|\mathbf{x}|^3} - m_y\frac{\partial}{\partial y}\frac{\mathbf{x}}{|\mathbf{x}|^3} - m_z\frac{\partial}{\partial z}\frac{\mathbf{x}}{|\mathbf{x}|^3}\right]$$

$$= \frac{\mu_0}{4\pi}\left[-m_x\left(\frac{\mathbf{e}_x}{|\mathbf{x}|^3} + \mathbf{x}\frac{-3x}{|\mathbf{x}|^5}\right) - (y) - (z)\right]$$

$$= \frac{\mu_0}{4\pi}\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{m}) - \mathbf{m}}{|\mathbf{x}|^3} \qquad \mathbf{n} = \frac{\mathbf{x}}{|\mathbf{x}|} \qquad \text{[magnetic dipole field]} \qquad (5.56)$$

As in the case of the electric dipole moment, here we characterize a localized current distribution by a constant quantity, the magnetic moment **m**, which turns an otherwise complicated field calculation (see, for example, Sec. 5.5) into a simple one (with limited validity.)

Consider, for example, a circular loop carrying current I. Using (5.57), we have $\mathbf{m} = I\pi a^2 \mathbf{e}_z$ (see figure). Hence, the dipole field is

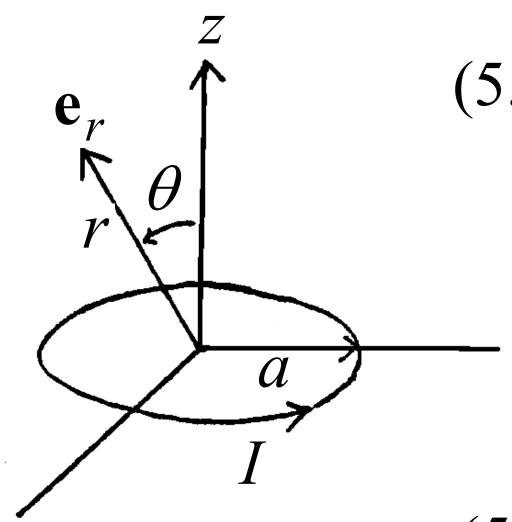
$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{m})-\mathbf{m}}{|\mathbf{x}|^3} \quad (5.56)$$

$\mathbf{n} = \mathbf{e}_r$
$\mathbf{m} = I\pi a^2 \mathbf{e}_z$

$$= \frac{\mu_0}{4\pi} I\pi a^2 \frac{3\mathbf{e}_r(\mathbf{e}_r\cdot\mathbf{e}_z)-\mathbf{e}_z}{r^3}$$

$$(\mathbf{e}_z = \mathbf{e}_r\cos\theta - \mathbf{e}_\theta \sin\theta)$$

$$= \frac{\mu_0}{4\pi} I\pi a^2 \frac{2\cos\theta\mathbf{e}_r + \sin\theta\mathbf{e}_\theta}{r^3} \quad (5.41)$$



When $r \gg a$, the dipole field is a good approximation of the total field [see Jackson (5.40).]

5.7 Forces and Torque on and Energy of a Localized Current Distribution in an External Magnetic Induction

Magnetic Force in External Field :

$$\mathbf{F} = \int \mathbf{J}(\mathbf{x}') \times \mathbf{B}(\mathbf{x}') d^3x' \qquad (5.12)$$

Expanding $\mathbf{B}$: [see lecture notes, Ch. 4, Appendix A, Eq. (A.4)]

$$\mathbf{B}(\mathbf{x}) = \mathbf{B}(0) + \underbrace{(\mathbf{x}\cdot\nabla)\mathbf{B}(0)} + \cdots$$

This implies "After differention of $\mathbf{B}(\mathbf{x})$, set $\mathbf{x}$ in results to 0," i.e.

$$(\mathbf{x}\cdot\nabla)\mathbf{B}(0) = x\left[\frac{\partial}{\partial x}\mathbf{B}(\mathbf{x})\right]_{\mathbf{x}=0} + y\left[\frac{\partial}{\partial y}\mathbf{B}(\mathbf{x})\right]_{\mathbf{x}=0} + z\left[\frac{\partial}{\partial z}\mathbf{B}(\mathbf{x})\right]_{\mathbf{x}=0}$$

$$\Rightarrow \mathbf{F} = [\underbrace{\int \mathbf{J}(\mathbf{x}') d^3x'}_{=0 \text{ (proved in Sec. 5.6)}}] \times \mathbf{B}(0) + \int \mathbf{J}(\mathbf{x}') \times \left[(\mathbf{x}'\cdot\nabla')\mathbf{B}(0)\right] d^3x' + \cdots$$

$$= \int \mathbf{J}(\mathbf{x}') \times \left[(\mathbf{x}'\cdot\nabla')\mathbf{B}(0)\right] d^3x' + \cdots = \nabla(\mathbf{m}\cdot\mathbf{B})$$

$$= -\nabla U, \qquad (5.69)$$

See derivation on pp.188-189

where $U = -\mathbf{m}\cdot\mathbf{B}$ = potential energy. (5.72)

Magnetic Torque in External Field:

$$\mathbf{N} = \int \mathbf{x}' \times \mathbf{f}(\mathbf{x}') d^3x'$$

$$= \int \mathbf{x}' \times [\mathbf{J}(\mathbf{x}') \times \underbrace{\mathbf{B}(\mathbf{x}')}]d^3x'$$

$\mathbf{B}(0) + (\mathbf{x}' \cdot \nabla')\mathbf{B}(0) + \cdots \approx \mathbf{B}(0)$

$$\approx \int \mathbf{x}' \times [\mathbf{J}(\mathbf{x}') \times \mathbf{B}(0)]d^3x'$$

$$= \int [\mathbf{B}(0) \cdot \mathbf{x}']\mathbf{J}(\mathbf{x}')d^3x' - \mathbf{B}(0)\int \overbrace{\mathbf{x}' \cdot \mathbf{J}(\mathbf{x}')} d^3x'$$

$\nabla' \cdot [|\mathbf{x}'|^2 \mathbf{J}(\mathbf{x}')]$

$= \mathbf{J}(\mathbf{x}') \cdot \overbrace{\nabla'|\mathbf{x}'|^2}^{2\mathbf{x}'} + |\mathbf{x}'|^2 \overbrace{\nabla' \cdot \mathbf{J}(\mathbf{x}')}^{0}$ (5.13)

$= 2\mathbf{x}' \cdot \mathbf{J}(\mathbf{x}')$

$$= \underbrace{\int [\mathbf{B}(0) \cdot \mathbf{x}']\mathbf{J}(\mathbf{x}')d^3x'} - \tfrac{1}{2}\mathbf{B}(0)\underbrace{\int \nabla' \cdot [|\mathbf{x}'|^2 \mathbf{J}(\mathbf{x}')]d^3x'}$$

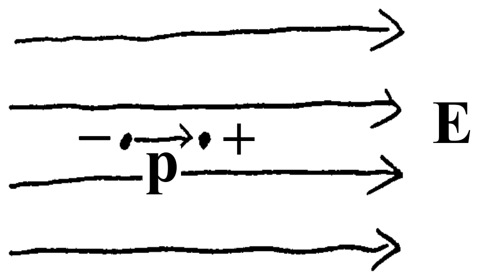
$= -\frac{1}{2}\int \mathbf{B}(0) \times [\mathbf{x}' \times \mathbf{J}(\mathbf{x}')]d^3x'$
[Using the formula at the bottom of p.185, replacing $\mathbf{x}$ with $\mathbf{B}(0)$.]

$= \oint_S |\mathbf{x}'|^2 \mathbf{J}(\mathbf{x}') \cdot d\mathbf{a}' = 0$
$\mathbf{J}$ is localized.
$\Rightarrow \mathbf{J} = 0$ on surface

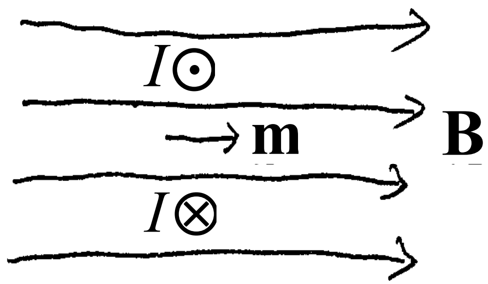
$$= \mathbf{m} \times \mathbf{B}(0) \quad (5.71)$$

A Comparison between Electric and Magnetic Potential Energy, Force, and Torque in External Field :

Potential energy	*Force*	*Torque*
$U = -\mathbf{p}\cdot\mathbf{E}$ (4.24)	$\mathbf{F} = -\nabla U$	$\mathbf{N} = \mathbf{p}\times\mathbf{E}$
$U = -\mathbf{m}\cdot\mathbf{B}$ (5.72)	$\mathbf{F} = -\nabla U$	$\mathbf{N} = \mathbf{m}\times\mathbf{B}$



Both **p** and **m** tend to orient along the positive field direction under the action of the torque (see figures on the right). This results in a state of minimum potential energy. In this state, **p** *reduces* **E**, whereas **m** *enhances* **B**.



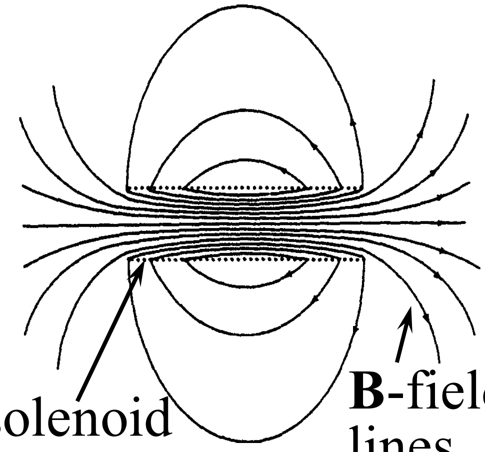
Questions:

(1) How does a permanent magnet attract a piece of iron?

(2) How does it attract another permanent magnet?

Force in Self-Consistent Field; Magnetic Pressure and Tension :

A self-consistent field is the combined field generated by the source **J** under consideration and the external source (if present). Thus, using $\nabla\times\mathbf{B}=\mu_0\mathbf{J}$, we may express the magnetic force density **f** (force / unit volume) in terms of **B**.



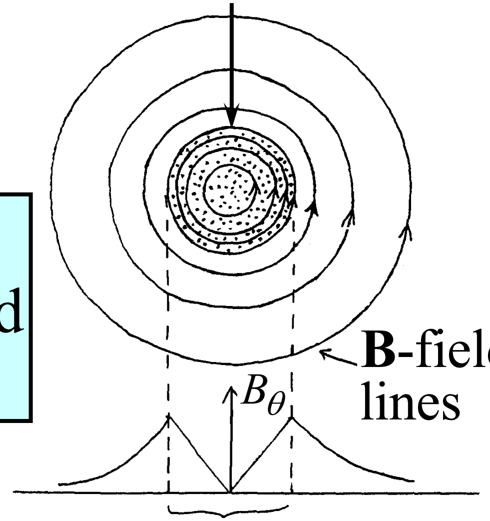
$$\mathbf{f}=\mathbf{J}\times\mathbf{B}=\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}$$

$$\boxed{\nabla(\mathbf{a}\cdot\mathbf{b})=(\mathbf{a}\cdot\nabla)\mathbf{b}+(\mathbf{b}\cdot\nabla)\mathbf{a}+\mathbf{a}\times(\nabla\times\mathbf{b})+\mathbf{b}\times(\nabla\times\mathbf{a})}$$

$$=\underbrace{-\nabla\frac{B^2}{2\mu_0}}+\underbrace{\frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B}}$$ [see p. 320)]

magnetic pressure force density

magnetic tension force density, as if a curved **B**-field line tended to become a straight line.

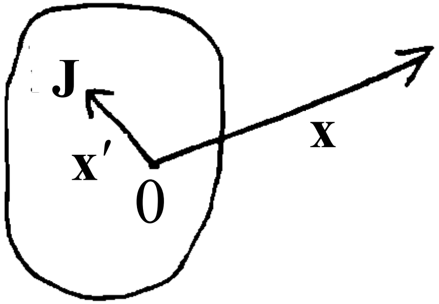


In regions where $\mathbf{J}=0$, we have $\mathbf{f}=0$ [pressure and tension force densities cancel out].

5.8 Macroscopic Equations, Boundary Conditions on B and H

Macroscopic Equations : To be more general, we move the point of reference for $\mathbf{m}$ from $\mathbf{x}=0$ to $\mathbf{x}=\mathbf{x}_0$ and write

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{1}{|\mathbf{x}-\mathbf{x}_0|}+\frac{(\mathbf{x}'-\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^3}+\cdots \quad \text{[See Sec. 4.1]}$$



Sub. this relation into

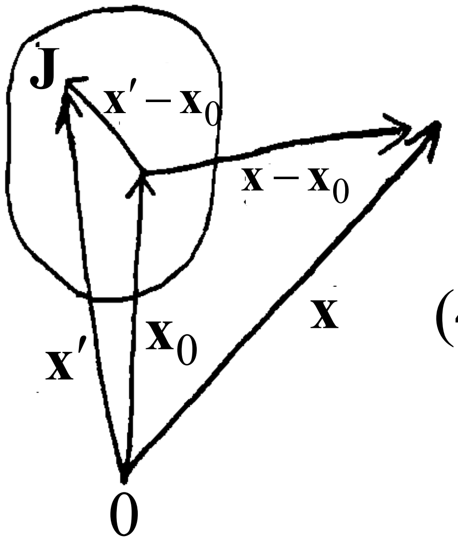
How about $\mathbf{J}_{free}$?

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \quad [(5.32)]$$

we obtain

vanishes only if $\mathbf{J}$ is localized within the volume of integration.

$$\mathbf{A}=\frac{\mu_0}{4\pi}\frac{\overbrace{\int\mathbf{J}(\mathbf{x}')d^3x'}}{|\mathbf{x}-\mathbf{x}_0|}+\frac{\mu_0}{4\pi}\frac{\mathbf{m}(\mathbf{x}_0)\times(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^3}+\cdots, \quad (4)$$



where $\mathbf{m}(\mathbf{x}_0)=\frac{1}{2}\int(\mathbf{x}'-\mathbf{x}_0)\times\mathbf{J}(\mathbf{x}')d^3x'$.

To proceed, we consider the orbital motion of atomic/molecular electrons, which can collectively give rise to a permanent or induced magnetization **M** (total magnetic moment / unit volume) given by

$$\mathbf{M}(\mathbf{x}) = \sum_i N_i < \mathbf{m}_i > \qquad (5.76)$$

N_i: volume density of type *i* molecules

$< \mathbf{m}_i >$: magnetic moment per type *i* molecule averaged over a small volume

As will be shown in (5.79), a current density ($\mathbf{J}_M$) is associated with **M**. In addition, there is also a current density due to the flow of *free* charges, which we denote by $\mathbf{J}_{free}$ (Jackson denotes it by **J** in Sec. 5.8). By the principle of linear superposition, we may write

$$\mathbf{A}(\mathbf{x}) = \mathbf{A}_{free}(\mathbf{x}) + \mathbf{A}_M(\mathbf{x}),$$

where $\mathbf{A}_{free}$ and $\mathbf{A}_M$ are due to $\mathbf{J}_{free}$ and $\mathbf{J}_M$, respectively.

Obviously, $\mathbf{A}_{free}(\mathbf{x}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\mathbf{J}_{free}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$

For $\mathbf{A}_M$, we have the expression for $\mathbf{M}$, but not yet for $\mathbf{J}_M$. So we approximate $\mathbf{A}_M$ by the dipole term in (4).

$$\mathbf{A}_M(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\cancel{\int \mathbf{J}_M(\mathbf{x}')d^3x'}}{|\mathbf{x}-\mathbf{x}_0|} + \frac{\mu_0}{4\pi}\frac{\mathbf{m}(\mathbf{x}_0)\times(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|^3} + \cdots,$$

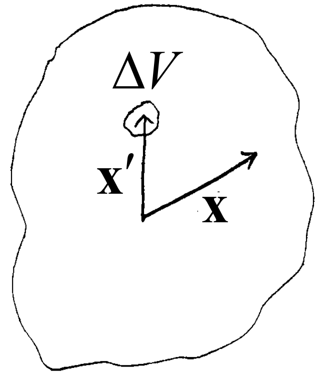
where we have set $\int \mathbf{J}_M(\mathbf{x}')d^3x' = 0$ because $\mathbf{J}_M$ is formed of current loops of atomic dimensions ($\ll$ volume of integration). Under this condition, $\mathbf{m}$ is independent of the point of reference because

$$\mathbf{m}(\mathbf{x}_0) = \tfrac{1}{2}\int(\mathbf{x}'-\mathbf{x}_0)\times\mathbf{J}_M(\mathbf{x}')d^3x'$$

$$= \tfrac{1}{2}\int \mathbf{x}'\times\mathbf{J}_M(\mathbf{x}')d^3x' - \tfrac{1}{2}\mathbf{x}_0\times\overbrace{\int \mathbf{J}_M(\mathbf{x}')d^3x'}^{0} = \mathbf{m}(0). \quad \text{(5.54)}$$

To represent $\mathbf{A}_M$ by the dipole term, we must have $|\mathbf{x}| \gg$ the dimension of the dipole. So, we divide the source into infinistesimal volumes. In each small volume ΔV, the dipole moment is $\mathbf{M}\Delta V$, which generates a small $\Delta\mathbf{A}_M$ at $\mathbf{x}$ given by

$$\Delta\mathbf{A}_M(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\mathbf{M}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')\Delta V}{|\mathbf{x}-\mathbf{x}'|^3}$$



where we have replaced the notation $\mathbf{x}_0$ with $\mathbf{x}'$. This

gives $$\mathbf{A}_M(\mathbf{x}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{M}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3x'$$

$$= \frac{\mu_0}{4\pi}\int\mathbf{M}(\mathbf{x}')\times\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

Volume of integration includes all sources.

$$= \frac{\mu_0}{4\pi}\int\frac{\nabla'\times\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' - \frac{\mu_0}{4\pi}\underbrace{\int\nabla'\times\frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'}$$

$\nabla\times\psi\mathbf{a} = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$

$\int_V\nabla\times\mathbf{A}d^3x = \oint_S\mathbf{n}\times\mathbf{A}da$

$$= \oint_S\mathbf{n}\times\frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}da = 0$$

($\mathbf{M} = 0$ on S)

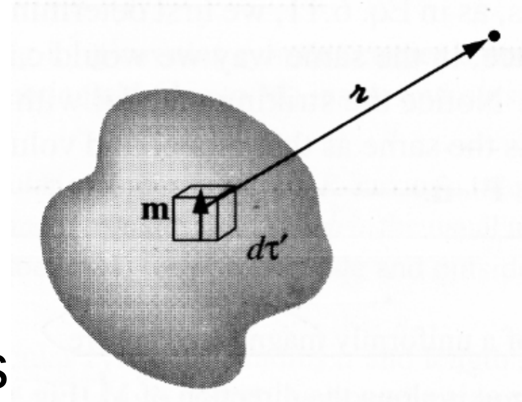
$$= \frac{\mu_0}{4\pi}\int\frac{\nabla'\times\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

What if $\mathbf{M}\neq 0$ on S?

Question: Does this relation still hold as $\mathbf{x}\to\mathbf{x}'$?

Griffith
1/4

6.2 The Field of a Magnetized Object
6.2.1 Bound Currents



Suppose we have a piece of magnetized material (i.e. $\mathbf{M}$ is given). What field does this object produce?

The vector potential of a single dipole $\mathbf{m}$ is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

In the magnetized object, each volume element carries a dipole moment $\mathbf{M}d\tau'$, so the total vector potential is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{M}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

Griffith 2/4

Vector potential and Bound Currents

Can the equation be expressed in a more illuminating form, as in the electrical case? Yes!

By exploiting the identity,

$$\nabla' \frac{1}{\mathscr{r}} = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$(\hat{\mathbf{x}}'\frac{\partial}{\partial x'} + \hat{\mathbf{y}}'\frac{\partial}{\partial y'} + \hat{\mathbf{z}}'\frac{\partial}{\partial z'})\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$= \frac{\hat{\mathbf{x}}'(x-x')+\hat{\mathbf{y}}'(y-y')+\hat{\mathbf{z}}'(z-z')}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}} = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

The vector potential is $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \mathbf{M}(\mathbf{r}') \times (\nabla' \frac{1}{\mathscr{r}}) d\tau'$

Using the product rule $\nabla \times (f\mathbf{A}) = \nabla f \times \mathbf{A} + f(\nabla \times \mathbf{A})$

and integrating by part, we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\int \frac{1}{\mathscr{r}}[\nabla' \times \mathbf{M}(\mathbf{r}')]d\tau' - \int \nabla' \times [\frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}}]d\tau'\right\}$$

↓ How? See next page.

$$= \frac{\mu_0}{4\pi}\left\{\int \frac{1}{\mathscr{r}}[\nabla' \times \mathbf{M}(\mathbf{r}')]d\tau'\right\} + \frac{\mu_0}{4\pi}\oint \frac{1}{\mathscr{r}}[\mathbf{M}(\mathbf{r}') \times \hat{\mathbf{n}}']da'$$

Griffith 3/4

Problem 1.60 Although the gradient, divergence, and curl theorems are the fundamental integral theorems of vector calculus, it is possible to derive a number of corollaries from them. Show that:

(a) $\int_{\mathcal{V}}(\nabla T)\,d\tau = \oint_{\mathcal{S}} T\,d\mathbf{a}$. [*Hint:* Let $\mathbf{v} = \mathbf{c}T$, where $\mathbf{c}$ is a constant, in the divergence theorem; use the product rules.]

(b) $\int_{\mathcal{V}}(\nabla \times \mathbf{v})\,d\tau = -\oint_{\mathcal{S}} \mathbf{v} \times d\mathbf{a}$. [*Hint:* Replace $\mathbf{v}$ by $(\mathbf{v} \times \mathbf{c})$ in the divergence theorem.]

(c) $\int_{\mathcal{V}}[T\nabla^2 U + (\nabla T)\cdot(\nabla U)]\,d\tau = \oint_{\mathcal{S}}(T\nabla U)\cdot d\mathbf{a}$. [*Hint:* Let $\mathbf{v} = T\nabla U$ in the divergence theorem.]

Gauss's law $\int_v (\nabla\cdot\mathbf{E})d\tau = \oint_S \mathbf{E}\cdot d\mathbf{a}$

Let $\mathbf{E} = \mathbf{v}\times\mathbf{c}$, $\begin{cases} \int_v (\nabla\cdot(\mathbf{v}\times\mathbf{c}))d\tau = \mathbf{c}\cdot\int_v (\nabla\times\mathbf{v})d\tau \\ \oint_S (\mathbf{v}\times\mathbf{c})\cdot d\mathbf{a} = -\mathbf{c}\cdot\oint_S \mathbf{v}\times d\mathbf{a} \end{cases}$

Since $\mathbf{c}$ is a constant vector, so $\int_v (\nabla\times\mathbf{v})d\tau = -\oint_S \mathbf{v}\times d\mathbf{a}$

Griffith 4/4

Vector potential and Bound Currents

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{1}{\mathscr{r}}\underbrace{[\nabla' \times \mathbf{M}(\mathbf{r}')]}\, d\tau' + \frac{\mu_0}{4\pi}\oint \frac{1}{\mathscr{r}}\underbrace{[\mathbf{M}(\mathbf{r}') \times \hat{\mathbf{n}}']}\, da'$$

$$\mathbf{J}_b = \nabla' \times \mathbf{M}(\mathbf{r}') \qquad \mathbf{K}_b = \mathbf{M}(\mathbf{r}') \times \hat{\mathbf{n}}'$$

volume current surface current

bound currents

With these definitions,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_v \frac{\mathbf{J}_b}{\mathscr{r}}\, d\tau' + \frac{\mu_0}{4\pi}\oint_S \frac{\mathbf{K}_b}{\mathscr{r}}\, da'$$

The electrical analogy

volume charge density $\rho_b = -\nabla \cdot \mathbf{P}$

surface charge density $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$

Thus, $$\mathbf{A}(\mathbf{x}) = \mathbf{A}_{free}(\mathbf{x}) + \mathbf{A}_M(\mathbf{x})$$

$$= \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}_{free}(\mathbf{x}') + \nabla' \times \mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \qquad (5.78)$$

For comparison, in Sec. 5.3, we have $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$, (5.31)

which has the solution: $$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \qquad (5.32)$$

In (5.31) and (5.32), $\mathbf{J}$ represents the current due to both free and bound (atomic) electrons, whereas in (5.78) contributions from free and bound electrons are separated into two terms.

Comparing (5.78) and (5.32), we find that the bound electrons contribute to $\mathbf{A}(\mathbf{x})$ through a magnetization current density ($\mathbf{J}_M$) given by $$\mathbf{J}_M = \nabla \times \mathbf{M}, \qquad (5.79)$$
which is the macroscopic exhibition of the atomic currents.

Hence, by separating free and bound electrons, $\nabla \times \mathbf{B}(\mathbf{x}) = \mu_0 \mathbf{J}(\mathbf{x})$ [(5.22)] can be written

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J}_{free} + \nabla \times \mathbf{M}) \qquad (5.80)$$

Defining a new quantity called the magnetic field $\mathbf{H}$:

$$\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}, \quad \left[\begin{array}{l}\Rightarrow \text{Effects of the atomic} \\ \text{currents are implicit in } \mathbf{H}.\end{array}\right] \qquad (5.81)$$

we obtain from (5.80) the macroscopic version of (5.22):

$$\nabla \times \mathbf{H} = \mathbf{J}_{free} \qquad (5.82)$$

Question: Does $\mathbf{H}$ have a physical meaning?

Diamagnetic, Paramagnetic, and Ferromagnetic Substances :

The counterpart of (5.81) in electrostatics is $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ [(4.34)]. In Sec. 4.3, it is shown that, for small displacement of the bound electrons, we have the linear relations:

$$\begin{cases} \mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} & (4.36) \\ \mathbf{D} = \varepsilon \mathbf{E}, \text{ with } \varepsilon = \varepsilon_0 (1 + \chi_e) & (4.37), (4.38) \end{cases}$$

However, the magnetic properties of materials are such that $\mathbf{M}$ is not always proportional to $\mathbf{B}$, depending on the type of the material. We summarize, without derivation, possible relations between $\mathbf{B}$ and $\mathbf{H}$.

1. For diamagnetic and paramagnetic substances, $\mathbf{M}$ is proportional to $\mathbf{B}$ and we express the linear relation as

$$\mathbf{M} = \frac{\mu - \mu_0}{\mu\mu_0}\mathbf{B} \quad \begin{bmatrix} \mu > \mu_0 \Rightarrow \mathbf{M} \uparrow\uparrow \mathbf{B}, & \text{paramagnetic} \\ \mu < \mu_0 \Rightarrow \mathbf{M} \uparrow\downarrow \mathbf{B}, & \text{diamagnetic} \end{bmatrix} \tag{5}$$

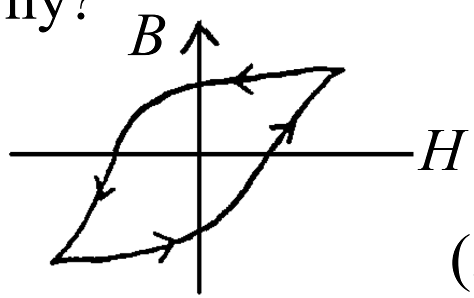
Substituting $\mathbf{M}$ into $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$, we get the linear relation:

$$\mathbf{B} = \mu\mathbf{H}, \tag{5.84}$$

where μ is called the magnetic permeability.

Question: The plasma is diamagnetic. Why?

2. For the ferromagnetic substance we have a nonlinear relation (see figure):



$$\mathbf{B} = \mathbf{F}(\mathbf{H}), \tag{5.85}$$

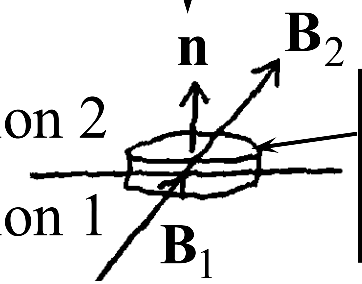
which exhibits the hysteresis phenomenon shown in the figure.

Boundary Conditions :

n : unit normal pointing from region 1 into region 2

(i) $\nabla\cdot\mathbf{B}=0\Rightarrow\int_v\nabla\cdot\mathbf{B}d^3x=\oint_S\mathbf{B}\cdot d\mathbf{a}=0$

$\Rightarrow(\mathbf{B}_2-\mathbf{B}_1)\cdot\mathbf{n}\Rightarrow B_{\perp1}=B_{\perp2}$ (5.86)



pillbox of infinitesimal thickness

(ii) $\nabla\times\mathbf{H}=\mathbf{J}_{free}$

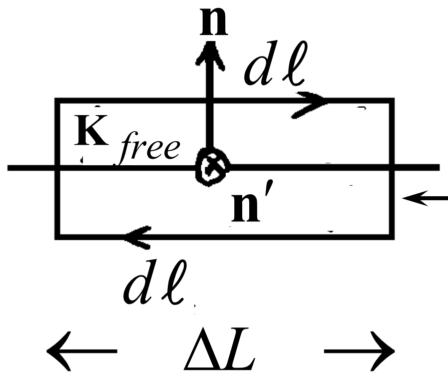
$\Rightarrow\int\nabla\times\mathbf{H}\cdot d\mathbf{a}=\int\mathbf{J}_{free}\cdot d\mathbf{a}$

$(LHS)=\oint\mathbf{H}\cdot d\ell$ (see lower figure)

$=(\mathbf{H}_2-\mathbf{H}_1)\cdot(\mathbf{n}'\times\mathbf{n})\Delta L$

$=\mathbf{n}'\cdot[\mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)]\Delta L$

$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$



rectangular loop of infinitesimal height

$(RHS)=\int\mathbf{J}_{free}\cdot\mathbf{n}'da=\mathbf{K}_{free}\cdot\mathbf{n}'\Delta L$

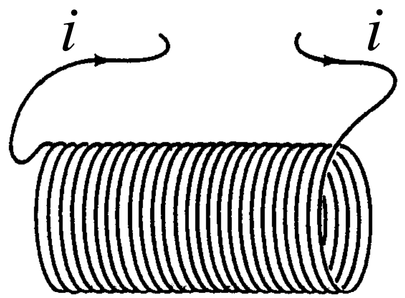
$\Rightarrow\mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)=\mathbf{K}_{free}$ (5.87)

$\mathbf{K}_{free}$: surface current of *free* charges (unit: A/m)

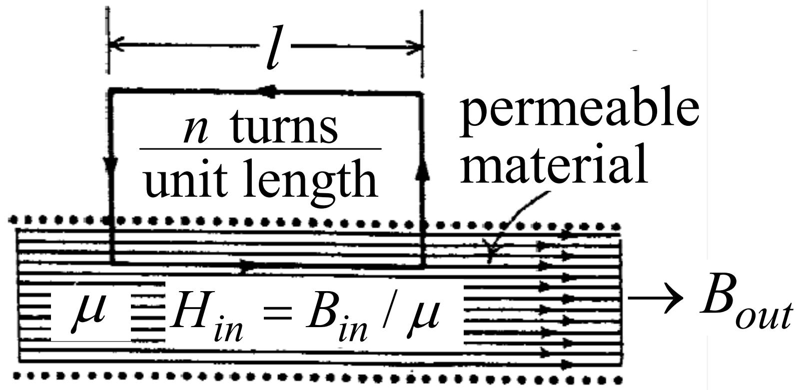
Special case: $\mathbf{K}_{free}=0\Rightarrow\mathbf{H}_{t2}=\mathbf{H}_{t1}$ (6)

t: tangential to surface

Application to a Solenoid :



A solenoid.

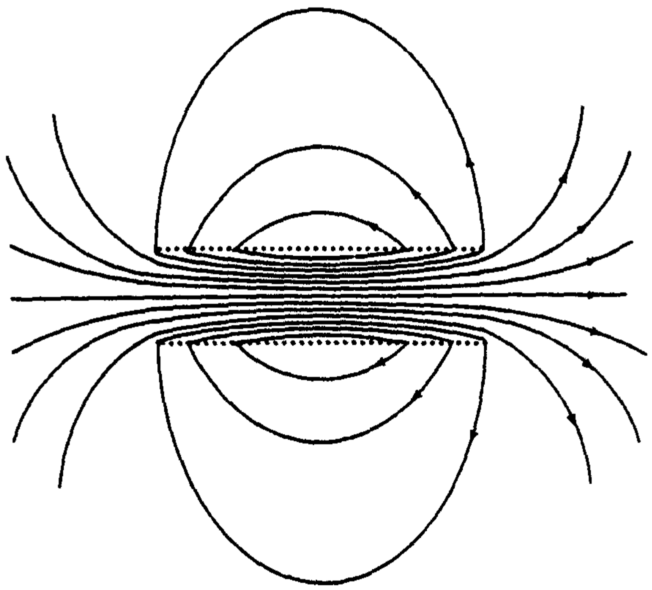


Approximate the manetic field by that of an infinite solenoid. So, H_{in}= constant.

$$\oint \mathbf{H}\cdot d\mathbf{l} = I_{free} \Rightarrow H_{in} l = nil$$

$$\Rightarrow H_{in} = ni \Rightarrow B_{in} = \mu H_{in} = \mu ni$$

$$\Rightarrow B_{out} = B_{in} = \mu ni$$



B-field lines

Question : "$B_{out} = \mu ni$" implies that filling the solenoid core with $\mu \gg \mu_0$ material (while keeping i at a constant value) can greatly enhance B_{out}. Why?

5.9 Methods of Solving Boundary-Value Problems in Magnetostatics

We put the basic equations: $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{H} = \mathbf{J}_{free}$ (5.90)
in forms suitable for 2 types of boundary - value problems.

Type 1: Linear medium with $\mu = \text{const}$ (in each region).

(a) Equation for vector potential (with or without $\mathbf{J}_{free}$)

$$\mathbf{B} = \mu \mathbf{H} = \nabla \times \mathbf{A} \Rightarrow \mathbf{H} = \tfrac{1}{\mu} \nabla \times \mathbf{A}$$

$$\Rightarrow \nabla \times \mathbf{H} = \tfrac{1}{\mu} \nabla \times \nabla \times \mathbf{A} = \tfrac{1}{\mu} [\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}] = \mathbf{J}_{free}$$

$$\Rightarrow \nabla^2 \mathbf{A} = -\mu \mathbf{J}_{free} \quad [\text{use Coulomb gauge, } \nabla \cdot \mathbf{A} = 0] \qquad (7)$$

(b) Equation for scalar potential (only for $\mathbf{J}_{free} = 0$)

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \mu \nabla \cdot \mathbf{H} = 0 \text{ and } \nabla \times \mathbf{H} = 0 \Rightarrow \mathbf{H} = -\nabla \Phi_M \qquad (5.93)$$

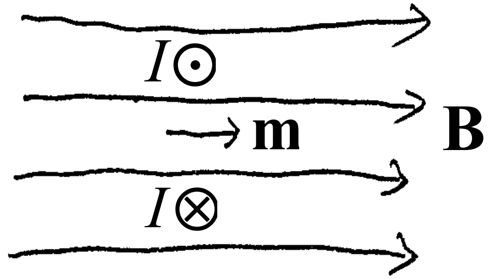
$$\Rightarrow \nabla^2 \Phi_M = 0 \qquad (8)$$

Typically, we use (7) or (8) to solve for $\mathbf{A}$ or Φ_M in each uniform region and find the coefficients by applying conditions (5.86) and (5.87) on the boundary. An example will be provided in Sec. 5.12.

Discussion:

In a vacuum medium, we have

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}_{free} \qquad \text{5.31)}$$



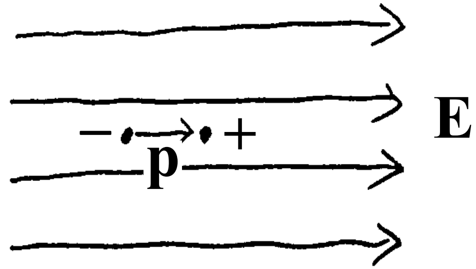
In a uniform-μ medium, we have

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}_{free}. \qquad (7)$$

Hence, the effect of $\mu > \mu_0$ medium is to *increase* the ability of $\mathbf{J}_{free}$ to produce **B** by a factor of μ / μ_0 (see figure above).

In electrostatics, we have

$$\nabla^2 \Phi = -\frac{\rho_{free}}{\varepsilon_0} \quad \text{(vacuum medium)} \qquad (1.13)$$



and

$$\nabla^2 \Phi = -\frac{\rho_{free}}{\varepsilon} \quad \text{(uniform dielectric medium)} \qquad (4.39)$$

Hence, an $\varepsilon > \varepsilon_0$ medium *reduces* the ability of ρ_{free} to produce **E** by a factor of $\varepsilon / \varepsilon_0$ (see figure above).

Type 2 : Hard ferromagnets (permanent magnet, $\mathbf{M}$ given, $\mathbf{J}_{free} = 0$)

(a) Vector potential

$$\nabla \times \mathbf{H} = \nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = 0$$

real current

$$\Rightarrow \nabla \times \mathbf{B} = \mu_0 \nabla \times \mathbf{M} = \mu_0 \mathbf{J}_M, \text{ where } \mathbf{J}_M \equiv \nabla \times \mathbf{M} \text{ [see (5.79)]}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \Rightarrow \nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}_M$$

$$\Rightarrow \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}_M \Rightarrow \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}_M(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x', \qquad (5.102)$$

(b) Scalar potential

ρ_M is a mathematical tool, not real charge.

$$\nabla \times \mathbf{H} = 0 \Rightarrow \mathbf{H} = -\nabla \Phi_M$$

$$\nabla \cdot \mathbf{B} = \mu_0 \nabla \cdot (\mathbf{H} + \mathbf{M}) = 0 \Rightarrow \nabla^2 \Phi_M = \nabla \cdot \mathbf{H} = -\rho_M \qquad (5.95)$$

where $\rho_M \equiv -\nabla \cdot \mathbf{M}$ (effective magnetic charge density) (5.96)

$$\Rightarrow \Phi_M = \frac{1}{4\pi} \int \frac{\rho_M(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' = -\frac{1}{4\pi} \int \frac{\nabla' \cdot \mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'$$

$\nabla \cdot \psi \mathbf{a} = \psi \nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla \psi$

$$= \frac{1}{4\pi} \int \mathbf{M}(\mathbf{x}') \cdot \nabla' \frac{1}{|\mathbf{x}-\mathbf{x}'|} d^3x' = -\frac{1}{4\pi} \nabla \cdot \int \frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \qquad (5.98)$$

Effective magnetic surface charge density σ_M:

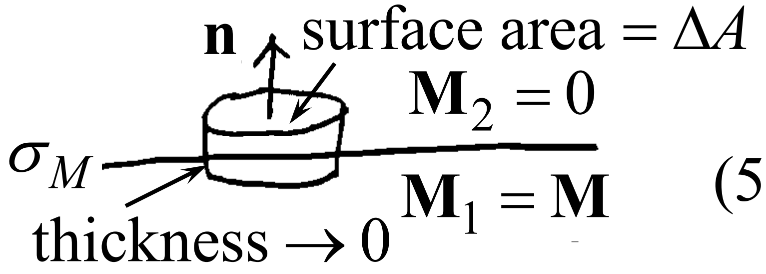
Rewrite (5.96): $\nabla \cdot \mathbf{M} = -\rho_M$ (5.96)

$\Rightarrow \int_v \nabla \cdot \mathbf{M} d^3x = \oint_S \mathbf{M} \cdot d\mathbf{a} = -\int_v \rho_M d^3x$ (see pillbox below)

$\Rightarrow (\underbrace{\mathbf{M}_2}_{=0} - \underbrace{\mathbf{M}_1}_{=\mathbf{M}}) \cdot \mathbf{n} \Delta A = -\sigma_M \Delta A$

$\Rightarrow \sigma_M = \mathbf{n} \cdot \mathbf{M}$ (5.99)

a mathematical tool



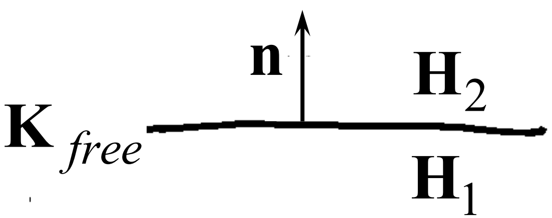
Surface current density $\mathbf{K}_M$ *due to magnetization* $\mathbf{M}$:

In Sec. 5.8,

$\nabla \times \mathbf{H} = \mathbf{J}_{free}$

real current

$\Rightarrow \mathbf{K}_{free} = \mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1)$

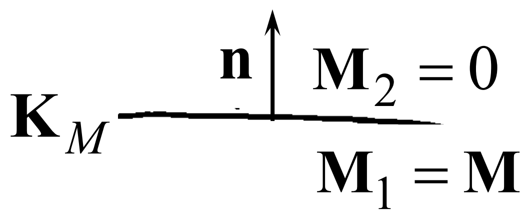


Here (by the same algebra),

$\nabla \times \mathbf{M} = \mathbf{J}_M$

real current

$\Rightarrow \mathbf{K}_M = \mathbf{n} \times (\underbrace{\mathbf{M}_2}_{=0} - \underbrace{\mathbf{M}_1}_{=\mathbf{M}}) = \mathbf{M} \times \mathbf{n}$



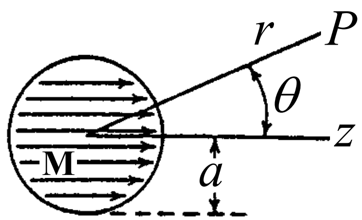
5.10 Uniformly Magnetized Sphere

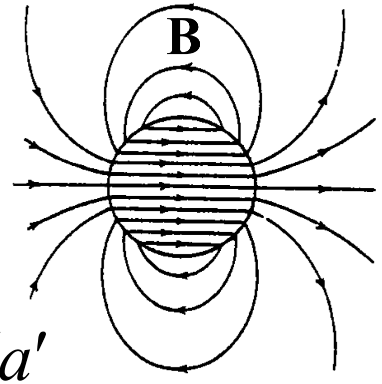
Consider a permanent magnet with magnetization :

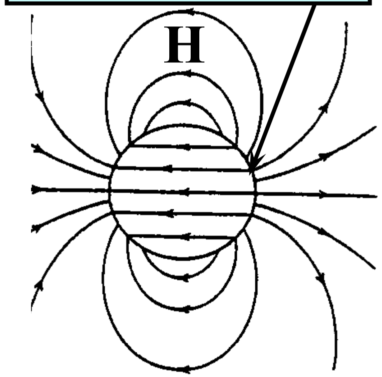
discontinuous!

$$\mathbf{M} = \begin{cases} M_0 \mathbf{e}_z \ , & r \le a \\ 0 \ , & r > a \end{cases}$$

ρ_M vanishes everywhere except on the surface.







$$\Phi_M = \frac{1}{4\pi}\int \frac{\rho_M(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' = \frac{1}{4\pi}\oint_S \frac{\mathbf{n}\cdot\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} da'$$

by (5.95)

$$= \frac{M_0 a^2}{4\pi}\int d\Omega' \frac{\overbrace{\cos\theta'}^{=\sqrt{\frac{4\pi}{3}}Y_{10}(\theta',\varphi')}}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{3} M_0 a^2 \frac{r_<}{r_>^2}\cos\theta$$

$$= \begin{cases} \frac{1}{3} M_0 r \cos\theta = \frac{1}{3} M_0 z, & r \le a \\ \frac{1}{3} M_0 a^3 \frac{\cos\theta}{r^2}, & r > a \end{cases} \qquad (5.104)$$

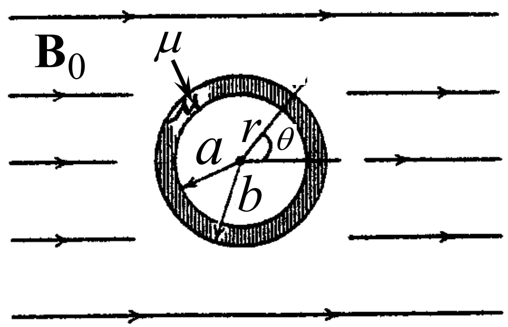
(3.70)

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} \doteq \frac{1}{r_>} + \frac{4\pi}{3}\frac{r_<}{r_>^2}\cdot [Y^*_{1,-1}(\theta',\varphi')Y_{1,-1}(\theta,\varphi) + Y^*_{10}(\theta',\varphi')\underbrace{Y_{10}(\theta,\varphi)}_{=\sqrt{\frac{3}{4\pi}}\cos\theta} + Y^*_{11}(\theta',\varphi')Y_{11}(\theta,\varphi)] + \cdots$$

$$\Rightarrow \begin{cases} \text{Inside: } \mathbf{H}_{in} = -\frac{1}{3}\mathbf{M} \\ \qquad \mathbf{B}_{in} = \mu_0 \mathbf{H}_{in} + \mu_0 \mathbf{M} = \frac{2}{3}\mu_0 \mathbf{M} \ (\Rightarrow \mathbf{H}_{in} \uparrow\downarrow \mathbf{B}_{in}) \\ \text{Outside: dipole field with dipole moment } \mathbf{m} = \frac{4\pi a^3}{3}\mathbf{M}. \end{cases}$$

5.12 Magnetic Shielding, Spherical Shell of Permeable Material in a Uniform Field

Consider a spherical μ-shell in an external $\mathbf{B}_0$.



$$\underbrace{\nabla^2\Phi_M = 0}_{\text{Eq. (8)}} \Rightarrow \Phi_M = \begin{Bmatrix} r^l \\ r^{-l-1} \end{Bmatrix} \begin{Bmatrix} P_l^m(\cos\theta) \\ Q_l^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$$

$$\Rightarrow \Phi_M = \begin{cases} -H_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{\alpha_l}{r^{l+1}} P_l(\cos\theta), & r > b \quad (5.117) \\ \sum_{l=0}^{\infty} (\beta_l r^l + \gamma_l \frac{1}{r^{l+1}}) P_l(\cos\theta), & a < r < b \quad (5.118) \\ \sum_{l=0}^{\infty} \delta_l r^l P_l(\cos\theta), & r < a \quad (5.119) \end{cases}$$

$-H_0 r\cos\theta$ gives the external $\mathbf{B}_0$.

boundary conditions

$$\begin{Bmatrix} \mathbf{H} = -\nabla\Phi_M \ (5.93) \\ \mathbf{B} = \mu_0\mathbf{H}\ \text{(outside)} \\ \mathbf{B} = \mu\mathbf{H}\ \text{(inside)} \end{Bmatrix} + \begin{Bmatrix} \mathbf{H}_{t2} = \mathbf{H}_{t1} \quad (6) \\ B_{\perp 1} = B_{\perp 2} \ (5.86) \end{Bmatrix} \Rightarrow \begin{cases} \left.\frac{\partial\Phi_M}{\partial\theta}\right|_{b^+} = \left.\frac{\partial\Phi_M}{\partial\theta}\right|_{b^-} \\ \left.\frac{\partial\Phi_M}{\partial\theta}\right|_{a^+} = \left.\frac{\partial\Phi_M}{\partial\theta}\right|_{a^-} \\ \mu_0 \left.\frac{\partial\Phi_M}{\partial r}\right|_{b^+} = \mu \left.\frac{\partial\Phi_M}{\partial r}\right|_{b^-} \\ \mu \left.\frac{\partial\Phi_M}{\partial r}\right|_{a^+} = \mu_0 \left.\frac{\partial\Phi_M}{\partial r}\right|_{a^-} \end{cases}$$

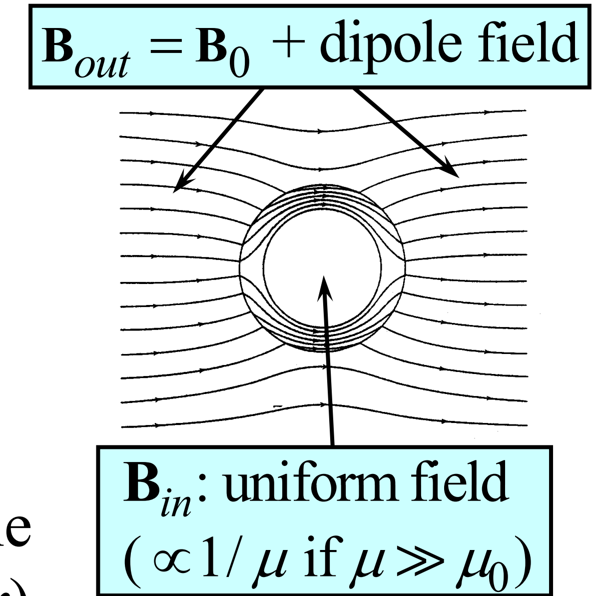
The shell is assumed to be a linear medium.

Boundary conditions result in solutions for the coefficients:

$$\begin{cases} \alpha_l = \beta_l = \gamma_l = \delta_l = 0 \text{ if } l \neq 1 \\ \alpha_1 = \dfrac{(2\frac{\mu}{\mu_0}+1)(\frac{\mu}{\mu_0}-1)(b^3-a^3)}{(2\frac{\mu}{\mu_0}+1)(\frac{\mu}{\mu_0}+2)-2\frac{a^3}{b^3}(\frac{\mu}{\mu_0}-1)^2} H_0 \approx b^3 H_0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \mu \gg \mu_0 \\ \delta_1 = \dfrac{-9\frac{\mu}{\mu_0}}{(2\frac{\mu}{\mu_0}+1)(\frac{\mu}{\mu_0}+2)-2\frac{a^3}{b^3}(\frac{\mu}{\mu_0}-1)^2} H_0 \approx \dfrac{-9\mu_0}{2\mu(1-\frac{a^3}{b^3})} H_0 \end{cases} \quad \begin{bmatrix} (5.121) \\ \& \\ (5.122) \end{bmatrix}$$

$\mathbf{B}_{in} \searrow$ as $\frac{\mu}{\mu_0} \nearrow$, implying that $\mu > \mu_0$ materials tend to "absorb" **B**-field lines and thereby provide the shielding effect. High-μ materials can have μ/μ_0 as high as $10^3 - 10^6$.

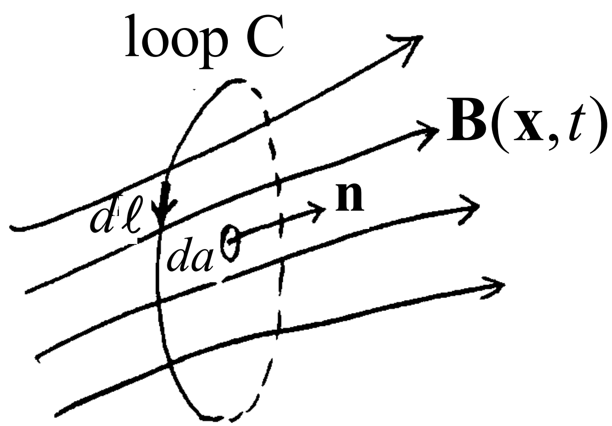
When $\mu = \mu_0$, **B** reduces to $\mathbf{B}_0$ everywhere, i.e. a static megnetic field penetrates into the shell as if there were no shell present (even if the shell is made of good conductor, such as copper).



5.15 Faraday's Law of Induction

The Biot-Savart (or Ampere's) law relates the magnetic field to electrical current. Faraday then discovered experimantally that time-varying magnetic flux through an electrical circuit could induce an electric field around the circuit. This not only provided the first link between electric and magnetic fields, but also led to a new mechanism to generate the **E**-field, i.e. a time-varying **B**-field.

Referring to the figure, let loop C be an electrical circuit (as in Faraday's original experiment) or any closed path in space (a generalization of the original observation with immense consequences). Faraday's law states

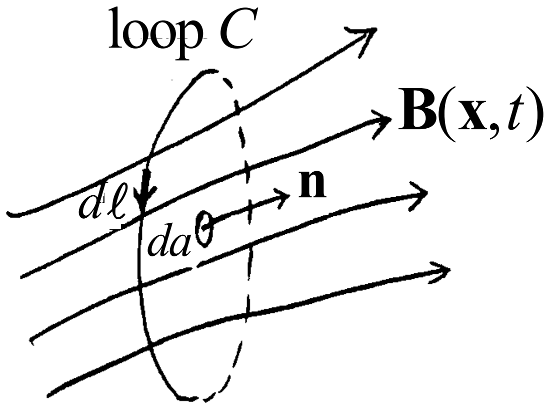


$$\oint_C \mathbf{E}' \cdot d\ell = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, da, \quad \left[\begin{array}{r} S\text{: an arbitrary surface} \\ \text{bounded by loop } C \end{array}\right] \quad (5.141)$$

where $\mathbf{E}'$ is the electric field at $d\ell$ in the frame in which $d\ell$ is at rest, and $\mathbf{B}$ is the magnetic induction in the lab frame.

Rewrite (5.141):

$$\oint_C \mathbf{E}' \cdot d\ell = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} da \qquad (5.141)$$



Assume loop C is at rest in the lab frame, then $\mathbf{E}' = \mathbf{E}$ (electric field in the lab frame) and (5.141) becomes

$$\oint_C \mathbf{E} \cdot d\ell = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} da \quad \left[\text{integral form of Faraday's law}\right] \qquad (9)$$

where both $\mathbf{E}$ and $\mathbf{B}$ are lab-frame quantities.

(9) can be written (by Stokes's theorem: $\oint_C \mathbf{E} \cdot d\ell = \int_S \nabla \times \mathbf{E} \cdot \mathbf{n} da$)

$$\int_S \nabla \times \mathbf{E} \cdot \mathbf{n} da = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} da$$

Thus,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \left[\text{differential form of Faraday's law)}\right] \qquad (5.143)$$

5.16 Energy in the Magnetic Field

To find the energy associated with a magnetic field, we evaluate the work needed to establish the current $\mathbf{J}(\mathbf{x})$, which produces the magnetic field. We break up $\mathbf{J}(\mathbf{x})$ into a network of thin loops. In the build-up process, an $\mathbf{E}$ field will be induced by $\partial\mathbf{B}/\partial t$. The rate of work done by $\mathbf{E}$ within each loop is

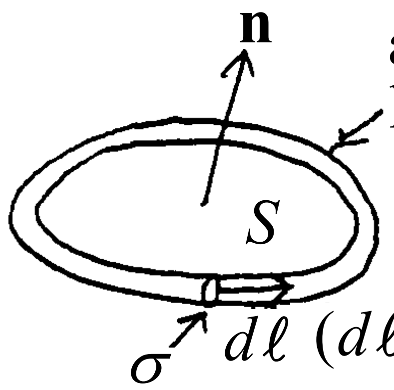
σ is the cross section of the loop (same as Jackson's $\Delta\sigma$). J & σ may vary along $d\ell$.

integration over the area encircled by the loop

$$\frac{dW_{loop}}{dt} = -\oint \overbrace{J\sigma}\, \mathbf{E}\cdot d\boldsymbol{\ell} = -\int_s J\sigma \underbrace{\nabla\times\mathbf{E}}_{-\partial\mathbf{B}/\partial t} \cdot \mathbf{n}da$$

$$= \int_s J\sigma\mathbf{n}\cdot\frac{\partial\mathbf{B}}{\partial t}da$$

Stokes's thm.



σ : cross section of loop

Work done within each loop to generate $\delta\mathbf{B}$:

$$\delta W_{loop} = \int_s J\sigma\mathbf{n}\cdot\underbrace{\delta\mathbf{B}}_{\nabla\times\delta\mathbf{A}} da = \int_s J\sigma\nabla\times\delta\mathbf{A}\cdot\mathbf{n}da$$

$$= \oint J\sigma\delta\mathbf{A}\cdot d\boldsymbol{\ell}$$

$$= \int_{loop} \delta\mathbf{A}\cdot\mathbf{J}d^3x \qquad (10)$$

$$J\sigma d\boldsymbol{\ell} = \mathbf{J}\underbrace{\sigma d\ell}_{d^3x} = \mathbf{J}d^3x$$

As shown in $\delta W_{loop} = \int_{loop} \delta \mathbf{A} \cdot \mathbf{J} d^3x$ [(10)], the work done within each loop is an integral over the volume of the loop. Thus, an integration over all space gives the total work done to generate $\delta \mathbf{B}$:

Assume the rate of build-up $\to 0 \Rightarrow \mathbf{H}$ obeys the static law $\nabla \times \mathbf{H} = \mathbf{J}$. Otherwise, the static law breaks down and there will be radiation loss.

$$\delta W = \int \delta \mathbf{A} \cdot \mathbf{J} d^3x \overset{\downarrow}{=} \int \delta \mathbf{A} \cdot (\nabla \times \mathbf{H}) d^3x \tag{5.144}$$

$$= \int \mathbf{H} \cdot \underbrace{(\nabla \times \delta \mathbf{A})}_{\delta \mathbf{B}} d^3x + \underbrace{\int \nabla \cdot (\mathbf{H} \times \delta \mathbf{A}) d^3x}_{= \oint_S (\mathbf{H} \times \delta \mathbf{A}) \cdot d\mathbf{a} = 0}$$

$$= \int \mathbf{H} \cdot \delta \mathbf{B} d^3x \underset{\uparrow}{=} \tfrac{1}{2} \int \delta (\mathbf{H} \cdot \mathbf{B}) d^3x$$

For this integral to vanish, the volume of integration must be ∞.

Assume linear medium: $\mathbf{B} = \mu \mathbf{H}$ or $\mathbf{B} = \overleftrightarrow{\boldsymbol{\mu}} \cdot \mathbf{H}$

Total work done to bring the field up from 0 to the final value $\mathbf{B}$:

$$W = \tfrac{1}{2} \int (\mathbf{H} \cdot \mathbf{B}) d^3x \quad \left[\begin{array}{l}\text{By conservation of energy, this is} \\ \text{the total magnetic field energy.}\end{array}\right] \tag{5.148}$$

$$\Rightarrow w = \tfrac{1}{2} \mathbf{H} \cdot \mathbf{B} \quad \text{[field energy density]} \tag{11}$$

Note: $w = \tfrac{1}{2} \mathbf{H} \cdot \mathbf{B} = \tfrac{1}{2} (\sum_j \mathbf{H}_j) \cdot (\sum_j \mathbf{B}_j) \neq \tfrac{1}{2} \sum_j (\mathbf{H}_j \cdot \mathbf{B}_j)$

5.17 Energy and Self- and Mutual Inductances

Assume linear relation between **J** and **A**

$$\delta W = \int \delta\mathbf{A}\cdot\mathbf{J}d^3x = \frac{1}{2}\int \delta(\mathbf{A}\cdot\mathbf{J})d^3x \qquad (5.144)$$

for nonpermeable ($\mu = \mu_0$) medium

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \quad (5.32)$$

$$\Rightarrow W = \frac{1}{2}\int \mathbf{A}\cdot\mathbf{J}d^3x \qquad (5.149)$$

$$= \frac{\mu_0}{8\pi}\int d^3x \int d^3x' \frac{\mathbf{J}(\mathbf{x})\cdot\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \qquad (5.153)$$

for N current-carrying circuits

$$= \frac{\mu_0}{8\pi}\sum_{i=1}^{N}\int d^3x_i \sum_{j=1}^{N}\int d^3x'_j \frac{\mathbf{J}(\mathbf{x}_i)\cdot\mathbf{J}(\mathbf{x}'_j)}{|\mathbf{x}_i-\mathbf{x}'_j|} = \frac{1}{2}\sum_{i=1}^{N} L_i I_i^2 + \sum_{i=1}^{N}\sum_{j>i}^{N} M_{ij}I_iI_j, \quad (5.152)$$

where

self-inductance

for a thin wire

$$L_i = \frac{\mu_0}{4\pi I_i^2}\int_{C_i} d^3x_i \int_{C_i} d^3x_i \frac{\mathbf{J}(\mathbf{x}_i)\cdot\mathbf{J}(\mathbf{x}'_i)}{|\mathbf{x}_i-\mathbf{x}'_i|}\left[= \frac{\mu_0}{4\pi}\oint_{C_i}\oint_{C_i}\frac{d\ell_i\cdot d\ell'_i}{|\mathbf{x}_i-\mathbf{x}'_i|}\right] \qquad (5.154)$$

$$M_{ij} = \frac{\mu_0}{4\pi I_i I_j}\int_{C_i} d^3x_i \int_{C_j} d^3x'_j \frac{\mathbf{J}(\mathbf{x}_i)\cdot\mathbf{J}(\mathbf{x}'_j)}{|\mathbf{x}_i-\mathbf{x}'_j|}\left[= \frac{\mu_0}{4\pi}\oint_{C_i}\oint_{C_j}\frac{d\ell_i\cdot d\ell'_j}{|\mathbf{x}_i-\mathbf{x}'_j|}\right] \qquad (5.155)$$

mutual inductance ($M_{ij} = M_{ji}$)

for thin wires

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \qquad (5.32)$$

$\Rightarrow$ Vector potential at circuit i due to current in circuit j :

$$\mathbf{A}_{ij}(\mathbf{x}_i) = \frac{\mu_0}{4\pi}\oint_{C_j} \frac{\mathbf{J}(\mathbf{x}'_j)}{|\mathbf{x}_i-\mathbf{x}'_j|} d^3x'_j \qquad (12)$$

From (12) and (5.155), we obtain $M_{ij} = \frac{1}{I_i I_j}\int_{C_i} \mathbf{A}_{ij}(\mathbf{x}_i)\cdot\mathbf{J}(\mathbf{x}_i)d^3x_i$

Assume $\mathbf{J}$ flows along wire $d\ell$ of negligible cross section da

$\Rightarrow$ $\mathbf{J}(\mathbf{x}_i)d^3x_i = J_{\|}da\,d\ell = I_i d\ell$

magnetic flux from circuit j passing through circuit i

$$\Rightarrow M_{ij} = \frac{1}{I_j}\oint_{C_i} \mathbf{A}_{ij}\cdot d\ell = \frac{1}{I_j}\oint_{S_i} \overbrace{(\nabla\times\mathbf{A}_{ij})}^{\mathbf{B}_{ij}}\cdot \mathbf{n}da = \frac{1}{I_j}F_{ij} \qquad (5.156)$$

$$\Rightarrow \varepsilon_{ij} \equiv -\frac{d}{dt}F_{ij} = -M_{ij}\frac{d}{dt}I_j$$

ε_{ij}: induced voltage in circuit i due to current variation in circuit j.

The "–" sign implies that the induced ε_{ij} tends to drive a current in circuit i to *inhibit* the flux change caused by circuit j (Lenz's law).

Homework of Chap. 5

Problems: 1, 3, 6, 11, 13,
18, 19, 20, 22, 30