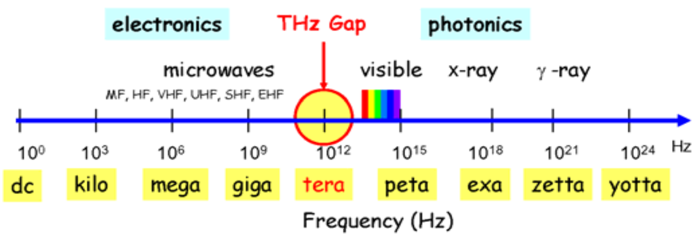
Chapter 7: Plane Electromagnetic Waves and Wave Propagation

T-Ray: Next frontier in Science and Technology

Terahertz wave (or T-ray), which is electromagnetic radiation in a frequency interval from 0.1 to 10 THz, lies a frequency range with rich science but limited technology.

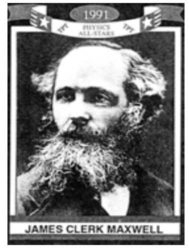


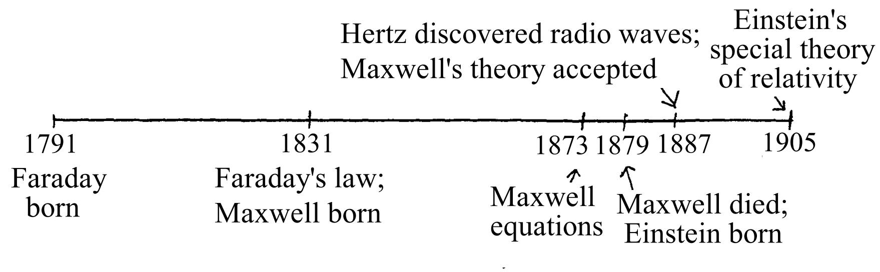
1 THz ~ 1 ps ~ 300 μm ~ 33 cm^{-1} ~ 4.1 meV ~ 47.6 °K

An Historical Perspective:

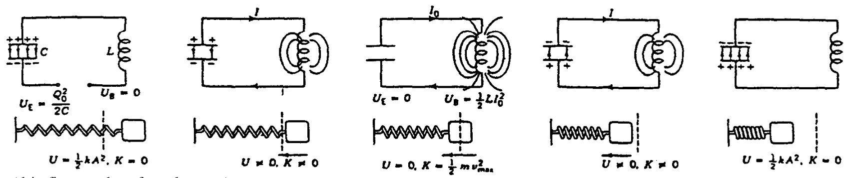
Faraday : Time-varying magnetic field generates electric field.

Maxwell : Time-varying electric field generates magnetic field.





A Note about Oscillatory Behavior:



(this figure taken from benson)

Common feature of oscillatory behavior: $\boxed{\text{energy type 1}} \Leftrightarrow \boxed{\text{energy type 2}}$

$\Rightarrow$ Oscillations require $\begin{cases} \text{energy storing mechanisms} \\ \text{energy exchange mechanism(s)} \end{cases}$

example	energy storing mechanisms	energy exchange mechanism(s)	medium
mass-spring system	$\frac{1}{2}mv^2,\ \frac{1}{2}kx^2$	restoring force	mass & spring
LC oscillator	$\frac{1}{2}LI^2,\ \frac{1}{2}CV^2$	$Q,\ I$	L, C, & wire
EM wave	$\frac{B^2}{2\mu},\ \frac{\varepsilon E^2}{2}$	$\frac{dB}{dt},\ \frac{dE}{dt}$	not required

Organization of Lecture Notes on Ch. 7:

In Jackson, plane waves in dielectric media are treated in Secs. 7.1 and 7.2. Various special cases (plasma medium and high-frequency limit) are treated in Sec. 7.5. Plane waves in conductors are treated in Sec. 5.18 [e.g. Eqs. (5.163)-(5.169)] and Sec. 8.1 [e.g. Eqs. (8.9), (8.10), (8.12), (8.14), and (8.15)] by methods different from those in Secs. 7.1 and 7.2.

Here, we will cover these sections in Jackson with a unified treatment of plane waves in both dielectrics and conductors, and at all frequencies. Equations in Jackson will be examined in greater detail, but in somewhat different order. So, in the lecture notes, the three sections on these materials will be numbered Secs. I, II, and III rather than following Jackson's section numbers. However, Secs. 7.3, 7.4, 7.8, and 7.9 of Jackson will be followed closely in subsequent lecture notes (and numbered as in Jackson) .

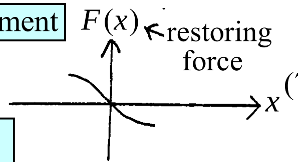
We begin with a derivation of the generalized dielectric constant $\varepsilon/\varepsilon_0$, which is applicable to both dielectric and conducting media.

I. Derivation of the Generalized Dielectric Constant $\varepsilon/\varepsilon_0$ [Sec. 7.5 (part A)]

Dipole Moment of a Single Electron: The equation of motion for an atomic or molecular electron with mass m and charge $-e$ in the presence of an external electric field $\mathbf{E}(\mathbf{x},t)$ can be written:

restoring force due to electron displacement

$$m\ddot{\mathbf{x}} = -e\mathbf{E}(\mathbf{x},t) \underbrace{-\gamma m\dot{\mathbf{x}}} \overbrace{-m\omega_0^2\mathbf{x}} \quad (7.49)$$



$\mathbf{x}$: displacement of the electron from its equilibrium position $\mathbf{x} = 0$.

γ: electron collision frequency

$-\gamma m\dot{\mathbf{x}}$: damping force (rate of change of electron momentum due to collisions)

$$F(x) = \underbrace{F(0)}_{0} + \underbrace{F'(0)}_{-m\omega_0^2 x}\, x + \cdots$$

As in Sec. 4.6, we neglect higher-order terms.

The "binding frquency" ω_0 is the natural oscillation frequency of the electron if it is set to oscillate about $\mathbf{x} = 0$ under the restoring force $-m\omega_0^2$. Since $\omega_0^2 \propto 1/m$, the restoring force is independent of m.

Rewrite (7.49), $m\ddot{\mathbf{x}} = -e\mathbf{E}(\mathbf{x},t) - \gamma m\dot{\mathbf{x}} - m\omega_0^2\mathbf{x}$, as

$$m(\ddot{\mathbf{x}} + \gamma\dot{\mathbf{x}} + \omega_0^2\mathbf{x}) = -e\mathbf{E}(\mathbf{x},t)$$

Let* $\mathbf{E}(\mathbf{x},t) = \mathbf{E}(\mathbf{x})e^{-i\omega t}$ and expand $\mathbf{E}(\mathbf{x})$ about the equilibrium position $\mathbf{x} = 0$, we obtain $\mathbf{E}(\mathbf{x}) = \mathbf{E}(0) + \underbrace{(\mathbf{x}\cdot\nabla)\mathbf{E}(0)}_{\text{of the order of } \frac{x}{\lambda}\mathbf{E}(0)} + \cdots \approx \mathbf{E}(0)$, $\boxed{\text{if } \frac{x}{\lambda} \ll 1}$

where λ is the scale length of $\mathbf{E}(\mathbf{x})$. For example, if $\mathbf{E}(\mathbf{x})$ is a wave field, then $\lambda \approx$ *wavelength*. By neglecting $(\mathbf{x}\cdot\nabla)\mathbf{E}(0)$, we have assumed that the electron displacement is too small for the electron to see any spatial field variation. Thus, we assume that the electron is acted on by a *spatially uniform* field:

$$\mathbf{E}(\mathbf{x},t) \approx \mathbf{E}(0)e^{-i\omega t},$$

and it is understood that $\mathbf{E}(\mathbf{x},t)$ is given by the real part of the RHS.

*This is equivalent to a Fourier trnasformation to the ω space and $\mathbf{E}(\mathbf{x})$ is a complex quantity called the phasor [see Appendix A]

Let $\mathbf{x}(t) = \mathbf{x}_0 e^{-i\omega t}$ and substitute

$$\begin{cases} \mathbf{x}(t) = \mathbf{x}_0 e^{-i\omega t} \\ \mathbf{E}(\mathbf{x},t) = \mathbf{E}(0)e^{-i\omega t} \end{cases} \text{ into } \quad m(\ddot{\mathbf{x}} + \gamma\dot{\mathbf{x}} + \omega_0^2\mathbf{x}) = -e\mathbf{E}(\mathbf{x},t),$$

we obtain $m(-\omega^2 - i\omega\gamma + \omega_0^2)\mathbf{x}_0 = -e\mathbf{E}(0)$ with the solution:

$$\mathbf{x}_0 = -\frac{e}{m}\frac{\mathbf{E}(0)}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

$$\Rightarrow \qquad \mathbf{x}(t) = -\frac{e}{m}\frac{\mathbf{E}(0)e^{-i\omega t}}{\omega_0^2 - \omega^2 - i\omega\gamma} \qquad (1)$$

(1) represents the *forced* oscillation of a simple harmonic oscillator with natural oscillation frequency ω_0. The time-dependent $\mathbf{x}(t)$ results in a time-dependent dipole moment at $\mathbf{x} = 0$ given by

$$\mathbf{p}(t) = \mathbf{p}_0 e^{-i\omega t},$$

where $\quad \mathbf{p}_0 = -e\mathbf{x}_0 = \frac{e^2}{m}\frac{\mathbf{E}(0)}{\omega_0^2 - \omega^2 - i\omega\gamma}$ [This reduces to (4.72) in the static limit: $\omega = 0$.]

$$\text{Rewrite } \begin{cases} \mathbf{E}(\mathbf{x},t) = \mathbf{E}(0)e^{-i\omega t} \\ \mathbf{x}(t) = \mathbf{x}_0 e^{-i\omega t} \\ \mathbf{p}(t) = \mathbf{p}_0 e^{-i\omega t} \end{cases} \text{ and } \begin{cases} \mathbf{x}_0 = -\frac{e}{m}\frac{\mathbf{E}(0)}{\omega_0^2 - \omega^2 - i\omega\gamma} \\ \mathbf{p}_0 = -e\mathbf{x}_0 = \frac{e^2}{m}\frac{\mathbf{E}(0)}{\omega_0^2 - \omega^2 - i\omega\gamma} \end{cases}$$

In these equations, $\mathbf{E}(0)$, $\mathbf{x}_0$, and $\mathbf{p}_0$ are phasors containing phase and amplitude information of $\mathbf{E}(\mathbf{x},t)$, $\mathbf{x}(t)$, and $\mathbf{p}(t)$, respectively. The subscript "0" in $\mathbf{x}_0$ and $\mathbf{p}_0$ refers to the fact that the oscillation is centered at $\mathbf{x} = 0$, where $\mathbf{E}(\mathbf{x},t)$ is approximated by a spatially uniform field $\mathbf{E}(0)e^{-i\omega t}$ (its value at $\mathbf{x} = 0$). If the oscillation is centered at an arbitrary point $\mathbf{x}$, the only difference is that the electron would see a sptially constant field given by $\mathbf{E}(\mathbf{x})e^{-i\omega t}$.

Thus, in general, $\mathbf{p}(t) = \mathbf{p}e^{-i\omega t}$ with $\quad \mathbf{p} = \frac{e^2}{m}\frac{\mathbf{E}(\mathbf{x})}{\omega_0^2 - \omega^2 - i\omega\gamma} \qquad (7.50)$

Note that, in (7.50), $\mathbf{x}$ is a spatial variable (not the electron displacement), and $\mathbf{p}$ and $\mathbf{E}(\mathbf{x})$ are phasors.

The Generalized Dielectric Constant : Assume there are N molecules per unit volume and Z electrons per molecule. Divide the electrons of a molecule into groups, each with electron number f_j ($\sum f_j = Z$), binding frequency ω_j, and collision frequency γ_j [There may be one or more free electrons ($\omega_j = 0$) per molecule.] Then, the electric polarization (total dipole moment per unit volume) is

$$\mathbf{P}(\mathbf{x}) = N\sum_j f_j \mathbf{p}_j = \underbrace{\frac{Ne^2}{m}\sum_j \frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}}_{\varepsilon_0\chi_e}\mathbf{E}(\mathbf{x}) = \varepsilon_0\chi_e\mathbf{E}(\mathbf{x})$$

a macroscopic quantity

(7.50)

(4.36)

a spatial variable

Extending the definitions of the static electric displacement ($\mathbf{D}$) and permittivity (ε):

$$\begin{cases} \mathbf{D}(\mathbf{x}) \equiv \varepsilon_0\mathbf{E}(\mathbf{x}) + \mathbf{P}(\mathbf{x}) = \varepsilon\mathbf{E}(\mathbf{x}) & (4.34)\ (4.37) \\ \varepsilon = \varepsilon_0(1+\chi_e) & (4.38) \end{cases}$$

to fields with $\exp(-i\omega t)$ dependence, we obtain $\mathbf{D}(\mathbf{x}) = \varepsilon\mathbf{E}(\mathbf{x})$ (2)

with $\frac{\varepsilon}{\varepsilon_0} = 1+\chi_e = 1+\frac{Ne^2}{\varepsilon_0 m}\sum_j \frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}$ [generalized dielectric constant] (7.51)

Divide the electrons in the medium into

$$\begin{cases} \text{bound electrons: } \omega_j \neq 0 \\ \text{free electrons: } \omega_j = 0,\ f_j = f_0,\ \gamma_j = \gamma_0 \end{cases}$$

For copper, $f_0 \approx 1$ and $\gamma_0 \approx 4\times10^{13}/s.$

$$(7.51) \Rightarrow \varepsilon = \underbrace{\varepsilon_0 + \frac{Ne^2}{m}\sum_{j\ (\text{bound})}\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}}_{\varepsilon_b} + i\underbrace{\frac{Ne^2 f_0}{m\omega(\gamma_0-i\omega)}}_{\sigma/\omega}$$

due to free electrons

$$= \varepsilon_b + i\frac{\sigma}{\omega} \qquad (7.56)$$

where $\sigma \equiv \frac{f_0Ne^2}{m(\gamma_0-i\omega)}$ [Drude model for the electrical conductivity] (7.58)

In general, $\omega_j > \gamma_j$ (see p. 310). Hence, ε_b is predominantly real. When $\omega \approx \omega_j$, $\text{Im}\,\varepsilon_b$ becomes large. $\Rightarrow$ resonant absorption

Questions:

1. $\varepsilon \to \infty$ as $\omega \to 0$. Hence, the derivation breaks down. Why?
2. What makes the medium dispersive (i.e. ε depends on ω)?

Discussion :

(i) $\mathbf{D} = \varepsilon\mathbf{E}$ implies a linear relation between $\mathbf{D}$ and $\mathbf{E}$. The linearity results from the assumption that the electron displacement x is sufficient samll so that, in (7.49), $f(\mathbf{x}) \propto x$ and $\mathbf{E}(\mathbf{x})$ can be approximated by a constant $\mathbf{E}(0)$.

(ii) $\varepsilon/\varepsilon_0$ in (7.51) or (7.56) is a *generalized* dielectric constant, which includes contributions from both bound and free electrons. It is thus applicable to both insulating and conducting materials. In the wave fields, free electrons oscillate about an equilibrium position just like the bound electron. Hence, both types of electrons can be treated on equal footing. The generalized ε is an extremely useful quantity. As will be shown, it allows a unified treatment of EM waves in both insulating and conducting materials.

(iii) Write $\varepsilon = \varepsilon' + i\varepsilon''$ [$\varepsilon' = \text{Re}(\varepsilon)$, $\varepsilon'' = \text{Im}(\varepsilon)$]. From (7.56), it can be seen that ε'' is due to γ [i.e. the damping term in (7.49)]. Hence, ε'' is resposible for the attenuation of EM waves in the material. For the insulating material, $\varepsilon'' \ll \varepsilon'$, the attenuation constant is given by Jackson (7.55) in terms of ε''. For a good conductor, $\varepsilon'' \gg \varepsilon'$, the attenuation constant is given by Jackson (5.164) in terms of σ. The attenuation constants in dielectric and conducting materials will be derived later in this chapter.

Note that both bound and free electrons contribute to ε'' [see (7.56)], but contribution from free electrons is usually far more important than bound electrons (why?). Even the insulating material contains a small number of free electrons to give the material a small conductivity.

(iv) ε is derived in the ω-space for a harmonic field of arbitrary frequency. Hence, $\mathbf{D}(\omega)=\varepsilon(\omega)\mathbf{E}(\omega)$ is a constitutive relation in ω-space valid for all ω. For multi-frequency fields, we may obtain the t-space $\mathbf{D}$ through a Fourier transformation

$$\mathbf{D}(t)=\tfrac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{D}(\omega)e^{-i\omega t}d\omega$$

$$=\tfrac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\varepsilon(\omega)}_{\boxed{\text{in general}}}\mathbf{E}(\omega)e^{-i\omega t}d\omega\ [\neq\varepsilon\mathbf{E}(t)] \quad (3)$$

$$\boxed{\varepsilon(\omega)=\varepsilon_0+\frac{Ne^2}{m}\sum_{j\ (\text{bound})}\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}+i\frac{Ne^2 f_0}{m\omega(\gamma_0-i\omega)}}$$

Since $\mathbf{E}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{E}(\omega)e^{-i\omega t}d\omega$, we find from (3) that, in general, $\mathbf{D}(t)\neq\varepsilon\mathbf{E}(t)$ because ε is a function of ω. There are, however, 2 special cases for which (3) will yield $\mathbf{D}=\varepsilon\mathbf{E}$ in t-space, as discussed in (v) and (vi) below.

(v) Consider a static ($\omega=0$) electric field $\mathbf{E}$ in a dielectric medium without free electrons ($f_0=0$), we have

$$\mathbf{E}(\omega)=\int_{-\infty}^{\infty}\mathbf{E}e^{i\omega t}dt=\mathbf{E}\int_{-\infty}^{\infty}e^{i\omega t}dt=2\pi\mathbf{E}\delta(\omega)$$

$$\varepsilon(\omega)=\varepsilon_0+\frac{Ne^2}{m}\sum_{j\ (\text{bound})}\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}+i\frac{Ne^2 f_0}{m\omega(\gamma_0-i\omega)}$$

$$\xrightarrow{\boxed{\omega=0,\ f_0=0}}\ =\varepsilon_0+\frac{Ne^2}{m}\sum_{j\ (\text{bound})}\frac{f_j}{\omega_j^2}$$

$$=\varepsilon_b\quad[\varepsilon_b\text{ is real.}]$$

Thus, in t-space, we have a static $\mathbf{D}$ given by

$$\mathbf{D}=\tfrac{1}{2\pi}\int_{-\infty}^{\infty}\varepsilon(\omega)\mathbf{E}(\omega)e^{-i\omega t}d\omega=\tfrac{\varepsilon_b}{2\pi}\int_{-\infty}^{\infty}2\pi\mathbf{E}\delta(\omega)e^{-i\omega t}d\omega$$

$$=\varepsilon_b\mathbf{E},$$

This recovers the static relation in (4.37) without making any approximation.

(vi) For time-dependent fields in a medium with negligible dispersion [i.e. $\varepsilon(\omega)\approx\varepsilon(\omega_0)$] and negligible loss (i.e. $\gamma_j\approx 0$), we have

$$\mathbf{D}(t)=\tfrac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{D}(\omega)e^{-i\omega t}d\omega\approx\tfrac{1}{2\pi}\varepsilon(\omega_0)\int_{-\infty}^{\infty}\mathbf{E}(\omega)e^{-i\omega t}d\omega=\varepsilon(\omega_0)\mathbf{E}(t),$$

where $\varepsilon(\omega_0)=\varepsilon_0+\frac{Ne^2}{m}\sum_j\frac{f_j}{\omega_j^2-\omega_0^2-i\omega\gamma_j}\approx\varepsilon_0+\frac{Ne^2}{m}\sum_j\frac{f_j}{\omega_j^2-\omega_0^2}$

This explains assumption (1) on p. 259 for the derivation of (6.107); namely, the macroscopic medium is linear in its electrical property and it has negligible dispersion and negligible loss. Under this assumption, we may write $\mathbf{D}(t)=\varepsilon\mathbf{E}(t)$. Hence, in (6.105), we have $\mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{D}=\varepsilon\mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{E}=\frac{\varepsilon}{2}\frac{\partial}{\partial t}\mathbf{E}\cdot\mathbf{E}=\frac{1}{2}\frac{\partial}{\partial t}\mathbf{E}\cdot\mathbf{D}$.

Questions:

1. Assume an electromagnetic signal is propagating in the medium. What is the condition on the signal in order for $\varepsilon(\omega)\approx\varepsilon(\omega_0)$?
2. Why is the assumption of "negligible loss" also required?

A note about terminology: In general, the electric permitivity is a tensor (denote it by $\ddot{\boldsymbol{\varepsilon}}$) and we may write

$$\mathbf{D}=\ddot{\boldsymbol{\varepsilon}}\cdot\mathbf{E},\quad\text{where }\ddot{\boldsymbol{\varepsilon}}=\begin{bmatrix}\varepsilon_{11}&\varepsilon_{12}&\varepsilon_{13}\\ \varepsilon_{21}&\varepsilon_{22}&\varepsilon_{23}\\ \varepsilon_{31}&\varepsilon_{32}&\varepsilon_{33}\end{bmatrix}$$

The electrical property of the medium is	if
uniform (or homogeneous)	$\ddot{\boldsymbol{\varepsilon}}$ is indept. of $\mathbf{x}$
linear	$\ddot{\boldsymbol{\varepsilon}}$ is indept. of $\mathbf{E}$
nondispersive	$\ddot{\boldsymbol{\varepsilon}}$ is indept. of ω
isotropic	$\varepsilon_{11}=\varepsilon_{22}=\varepsilon_{33}$, $\varepsilon_{ij}=0$ if $i\neq j$

II. Plane Wave Equations in Dielectrics and Conductors - A Unified Formalism

Basic Equations:

Macroscopic Maxwell equations:

$$\begin{cases} \nabla\cdot\mathbf{D}(\mathbf{x},t)=\rho_{free}(\mathbf{x},t) \\ \nabla\cdot\mathbf{B}(\mathbf{x},t)=0 \\ \nabla\times\mathbf{E}(\mathbf{x},t)=-\frac{\partial}{\partial t}\mathbf{B}(\mathbf{x},t) \\ \nabla\times\mathbf{H}(\mathbf{x},t)=\mathbf{J}_{free}(\mathbf{x},t)+\frac{\partial}{\partial t}\mathbf{D}(\mathbf{x},t) \end{cases} \quad (4)$$

ρ_{free}, $\mathbf{J}_{free}$ are due to free electrons. They are neglected in (7.1).
$\mathbf{E}(\mathbf{x},t)$, $\mathbf{D}(\mathbf{x},t)$, $\mathbf{B}(\mathbf{x},t)$, and $\mathbf{H}(\mathbf{x},t)$ here are **E**, **D**, **B**, and **H** in (7.1).

Equation of continuity (conservation of free charges):

$$\frac{\partial}{\partial t}\rho_{free}(\mathbf{x},t)+\nabla\cdot\mathbf{J}_{free}(\mathbf{x},t)=0 \quad (5)$$

As discussed earlier, the constitutive relations $\mathbf{D}=\varepsilon_b\mathbf{E}$ (for bound electrons) and $\mathbf{D}=\varepsilon\mathbf{E}$ (for both bound and free electrons) are in general applicable only in the ω-space. Similarly, $\mathbf{B}=\mu\mathbf{H}$ and $\mathbf{J}=\sigma\mathbf{E}$ are also ω-space relations. To utilize these relation, we go to the ω-space by assuming harmonic time dependence for the fields.

Assumption 1: harmonic time dependence (ω: real and positive)

$$\text{Let}\ \underbrace{\begin{Bmatrix} \mathbf{E}(\mathbf{x},t) \\ \mathbf{D}(\mathbf{x},t) \\ \mathbf{B}(\mathbf{x},t) \\ \mathbf{H}(\mathbf{x},t) \\ \mathbf{J}(\mathbf{x},t) \\ \rho(\mathbf{x},t) \end{Bmatrix}}_{\text{real}} = \text{Re}\left[\begin{Bmatrix} \mathbf{E}(\mathbf{x}) \\ \mathbf{D}(\mathbf{x}) \\ \mathbf{B}(\mathbf{x}) \\ \mathbf{H}(\mathbf{x}) \\ \mathbf{J}(\mathbf{x}) \\ \rho(\mathbf{x}) \end{Bmatrix} e^{-i\omega t}\right] \ \text{with .}$$

real complex (called the phasor)

By convention, the LHS is the real part of the RHS.

$\mathbf{E}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$ here are **E**, **B** in (7.2) and (7.3)

$$\begin{cases} \nabla\cdot\mathbf{D}(\mathbf{x},t)=\rho_{free}(\mathbf{x},t) \\ \nabla\cdot\mathbf{B}(\mathbf{x},t)=0 \\ \nabla\times\mathbf{E}(\mathbf{x},t)=-\frac{\partial}{\partial t}\mathbf{B}(\mathbf{x},t) \\ \nabla\times\mathbf{H}(\mathbf{x},t)=\mathbf{J}_{free}(\mathbf{x},t)+\frac{\partial}{\partial t}\mathbf{D}(\mathbf{x},t) \end{cases} \Rightarrow \begin{cases} \nabla\cdot\mathbf{D}(\mathbf{x})=\rho_{free}(\mathbf{x}) \\ \nabla\cdot\mathbf{B}(\mathbf{x})=0 \\ \nabla\times\mathbf{E}(\mathbf{x})=i\omega\mathbf{B}(\mathbf{x}) \\ \nabla\times\mathbf{H}(\mathbf{x})=\mathbf{J}_{free}(\mathbf{x})-i\omega\mathbf{D}(\mathbf{x}) \end{cases} \quad (6)$$

$$\frac{\partial}{\partial t}\rho_{free}(\mathbf{x},t)+\nabla\cdot\mathbf{J}_{free}(\mathbf{x},t)=0 \ \Rightarrow\ -i\omega\rho_{free}(\mathbf{x})+\nabla\cdot\mathbf{J}_{free}(\mathbf{x})=0 \quad (7)$$

Ohm's law: (5.159) and P. 320

Assumption 2: linear and isotropic medium, i.e.

$\mathbf{D}(\mathbf{x})=\varepsilon_b\mathbf{E}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})=\mu\mathbf{H}(\mathbf{x})$, $\mathbf{J}_{free}(\mathbf{x})=\sigma\mathbf{E}(\mathbf{x})$ or $(=\rho\mathbf{v})$.

Note: We have used 2 definitions of **D**. Here, $\mathbf{D}=\varepsilon_b\mathbf{E}$. In (2), $\mathbf{D}=\varepsilon\mathbf{E}=(\varepsilon_b+i\frac{\sigma}{\omega})\mathbf{E}$. (**D** has no physical significance.)

Rewrite (7): $-i\omega\rho_{free}(\mathbf{x})+\nabla\cdot\mathbf{J}_{free}(\mathbf{x})=0$

$$\Rightarrow\quad -i\omega\rho_{free}(\mathbf{x})+\nabla\cdot\sigma\mathbf{E}(\mathbf{x})=0\ \Rightarrow \rho_{free}(\mathbf{x})=\frac{\nabla\cdot\sigma\mathbf{E}(\mathbf{x})}{i\omega}$$

Hence, $\nabla\cdot\mathbf{D}(\mathbf{x})=\rho_{free}(\mathbf{x})\Rightarrow\nabla\cdot\varepsilon_b\mathbf{E}(\mathbf{x})=\frac{1}{i\omega}\nabla\cdot\sigma\mathbf{E}(\mathbf{x})$

$$\Rightarrow\quad \nabla\cdot(\varepsilon_b+i\tfrac{\sigma}{\omega})\mathbf{E}(\mathbf{x})=0\Rightarrow\nabla\cdot\varepsilon\mathbf{E}(\mathbf{x})=0, \quad (8)$$

where $\varepsilon\equiv\varepsilon_b+i\frac{\sigma}{\omega}$ takes the form of the generalized ε derived in (7.51) and (7.56). Similarly, $\nabla\times\mathbf{H}(\mathbf{x})=\mathbf{J}_{free}(\mathbf{x})-i\omega\mathbf{D}(\mathbf{x})$ gives

$$\nabla\times\mathbf{H}(\mathbf{x})=\sigma\mathbf{E}(\mathbf{x})-i\omega\varepsilon_b\mathbf{E}(\mathbf{x})=-i\omega[\varepsilon_b+i\tfrac{\sigma}{\omega}]\mathbf{E}(\mathbf{x})=-i\omega\varepsilon\mathbf{E}(\mathbf{x}), \quad (9)$$

where again ε_b and σ are combined in the same manner as in (8).

This gives an alternative derivation of the generalized ε. However, ε in (7.51) and (7.56) gives the explicit expressions for ε_b and σ.

Using (8) and (9), we write the macroscopic Maxwell equations for harmonic fields in a linear and isotropic medium in terms of phasor fields and the generalized ε:

$$\begin{cases} \nabla\cdot\varepsilon\mathbf{E}(\mathbf{x})=0 \\ \nabla\cdot\mathbf{B}(\mathbf{x})=0 \\ \nabla\times\mathbf{E}(\mathbf{x})=i\omega\mathbf{B}(\mathbf{x}) \\ \nabla\times\mathbf{H}(\mathbf{x})=-i\omega\varepsilon\mathbf{E}(\mathbf{x}) \end{cases} \quad (10)$$

Discussion:

(i) Bound electrons and free electrons are separated in the Maxwell equations in (4) and (6), where ε_b contains the effects of bound electrons and σ contains the effects of free electrons.

(ii) Bound electrons and free electrons are combined in the Maxwell equations in (10), where $\varepsilon(=\varepsilon_b+i\frac{\sigma}{\omega})$ contains the effects of both bound and free electrons.

Assumption 3 : uniform medium (i.e. ε, μ independent of $\mathbf{x}$)

$$\begin{cases} \nabla\cdot\varepsilon\mathbf{E}(\mathbf{x})=0 \\ \nabla\cdot\mathbf{B}(\mathbf{x})=0 \\ \nabla\times\mathbf{E}(\mathbf{x})=i\omega\mathbf{B}(\mathbf{x}) \\ \nabla\times\mathbf{H}(\mathbf{x})=-i\omega\varepsilon\mathbf{E}(\mathbf{x}) \end{cases} \Rightarrow \begin{cases} \nabla\cdot\mathbf{E}(\mathbf{x})=0 & (11) \\ \nabla\cdot\mathbf{B}(\mathbf{x})=0 & (12) \\ \nabla\times\mathbf{E}(\mathbf{x})=i\omega\mathbf{B}(\mathbf{x}) & (13) \\ \nabla\times\mathbf{B}(\mathbf{x})=-i\omega\mu\varepsilon\mathbf{E}(\mathbf{x}) & (14) \end{cases}$$

$$\nabla\times\begin{Bmatrix}(13)\\(14)\end{Bmatrix} \Rightarrow \nabla^2\begin{Bmatrix}\mathbf{E}(\mathbf{x})\\ \mathbf{B}(\mathbf{x})\end{Bmatrix}+\mu\varepsilon\omega^2\begin{Bmatrix}\mathbf{E}(\mathbf{x})\\ \mathbf{B}(\mathbf{x})\end{Bmatrix}=0 \qquad (15)$$

(15) has the same form as (7.3), which is derived from the source-free Maxwell equations [(7.1)] for a non-conducting medium ($\sigma=0$). However, (15) is applicable to both dielectric and conducting media. In (7.3), $\varepsilon=\varepsilon_b$. In (15), $\varepsilon=\varepsilon_b+i\frac{\sigma}{\omega}$. Solution for (15) and (7.3) takes the same algebraic steps. But with $\varepsilon=\varepsilon_b+i\frac{\sigma}{\omega}$, the solution for (15) will be applicable to both dielectric and conducting media.

Assumption 4 : $\begin{Bmatrix}\mathbf{E}(\mathbf{x})\\ \mathbf{B}(\mathbf{x})\end{Bmatrix}=\begin{Bmatrix}\mathbf{E}_0\\ \mathbf{B}_0\end{Bmatrix}e^{i\mathbf{k}\cdot\mathbf{x}}$ $\boxed{\mathbf{E}_0,\ \mathbf{B}_0 \text{ here are } \mathfrak{E},\ \mathfrak{B} \text{ in (7.8)-(7.12)}}$

$$\nabla^2\begin{Bmatrix}\mathbf{E}(\mathbf{x})\\ \mathbf{B}(\mathbf{x})\end{Bmatrix}+\mu\varepsilon\omega^2\begin{Bmatrix}\mathbf{E}(\mathbf{x})\\ \mathbf{B}(\mathbf{x})\end{Bmatrix}=0 \Rightarrow \left(-k^2+\mu\varepsilon\omega^2\right)\begin{Bmatrix}\mathbf{E}_0\\ \mathbf{B}_0\end{Bmatrix}=0$$

$$\Rightarrow k=\sqrt{\mu\varepsilon}\,\omega \quad \text{(dispersion relation)} \qquad (16)$$

Note: 1. $k^2\equiv\mathbf{k}\cdot\mathbf{k}$; $|\mathbf{k}|^2\equiv\mathbf{k}\cdot\mathbf{k}^*$.
2. $k^2\neq|\mathbf{k}|^2$ and $k\neq|\mathbf{k}|$ unless $\mathbf{k}$ is real.
3. k can be complex, but $|\mathbf{k}|$ is always real and positive.

$$(11)\text{-}(13)\Rightarrow\begin{cases}\mathbf{k}\cdot\mathbf{E}_0=0 & (17)\\ \mathbf{k}\cdot\mathbf{B}_0=0 & (18)\\ \mathbf{B}_0=\frac{1}{\omega}\mathbf{k}\times\mathbf{E}_0=\sqrt{\mu\varepsilon}\,\frac{\mathbf{k}\times\mathbf{E}_0}{k} & (19)\end{cases}$$

Note: (14) gives $\mathbf{E}_0=-\frac{1}{\omega\mu\varepsilon}\mathbf{k}\times\mathbf{B}_0$, which is implicit in (17) and (19).

$\langle\mathbf{S}\rangle_t$ = time averaged power flow per unit area (called <u>intensity</u>)

$=\langle\mathbf{E}(\mathbf{x},t)\times\mathbf{H}(\mathbf{x},t)\rangle_t$

real quantities

phasors

$$\boxed{\begin{aligned}&\mathbf{E}(\mathbf{x})=\mathbf{E}_0e^{i\mathbf{k}\cdot\mathbf{x}}\\ &\mathbf{H}(\mathbf{x})=\mathbf{H}_0e^{i\mathbf{k}\cdot\mathbf{x}}\\ &\mathbf{H}_0=\frac{\mathbf{B}_0}{\mu}=\sqrt{\frac{\varepsilon}{\mu}}\frac{\mathbf{k}\times\mathbf{E}_0}{k}\quad(19)\end{aligned}}$$

$=\frac{1}{2}\mathrm{Re}\left[\mathbf{E}^*(\mathbf{x})\times\mathbf{H}(\mathbf{x})\right]$

$=\frac{1}{2}\mathrm{Re}\left[\sqrt{\frac{\varepsilon}{\mu}}\frac{1}{k}\mathbf{E}_0^*\times(\mathbf{k}\times\mathbf{E}_0)e^{i(\mathbf{k}-\mathbf{k}^*)\cdot\mathbf{x}}\right]$

$\boxed{|\mathbf{E}_0|^2\equiv\mathbf{E}_0\cdot\mathbf{E}_0^*}$

$$=\begin{cases}\frac{1}{2}\mathrm{Re}\left\{\sqrt{\frac{\varepsilon}{\mu}}\frac{1}{k}\left[\mathbf{k}|\mathbf{E}_0|^2-\mathbf{E}_0\left(\mathbf{k}\cdot\mathbf{E}_0^*\right)\right]e^{i(\mathbf{k}-\mathbf{k}^*)\cdot\mathbf{x}}\right\} & (20)\\ \frac{1}{2\omega}\mathrm{Re}\left\{\frac{1}{\mu}\left[\mathbf{k}|\mathbf{E}_0|^2-\mathbf{E}_0\left(\mathbf{k}\cdot\mathbf{E}_0^*\right)\right]e^{i(\mathbf{k}-\mathbf{k}^*)\cdot\mathbf{x}}\right\} & (20')\end{cases}$$

Note: $\langle\mathbf{E}(\mathbf{x},t)\times\mathbf{H}(\mathbf{x},t)\rangle_t=\frac{1}{2}\mathrm{Re}[\mathbf{E}^*(\mathbf{x})\times\mathbf{H}(\mathbf{x})]$ is derived in Sec. 6.9 of lecture notes.

Discussion:

(i) Assuming μ, ε are given, (16)-(19) are conditions imposed on ω, $\mathbf{k}$, $\mathbf{E}_0$, $\mathbf{B}_0$ by the Maxwell equations.

(ii) The derivation of (16)-(19) only requires μ, ε, ω, $\mathbf{k}$, $\mathbf{E}_0$, and $\mathbf{B}_0$ to be constants, but not necessarily real (we have assumed ω to be real). Thus, any set of complex constants μ, ε, ω, $\mathbf{k}$, $\mathbf{E}_0$, and $\mathbf{B}_0$ can be a valid solution of the Maxwell equations provided they satisfy (16)-(19) and the boundary conditions (if applicable).

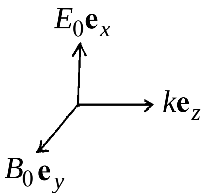
(iii) The generalized ε is in general a complex number. μ can also be a complex number. Either complex ε or complex μ can lead to complex solutions for $\mathbf{k}$, $\mathbf{E}_0$, and $\mathbf{B}_0$. Even when ε and μ are real, boundary conditions (if applicable) can lead to complex solutions for $\mathbf{k}$, $\mathbf{E}_0$, and $\mathbf{B}_0$ [to be shown in Sec. 7.4, Eq. (48)].

(iv) Under assumptions 1 and 4, the fields ($\sim e^{-i\omega t+i\mathbf{k}\cdot\mathbf{x}}$) are those of a plane wave; namely, the surface of constant phase is a plane (see following examples). There are 2 types of plane waves depending on the form of the wave vector $\mathbf{k}$ (also called the propagation constant).

a. Homogeneous plane wave

Consider the solution:

$$\begin{cases} \mathbf{k} = k\mathbf{e}_z \\ \mathbf{E}_0 = E_0\mathbf{e}_x \\ \mathbf{B}_0 = B_0\mathbf{e}_y \end{cases} \quad \text{with} \quad \begin{cases} B_0 = \sqrt{\mu\varepsilon}E_0 \\ k = \sqrt{\mu\varepsilon}\,\omega \end{cases}$$



where $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are real unit vectors, but E_0, B_0, and k can all be complex. This clearly satisfies (16)-(19) and is the most familiar type of plane waves. Any plane perpendicular to the z-axis is a plane of constant phase.

Optional

b. Inhomogeneous plane wave

Consider another solution satisfying (16)-(19):

$$\begin{cases} \mathbf{k} = k_x\mathbf{e}_x + ik_z\mathbf{e}_z \\ \mathbf{E}_0 = E_{0x}\mathbf{e}_x + iE_{0z}\mathbf{e}_z \\ \mathbf{B}_0 = iB_{0y}\mathbf{e}_y \end{cases} \text{with} \begin{cases} k^2 = \mathbf{k}\cdot\mathbf{k} = k_x^2 - k_z^2 = \mu\varepsilon\omega^2 \\ \mathbf{k}\cdot\mathbf{E}_0 = k_xE_{0x} - k_zE_{0z} = 0 \\ B_{0y} = (-k_xE_{0z} + k_zE_{0x})/\omega \end{cases} \quad (21)$$

where k_x, k_z, E_{0x}, E_{0z}, and B_{0y} are all real constants.

> $\mathbf{k} = k_x\mathbf{e}_x + ik_z\mathbf{e}_z$ defined here can be converted to the form $\mathbf{k} = k\mathbf{n} = k(\mathbf{n}_R + i\mathbf{n}_I)$ as used on p. 298 of Jackson. Here, we reserve the notation $\mathbf{n}$ for later use as a *real* unit vector.

The physical meaning of such a solution becomes clear when we construct the physical quantity $\mathbf{E}(\mathbf{x},t)$ from the phasor $\mathbf{E}(\mathbf{x})$.

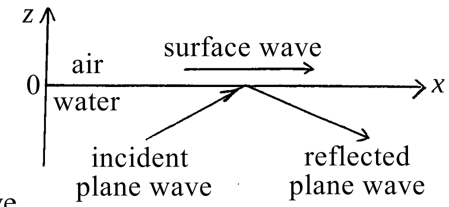
$$\mathbf{E}(\mathbf{x},t) = \mathrm{Re}\left[\mathbf{E}_0e^{i\mathbf{k}\cdot\mathbf{x}}e^{-i\omega t}\right] = \mathrm{Re}\left[\left(E_{0x}\mathbf{e}_x + iE_{0z}\mathbf{e}_z\right)e^{-i\omega t+ik_xx-k_zz}\right]$$
$$= \left[E_{0x}\cos\left(\omega t - k_xx\right)\mathbf{e}_x + E_{0z}\sin\left(\omega t - k_xx\right)\mathbf{e}_z\right]e^{-k_zz}$$

Interesting phenomenon

Rewrite $\mathbf{E}(\mathbf{x},t) = \left[E_{0x}\cos\left(\omega t - k_xx\right)\mathbf{e}_x + E_{0z}\sin\left(\omega t - k_xx\right)\mathbf{e}_z\right]e^{-k_zz}$

This represents a surface wave in the $z \geq 0$ half space. It propagates along the x-direction with an amplitude peaking at $z = 0$ and decreasing exponentially along the positive z-direction. The surface wave is also called an inhomogeneous plane wave (p.298). Any plane perpendicular to the x-axis is a plane of constant phase.

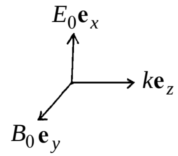
When a plane wave incident from a dense medium onto a tenuous medium (e.g. water to air) is totally reflected from the interface, fields in the tenuous medium form such a surface wave due to boundary conditions at $z = 0$. This will be discussed in Sec. 7.4.



(v) Orthogonality of vectors **k,** $\mathbf{E}_0$, and $\mathbf{B}_0$ in (17)-(19)

$$(17)\text{-}(19) \Rightarrow \begin{Bmatrix} \mathbf{k}\cdot\mathbf{E}_0 = 0 \\ \mathbf{k}\cdot\mathbf{B}_0 = 0 \\ \mathbf{E}_0\cdot\mathbf{B}_0 = 0 \end{Bmatrix} \Rightarrow \left[\begin{matrix}\mathbf{E}_0,\ \mathbf{B}_0,\ \text{and } \mathbf{k} \text{ are } \textit{algebraically} \\ \text{orthogonal to one another}\end{matrix}\right]$$

For the homogeneous plane wave, $\mathbf{E}_0(=E_0\mathbf{e}_x)$, $\mathbf{B}_0(=B_0\mathbf{e}_y)$, and $\mathbf{k}(=k\mathbf{e}_z)$ are also *geometrically* orthogonal.



For the inhomogeneous plane wave, the algebraic orthogonality of $\mathbf{k}(=k_x\mathbf{e}_x + ik_z\mathbf{e}_z)$, $\mathbf{E}_0(=E_{0x}\mathbf{e}_x + iE_{0z}\mathbf{e}_z)$, and $\mathbf{B}_0(=iB_{0y}\mathbf{e}_y)$ does not imply geometric orthogonality because $\mathbf{k}$ and $\mathbf{E}_0$ do not have clear geometric directions. In t-space, we have just shown

$$\mathbf{E}(\mathbf{x},t) = \left[E_{0x}\cos\left(\omega t - k_xx\right)\mathbf{e}_x + E_{0z}\sin\left(\omega t - k_xx\right)\mathbf{e}_z\right]e^{-k_zz},$$

which shows that the wave propagates along the x-direction, but its **E**-field also has an x-component.

(vi) $\mathbf{k}\cdot\mathbf{E}_0 = 0$ does not necessarily imply $\mathbf{k}\cdot\mathbf{E}_0^* = 0$.

(A similar comment is made in Jackson, see footnote on p. 298.)

For the homogeneous plane wave ($\mathbf{k} = k\mathbf{e}_z$, $\mathbf{E}_0 = E_0\mathbf{e}_x$),

$$\mathbf{k}\cdot\mathbf{E}_0 = 0$$

$$\Rightarrow \mathbf{k}\cdot\mathbf{E}_0^* = 0$$

But for the inhomogeneous plane wave: $\begin{cases}\mathbf{k} = k_x\mathbf{e}_x + ik_z\mathbf{e}_z \\ \mathbf{E}_0 = E_{0x}\mathbf{e}_x + iE_{0z}\mathbf{e}_z\end{cases}$

$$\mathbf{k}\cdot\mathbf{E}_0 = 0$$

$$\Rightarrow k_xE_{0x} - k_zE_{0z} = 0$$

$$\Rightarrow k_xE_{0x} = k_zE_{0z}$$

$$\Rightarrow \mathbf{k}\cdot\mathbf{E}_0^* = k_xE_{0x} + k_zE_{0z} = 2k_zE_{0z} \neq 0$$

Thus, in general, the $\mathbf{k}\cdot\mathbf{E}_0^*$ term must be kept in (20) [see Eqs. (53) and (54) in Sec. 7.4.]

(vii) Rewrite (16)-(19): $\begin{cases} k = \sqrt{\mu\varepsilon}\,\omega & (16) \\ \mathbf{k}\cdot\mathbf{E}_0 = 0 & (17) \\ \mathbf{k}\cdot\mathbf{B}_0 = 0 & (18) \\ \mathbf{B}_0 = \frac{1}{\omega}\mathbf{k}\times\mathbf{E}_0 = \sqrt{\mu\varepsilon}\,\frac{\mathbf{k}\times\mathbf{E}_0}{k} & (19)\end{cases}$

This set of equations is equivalent to (7.9)-(9.11) in Jackson, with ε in (7.9)-(7.11) interpreted as the generalized ε. The difference is in notations. In (7.9)-(7.11), $\mathbf{n}$ is in general a complex unit vector subject to the condition $\mathbf{n}\cdot\mathbf{n} = 1$, which leads to condition (7.15). Here, we treat $\mathbf{k}$ as complex vector [as in (21)] without any additional condition except for those imposed by the Maxwell equations [(16)-(19)]. Thus, the complex $\mathbf{k}$ is more convenient to use, as has been demonstrated in (21) and will be seen again in Sec. 7.4.

Assumption 5: $\mathbf{k} = k\mathbf{n} = (k_r + ik_i)\mathbf{n}$

k: complex constant
$\mathbf{n}$: real unit vector

Then, (17)-(19) can be written

$$\begin{cases}\mathbf{n}\cdot\mathbf{E}_0 = 0 & (22) \\ \mathbf{n}\cdot\mathbf{B}_0 = 0 & (23) \\ \mathbf{B}_0 = \sqrt{\mu\varepsilon}\,\mathbf{n}\times\mathbf{E}_0 & (24)\end{cases}$$

(16), (22)-(24) here are equivalent to (7.9)-(7.11) when $\mathbf{n}$ in (7.9)-(7.11) is a real unit vector and ε in (7.9)-(7.11) is interpreted as the generalized ε.

and $\mathbf{k}\cdot\mathbf{E}_0 = 0 \;\Rightarrow \mathbf{k}\cdot\mathbf{E}_0^* = 0$. Thus, (20) reduces to

$$\langle\mathbf{S}\rangle_t = \tfrac{1}{2}\mathrm{Re}\left[\sqrt{\tfrac{\varepsilon}{\mu}}\,|\mathbf{E}_0|^2\, e^{-2k_i\mathbf{n}\cdot\mathbf{x}}\right]\mathbf{n} \qquad (25)$$

Under assumption 5, the wave vector $\mathbf{k}$ has a geometric direction ($\mathbf{n}$). Hence, (22)-(24) now represent *homogeneous* plane waves with *geometrically* orthogonal $\mathbf{k}$, $\mathbf{E}_0$, and $\mathbf{B}_0$.

In $\mathbf{k} = (k_r + ik_i)\mathbf{n}$, $k_r\,(=\frac{2\pi}{\lambda})$ gives the wavelength, k_i gives the rate of attenuation, and $\mathbf{n}$ gives the direction of wave propagation.

See Chap. 6.9

Definition of impedance and admittance of the medium:

Rewrite $\quad \mathbf{B}_0 = \sqrt{\mu\varepsilon}\,\mathbf{n}\times\mathbf{E}_0 \qquad (24)$

In engineering literature, this equation is often written

$$\mathbf{H}_0 = \frac{\mathbf{B}_0}{\mu} = \frac{\mathbf{n}\times\mathbf{E}_0}{Z}, \qquad (7.11)$$

where $Z \equiv \sqrt{\frac{\mu}{\varepsilon}}$ is the impedance of the medium (p. 297). The admittance of the medium is defined as $Y \equiv \frac{1}{Z} = \sqrt{\frac{\varepsilon}{\mu}}$. Z and Y are *intrinsic* properties of the medium.

Let $\mathbf{E}_0 = E_0\boldsymbol{\varepsilon}_1$ and $\mathbf{B}_0 = B_0\boldsymbol{\varepsilon}_2$. Because $\mathbf{n}$, $\boldsymbol{\varepsilon}_1$, and $\boldsymbol{\varepsilon}_2$ are mutually perpendicular, we have $\quad Z = E_0 / H_0$

$\Rightarrow Z$ is the (complex) amplitude ratio of E_0 and H_0 in the medium (The definition is valid even if μ, ε are complex). In vacuum,

$$Z = Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7\ \Omega$$

III. Properties of Plane Waves in Dielectrics and Conductors [A unified treatment of Secs. 5.18, 7.1, 7.2, 7.5, and 8.1 using the generalized ε in (7.51)]

In Sec. II, under assumptions 1-5, we have obtained the familiar homogeneous plane-wave equations:

$$\begin{cases} k=\sqrt{\mu\varepsilon}\,\omega \ \left[k:\ \underline{\text{wave number}} \text{ or } \underline{\text{propagation constant}}\right] & (16) \\ \mathbf{n}\cdot\mathbf{E}_0=0 & (22) \\ \mathbf{n}\cdot\mathbf{B}_0=0 & (23) \\ \mathbf{B}_0=\sqrt{\mu\varepsilon}\,\mathbf{n}\times\mathbf{E}_0 & (24) \\ \langle\mathbf{S}\rangle_t=\frac{1}{2}\mathrm{Re}\left[\sqrt{\frac{\varepsilon}{\mu}}\,|\mathbf{E}_0|^2\,e^{-2k_i\mathbf{n}\cdot\mathbf{x}}\right]\mathbf{n} & (25) \end{cases}$$

for a uniform and isotropic medium, where $\mathbf{E}_0$ and $\mathbf{B}_0$ are (complex) amplitude constants of the fields: $\begin{pmatrix}\mathbf{E}(\mathbf{x},t)\\ \mathbf{B}(\mathbf{x},t)\end{pmatrix}=\mathrm{Re}\left[\begin{pmatrix}\mathbf{E}_0\\ \mathbf{B}_0\end{pmatrix}e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}\right]$

and $\mathbf{n}$ is a (real) direction unit vector of the (complex) wave vector or propagation vector: $\mathbf{k}=k\mathbf{n}=(k_r+ik_i)\mathbf{n}$

On the basis of these equations, we consider below 4 radically different cases which are distinguishable by the wave frequency and the medium property characterized by the generalized permittivity:

$$\varepsilon=\underbrace{\varepsilon_0+\frac{Ne^2}{m}\sum_{j\text{ (bound)}}\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}}_{\varepsilon_b}+i\underbrace{\frac{Ne^2 f_0}{m\omega(\gamma_0-i\omega)}}_{\sigma/\omega} \qquad (7.51),\ (7.56)$$

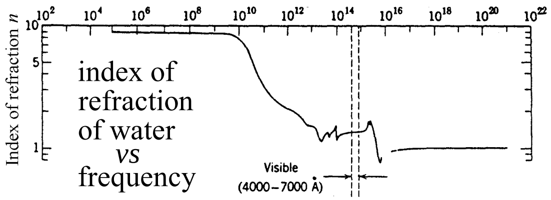
Case 1. Waves in a dielectric medium
Case 2. Waves in a good conductor
Case 3. Waves at optical frequencies and beyond
Case 4. Waves in a plasma

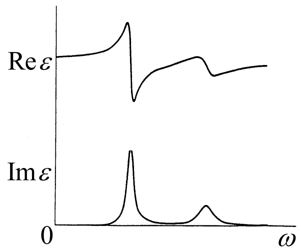
Case 1: Waves in a dielectric medium [§ 7.1, § 7.2, § 7.5 (Part B)]

$$\varepsilon=\varepsilon_0+\frac{Ne^2}{m}\sum_{j\text{ (bound)}}\frac{f_j}{\omega_j^2-\omega^2-i\omega\gamma_j}+i\underbrace{\cancel{\frac{Ne^2 f_0}{m\omega(\gamma_0-i\omega)}}}_{\text{negligible }(\because\ f_0\,=\,0\text{ or very small})} \qquad (7.51)$$

Properties of ε:

1. In general, $\gamma_j << \omega_j$ (see p.310), hence Imε << Reε.
2. When ω is near each ω_j (binding frequency of the j^{th} group of electrons), ε exhibits resonant behavior in the form of anomalous dispersion and resonant absorption.
3. As ω passes more ω_j's, Reε decreases.





Case 1.1: Lossless dielectric (μ and ε are real. Secs. 7.1 and 7.2)

Plane waves in a dielectric medium governed by Eqs. (16), (22)-(25) are best examplified by the simple case of no medium loss (i.e. μ and ε are both real).

Time-averaged quantities:

$$(25)\Rightarrow\langle\mathbf{S}\rangle_t=\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2\,\mathbf{n}\ \left[\frac{\text{intensity: time averaged}}{\text{Poynting vector}}\right] \qquad (7.13)$$

$$\mathbf{E}(\mathbf{x})=\mathbf{E}_0e^{i\mathbf{k}\cdot\mathbf{x}},\ \mathbf{B}(\mathbf{x})=\mathbf{B}_0e^{i\mathbf{k}\cdot\mathbf{x}}=\sqrt{\mu\varepsilon}\,\mathbf{n}\times\mathbf{E}_0e^{i\mathbf{k}\cdot\mathbf{x}}$$

$\Rightarrow\langle u\rangle_t$ = time averaged energy density

$$=\frac{1}{4}\underbrace{\left[\varepsilon\mathbf{E}(\mathbf{x})\cdot\mathbf{E}^*(\mathbf{x})+\frac{1}{\mu}\mathbf{B}(\mathbf{x})\cdot\mathbf{B}^*(\mathbf{x})\right]}=\frac{\varepsilon}{2}|\mathbf{E}_0|^2 \qquad (7.14)$$

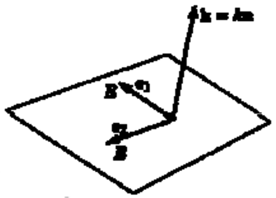
These 2 terms are equal [$\because\ \mathbf{B}_0=\sqrt{\mu\varepsilon}\,\mathbf{n}\times\mathbf{E}_0$ (24)].

$\Rightarrow$ equipartition of E-field and B-field energies

$$(7.13)\text{ and }(7.14)\Rightarrow\langle\mathbf{S}\rangle_t\cdot\mathbf{n}=\langle u\rangle_t v_g,\ \text{where } v_g=\frac{d\omega}{dk}=\frac{1}{\sqrt{\mu\varepsilon}}\ \left(=\frac{\omega}{k}\right)$$

Time-dependent fields:

Let $\begin{cases} \mathbf{E}(\mathbf{x}) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}} = E_0 e^{i\mathbf{k}\cdot\mathbf{x}} \boldsymbol{\varepsilon}_1 \\ \mathbf{B}(\mathbf{x}) = \sqrt{\mu\varepsilon}\,\mathbf{n}\times\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}} = \sqrt{\mu\varepsilon} E_0 e^{i\mathbf{k}\cdot\mathbf{x}} \boldsymbol{\varepsilon}_2 \end{cases}$



$\boldsymbol{\varepsilon}_1 \perp \boldsymbol{\varepsilon}_1,\ \mathbf{n} = \boldsymbol{\varepsilon}_1 \times \boldsymbol{\varepsilon}_1$

where $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$, $\mathbf{k}$ are mutually perpendicular and the fields are linearly polarized.

Further let $E_0 = |E_0| e^{i\theta}$, then

$$\begin{cases} \mathbf{E}(\mathbf{x},t) = \text{Re}[\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}] = |E_0| \cos(\mathbf{k}\cdot\mathbf{x} - \omega t + \theta)\boldsymbol{\varepsilon}_1 \\ \mathbf{B}(\mathbf{x},t) = \sqrt{\mu\varepsilon}\,\text{Re}[\mathbf{n}\times\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}] = \sqrt{\mu\varepsilon}|E_0| \cos(\mathbf{k}\cdot\mathbf{x} - \omega t + \theta)\boldsymbol{\varepsilon}_2 \end{cases}$$

μ and ε are real. $\Rightarrow \mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ are in phase.

$\mathbf{S}(\mathbf{x},t) = \mathbf{E}(\mathbf{x},t) \times \mathbf{H}(\mathbf{x},t) =$ instantaneous Poynting vector [(6.109)]

$$= \sqrt{\tfrac{\varepsilon}{\mu}} |E_0|^2 \cos^2(\mathbf{k}\cdot\mathbf{x} - \omega t + \theta)\mathbf{n}$$

$\Rightarrow$ At a fixed position, $\mathbf{S}$ varies between 0 and the maximum (positive) value at the frequency 2ω.

Two linearly polarized waves can be combined to give

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}_1(\mathbf{x},t) + \mathbf{E}_2(\mathbf{x},t) = (\boldsymbol{\varepsilon}_1 E_1 + \boldsymbol{\varepsilon}_2 E_2) e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \quad (7.19)$$

(7.19) consists of the following 3 cases:

1. (7.19) is a linearly polarized plane wave if $\mathbf{E}_1$ and $\mathbf{E}_2$ are in phase, i.e. if $E_1 = |E_1| e^{i\theta}$ and $E_2 = |E_2| e^{i\theta}$
2. (7.19) is an elliptically polarized plane wave if $\mathbf{E}_1$ and $\mathbf{E}_2$ are not in phase, i.e. if $E_1 = |E_1| e^{i\theta}$ and $E_2 = |E_2| e^{i(\theta+\varphi)}$.
3. (7.19) is a circularly polarized plane wave (a special case of elliptical polarization) if $|E_1| = |E_2| (= |E_0|)$ and $\varphi = \pm\pi/2$. Hence,

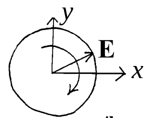
$$\mathbf{E}(\mathbf{x},t) = E_0(\boldsymbol{\varepsilon}_1 \pm i\boldsymbol{\varepsilon}_2) e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \quad (7.20)$$

For an alternative representation, we define $\boldsymbol{\varepsilon}_\pm \equiv \frac{1}{\sqrt{2}}(\boldsymbol{\varepsilon}_1 \pm i\boldsymbol{\varepsilon}_2)$, (7.22) where $\boldsymbol{\varepsilon}_\pm^* \cdot \boldsymbol{\varepsilon}_\pm = 1$ and $\boldsymbol{\varepsilon}_\pm^* \cdot \boldsymbol{\varepsilon}_\mp = 0$. Then, (7.19) [not (7.20)] can be written

$$\mathbf{E}(\mathbf{x},t) = (E_+ \boldsymbol{\varepsilon}_+ + E_- \boldsymbol{\varepsilon}_-) e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \quad (7.24)$$

A specific example of circularly polarized wave:

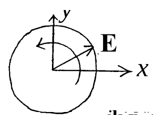
Rewrite (7.20): $\mathbf{E}(\mathbf{x},t) = \text{Re}\left[\mathbf{E}_0 (\boldsymbol{\varepsilon}_1 \pm i\boldsymbol{\varepsilon}_2) e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}\right]$



$E_0(\mathbf{e}_x - i\mathbf{e}_y)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}$ negative helicity

Let $\boldsymbol{\varepsilon}_1 = \mathbf{e}_x$, $\boldsymbol{\varepsilon}_2 = \mathbf{e}_y$, and $\mathbf{n} = \frac{\mathbf{k}}{k} = \mathbf{e}_z$. We have

$$\begin{cases} E_x(\mathbf{x},t) = E_0 \cos(kz - \omega t + \theta) \\ E_y(\mathbf{x},t) = \mp E_0 \sin(kz - \omega t + \theta) \end{cases}$$



$E_0(\mathbf{e}_x + i\mathbf{e}_y)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}$ positive helicity

Exercise: Show that the instantaneous Poynting vector of a circularly polarized plane wave is independent of time.

Medium property: $k = \sqrt{\mu\varepsilon}\,\omega$ [(16)] gives the phase velocity (v)

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}, \text{ where } n = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \text{ (index of refraction)} \quad (7.5)$$

Next, we consider plane waves in a lossy dielectric, where the fields differ only slightly from those in a lossless case dielectric (e.g. $\mathbf{E}$, $\mathbf{B}$ are slightly out of phase). However, as a qualitative difference, the medium absorbs the wave. So, our emphsis will be on the medium properties.

Case 1.2: Lossy dielectric [$\mu$and/or ε are complex , Sec. 7.5 (Part B)]

$k = \sqrt{\mu\varepsilon}\,\omega$ can be written: $k = \text{Re}\sqrt{\mu\varepsilon}\,\omega + i\,\text{Im}\sqrt{\mu\varepsilon}\,\omega = \beta + i\frac{\alpha}{2}$ (7.53)

where $\beta = k_r$ gives (for arbitrary μ and ε)

$$\begin{cases} \text{the wavelength } \lambda = \frac{2\pi}{\beta} \\ \text{the phase velocity } v = \frac{\omega}{\beta} = \frac{1}{\text{Re}\sqrt{\mu\varepsilon}} \\ \text{the index of refraction } n = \frac{c}{v} = \text{Re}\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{[used on p. 314.]} \end{cases}$$

To find the meaning of α, we set $k_i = \frac{\alpha}{2}$ and $\mathbf{n} = \mathbf{e}_z$ in

$$\langle \mathbf{S} \rangle_t = \tfrac{1}{2}\text{Re}\left[\sqrt{\tfrac{\varepsilon}{\mu}} |\mathbf{E}_0|^2 e^{-2k_i \mathbf{n}\cdot\mathbf{x}}\right]\mathbf{n} \quad (25)$$

$$\Rightarrow P = \langle \mathbf{S} \rangle_t \cdot \mathbf{n} = \tfrac{1}{2}\text{Re}\sqrt{\tfrac{\varepsilon}{\mu}} |\mathbf{E}_0|^2 e^{-\alpha z} \left[\begin{matrix}\text{intensity (average} \\ \text{power/unit area)}\end{matrix}\right],$$

Hence, α is the *power* attenuation constant given by

$$\alpha = -\tfrac{1}{P}\tfrac{\partial}{\partial z} P = 2k_i \ [= 2\,\text{Im}\sqrt{\mu\varepsilon}\,\omega] \quad \text{[used on p. 314].}$$

For the common case of weak attenuation, we let

$\mu = real,\ \varepsilon = \varepsilon' + i\varepsilon''$ with $\varepsilon' >> \varepsilon''$

$$\Rightarrow \sqrt{\varepsilon} = \sqrt{\varepsilon'}\left(1 + i\frac{\varepsilon''}{\varepsilon'}\right)^{\frac{1}{2}} \approx \sqrt{\varepsilon'}\left(1 + i\frac{\varepsilon''}{2\varepsilon'}\right)$$

$$\Rightarrow k = \mathrm{Re}\sqrt{\mu\varepsilon}\omega + i\,\mathrm{Im}\sqrt{\mu\varepsilon}\omega \approx \sqrt{\mu\varepsilon'}\omega + \frac{i}{2}\sqrt{\frac{\mu}{\varepsilon'}}\varepsilon''\omega \text{ (for real } \mu \text{ and small } \varepsilon'')$$

$$\Rightarrow \begin{cases} \beta = k_r \approx \sqrt{\mu\varepsilon'}\omega = \sqrt{\frac{\mu\varepsilon'}{\mu_0\varepsilon_0}}\frac{\omega}{c} & \text{(phase constant)} \\ v = \frac{\omega}{\beta} \approx \frac{1}{\sqrt{\mu\varepsilon'}} = \frac{c}{n} & \text{(phase velocity)} \\ n = \frac{c}{v} \approx \sqrt{\mu\varepsilon'}c = \sqrt{\frac{\mu\varepsilon'}{\mu_0\varepsilon_0}} & \text{(index of refraction)} \\ \alpha = 2k_i \approx i\sqrt{\frac{\mu}{\varepsilon'}}\varepsilon''\omega = \frac{\varepsilon''}{\varepsilon'}\beta & \text{(power attenuation constant)} \quad (7.55) \\ P = \frac{1}{2}\mathrm{Re}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2 e^{-\alpha z} \approx \frac{1}{2}\sqrt{\frac{\varepsilon'}{\mu}}|\mathbf{E}_0|^2 e^{-\alpha z} & \text{(intensity)} \end{cases}$$

β reduces to the expression on p. 311 when $\mu = \mu_0$.

In (7.55), $\frac{\varepsilon''}{\varepsilon'}(\equiv \tan\delta_l)$ is commonly referred to as the loss tangent.

$\varepsilon'(\mathrm{Re}\varepsilon)$ and loss tangent ($\tan\delta_l$ or $\frac{\varepsilon''}{\varepsilon'}$)
of some materials at different frequencies

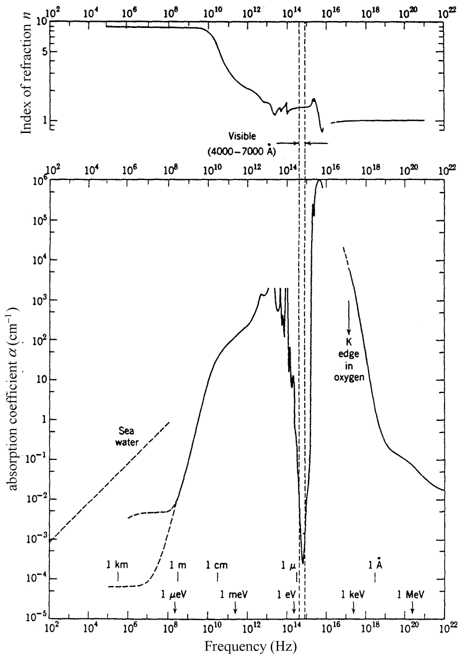
Material	ϵ'/ϵ_0			Loss tangent, $10^4\,\epsilon''/\epsilon'$		
	$f = 10^6$	$f = 10^8$	$f = 10^{10}$	$f = 10^6$	$f = 10^8$	$f = 10^{10}$
Glass, Corning 707	4.00	4.00	4.00	8	12	21
Fused quartz	3.78	3.78	3.78	2	1	1
Ruby mica	5.4	5.4	—	3	2	—
Ceramic Alsimag 393	4.95	4.95	4.95	10	10	9.7
Titania	100	100	—	3	2.5	—
Polystyrene	2.56	2.55	2.54	0.7	1	4.3
Neoprene	5.7	3.4	—	950	1600	—

from Ramo, Whinnery, and Van Duzer, p.334.

A miraculous property of water:

The index of refraction (top) and absorption coefficient (bottom) for liquid water as a function of frequency in Hz

[Sec. 7.5 (Part E)]



Case 2: Waves in a good conductor [Secs. 5.18 and 8.1, applicable to waves in metals under the condition $\omega << \gamma_0$ ($\sim 4\times10^{13}$/s. see p. 312), i.e. for very low frequency (e.g. 60 Hz) up to near terahertz frequencies]

Definition of good conductor:

$$\varepsilon = \underbrace{\varepsilon_0 + \frac{Ne^2}{m}\sum_{j\ \text{(bound)}}\frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}}_{\varepsilon_b} + i\underbrace{\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}}_{\frac{\sigma}{\omega}} \quad (7.51)$$

[In general, $\gamma_j < \omega_j$, see p. 310.
$\Rightarrow$ In general, $\mathrm{Re}(\varepsilon_b) >> \mathrm{Im}(\varepsilon_b)$]

$$\Rightarrow \varepsilon = \varepsilon_b + i\frac{\sigma}{\omega} \qquad \boxed{\sigma = \frac{Ne^2 f_0}{m(\gamma_0 - i\omega)}\ (7.58)} \qquad (7.56)$$

Up to low terahertz region, we have $\omega \ll \gamma_0$ (γ_0 is of the order of 4×10^{13} /s). Hence,

$$\sigma = \frac{Ne^2 f_0}{m(\gamma_0 - i\omega)} \approx \frac{Ne^2 f_0}{\gamma_0 m} = \frac{ne^2}{\gamma_0 m} \quad \left[\begin{array}{l}\Rightarrow \text{When } \omega \ll \gamma_0,\ \sigma \approx \text{real} \\ \text{and is independent of } \omega. \\ (n: \text{free electron density})\end{array}\right]$$

In $\varepsilon = \varepsilon_b + i\sigma/\omega$ [(7.56)], $\sigma/\omega \gg \text{Im}(\varepsilon_b)$. So we may assume ε_b to be real. A good conductor is defined by: $\frac{\sigma}{\omega\varepsilon_b} \gg 1$ (26)

Quantitative examples:

$$\begin{cases} \varepsilon_b \sim \varepsilon_0 = 8.85\times 10^{-12} \text{ farad/m} \\ \sigma_{copper} \approx 5.9\times 10^7 /\Omega\text{-m},\ \sigma_{graphite} \approx 6\times 10^4 /\Omega\text{-m} \\ \sigma_{sea\ water} \approx 6/\Omega\text{-m},\ \sigma_{ground} \approx 10^{-3}\text{-}3.5\times 10^{-2} /\Omega\text{-m} \\ f = \frac{\omega}{2\pi} = \begin{cases} 60 \text{ Hz for household current} \\ 0.3-300 \text{ GHz for microwaves} \end{cases} \end{cases}$$

Question: Why is it dangerous if an electrical appliance falls into your bath tub?

Fields in a good conductor: For a good conductor ($\frac{\sigma}{\omega\varepsilon_b} \gg 1$), we have $\sqrt{\varepsilon} = (\varepsilon_b + i\frac{\sigma}{\omega})^{\frac{1}{2}} \approx (i\frac{\sigma}{\omega})^{\frac{1}{2}} = \sqrt{\frac{\sigma}{2\omega}}(1+i)$ $\boxed{i^{\frac{1}{2}} = (e^{i\frac{\pi}{2}})^{\frac{1}{2}} = \frac{1}{\sqrt{2}}(1+i)}$

$$\Rightarrow k = \sqrt{\mu\varepsilon}\,\omega = \sqrt{\frac{\mu\sigma\omega}{2}}(1+i) = \frac{1+i}{\delta} \text{ (for forward wave)} \qquad (5.164)$$

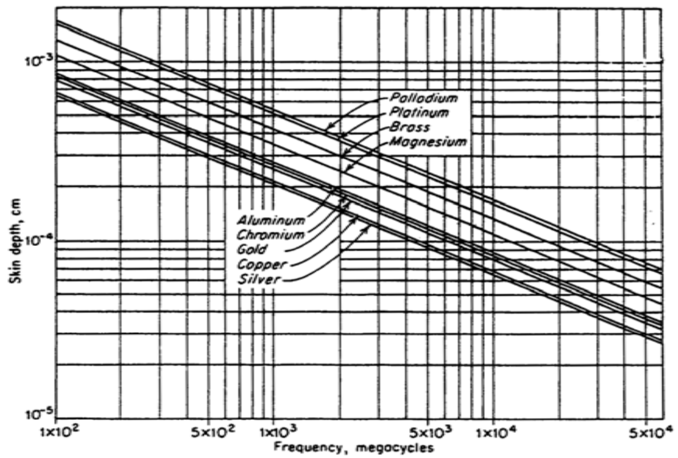
where $\delta \equiv \sqrt{\frac{2}{\mu\sigma\omega}}$ $\left[\begin{array}{l}\delta: \text{ skin depth} \\ \mu \text{ is real by assumption.}\end{array}\right]$ (5.165) and (8.8)

Let $\mathbf{E}_0 = E_0\mathbf{e}_x$, $\mathbf{n} = \mathbf{e}_z$. Then, $\mathbf{H}_0 = \sqrt{\frac{\varepsilon}{\mu}}\mathbf{n}\times\mathbf{E}_0 = \sqrt{\frac{\varepsilon}{\mu}}\mathbf{e}_z \times E_0\mathbf{e}_x = \sqrt{\frac{\varepsilon}{\mu}}E_0\mathbf{e}_y$

$$\Rightarrow \begin{cases} \mathbf{E}(\mathbf{x},t) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} = E_0 e^{ikz-i\omega t}\mathbf{e}_x = E_0 e^{-\frac{z}{\delta}} e^{i\left(\frac{z}{\delta}-\omega t\right)}\mathbf{e}_x & (27) \\ \mathbf{H}(\mathbf{x},t) = \mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} = \sqrt{\frac{\varepsilon}{\mu}}E_0\mathbf{e}_y e^{ikz-i\omega t} \\ \qquad = \sqrt{\frac{\sigma}{2\mu\omega}}(1+i)E_0 e^{-\frac{z}{\delta}} e^{i\left(\frac{z}{\delta}-\omega t\right)}\mathbf{e}_y & (28) \end{cases}$$

(27) and (28) are equivalent to (8.11) and (8.9).

Skin Depth. The skin depth δ is defined as the distance from the surface of a plane conductor at which the electric and magnetic fields have decreased to $1/e$ of their values at the surface.



Skin depth as a function of frequency for a few common metals. (*From T. Moreno, "Microwave Transmission Design Data," McGraw-Hill Book Company, Inc., New York,* 1948.)

Examples: $\delta_{copper} \approx \begin{cases} 0.85 \text{ cm at } f = 60 \text{ Hz (household current)} \\ 7\times 10^{-5} \text{ cm at } f = 10^{10} \text{ Hz (microwave)} \end{cases}$

Discussion:

(i) Rewrite

$$\begin{cases} \mathbf{E}(\mathbf{x},t) = E_0 e^{-\frac{z}{\delta}} e^{i\left(\frac{z}{\delta}-\omega t\right)}\mathbf{e}_x & (27) \\ \mathbf{H}(\mathbf{x},t) = \sqrt{\frac{\sigma}{2\mu\omega}}(1+i)E_0 e^{-\frac{z}{\delta}} e^{i\left(\frac{z}{\delta}-\omega t\right)}\mathbf{e}_y & (28) \end{cases}$$

$\Rightarrow$ Inside the good conductor, the wave has a wavelength of $\lambda = 2\pi\delta$ and it damps by a factor of $1/e$ over a distance of δ.

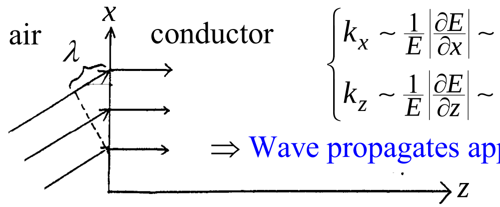
(ii) **E** and **H** in a good conductor are 45^{o} out of phase.

(iii) The fields in a good conductor are similar to those in a lossy dielectric in that they both represent an attenuated plane wave with **k**, **E**, **H**, mutually orthogonal. However, at the same frequency, the wavelength is much shorter and the attenuation constant much greater in the conductor than in the dielectric.

Examples: Let $f=\frac{\omega}{2\pi}=10^{10}$ Hz (typical microwave frequency)

glass ($\frac{\mu}{\mu_0}\approx 1,\ \frac{\varepsilon'}{\varepsilon_0}\approx 4,\ \frac{\varepsilon''}{\varepsilon'}\approx 2.1\times 10^{-4}$)	copper $\left(\delta\approx 7\times 10^{-5}\text{ cm}\right)$
$\lambda=\frac{2\pi}{\beta}=\frac{2\pi}{\sqrt{\mu\varepsilon'}\omega}\approx 1.5$ cm (Case 1.2)	$\lambda=2\pi\delta\approx 4.4\times 10^{-4}$ cm
$\alpha=\frac{2\pi}{\lambda}\frac{\varepsilon''}{\varepsilon'}\approx 8.8\times 10^{-4}\text{ cm}^{-1}$ (7.55)	$\alpha=-\frac{1}{P}\frac{dP}{dz}=\frac{2}{\delta}\approx 4.5\times 10^{3}\text{ cm}^{-1}$

(iv) A wave incident from the outside into a good conductor (at any incident angle) will propagate and attenuate inside the conductor approximately along the normal to the surface (see Jackson Sec. 8.1). The reason is shown in the figure below.



$$\begin{cases} k_x\sim\frac{1}{E}\left|\frac{\partial E}{\partial x}\right|\sim\frac{1}{\lambda} \\ k_z\sim\frac{1}{E}\left|\frac{\partial E}{\partial z}\right|\sim\frac{1}{\delta}\gg k_x \end{cases}$$

$\Rightarrow$ Wave propagates approximately along z.

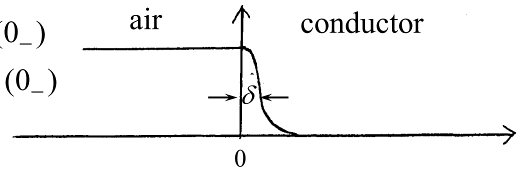
Hence, we may approximately write the wave fields inside the conductor as (27) and (28), i.e. **E** and **H** are parallel to the surface, even if the wave is incident at an oblique angle into the conductor.

Question: Does it make sense to use power lines of very large diameter (e.g.10 cm) in order to conduct higher current and hence transmit more power?

(v) The 2 homogeneous Maxwell equations require that $E_{\|}$ and $B_{\perp}$ be continuous across the conductor surface.

$E_{\|}$, $H_{\|}$(since $\delta\neq 0$, what happen to the surface current **K**?)

$$E_{\|}(0_+)=E_{\|}(0_-)$$
$$H_{\|}(0_+)=H_{\|}(0_-)$$



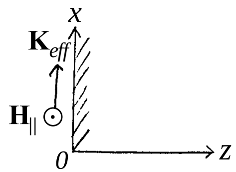
Note: The current density in a good conductor is finite unless $\delta=0$ (or $\sigma=\infty$, i.e. the current flows on the surface).

Surface current $\mathbf{K}_{eff}$ *on a good conductor* :

If $\delta\neq 0$, the "surface" current $\mathbf{K}_{eff}$ is not exactly on the surface, It is concentrated over a depth of ~ one skin depth. $\mathbf{K}_{eff}$ (unit: A/m) is an integrated value of $\mathbf{J}$ (unit: A/m^2) over the penetration depth.

$$\mathbf{K}_{eff}=\int_0^\infty \mathbf{J}dz=\sigma\int_0^\infty \mathbf{E}dz=\sigma E_0 e^{-i\omega t}\underbrace{\int_0^\infty e^{\frac{-1+i}{\delta}z}dz}_{=\frac{\delta}{1-i}=\frac{\delta(1+i)}{2}=\frac{(1+i)}{\sqrt{2\mu\sigma\omega}}}\mathbf{e}_x$$

$\boxed{\mathbf{E}=E_0e^{-\frac{z}{\delta}}e^{i\left(\frac{z}{\delta}-\omega t\right)}\mathbf{e}_x\ (27)}$



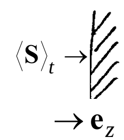
$$=\sqrt{\frac{\sigma}{2\mu\omega}}(1+i)E_0e^{-i\omega t}\mathbf{e}_x=-\mathbf{e}_z\times\mathbf{H}(z=0)=-\mathbf{e}_z\times\mathbf{H}_{\|}(z=0)\quad(29)$$

(29) here is (8.14) in Jackson; "$-\mathbf{e}_z$" in (29) is "**n**" in (8.14).

(29) shows that the surface current $\mathbf{K}_{eff}$ on a good conductor depends only on the $\mathbf{H}_{\|}$ on its surface. Physically, $\mathbf{K}_{eff}$ is the response of the conductor in order to shield its inside from $\mathbf{H}_{\|}$ (Faraday's law). Hence, $\mathbf{K}_{eff}$ is determined entirely by $\mathbf{H}_{\|}$.

Time-averaged power loss on the surface of a good conductor:

$$\frac{dP_{loss}}{da}=\frac{\text{power going into conductor}}{\text{unit area of conductor surface}}=\langle\mathbf{S}(z=0)\rangle_t\cdot\mathbf{e}_z$$



$$=\tfrac{1}{2}\mathrm{Re}\left[\mathbf{E}(z=0)\times\mathbf{H}^*(z=0)\right]\cdot\mathbf{e}_z$$

$\boxed{(27),(28)}\rightarrow$

$$=\frac{1}{2}\sqrt{\frac{\sigma}{2\mu\omega}}\left|\mathbf{E}(0)\right|^2=\frac{1}{2}\sqrt{\frac{\sigma}{2\mu\omega}}\left|\mathbf{E}_{\|}(0)\right|^2\quad\left[\mathbf{E}(0)\perp\mathbf{e}_z\right]\qquad(30)$$

$$(27),(28)\Rightarrow\left|\mathbf{E}_{\|}(0)\right|=\sqrt{\frac{\mu\omega}{\sigma}}\left|\mathbf{H}_{\|}(0)\right|\qquad(31)$$

Sub. (31) into (30) $\Rightarrow\frac{dP_{loss}}{da}=\frac{1}{2}\sqrt{\frac{\mu\omega}{2\sigma}}\left|\mathbf{H}_{\|}(0)\right|^2$ [useful form to explain induction heating]

$\boxed{\delta=\sqrt{\frac{2}{\mu\sigma\omega}}}\rightarrow$

$$=\frac{1}{4}\mu\omega\delta\left|\mathbf{H}_{\|}(0)\right|^2\qquad(8.12)$$

$\boxed{(29)}$

$$=\frac{1}{2}\frac{1}{\sigma\delta}\left|\mathbf{H}_{\|}(0)\right|^2=\frac{1}{2}\frac{1}{\sigma\delta}\left|\mathbf{K}_{eff}\right|^2\qquad(8.15)$$

Note: If there is reflection, $\left|\mathbf{H}_{\|}(0)\right|^2=\left|\mathbf{H}_{\|\,incident}(0)+\mathbf{H}_{\|\,reflected}(0)\right|^2$

dP_{loss}/da in (8.12), obtained by the Poynting vector method, can be shown to be exactly the Ohmic power dissipated inside the conductor.

$P_{resistive}$ = ohmic power in the conductor/unit volume

$$= \tfrac{1}{2}\mathrm{Re}\left[\mathbf{J}\cdot\mathbf{E}^*\right] = \tfrac{1}{2}\sigma|\mathbf{E}|^2$$

$$= \underset{(27)}{\tfrac{1}{2}}\sigma|\mathbf{E}_0|^2 e^{-\frac{2z}{\delta}} = \underset{(28)}{\tfrac{1}{2}}\mu\omega|\mathbf{H}_0|^2 e^{-\frac{2z}{\delta}} \qquad (5.169)$$

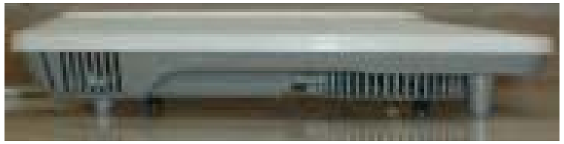
$P_{resistive}$

$\boxed{\mathbf{H}_0 = \mathbf{H}_{\parallel}(z=0)}$

$$\frac{dP_{loss}}{da} = \int_0^\infty P_{resistive}\,dz = \tfrac{1}{2}\mu\omega|\mathbf{H}_0|^2\int_0^\infty e^{-\frac{2z}{\delta}}dz = \tfrac{1}{4}\mu\omega\delta|\mathbf{H}_0|^2 \quad \begin{bmatrix}\text{same as}\\ (8.12)\end{bmatrix}$$

Questions: 1. Why does a microwave oven save energy?

2. How would you design an induction cooker? high μ and δ



Definitions: surface impedance Z_s, surface resistance R_s, and surface reactance X_s of metal

$$(27+29) \Rightarrow \mathbf{K}_{eff} = \sqrt{\tfrac{\sigma}{2\mu\omega}}(1+i)\mathbf{E}_{\parallel}(0) = \frac{\sigma\delta}{1-i}\mathbf{E}_{\parallel}(0) = \frac{\mathbf{E}_{\parallel}(0)}{Z_s} \quad \begin{bmatrix} Z_s\text{: ratio of} \\ E_{\parallel}(0) \text{ to } K_{eff}\end{bmatrix}$$

where $Z_s \equiv \dfrac{1-i}{\sigma\delta}$ [Jackson p. 356, bottom] is called the surface impedance. We may write

$$\begin{cases} Z_s = R_s - iX_s \\ \text{where } \underbrace{R_s}_{\boxed{\text{surface resistance}}} = \underbrace{X_s}_{\boxed{\text{surface reactance}}} = \dfrac{1}{\sigma\delta}, \end{cases} \qquad (32)$$

Example: R_s of copper $\approx 0.026\ \Omega$ at 10 GHz

The surface impedance Z_s is an intrinsic property (rather than surface property) of metal. It is in fact the impedance of a good conductor:

$$Z_s = \sqrt{\frac{\mu}{\varepsilon(\text{metal})}} = \frac{\sqrt{\mu}}{\sqrt{\frac{\sigma}{2\omega}}(1+i)} = \frac{1-i}{\sqrt{\frac{2\sigma}{\mu\omega}}} = \frac{1-i}{\sigma\delta}.$$

Case 3: Waves at optical frequencies and beyond [Sec. 7.5 (Part D)]

Case 3.1: $\omega >> \gamma_0$ but $\omega < \omega_j$ for all or some of the bound electrons [a subcase of Sec. 7.5 (Part D), pp. 313-4, total reflection of light off the mirror and ultraviolet transparency of metals)

$$\underbrace{\varepsilon = \varepsilon_0 + \frac{Ne^2}{m}\sum_{j\,(\text{bound})}\frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}}_{\varepsilon_b} + i\underbrace{\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}}_{\approx -\frac{Ne^2 f_0}{m\omega^2}} \qquad (7.51)$$

$$\begin{pmatrix}\text{In general, } \gamma_j < \omega_j\text{, see p. 310.} \\ \Rightarrow \text{In general, } \mathrm{Re}(\varepsilon_b) >> \mathrm{Im}(\varepsilon_b)\end{pmatrix} \qquad (\because\ \omega >> \gamma_0)$$

The free electron term is predominantly imaginary when $\omega << \gamma_0$. But, as shown above, when $\omega >> \gamma_0$, it becomes predominantly real, a qualitative departure from Case 2. This radically changes the metal response to EM waves. Examples are given below and in Case 3.2. *Question*: What is the physical reason for the free electron term to become predominantly real when $\omega >> \gamma_0$?

Let $n = Nf_0$ be the free electron density in the conductor ($f_0 \sim 1$, i.e. each atom in the conductor contains on average approximately one free electron, see p.312), we obtain from (7.51)

$$\varepsilon = \varepsilon_b - \frac{\omega_p^2}{\omega^2}\varepsilon_0$$

where ω_p is the plasma frequency of the conduction electrons

$$\omega_p^2 = \frac{ne^2}{m^*\varepsilon_0} \qquad \text{[See bottom of p.313.]}$$

and we have replaced m in (7.51) with the effective mass m^* of the conduction electrons to account for the effects of binding. For simplicity, we assume ε_b to be real by neglecting the weak damping effects of bound electrons.

Sub. $\varepsilon = \varepsilon_b - \omega_p^2 \varepsilon_0 / \omega^2$ into $k = \sqrt{\mu\varepsilon}\,\omega$, we obtain

$$k = \sqrt{\mu(\varepsilon_b - \frac{\omega_p^2 \varepsilon_0}{\omega^2})}\ \omega$$

Hence, k is either real (propagation without attenuation) or purely imaginary (evanescent fields) depending on the wave frequency.

When $\omega < \sqrt{\frac{\varepsilon_0}{\varepsilon_b}}\,\omega_p$, $\varepsilon < 0$ and $k = i\sqrt{\mu(\frac{\omega_p^2 \varepsilon_0}{\omega^2} - \varepsilon_b)}\,\omega = i|k|$. Then,

$$\begin{cases} \mathbf{E} = E_0 e^{ikz - i\omega t}\mathbf{e}_x = E_0 e^{-|k|z - i\omega t}\mathbf{e}_x & (33) \\ \mathbf{H} = \sqrt{\frac{\varepsilon}{\mu}}\mathbf{e}_z \times \mathbf{E} = i\sqrt{\frac{|\varepsilon|}{\mu}} E_0 e^{-|k|z - i\omega t}\mathbf{e}_y & (34) \end{cases}$$

$\boxed{(24)}$ $\boxed{\sqrt{\varepsilon} = \sqrt{-|\varepsilon|} = i\sqrt{|\varepsilon|}}$

$\mathbf{E} = E_0 e^{-|k|z - i\omega t}\mathbf{e}_x$ in (33) and $\mathbf{H} = i\sqrt{\frac{|\varepsilon|}{\mu}} E_0 e^{-|k|z - i\omega t}\mathbf{e}_y$ in (34) are evanescent fields which fall off exponentially inside the conductor. They do not constitute a propagating wave. This is because **E** and **H** are 90° out of phase. Hence, Re[**E** × **H***] = 0 ⇒ No power flow into the conductor. Thus, an incident wave will be totally reflected from the conductor surface, with (33) and (34) representing the shallow fringe fields inside the conductor. This is the principle of "light reflection off the mirror". By comparison, for microwave reflection off a good conductor (Case 2), **E** and **H** are 45° out of phase in the conductor ⇒ Some power flows into the conductor.

At higher frequencies ($\omega > \sqrt{\varepsilon_0 / \varepsilon_b}\,\omega_p$), $\varepsilon = \varepsilon_b - \omega_p^2 \varepsilon_0 / \omega^2 > 0$. Hence, $k (= \sqrt{\mu\varepsilon}\,\omega)$ becomes real. The wave can then propagate freely. This is the principle of "ultraviolet transparency of metals".

Question: Why can the wave propagate without attenuation in a conductor? (see discussion at the end of Case 3.2.)

Case 3.2: $\omega >> \gamma_j$ and $\omega >> \omega_j$ for all electrons in the medium

[a subcase of Sec. 7.5 (Part D), p. 313, applicable to X-ray frequencies and beyond]

Under the conditions $\omega >> \gamma_j$ (including γ_0) and $\omega >> \omega_j$, we may neglect γ_j and ω_j in (7.51),

$$\varepsilon = \varepsilon_0 + \underbrace{\frac{Ne^2}{m}\sum_{j\,(\text{bound})}\frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} + i\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}}_{\approx -\frac{NZe^2}{m\omega^2}\ (\text{use } \sum_{j(all)} f_j = Z)} \qquad (7.51)$$

$$\Rightarrow \ \frac{\varepsilon}{\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2}, \qquad (7.59)$$

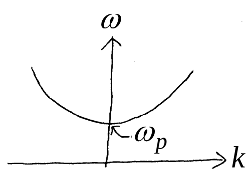
$$\text{where } \ \omega_p^2 \equiv \frac{NZe^2}{m\varepsilon_0} \quad \begin{bmatrix} NZ \text{ is the density of all electrons} \\ \text{(bound and free) in the medium.} \end{bmatrix} \qquad (7.60)$$

Sub. $\frac{\varepsilon}{\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2}$ into $k = \sqrt{\mu\varepsilon}\,\omega$ and assume $\mu = \mu_0$, we obtain

$$k^2 = \mu\varepsilon\omega^2 = \overbrace{\mu_0\varepsilon_0}^{1/c^2}(1 - \frac{\omega_p^2}{\omega^2})\omega^2$$

$$\Rightarrow \ \omega^2 = k^2c^2 + \omega_p^2 \qquad (7.61)$$

Although (7.61) predicts evanescent fields for $\omega < \omega_p$, the validity of (7.61) requires $\omega >> \gamma_j$ and $\omega >> \omega_j$ for all the electrons in the medium. This in turn requires $\omega >> \omega_p$. Hence, k is always real and the wave is always a propagating wave in the medium under the validity condition for (7.61).



The above treatment for Case 3.2 applies to both dielectric and conducting media.

Discussion: To examine the physical reason why we may neglect collisions and binding forces in (7.51) under the conditions $\omega >> \gamma_j$ and $\omega >> \omega_j$, we go back to the equation of motion for the electrons:

$$m(\ddot{\mathbf{x}} + \gamma_j \dot{\mathbf{x}} + \omega_j^2 \mathbf{x}) = -e\mathbf{E}(\mathbf{x},t) \qquad (7.49)$$

By assuming $\mathbf{E}(\mathbf{x},t) = \mathbf{E}(0)e^{-i\omega t}$, we obtain [see Eq. (1)]

$$\mathbf{x}(t) = -\frac{e}{m}\frac{\mathbf{E}(0)e^{-i\omega t}}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \quad \Rightarrow \quad \dot{\mathbf{x}}(t) = \frac{e}{m}\frac{i\omega\mathbf{E}(0)e^{-i\omega t}}{\omega_j^2 - \omega^2 - i\omega\gamma_j}$$

Thus, when $\omega >> \gamma_j$ and ω_j, we have $\mathbf{x}(t) \propto 1/\omega^2$ and $\dot{\mathbf{x}}(t) \propto 1/\omega$. This implies that, for the same $\mathbf{E}(0)$, the collisional damping force ($m\gamma_j \dot{\mathbf{x}} \propto 1/\omega$) and the binding force ($m\omega_j^2 \mathbf{x} \propto 1/\omega^2$) decrease with increasing ω and become negligible at a sufficiently large ω.
Exercise: Explain "$m\gamma_j \dot{\mathbf{x}} \propto 1/\omega$" and "$m\omega_j^2 \mathbf{x} \propto 1/\omega^2$" qualitatively from the simple case of constant acceleration a: $v = at$ and $x = \frac{1}{2}at^2$.

Case 4: Waves in plasmas [a subcase of Sec. 7.5 (Part D), p. 313]

The plasma is a partially ionized (e.g. ionosphere) or fully ionized (e.g. fusion plasmas) gas. In general, effects of neutral gas (if present) and collisions can both be neglected. Ion motion can also be neglected at sufficiently high frequencies. Then,

$$\varepsilon = \varepsilon_0 + \underbrace{\cancel{\frac{Ne^2}{m}\sum_{j\,(\text{bound})}\frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}}}_{\text{negligible}} + \underbrace{i\frac{Ne^2 f_0}{m\omega(\gamma_0 - i\omega)}}_{\approx -\frac{Ne^2 f_0}{m\omega^2}\ (\gamma_0 \to 0)} \qquad (7.51)$$

$$\Rightarrow \frac{\varepsilon}{\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2} \quad \begin{bmatrix}\text{same equation as (7.59) but} \\ \text{with a much smaller } \omega_p\end{bmatrix} \qquad (35)$$

where ω_p is the plasma frequency defined as

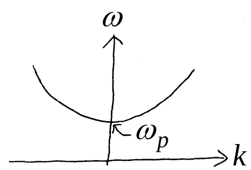
$$\omega_p^2 \equiv \frac{ne^2}{\varepsilon_0 m} \quad \begin{bmatrix} n = Nf_0 = \text{plasma electron density, normally} \\ \text{much smaller than the density of solids.}\end{bmatrix} \qquad (36)$$

Sub. $\frac{\varepsilon}{\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2}$ into $k = \sqrt{\mu\varepsilon}\,\omega$, we obtain

$$k^2 = \mu\varepsilon\omega^2 = \overbrace{\mu_0\varepsilon_0}^{1/c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)\omega^2 \quad (\mu = \mu_0 \text{ for plasmas})$$

$$\Rightarrow \omega^2 = k^2c^2 + \omega_p^2 \quad \begin{bmatrix}\text{same equation as (7.61) but} \\ \text{with a much smaller } \omega_p^2\end{bmatrix} \qquad (37)$$

(37) is the well known dispersion relation for electromagnetic waves in a plasma in the absence of an externally applied static magnetic field. (Sec. 7.6 considers the dispersion relation for a magnetized plasma.) When ω is extremely large (such as the gamma ray), all materials have a dispersion relation given by (37) (Case 3.2). But for the plasma, (37) is valid for all frequencies (e.g. MHz).



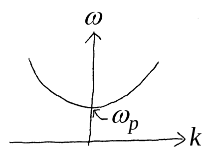
Rewrite $\qquad \omega^2 = k^2c^2 + \omega_p^2 \qquad (37)$

For $\omega < \omega_p$, k is purely imaginary ($k = i\,|k|$) and hence $\mathbf{E}$ and $\mathbf{H}$ are evanescent fields given by (33) and (34):

$$\mathbf{E} = E_0 e^{-|k|z - i\omega t}\mathbf{e}_x \ ; \ \mathbf{H} = i\sqrt{\frac{|\varepsilon|}{\mu}}E_0 e^{-|k|z - i\omega t}\mathbf{e}_y$$

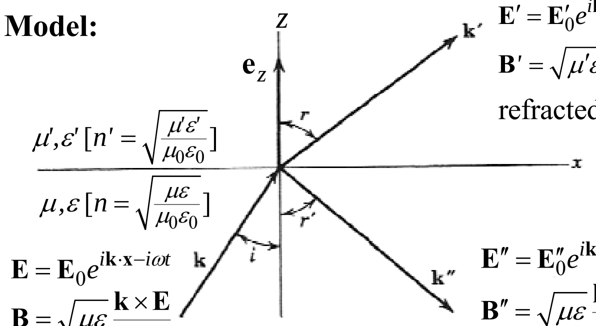
As in the case of light reflection off the mirror, an incident wave wave will be totally reflected [Shortwave broadcasting exploits the reflection of radio waves (~10 MHz) off the ionosphere].

For $\omega > \omega_p$, k is real. Hence, the wave will propagate in the plasma, but with a phase velocity greater than the speed of light [as can be seen from (37)]. This implies that the plasma has an index of refraction (n) less than 1. From (35), we have $\frac{\varepsilon}{\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2} < 1$. Thus, with $\mu = \mu_0$, we have $n = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} < 1$, as expected.



7.3 Reflection and Refraction of Electromagnetic Waves at a Plane Interface Between Dielectrics

Model:



$\mathbf{E}' = \mathbf{E}_0' e^{i\mathbf{k}'\cdot\mathbf{x} - i\omega t}$

$\mathbf{B}' = \sqrt{\mu'\varepsilon'} \dfrac{\mathbf{k}' \times \mathbf{E}'}{k'}$

refracted wave (assumed)

$\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$

$\mathbf{B} = \sqrt{\mu\varepsilon} \dfrac{\mathbf{k} \times \mathbf{E}}{k}$

incident wave (a given linearly polarized homogeneous plane wave)

$\mathbf{E}'' = \mathbf{E}_0'' e^{i\mathbf{k}''\cdot\mathbf{x} - i\omega t}$

$\mathbf{B}'' = \sqrt{\mu\varepsilon} \dfrac{\mathbf{k}'' \times \mathbf{E}''}{k''}$

reflected wave (assumed)

Note: In *Case* 1.2 of Part III, $n \equiv \frac{c}{v} = \mathrm{Re}\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$. Here, $n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$.

Kinematic properties: relations between angles of incidence, reflection, and refraction

Dynamic properties: intensity, phase, and polarization relations

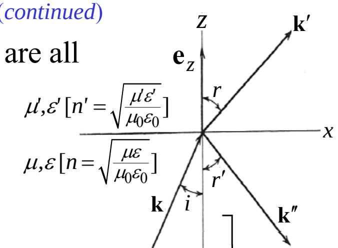
Kinematic Properties :

Boundary conditions for the fields at $z = 0$ have the form:

$$Xe^{ik_x x + ik_y y} + Ye^{ik_x'' x + ik_y'' y} = Ze^{ik_x' x + ik_y' y} \text{ at any } x \text{ and } y,$$

where X, Y, and Z are functions of the fields [see (7.37)]. Since $e^{ik_x x}$, $e^{ik_x' x}$, $e^{ik_x'' x}$ are linearly independent, we must have $k_x = k_x'' = k_x'$. Otherwise, we will have the trivial condition $X = Y = Z = 0$. For the same reason, $k_y = k_y'' = k_y'$. Hence, $\mathbf{k}$, $\mathbf{k}''$, and $\mathbf{k}'$ lie in the same plane.

Without loss of generality, we choose a convenient coordinate system in which $k_y = k_y'' = k_y' = 0$. Then, $\mathbf{k}$, $\mathbf{k}''$, and $\mathbf{k}'$ all lie in the x - z plane, which we call the plane of incidence.



Assume ε, ε', μ, and μ' (hence n and n') are all real numbers. Let

$$\begin{cases} \mathbf{k} = k \sin i \mathbf{e}_x + k \cos i \mathbf{e}_z \\ \mathbf{k}'' = k \sin r' \mathbf{e}_x - k \cos r' \mathbf{e}_z \\ \mathbf{k}' = k' \sin r \mathbf{e}_x + k' \cos r \mathbf{e}_z \end{cases}$$

$$(16) \Rightarrow \begin{cases} k = \sqrt{\mu\varepsilon}\,\omega = \frac{\omega}{c} n \\ k' = \sqrt{\mu'\varepsilon'}\,\omega = \frac{\omega}{c} n' \end{cases} \quad \left[\begin{array}{l} c = 1/\sqrt{\mu_0\varepsilon_0}, \\ n = \sqrt{\mu\varepsilon/\mu_0\varepsilon_0},\ n' = \sqrt{\mu'\varepsilon'/\mu_0\varepsilon_0} \end{array} \right]$$

$$k_x = k_x' = k_x'' \Rightarrow \begin{cases} i = r' \text{ (angle of incidence = angle of reflection)} \\ \dfrac{\sin i}{\sin r} = \dfrac{k'}{k} = \dfrac{n'}{n} \quad \text{(\underline{Snell's law})} \qquad (7.36) \end{cases}$$

A note on Jackson (7.33):

$$\begin{cases} k^2 \equiv \mathbf{k}\cdot\mathbf{k} \\ |\mathbf{k}|^2 \equiv \mathbf{k}\cdot\mathbf{k}^* \end{cases} \Rightarrow \text{ In general, } \begin{cases} k^2 \neq |\mathbf{k}|^2 \text{ and } k \neq |\mathbf{k}| \\ k \text{ can be complex, but } |\mathbf{k}| \text{ is} \\ \text{always real and positive.} \end{cases}$$

Thus, Jackson's formula $k = |\mathbf{k}|$ in (7.33) is valid only when $\mathbf{k}$ is real.

Dynamic Properties :

Information concerning the *intensity*, *phase*, and *polarization* is contained in the complex $\mathbf{E}_0$, $\mathbf{E}_0'$, and $\mathbf{E}_0''$. The intensity, phase, and polarization of reflected and refracted waves with respect to those of the incident wave can be obtained from the boundary conditions at $z = 0$:

$$\begin{cases} D_\perp \text{ continuous} \Rightarrow [\varepsilon(\mathbf{E}_0 + \mathbf{E}_0'') - \varepsilon'\mathbf{E}_0'] \cdot \mathbf{e}_z = 0 & (39) \\ B_\perp \text{ continuous} \Rightarrow [\mathbf{k}\times\mathbf{E}_0 + \mathbf{k}''\times\mathbf{E}_0'' - \mathbf{k}'\times\mathbf{E}_0'] \cdot \mathbf{e}_z = 0 & (40) \\ E_\parallel \text{ continuous} \Rightarrow [\mathbf{E}_0 + \mathbf{E}_0'' - \mathbf{E}_0'] \times \mathbf{e}_z = 0 & (41) \\ H_\parallel \text{ continuous} \Rightarrow [\frac{1}{\mu}(\mathbf{k}\times\mathbf{E}_0 + \mathbf{k}''\times\mathbf{E}_0'') - \frac{1}{\mu'}(\mathbf{k}'\times\mathbf{E}_0')] \times \mathbf{e}_z = 0 & (42) \end{cases}$$

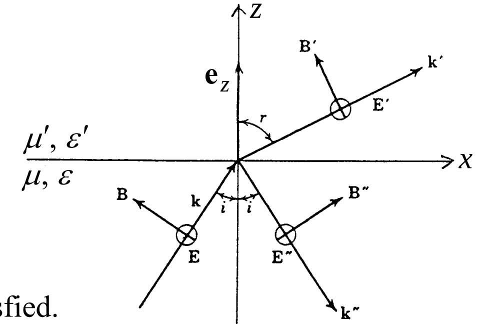
Note: (1) Here, ε, ε', μ, and μ' (hence n and n') are in general complex numbers (see first paragraph of Jackson, p. 306.) We assume that ε (or ε') is the generalized electric permittivity. Hence, the results derived below apply to any media (including metal).

(2) For a complex n (or n'), the phase velocity is the speed of light divided by Re[n]. [See lecture notes, the equation before (25)].

Case 1: $\mathbf{E}_0 \perp$ plane of incidence (the x - z plane)

$$\begin{cases} \mathbf{k} = k_x \mathbf{e}_x + k_z \mathbf{e}_z \\ \mathbf{k}' = k_x \mathbf{e}_x + k_z' \mathbf{e}_z \\ \mathbf{k}'' = k_x \mathbf{e}_x - k_z \mathbf{e}_z \end{cases}$$

$$\begin{cases} \mathbf{E}_0 = E_0 \mathbf{e}_y \\ \mathbf{E}_0' = E_0' \mathbf{e}_y \\ \mathbf{E}_0'' = E_0'' \mathbf{e}_y \end{cases}$$



(39) is automatically satisfied.

$$(40) \Rightarrow \left(k_x E_0 \mathbf{e}_z - k_z E_0 \mathbf{e}_x\right)\cdot \mathbf{e}_z + \left(k_x E_0'' \mathbf{e}_z + k_z E_0'' \mathbf{e}_x\right)\cdot \mathbf{e}_z - \left(k_x E_0' \mathbf{e}_z - k_z' E_0' \mathbf{e}_x\right)\cdot \mathbf{e}_z = 0$$

$$\Rightarrow E_0 + E_0'' - E_0' = 0 \qquad (43)$$

(41) also gives (43).

$$(42) \Rightarrow \tfrac{1}{\mu}\left(k_x E_0 \mathbf{e}_z - k_z E_0 \mathbf{e}_x\right)\times \mathbf{e}_z + \tfrac{1}{\mu}\left(k_x E_0'' \mathbf{e}_z + k_z E_0'' \mathbf{e}_x\right)\times \mathbf{e}_z - \tfrac{1}{\mu'}\left(k_x E_0' \mathbf{e}_z - k_z' E_0' \mathbf{e}_x\right)\times \mathbf{e}_z = 0$$

$$\Rightarrow \tfrac{1}{\mu} k_z \left(E_0 - E_0''\right) - \tfrac{1}{\mu'} k_z' E_0' = 0 \quad \longleftarrow \quad \boxed{\begin{aligned} k_z &= k\cos i = \tfrac{\omega}{c} n \cos i \\ k_z' &= k' \cos r = \tfrac{\omega}{c} n' \cos r \end{aligned}}$$

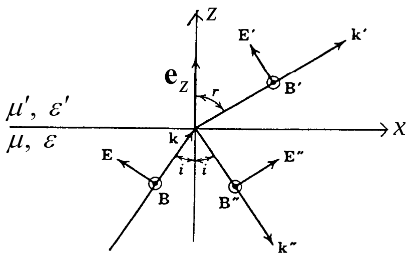
$$\Rightarrow \tfrac{n}{\mu}\left(E_0 - E_0''\right)\cos i - \tfrac{n'}{\mu'} E_0' \cos r = 0 \qquad (44)$$

$$\left.\begin{matrix}(43)\\(44)\end{matrix}\right\} \Rightarrow \begin{cases} \dfrac{E_0'}{E_0} = \dfrac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2 \sin^2 i}} \\[2ex] \dfrac{E_0''}{E_0} = \dfrac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2 \sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2 \sin^2 i}} \end{cases} \qquad (7.39)$$

Case 2: $\mathbf{E}_0 \parallel$ plane of incidence

$$\begin{cases} \mathbf{k} = k\left(\sin i\mathbf{e}_x + \cos i\mathbf{e}_z\right) \\ \mathbf{k}' = k'\left(\sin r\mathbf{e}_x + \cos r\mathbf{e}_z\right) \\ \mathbf{k}'' = k''\left(\sin i\mathbf{e}_x - \cos i\mathbf{e}_z\right) \end{cases} \qquad (45)$$

$$\begin{cases} \mathbf{E}_0 = E_0\left(-\cos i\mathbf{e}_x + \sin i\mathbf{e}_z\right) \\ \mathbf{E}_0' = E_0'\left(-\cos r\mathbf{e}_x + \sin r\mathbf{e}_z\right) \\ \mathbf{E}_0'' = E_0''\left(\cos i\mathbf{e}_x + \sin i\mathbf{e}_z\right) \end{cases} \qquad (46)$$

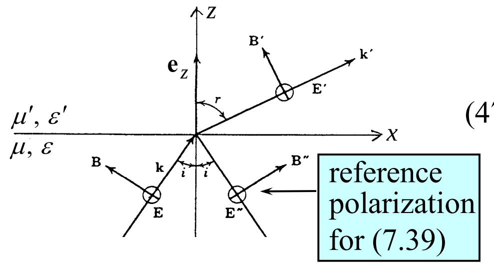


Sub. (45) and (46) into (39)-(42) yields

$$\begin{cases} \dfrac{E_0'}{E_0} = \dfrac{2nn'\cos i}{\frac{\mu}{\mu'} n'^2 \cos i + n\sqrt{n'^2 - n^2 \sin^2 i}} \\[2ex] \dfrac{E_0''}{E_0} = \dfrac{\frac{\mu}{\mu'} n'^2 \cos i - n\sqrt{n'^2 - n^2 \sin^2 i}}{\frac{\mu}{\mu'} n'^2 \cos i + n\sqrt{n'^2 - n^2 \sin^2 i}} \end{cases} \qquad (7.41)$$

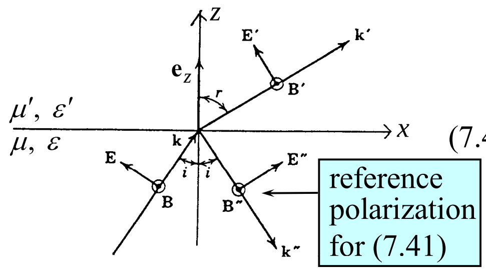
For normal incidence ($i = r = 0$), (7.39) reduces to

$$\begin{cases} \dfrac{E_0'}{E_0} = \dfrac{2}{1+\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}} \xrightarrow[\mu=\mu']{} \dfrac{2n}{n+n'} \\[2ex] \dfrac{E_0''}{E_0} = \dfrac{1-\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}}{1+\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}} \xrightarrow[\mu=\mu']{} \dfrac{n-n'}{n+n'} \end{cases} \qquad (47)$$



and (7.41) reduces to

$$\begin{cases} \dfrac{E_0'}{E_0} = \dfrac{2}{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}+1} \xrightarrow[\mu=\mu']{} \dfrac{2n}{n'+n} \\[2ex] \dfrac{E_0''}{E_0} = \dfrac{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}-1}{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}+1} \xrightarrow[\mu=\mu']{} \dfrac{n'-n}{n'+n} \end{cases} \qquad (7.42)$$

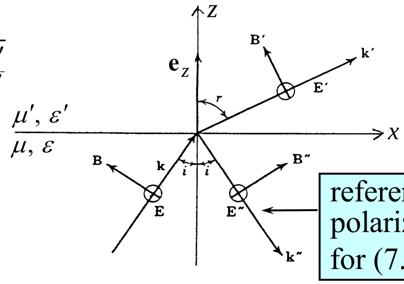


These two limiting results are identical and show that, if $n' > n$, there is a phase reversal of the reflected wave at the interface.

self-study

The results for normal incidence ($i = r = 0$) can be expressed in terms of the impedance of the two media [The impedance of the medium is defined on p. 297 and in the lecture notes following (7.11)]:

$$\begin{cases} Z \text{ (lower medium)} = \sqrt{\dfrac{\mu}{\varepsilon}} \\ Z' \text{ (upper medium)} = \sqrt{\dfrac{\mu'}{\varepsilon'}} \end{cases}$$



reference polarization for (7.39)

Thus, (7.39) reduces to

$$\begin{cases} \dfrac{E_0'}{E_0} = \dfrac{2Z'}{Z'+Z} \\ \dfrac{E_0''}{E_0} = \dfrac{Z'-Z}{Z'+Z} \end{cases}$$

If the lower medium is vacuum and the upper medium is copper, we have $\begin{cases} Z = Z_0 = 376.7\ \Omega \text{ [lecture notes following (7.11)]} \\ Z' = Z_s \approx (0.026 - i0.026)\ \Omega \text{ for copper at 10 GHz [(32)]} \end{cases}$

Thus, $E''/E_0 \approx -1$, i.e. almost all of the incident wave will be reflected with a phase reversal of the reflected wave at the interface.

self-study

Discussion: Sources of electromagnetic fields in dielectrics

The source-free macroscopic Maxwell equations [(7.1)] can be converted into the microscopic form as follows:

$$\begin{cases} \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{D} = 0 \\ \nabla \times \mathbf{H} = \dfrac{\partial \mathbf{D}}{\partial t} \end{cases} \begin{bmatrix} \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \dfrac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{bmatrix} \Rightarrow \begin{cases} \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{E} = -\dfrac{\nabla \cdot \mathbf{P}}{\varepsilon_0} = \dfrac{1}{\varepsilon_0}\rho_{pol} \\ \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} + \mu_0 \underbrace{\nabla \times \mathbf{M}}_{\mathbf{J}_M\ [(5.79)]} + \mu_0 \overbrace{\dfrac{\partial \mathbf{P}}{\partial t}} \end{cases}$$

Jackson p.156 and lecture notes Ch. 4

$\mathbf{J}_{pol}$ [lecture notes, Ch. 4]

$$= \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}_M + \mu_0 \mathbf{J}_{pol}$$

self-study

We see from the microscopic Maxwell equations that, upon action by the electromagnetic fields, bound electrons of atoms/molecules in a dielectric ($\varepsilon \neq \varepsilon_0, \mu \neq \mu_0$) will produce polarization charge and current densities (ρ_{pol} and $\mathbf{J}_{pol}$) and magnetization current density ($\mathbf{J}_M$), through which the dielectric will generate its own fields. In the macroscopic Maxwell equations, ρ_{pol}, $\mathbf{J}_{pol}$, and $\mathbf{J}_M$ are hidden in **D** and **H**, but the fields they generate will appear in the solutions. For example, as a wave is incident from a vacuum into an $\varepsilon \neq \varepsilon_0$ medium, it will induce ρ_{pol} and $\mathbf{J}_{pol}$ ($\rho_{pol} = 0$ inside a uniform medium, whereas $\mathbf{J}_{pol}$ is always present). ρ_{pol} and $\mathbf{J}_{pol}$ are the sources which generate the reflected wave and cause refraction of the transmitted wave.

Similarly, in the case of a charged particle traveling in a dielectric medium at a speed greater than the speed of light in that medium, the ρ_{pol} and $\mathbf{J}_{pol}$ induced by the fields of the charged particle will generate the Cherenkov radiation (treated in Jackson, Sec. 13.4).

7.4. Polarization by Reflection and Total Internal Reflection

Brewster's Angle $\mathbf{i_B}$: (for $\mathbf{E}_0 \parallel$ plane of incidence)

$$\text{Rewrite} \begin{cases} \dfrac{E_0'}{E_0} = \dfrac{2nn'\cos i}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}} \\ \dfrac{E_0''}{E_0} = \dfrac{\frac{\mu}{\mu'}n'^2\cos i - n\sqrt{n'^2 - n^2\sin^2 i}}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}} \end{cases} \quad (7.41)$$

Assume ε, ε', μ, and μ' (hence n and n') are all real numbers. Let $\mu = \mu'$. We see that, if $i = i_B$, where i_B satisfies

$$n'^2 \cos i_B = n\sqrt{n'^2 - n^2 \sin^2 i_B}$$

then $E_0'' = 0$, i.e. there will be no reflected wave. Consequently, upon reflection at the incident angle $i = i_B$, waves with mixed polarization become linearly polarized with $\mathbf{E}_0 \perp$ plane of incidence.

Calculation of i_B :

Rewrite $\quad n'^2 \cos i_B = n\sqrt{n'^2 - n^2 \sin^2 i_B}$

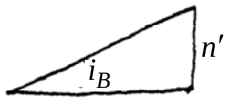
$$\Rightarrow n'^4 \cos^2 i_B = n^2 \left(n'^2 - n^2 \sin^2 i_B\right)$$

$$\Rightarrow n'^4 \left(1 - \sin^2 i_B\right) = n^2 n'^2 - n^4 \sin^2 i_B$$

$$\Rightarrow \left(n^4 - n'^4\right) \sin^2 i_B = n'^2 \left(n^2 - n'^2\right)$$

$$\Rightarrow \sin^2 i_B = \frac{n'^2}{n^2 + n'^2}$$

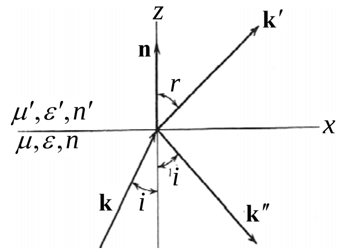
$$\Rightarrow \tan i_B = \frac{n'}{n} \qquad (7.43)$$



Total Internal Reflection: (occurs only when $n > n'$)

Assume ε, ε', μ, and μ' (hence n and n') are all real and $n > n'$.

Let $\begin{cases} \mathbf{k} = k \sin i \mathbf{e}_x + k \cos i \mathbf{e}_z \\ \mathbf{k}' = k' \sin r \mathbf{e}_x + k' \cos r \mathbf{e}_z \end{cases}$



Snell's law, $\frac{\sin i}{\sin r} = \frac{n'}{n}$ [(7.36)], can be written: $\sin r = \frac{\sin i}{\sin i_0}$,

where $i_0 \equiv \sin^{-1} \frac{n'}{n}$ [$< 90^\circ$, $\because n > n'$].

See p. 27

Thus, if $i > i_0$, we have

$$\sin r = \frac{\sin i}{\sin i_0} > 1 \;\Rightarrow\; \cos r = [\underbrace{1 - \sin^2 r}_{<0}]^{1/2} = i[(\frac{\sin i}{\sin i_0})^2 - 1]^{1/2}$$

$\Rightarrow$ The propagation factor ($e^{i\mathbf{k}'\cdot\mathbf{x}}$) of the refracted wave behaves as

$$e^{i\mathbf{k}'\cdot\mathbf{x}} = e^{ik'(x \sin r + z \cos r)} = \underbrace{e^{-k'[(\frac{\sin i}{\sin i_0})^2 - 1]^{1/2} z}\, e^{ik'\frac{\sin i}{\sin i_0} x}}_{\text{surface wave}} \qquad (7.46)$$

self-study

Wave vector and fields of the refracted wave:

Rewrite (7.46): $e^{i\mathbf{k}'\cdot\mathbf{x}} = e^{-k'[(\frac{\sin i}{\sin i_0})^2 - 1]^{1/2} z}\, e^{ik'\frac{\sin i}{\sin i_0} x}$

We see that $\mathbf{k}'$ (of the refracted wave) may be expressed as

$$\mathbf{k}' = k'_x \mathbf{e}_x + i k'_z \mathbf{e}_z \qquad (48)$$

where $k'_x = k' \frac{\sin i}{\sin i_0}$, $k'_z = k'[(\frac{\sin i}{\sin i_0})^2 - 1]^{1/2}$ and both k'_x and k'_z are real and positive quantities determined by the incident angle i. Note that (48) satisfies (16), i.e. $\mathbf{k}'\cdot\mathbf{k}' = k'^2_x - k'^2_z = k'^2 = \sqrt{\mu'\varepsilon'}\omega^2$.

Consider the case with $\mathbf{E}'_0 \parallel$ plane of incidence and write

$$\mathbf{E}'_0 = E'_{0x} \mathbf{e}_x + i E'_{0z} \mathbf{e}_z \qquad (49)$$

Then, $\begin{cases} \mathbf{k}'\cdot\mathbf{E}'_0 = 0 \;[(17)] \;\Rightarrow\; k'_x E'_{0x} - E'_{0z} k'_z = 0 & (50) \\ \mathbf{B}'_0 = \sqrt{\mu'\varepsilon'} \frac{\mathbf{k}' \times \mathbf{E}'_0}{k'} \;[(19)] \;\Rightarrow\; \mathbf{B}'_0 = i \frac{-k'_x E'_{0z} + k'_z E'_{0x}}{\omega} \mathbf{e}_y & (51) \end{cases}$

(48)-(51) give the surface wave solution discussed earlier in (21).

self-study

Poynting vector: (Consider $\mathbf{E}'_0 \parallel$ plane of incidence as an example)

Rewrite (20'): $\langle \mathbf{S} \rangle_t = \frac{1}{2\omega} \mathrm{Re}\left\{ \frac{1}{\mu'} \left[\mathbf{k}' |\mathbf{E}'_0|^2 - \mathbf{E}'_0 (\mathbf{k}' \cdot \mathbf{E}'^*_0) \right] e^{i(\mathbf{k}' - \mathbf{k}'^*)\cdot\mathbf{x}} \right\}$

$$\mathbf{k}' = k'_x \mathbf{e}_x + i k'_z \mathbf{e}_z \Rightarrow e^{i(\mathbf{k}' - \mathbf{k}'^*)\cdot\mathbf{x}} = e^{-2k'_z z} \qquad (52)$$

$$\mathbf{E}'_0 = E'_{0x} \mathbf{e}_x + i E'_{0z} \mathbf{e}_z \Rightarrow \mathbf{k}' \cdot \mathbf{E}'^*_0 = k'_x E'_{0x} + k'_z E'_{0z} = 2 k'_x E'_{0x} \qquad (53)$$

Sub. (52), (53), $\mathbf{k}' = k'_x \mathbf{e}_x + i k'_z \mathbf{e}_z$, and $\mathbf{E}'_0 = E'_{0x} \mathbf{e}_x + i E'_{0z} \mathbf{e}_z$ into (20′)

$$\langle \mathbf{S} \rangle_t = \frac{1}{2\omega\mu'} [k'_x \overbrace{|\mathbf{E}'_0|^2}^{\boxed{= |E'_{0x}|^2 + |E'_{0z}|^2}} - 2k'_x |E'_{0x}|^2] e^{-2k'_z z} \mathbf{e}_x = \frac{k'_x}{2\omega\mu'} [|E'_{0z}|^2 - \overbrace{|E'_{0x}|^2}^{\boxed{= k'^2_z |E'_{0z}|^2 / k'^2_x \text{ [from (50)]}}}] e^{-2k'_z z} \mathbf{e}_x$$

$$= \frac{k'_x}{2\omega\mu'} [|E'_{0z}|^2 - \frac{k'^2_z}{k'^2_x} |E'_{0z}|^2] e^{-2k'_z z} \mathbf{e}_x = \frac{1}{2\omega\mu' k'_x} |E'_{0z}|^2 \underbrace{(k'^2_x - k'^2_z)}_{\boxed{= k'^2}} e^{-2k'_z z} \mathbf{e}_x$$

$$= \frac{k'^2}{2\omega\mu' k'_x} |E'_{0z}|^2 e^{-2k'_z z} \mathbf{e}_x = \frac{\varepsilon'\omega}{2k'_x} |E'_{0z}|^2 e^{-2k'_z z} \mathbf{e}_x \qquad (54)$$

$\Rightarrow$ Power flows along the x-direction. There is no power flowing from the $z < 0$ region into the $z > 0$ region $\Rightarrow$ total reflection as expected.

7.8 Superposition of Waves in One Dimension; Group Velocity

Superposition of 2 Waves: Consider 2 waves (Fig. 1), $\cos(\omega_1 t - k_1 x)$ and $\cos(\omega_2 t - k_2 x)$, in a dispersive medium characterized by $\omega = \omega(k)$. Assume $\omega_1 \approx \omega_2$ and $k_1 \approx k_2$, then $\frac{\omega_1}{k_1} \approx \frac{\omega_2}{k_2}$ gives the approximate phase velocity (v_{ph}) of the superposed wave (Fig.2). The difference in wavelengths results in alternating regions of constructive/destructive interferences, or spatial modulations of the superposed wave (Fig. 2). In addition, because of the difference in phase velocities, regions of constructive interference, which carry the field energy, will be at different positions at different times, moving at the group velocity (v_g).

$\cos(\omega_1 t - k_1 x) \rightarrow$

$\cos(\omega_2 t - k_2 x) \rightarrow$

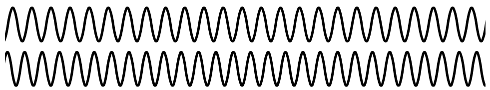


Fig. 1

$\begin{cases}\cos(\omega_1 t - k_1 x) \\ +\cos(\omega_2 t - k_2 x)\end{cases} \rightarrow$

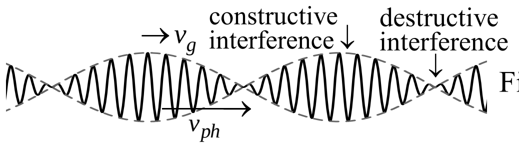


Fig. 2

The above qualitative picture can be analyzed as follows.

$$\cos(\omega_1 t - k_1 x) + \cos(\omega_2 t - k_2 x)$$

$$= 2\cos\left(\frac{\omega_1-\omega_2}{2}t - \frac{k_1-k_2}{2}x\right)\cos\left(\frac{\omega_1+\omega_2}{2}t - \frac{k_1+k_2}{2}x\right)$$

$$\approx \underbrace{2\cos\left(\frac{\omega_1-\omega_2}{2}t - \frac{k_1-k_2}{2}x\right)}_{(A)}\underbrace{\cos(\omega t - kx)}_{(B)},$$

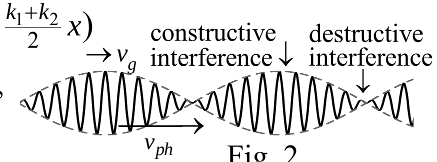


Fig. 2

where $\omega = \frac{\omega_1+\omega_2}{2} (\approx \omega_1 \approx \omega_2)$ and $k = \frac{k_1+k_2}{2} (\approx k_1 \approx k_2)$.

Factor (A) is the envelope function of the modulated wave (Fig. 2), which divides the wave into packets, each propagating at the speed

$$v_g = \frac{\frac{\omega_1-\omega_2}{2}}{\frac{k_1-k_2}{2}} = \frac{\omega_1-\omega_2}{k_1-k_2} \approx \frac{d\omega}{dk} \quad \text{(group velocity)}$$

Factor (B) gives the phase speed of the wave within each packet,

$$v_{ph} = \frac{\omega}{k} \quad \text{(phase velocity)}$$

Superposition of an Infinite Number of Waves: When an infinite number of waves (centered around ω_0, k_0 with a spread Δk, see Fig. 4) are superposed, interferences can result in cancellation everywhere except for a region of length Δx (Fig. 3), where the waves are constructively superposed into a wave packet.

$$k = \sqrt{\mu\varepsilon}\,\omega = \sqrt{\mu_r\varepsilon_r}\sqrt{\mu_0\varepsilon_0}\,\omega = \frac{n\omega}{c}$$

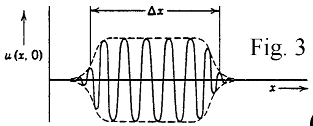


Fig. 3

Phase velocity: $v_p = \dfrac{\omega}{k} = \dfrac{c}{n}$ (7.88)

Group velocity: $v_g = \dfrac{d\omega}{dk} = \dfrac{c}{n + \omega(dn/d\omega)}$ (7.89)

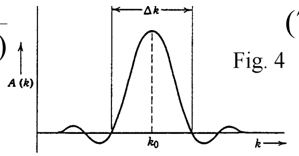


Fig. 4

Group delay: $\tau_g = \dfrac{L}{v_g} = \dfrac{dkL}{d\omega} = \dfrac{d\phi}{d\omega}$

Can a wave packet propagate at the group velocity faster than the speed of light?

Discussion:

(i) The pulse shape give by (7.85) is undistorted in time. However, if high order terms (e.g. $\frac{d^2\omega}{dk^2}$) are included in the expansion of $\omega(k)$ [(7.83)], the pulse will broaden with time.

Reason: $v_g = v_g(k) \Rightarrow \Delta v_g = \dfrac{dv_g}{dk}\Delta k = \dfrac{d^2\omega}{dk^2}\Delta k$

$\Rightarrow$ If $\dfrac{d^2\omega}{dk^2} \neq 0$, there is a spread in v_g

$\Rightarrow$

(ii) $\Delta k \Delta x \geq \frac{1}{2}$ $\Rightarrow$ A shorter wave packet has a greater spread in k (and v_g). Hence, it broadens faster than a longer pluse.

(iii) Wave packets in vacuum remain undistorted ($\omega = kc \Rightarrow \frac{d^2\omega}{dk^2} = 0$).

The following section gives a more rigorous treatment of the wave packet including pulse broadening.

7.9 Illustration of the Spreading of a Pulse as It Propagates in a Dispersive Medium

Rigorously, the real quantity $u(x,t)$, which we expressed in (7.80) as $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{ikx-i\omega(k)t}dk$, should be written*:

$$u(x,t)=\tfrac{1}{2}\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{ikx-i\omega(k)t}dk + c.c. \tag{7.90}$$

Assume (i) ω, k are both real, i.e. no dissipation.

(ii) The medium is isotropic, hence $\omega(-k)=\omega(k)$.

$$\Rightarrow u(x,t)=\tfrac{1}{2}\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{ikx-i\omega(k)t}dk+\tfrac{1}{2}\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A^*(k)e^{-ikx+i\omega(k)t}dk$$

$$\tfrac{\partial}{\partial t}u(x,t)=\tfrac{1}{2}\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} -i\omega(k)A(k)e^{ikx-i\omega(k)t}dk$$

$$+\tfrac{1}{2}\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} i\omega(k)A^*(k)e^{-ikx+i\omega(k)t}dk$$

*Note: In (7.90), $A(k)$ is not the Fourier transform of $u(x,t)$. Hence, the "reality condition" $A(k)=A^*(-k)$ [see Sec. 2.8 of lecture notes] is not applicable.

$$\begin{cases} u(x,0)=\tfrac{1}{2}\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{ikx}dk+\tfrac{1}{2}\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A^*(k)e^{-ikx}dk \\ \tfrac{\partial}{\partial t}u(x,0)=\tfrac{1}{2}\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} -i\omega(k)A(k)e^{ikx}dk+\tfrac{1}{2}\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} i\omega(k)A^*(k)e^{-ikx}dk \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-ik'x}u(x,0)dx=\tfrac{1}{2}\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(k)e^{i(k-k')x}dkdx$$

$$\boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm ixy}dx=\delta(y)} \rightarrow \quad +\tfrac{1}{2}\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A^*(k)e^{-i(k+k')x}dkdx$$

$$=\tfrac{\sqrt{2\pi}}{2}[A(k')+A^*(-k')] \tag{56}$$

$$\int_{-\infty}^{\infty} e^{-ik'x}\tfrac{\partial}{\partial t}u(x,0)dx=\tfrac{\sqrt{2\pi}}{2}[-i\omega(k')A(k')+i\,\underbrace{\omega(-k')}_{\omega(k')}A^*(-k')] \tag{57}$$

$(56)-\frac{1}{i\omega(k')}(57)$ $\qquad$ $\boxed{\text{by assumption (ii)}}$

$$\Rightarrow A(k)=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\left[u(x,0)+\tfrac{i}{\omega(k)}\tfrac{\partial}{\partial t}u(x,0)\right]dx \tag{7.91}$$

Example:

$$\begin{cases} u(x,0)=\exp(-\frac{x^2}{2L^2})\cos k_0x & (7.92) \\ \frac{\partial}{\partial t}u(x,0)=0 & (7.93) \\ \omega(k)=\nu[1+\frac{a^2k^2}{2}] & (7.95) \end{cases}$$

(7.92) and (7.93): initial conditions

$$\boxed{\frac{dv_g}{dk}=\frac{d^2\omega}{dk^2}=\nu a^2\neq 0 \quad \Rightarrow \text{Expect spreading of pulse.}}$$

$$\Rightarrow A(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}[u(x,0)+\frac{i}{\omega(k)}\frac{\partial}{\partial t}u(x,0)]dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}e^{-x^2/2L^2}\cos k_0x\,dx$$

$$=\frac{L}{2}\left\{\exp[(-\frac{L^2}{2})(k-k_0)^2]+\exp[(-\frac{L^2}{2})(k+k_0)^2]\right\} \tag{7.94}$$

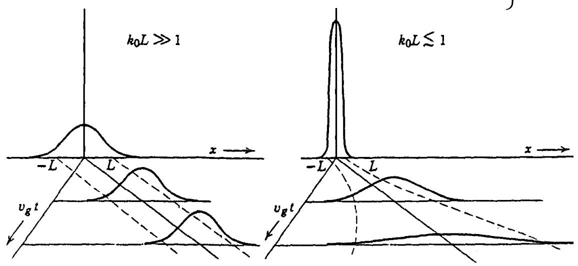
$$\Rightarrow u(x,t)=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{ikx-i\omega(k)t}dk+c.c.$$

$$=\frac{L}{2\sqrt{2\pi}}\mathrm{Re}\int_{-\infty}^{\infty}[e^{-\frac{L^2}{2}(k-k_0)^2}+e^{-\frac{L^2}{2}(k+k_0)^2}]e^{ikx-i\nu t(1+\frac{a^2k^2}{2})}dk$$

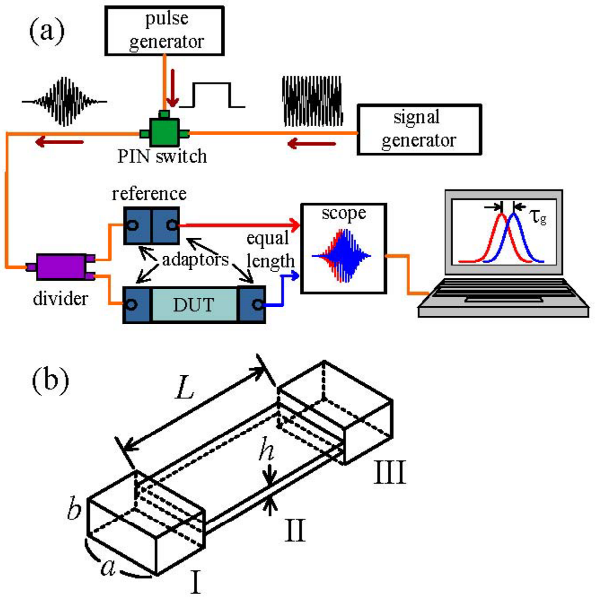
$$=\frac{1}{2}\mathrm{Re}\left\{\begin{array}{l} \frac{1}{(1+\frac{ia^2\nu t}{L^2})^{\frac{1}{2}}}\exp[-\frac{(x-\nu a^2k_0t)^2}{2L^2(1+\frac{ia^2\nu t}{L^2})}]\cdot\exp[ik_0x-i\nu(1+\frac{a^2k_0^2}{2})t] \\ \qquad\qquad \uparrow \text{a wave packet propagating forward} \\ +\ (k_0\rightarrow -k_0) \ \leftarrow \text{a wave packet propagating backward} \end{array}\right\} \tag{7.98}$$

where L is a function of t given by (7.99):

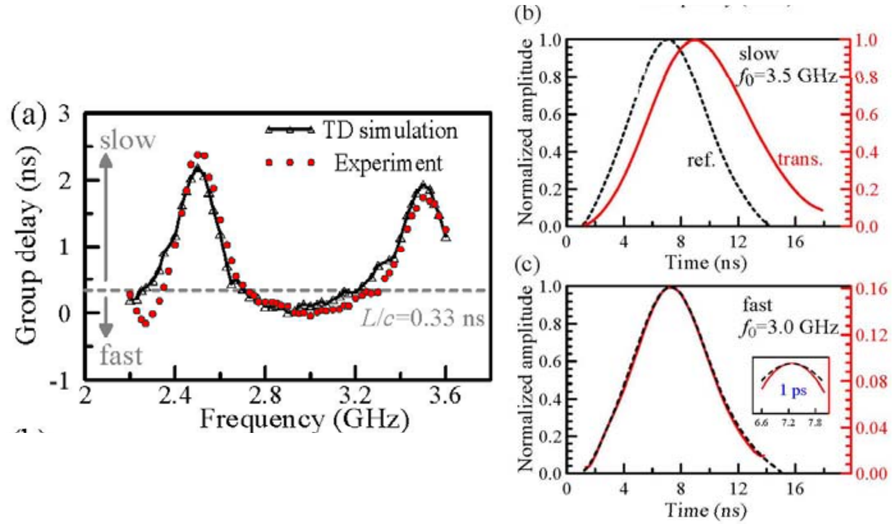
$$L(t)=[L^2+(\tfrac{a^2\nu t}{L})^2]^{\frac{1}{2}}$$



Superluminal Effect



Experimental results



Homework of Chap. 7

Problems: 2, 3, 4, 6, 13,
14, 19, 20, 21, 28

Optional: 1, 22, 23, 27,