Chapter 9: Radiating Systems, Multipole Fields and Radiation

An Overview of Chapters on EM Waves : (covered in this course)

	source term in wave equation	boundary
Ch. 7	none	plane wave in ∞ space or in two semi-∞ spaces separated by the x - y plane
Ch. 8	none	conducting walls
Ch. 9	$\mathbf{J}, \rho \sim e^{-i\omega t}$ prescribed, as in an antenna	outgoing wave to ∞
Ch. 10	$\mathbf{J}, \rho \sim e^{-i\omega t}$ induced by incident EM waves, as in the case of scattering of a plane wave by a dielectric object.	outgoing wave to ∞
Ch. 14	moving charges, such as electrons in a synchrotron	outgoing wave to ∞

9.6 Spherical Wave Solutions of the Scalar Wave Equation

Spherical Bessel Functions and Hankel functions : Although this chapter deals with radiating systems, here we first solve the scalar source-free wave equation in the spherical coordinate syatem. The purpose is to obtain a complete set of spherical Bessel funtions and Hankel functions, with which we will expand the fields produced by the sources.

The scalar source-free wave equation is [see (6.32)]

$$\nabla^2\psi(\mathbf{x},t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\mathbf{x},t)=0 \qquad (9.77)$$

Let $\psi(\mathbf{x},t)=\int_{-\infty}^{\infty}\psi(\mathbf{x},\omega)e^{-i\omega t}d\omega$ (9.78)

$\Rightarrow$ Each Fourier component satisfies the Helmholtz wave eq.

$$(\nabla^2+k^2)\psi(\mathbf{x},\omega)=0, \qquad (9.79)$$

where $k\equiv\frac{\omega}{c}$

In spherical coordinates, $(\nabla^2+k^2)\psi=0$ is written

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}+k^2\psi=0$$

Let $\psi=U(r)P(\theta)Q(\varphi)$, we obtain

$$PQ\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dU}{dr}\right)+UQ\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+UP\frac{1}{r^2\sin^2\theta}\frac{d^2Q}{d\varphi^2}+k^2UPQ=0$$

Multiply by $\frac{r^2\sin^2\theta}{UPQ}$

The only term with φ-dependence, so this term must be a constant. Let it be $-m^2$.

$$\sin^2\theta\Big[\underbrace{\frac{1}{U}\frac{d}{dr}\left(r^2\frac{dU}{dr}\right)+k^2r^2}_{=l(l+1)}+\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\Big]+\underbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}_{=-m^2}=0$$

Dividing all terms by $\sin^2\theta$, we see that this is the only term with r-dependence. So it must be a constant. Let it be $l(l+1)$.

$$\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\varphi}$$

Thus, as in Sec. 3.1 of lecture notes,

$$P=P_\ell^m(\cos\theta),\ Q_\ell^m(\cos\theta);\ Q=e^{im\varphi},e^{-im\varphi}\Rightarrow\ PQ=Y_{lm}(\theta,\varphi)$$

rejected because of divergence at $\theta=\pm\pi$

$U(r)$ is governed by $\frac{d}{dr}(r^2\frac{dU}{dr})+k^2r^2U=l(l+1)U$. Rewrite U as $f_l(r)$. Then, $\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+k^2-\frac{l(l+1)}{r^2}\right]f_l(r)=0$ (9.81)

Let $f_l(r)=\frac{1}{r^{1/2}}u_l(r)\Rightarrow\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+k^2-\frac{(l+1/2)^2}{r^2}\right]u_l(r)=0$ (9.83)

$\Rightarrow\ u_l(r)=J_{l+\frac{1}{2}}(kr),\ N_{l+\frac{1}{2}}(kr)$ [Bessel functions of fractional order]

$\Rightarrow\ f_l(r)=\frac{1}{r^{1/2}}J_{l+\frac{1}{2}}(kr),\ \frac{1}{r^{1/2}}N_{l+\frac{1}{2}}(kr)$

Define $\begin{cases} j_l(kr)=\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}}J_{l+\frac{1}{2}}(kr) \\ n_l(kr)=\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}}N_{l+\frac{1}{2}}(kr)\end{cases}$ and $\begin{cases} h_l^{(1)}(kr)=j_l(kr)+in_l(kr) \\ h_l^{(2)}(kr)=j_l(kr)-in_l(kr)\end{cases}$

spherical Bessel functions   Hankel functions

$$\Rightarrow\psi(\mathbf{x},\omega)=\sum_{lm}\left[A_{lm}^{(1)}h_l^{(1)}(kr)+A_{lm}^{(2)}h_l^{(2)}(kr)\right]Y_{lm}(\theta,\phi)\quad [k=\frac{\omega}{c}]\quad (9.92)$$

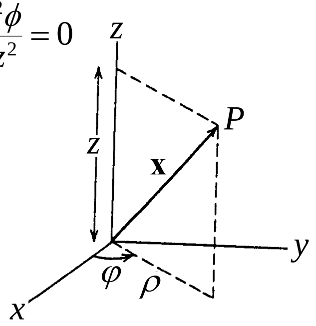
Review

3.7 Laplace Equation in Cylindrical Coordinates; Bessel Functions

$$\nabla^2\phi(\mathbf{x}) = 0 \Rightarrow \frac{\partial^2\phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

Let $\phi(\mathrm{x}) = R(\rho)Q(\varphi)Z(z)$

$$\Rightarrow \begin{cases} \frac{\partial^2 Z}{\partial z^2} - k^2 Z = 0 \Rightarrow Z = e^{\pm kz} \\ \frac{\partial^2 Q}{\partial\varphi^2} + \nu^2 Q = 0 \Rightarrow Q = e^{\pm i\nu\varphi} \\ \frac{\partial^2 R}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R = 0 \Rightarrow R = J_\nu(k\rho),\ N_\nu(k\rho) \end{cases}$$



where J_ν and N_ν are Bessel functions of the first and second kind, respectively (see following pages).

$$\Rightarrow \phi = \begin{Bmatrix} J_\nu(k\rho) \\ N_\nu(k\rho) \end{Bmatrix} \begin{Bmatrix} e^{i\nu\varphi} \\ e^{-i\nu\varphi} \end{Bmatrix} \begin{Bmatrix} e^{kz} \\ e^{-kz} \end{Bmatrix} \qquad (3)$$

Bessel Functions : If we let $x = k\rho$, the equation for R takes the standard form of the Bessel equation,

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0 \qquad (3.77)$$

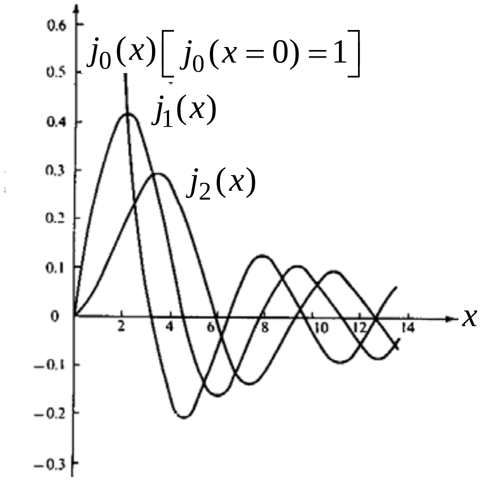
with solutions $J_\nu(x)$ and $N_\nu(x)$, from which we define the Hankel functions:

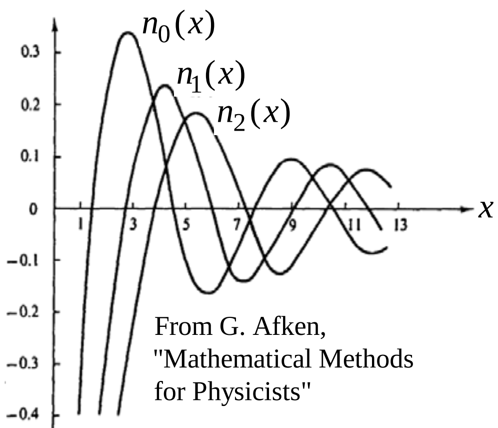
$$\begin{cases} H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x) \\ H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x) \end{cases} \qquad (3.86)$$

and the modified Bessel functions (Bessel functions of imaginary argument)

$$\begin{cases} I_\nu(x) = i^{-\nu} J_\nu(ix) & (3.100) \\ K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) & (3.101) \end{cases}$$

See Jackson pp. 112-116, Gradshteyn & Ryzhik, and Abramowitz & Stegun for properties of these special functions.





From G. Afken, "Mathematical Methods for Physicists"

$$j_l(x) \xrightarrow{x<<1,\ l} \frac{x^l}{(2l+1)!!}\left[1 - \frac{x^2}{2(2l+3)} + \cdots\right]$$

$$n_l(x) \xrightarrow{x<<1,\ l} -\frac{(2l-1)!!}{x^{l+1}}\left[1 - \frac{x^2}{2(1-2l)} + \cdots\right]$$

$$j_l(x) \xrightarrow{x>>l} \frac{1}{x}\sin\left(x - \frac{l\pi}{2}\right)$$

$$n_l(x) \xrightarrow{x>>l} -\frac{1}{x}\cos\left(x - \frac{l\pi}{2}\right)$$

$h_l^{(1)}(x) \xrightarrow{x>>l} (-i)^{l+1}\frac{e^{ix}}{x}$ [$\Rightarrow$ spatial dependence of spherical waves.]

See Jackson pp. 426-427 for further properties of j_l, n_l, $h_l^{(1)}$, and $h_l^{(2)}$.

Expansion of the Green function : Solution of the Green equation

$$(\nabla^2 + k^2)G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \qquad (6.36)$$

is given by (derived in Sec. 6.4.)

$$G(\mathbf{x},\mathbf{x}') = \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \quad \left[\begin{matrix}\text{in infinite space and for outgoing-} \\ \text{wave boundary condition.}\end{matrix}\right] \qquad (6.40)$$

We may solve (6.36) in the same way as in Sec. 3.9, i.e. write

$$G(\mathbf{x},\mathbf{x}') = \sum_{lm} g_l(r,r')Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi),$$

solve for $g_l(r,r')$ for $r > r'$ and $r < r'$ [where $\delta(\mathbf{x}-\mathbf{x}') = 0$], and then apply boundary conditions at $r = 0$, $r = \infty$, and $r = r'$. The result is

$$G(\mathbf{x},\mathbf{x}') = 4\pi ik \sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>) \sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

Equating the two expressions above for $G(\mathbf{x},\mathbf{x}')$, we obtain

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} = 4\pi ik \sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>) \sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi), \qquad (9.98)$$

where $r_<$ and $r_>$ are, respectively, the smaller and larger of r and r'.

Summary of Differential Equations and Solutions:

Source-free D.E.	Laplace eq. $\nabla^2\phi=0$	Helmholtz eq. $\left(\nabla^2+k^2\right)\psi=0$
Solutions Cartesian cylindrical spherical	$e^{i\alpha x},\ e^{i\beta y},\ e^{\sqrt{\alpha^2+\beta^2}z}$, etc. (Sec. 2.9) $J_m(kr),\ e^{im\theta},\ e^{kz}$, etc. (Sec. 3.7) $Y_{lm}(\theta,\phi),\ r^l$, etc. (Secs. 3.1, 3.2)	$e^{ik_x x},\ e^{ik_y y},\ e^{ik_z z}$, etc. (Sec. 8.4) $J_m\left(\sqrt{\frac{\omega^2}{c^2}-k_z^2}\ r\right), e^{im\theta}, e^{ik_z z}$, etc. (Sec. 8.7) $Y_{lm}(\theta,\phi),\ j_l(kr),\ n_l(kr)$, etc. (Sec. 9.6)
D.E. with a point source	$\nabla^2 G(\mathbf{x},\mathbf{x}')=-4\pi\delta(\mathbf{x}-\mathbf{x}')$ b.c.: $G(\infty)=0$	$(\nabla^2+k^2)G(\mathbf{x},\mathbf{x}')=-4\pi\delta(\mathbf{x}-\mathbf{x}')$ b.c.: outgoing wave
Solutions (Green functions)	$G=\frac{1}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$	$G=\frac{e^{ik\lvert\mathbf{x}-\mathbf{x}'\rvert}}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$ [Eq. (6.40)]
Series expansin of Green function	Eqs. (3.70), (3.148), (3.168)	Eq. (9.98)

Summary of Differential Equations and Solutions:

Source-free D.E.	Helmholtz eq. $\left(\nabla^2+k^2\right)\psi=0$	Wave Eq. $\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi=0$
Solutions Cartesian cylindrical spherical	$e^{ik_x x},\ e^{ik_y y},\ e^{ik_z z}$, etc. (Sec. 8.4) $J_m\left(\sqrt{\frac{\omega^2}{c^2}-k_z^2}\ r\right), e^{im\theta}, e^{ik_z z}$, etc. (Sec. 8.7) $Y_{lm}(\theta,\phi),\ j_l(kr),\ n_l(kr)$, etc. (Sec. 9.6)	$\begin{Bmatrix}\mathbf{A}(\mathbf{x},t)\\ \Phi(\mathbf{x},t)\end{Bmatrix}=\iint d^3x'dt'\ \frac{\delta\left[t'-\left(t-\frac{\lvert\mathbf{x}-\mathbf{x}'\rvert}{c}\right)\right]}{4\pi\lvert\mathbf{x}-\mathbf{x}'\rvert}\begin{Bmatrix}\mu_0\mathbf{J}(\mathbf{x}',t')\\ \rho(\mathbf{x}',t')/\varepsilon_0\end{Bmatrix}$
D.E. with a point source	$(\nabla^2+k^2)G(\mathbf{x},\mathbf{x}')=-4\pi\delta(\mathbf{x}-\mathbf{x}')$ b.c.: outgoing wave	$(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2})G^+(\mathbf{x},t,\mathbf{x}',t')$ $=-4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$ b.c.: outgoing wave
Solutions (Green functions)	$G=\frac{e^{ik\lvert\mathbf{x}-\mathbf{x}'\rvert}}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$ [Eq. (6.40)]	$G^+(\mathbf{x},t,\mathbf{x}',t')=\frac{\delta\left[t'-\left(t-\frac{\lvert\mathbf{x}-\mathbf{x}'\rvert}{c}\right)\right]}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$ [Eq. (6.44)]
Series expansin of Green function	Eq. (9.98)	

9.1 Radiation of a Localized Oscillating Source

Review of Inhomogeneous Wave Equations and Solutions:

$$\begin{cases}\nabla^2\Phi-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi=-\rho/\varepsilon_0 & (6.15)\\ \nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}=-\mu_0\mathbf{J} & (6.16)\end{cases}$$

[in free space, Φ and $\mathbf{A}$ satisfy Lorenz gauge.]

Basic structure of the inhomogenous wave equation:

$$\nabla^2\psi-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi=-4\pi f(\mathbf{x},t) \quad (6.32)$$

Solution of (6.32) with outgoing-wave b.c.:

$$\psi(\mathbf{x},t)=\underbrace{\psi_{in}(\mathbf{x},t)}_{\text{homogeneous solution}}+\int d^3x'\int dt' G^+(\mathbf{x},t,\mathbf{x}',t')f(\mathbf{x}',t') \quad (6.45)$$

where $G^+(\mathbf{x},t,\mathbf{x}',t')=\frac{\delta\left[t'-\left(t-\frac{\lvert\mathbf{x}-\mathbf{x}'\rvert}{c}\right)\right]}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$ (6.44)

$f(\mathbf{x}',t')$ in (6.45) is evaluated at the retarded time.

is the solution of

$$(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2})G^+(\mathbf{x},t,\mathbf{x}',t')=-4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \quad (6.41)$$

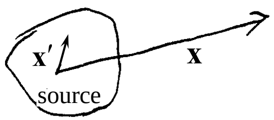
with outgoing wave b.c.

Using (6.45) (assume $\psi_{in}=0$) on (6.15) & (6.16), we obtain the gereral solutions for $\mathbf{A}$ and Φ, which are valid for arbitrary $\mathbf{J}$ and ρ.

$$\begin{Bmatrix}\mathbf{A}(\mathbf{x},t)\\ \Phi(\mathbf{x},t)\end{Bmatrix}=\frac{1}{4\pi}\int d^3x'\int dt'\ \frac{\delta\left[t'-\left(t-\frac{\lvert\mathbf{x}-\mathbf{x}'\rvert}{c}\right)\right]}{\lvert\mathbf{x}-\mathbf{x}'\rvert}\begin{Bmatrix}\mu_0\mathbf{J}(\mathbf{x}',t')\\ \rho(\mathbf{x}',t')/\varepsilon_0\end{Bmatrix} \quad (6.48),(9.2)$$

In general, the sources, $\mathbf{J}(\mathbf{x}',t')$ and $\rho(\mathbf{x}',t')$, contain a static part and a time dependent part. For static $\mathbf{J}(\mathbf{x})$ and $\rho(\mathbf{x})$, (9.2) gives the static $\mathbf{A}$ and Φ in Ch. 5 and Ch. 1, respecticely.

$$\mathbf{A}(\mathbf{x})=\mathbf{A}(\mathbf{x})=\frac{\mu_0}{4\pi}\int d^3x'\frac{\mathbf{J}(\mathbf{x}')}{\lvert\mathbf{x}-\mathbf{x}'\rvert} \quad (5.32)$$



$$\Phi(\mathbf{x})=\Phi(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int d^3x'\frac{\rho(\mathbf{x}')}{\lvert\mathbf{x}-\mathbf{x}'\rvert} \quad (1.17)$$

Question: It is stated on p. 408 that (9.2) is valid provided no boundary surfaces are present. Why? [See discussion below (6.47) in Ch. 6 of lectures notes.]

Fields by Harmonic Sources : Only time-dependent sources can radiate. Radiation from moving charges are treated in Ch. 13 and Ch. 14. Here, specialize to sources of the form (as in an antenna):

$$\begin{aligned}\rho(\mathbf{x},t) &= \rho(\mathbf{x})e^{-i\omega t}\\ \mathbf{J}(\mathbf{x},t) &= \mathbf{J}(\mathbf{x})e^{-i\omega t}\end{aligned} \qquad (9.1)$$

Sub. (9.1) into (9.2) and carry out the t'-integration, we obtain

$$\mathbf{A}(\mathbf{x},t) = \mathbf{A}(\mathbf{x})e^{-i\omega t} \text{ with } \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}'), \qquad (9.3)$$

where $k \equiv \frac{\omega}{c}$.

We shall assume that $\mathbf{J}(\mathbf{x})$ is independent of $\mathbf{A}(\mathbf{x})$, i.e. the source will not be affected by the fields they radiate. Otherwise, (9.3) is an integral equation for $\mathbf{A}(\mathbf{x})$.

A simpler derivation of (9.3): We specialize to harmonic sources from the outset. Then, only (6.16) is required.

$$\nabla^2\mathbf{A}(\mathbf{x},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}(\mathbf{x},t) = -\mu_0\mathbf{J}(\mathbf{x},t) \qquad (6.16)$$

Let $\mathbf{J}(\mathbf{x},t) = \mathbf{J}(\mathbf{x})e^{-i\omega t}$ and $\mathbf{A}(\mathbf{x},t) = \mathbf{A}(\mathbf{x})e^{-i\omega t}$

$\Rightarrow \left(\nabla^2 + k^2\right)\mathbf{A}(\mathbf{x}) = -\mu_0\mathbf{J}(\mathbf{x})$ [inhomogeneous Helmholtz wave eq.]

The Green equation for the above equation is

$$\left(\nabla^2 + k^2\right)G_k(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \qquad (6.36)$$

Solution of (6.36) with outgoing wave b.c.

$$G_k(\mathbf{x},\mathbf{x}') = \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \qquad (6.40)$$

$$\Rightarrow \mathbf{A}(\mathbf{x}) = \int d^3x' G_k(\mathbf{x},\mathbf{x}')\frac{\mu_0}{4\pi}\mathbf{J}(\mathbf{x}') = \frac{\mu_0}{4\pi}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}'),$$

which is (9.3).

Rewrite (9.3), $\quad \mathbf{A}(\mathbf{x}) = \dfrac{\mu_0}{4\pi}\displaystyle\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}'), \qquad (9.3)$

Maxwell eqs. give
$$\begin{cases}\mathbf{H} = \frac{1}{\mu_0}\nabla\times\mathbf{A} \quad \text{(everywhere)} & (9.4)\\ \mathbf{E} = \frac{iZ_0}{k}\nabla\times\mathbf{H} \quad \text{(outside the source)} & (9.5)\end{cases}$$

where $Z_0 = \sqrt{\mu_0/\varepsilon_0} = 377\ \Omega$ (impedance of free space, p. 297).

Thus, given the source function $\mathbf{J}(\mathbf{x})$, we may in principle evaluate $\mathbf{A}(\mathbf{x})$ from (9.3) and then obtain the fields $\mathbf{H}$ and $\mathbf{E}$ from (9.4) and (9.5).

Note that $e^{-i\omega t}$ dependence has been assumed for $\mathbf{J}$, hence for all other quantities which are expressed in terms of $\mathbf{J}$.

Note: The charge distribution ρ and scalar potential Φ are not required for the determination of $\mathbf{H}$ and $\mathbf{E}$? (why?)

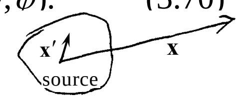
Near-Field Expansion of $\mathbf{A}(\mathbf{x}) = \dfrac{\mu_0}{4\pi}\displaystyle\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}') \qquad (9.3)$

Before going into algebraic details, we may readily observe some general properties of $\mathbf{A}(\mathbf{x})$ near the source ($r \ll \lambda$).

For $\mathbf{x}$ outside the source and $r \ll \lambda$ (or $kr \ll 1$), we let $e^{ik|\mathbf{x}-\mathbf{x}'|} \approx 1$

and use $\dfrac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi\displaystyle\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi). \qquad (3.70)$

Since $r > r'$, we have $r_> = r$ and $r_< = r'$.



$$\Rightarrow \mathbf{A}(\mathbf{x}) \underset{\boxed{kr\ll 1}}{\approx} \mu_0\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{1}{r^{l+1}}Y_{lm}(\theta,\varphi)\int d^3x'\mathbf{J}(\mathbf{x}')r'^lY_{lm}^*(\theta',\varphi') \qquad (9.6)$$

The integral in (9.6) yields multipole coefficients as in (4.2). Thus, (9.6) shows that, for $kr \ll 1$, $\mathbf{A}(\mathbf{x})$ can be decomposed into multipole fields, which fall off as $r^{-(l+1)}$ just as the static multipole fields, but with the $e^{-i\omega t}$ dependence. However, we will show later that, far from the source ($kr \gg 1$), $\mathbf{A}(\mathbf{x})$ behaves as an outgoing spherical wave.

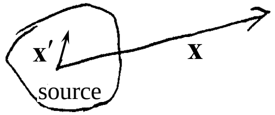
Full Expansion of $\mathbf{A}(\mathbf{x})$: We may in fact expand $\mathbf{A}(\mathbf{x})$, without approximations, by using (9.98). For $\mathbf{x}$ outside the source, we have $r_> = |\mathbf{x}| = r$, $r_< = |\mathbf{x}'| = r'$. Hence, (9.98) can be written

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} = 4\pi ik \sum_{l=0}^{\infty} j_l(kr')h_l^{(1)}(kr) \sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

Sub. this equation into $\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}')$, we obtain

$$\mathbf{A}(\mathbf{x}) = \mu_0 ik \sum_{l,m} h_l^{(1)}(kr)Y_{lm}(\theta,\phi)\int d^3x'\mathbf{J}(\mathbf{x}')j_l(kr')Y_{lm}^*(\theta',\phi'), \quad (9.11)$$

where $h_l^{(1)}(kr) = \frac{e^{ikr}(2l-1)!!}{i(kr)^{l+1}}\sum_{n=0}^{l} a_n(ikr)^n$

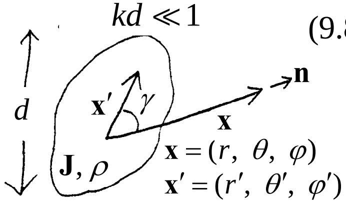


with $a_n = \frac{(-1)^n(2l-n)!}{(2l-1)!!(2l-2n)!!n!}$ $(a_0 = 1,\ a_1 = -1, \cdots)$

(See Abramowitz & Stegun, "Handbook of Mathematical Functions," p. 439.)

(9.11) is an exact expression for $\mathbf{A}(\mathbf{x})$. We now assume $kd \ll 1$ (i.e. source dimension $\ll$ wavelength). Then, $kr' \ll 1$ and $j_l(kr')$ reduces to

$$j_l(kr')\big|_{kr'\ll 1} = \frac{(kr')^l}{(2l+1)!!}$$



(9.88)

Sub. $h_l^{(1)}(kr) = \frac{e^{ikr}(2l-1)!!}{i(kr)^{l+1}}\sum_{n=0}^{l} a_n(ikr)^n$

and (9.88) into (9.11), we obtain

$$\mathbf{A}(\mathbf{x}) = \mu_0 \sum_{l,m}\left\{\begin{array}{l}\frac{1}{2l+1}Y_{lm}(\theta,\phi)\frac{e^{ikr}}{r^{l+1}}[1 + a_1(ikr) + a_2(ikr)^2 + \cdots + a_l(ikr)^l] \\ \cdot\int d^3x'\mathbf{J}(\mathbf{x}')r'^l Y_{lm}^*(\theta',\phi')\end{array}\right\} \quad (1)$$

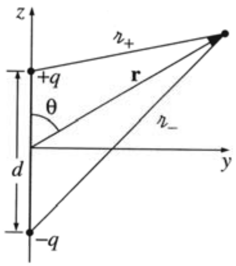
(1) is the combination of (9.6) and (9.12) in Jackson. It is valid for $kd \ll 1$ and any $\mathbf{x}$ outside the source. The region outside the source is commonly divided into 3 zones (by their different physical characters):

The near (static) zone: $d \ll r \ll \lambda$ $(\Rightarrow kr \ll 1)$
The intermediate (induction) zone: $d \ll r \sim \lambda$ $(\Rightarrow kr \sim 1)$
The far (radiation) zone: $d \ll \lambda \ll r$ $(\Rightarrow kr \gg 1)$

Griffiths

11.1.2 Electric Dipole Radiation

Consider two point charges of $+q$ and $-q$ separating by a distance $d(t)$. Assume $d(t)$ can be expressed in sinusoidal form.



The result is an oscillating electric dipole:

$$\mathbf{p}(t) = qd(t)\hat{\mathbf{z}} = qd\cos(\omega t)\hat{\mathbf{z}} = p_0\cos(\omega t)\hat{\mathbf{z}}, \quad \text{where } p_0 \equiv qd.$$

The retarded potential is:

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\left\{\frac{q_0\cos[\omega(t-\mathscr{r}_+/c)]}{\mathscr{r}_+} - \frac{q_0\cos[\omega(t-\mathscr{r}_-/c)]}{\mathscr{r}_-}\right\}$$

Griffiths

Electric Dipole Radiation: Approximations

Approximation #1: Make this physical dipole into a perfect dipole.

$$d << r$$

Estimate the spearation distances by the law of cosines.

$$\mathscr{r}_\pm = \sqrt{r^2 \mp rd\cos\theta + (d/2)^2} \cong r(1 \mp \frac{d}{2r}\cos\theta)$$

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}(1 \pm \frac{d}{2r}\cos\theta)$$

$$\cos[\omega(t-\mathscr{r}_\pm/c)] \cong \cos[\omega(t-\frac{r}{c}) \pm \frac{\omega d}{2c}\cos\theta]$$

$$= \cos[\omega(t-\frac{r}{c})]\cos(\frac{\omega d}{2c}\cos\theta) \mp \sin[\omega(t-\frac{r}{c})]\sin(\frac{\omega d}{2c}\cos\theta)$$

Approximation #2: The wavelength is much longer than the dipole size.

$$d << \frac{c}{\omega} = \frac{\lambda}{2\pi}$$

Griffiths

The Retarded Scalar Potential

$$\cos[\omega(t-\mathscr{r}_{\pm}/c)] \cong \cos[\omega(t-\frac{r}{c})]\underbrace{\cos(\frac{\omega d}{2c}\cos\theta)}_{=1} \mp \sin[\omega(t-\frac{r}{c})]\underbrace{\sin(\frac{\omega d}{2c}\cos\theta)}_{\frac{\omega d}{2c}\cos\theta}$$

$$= \cos[\omega(t-\frac{r}{c})] \mp \sin[\omega(t-\frac{r}{c})]\frac{\omega d}{2c}\cos\theta$$

The retarded scalar potential is:

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\left\{ \begin{array}{l} \left[\cos[\omega(t-\frac{r}{c})] - \sin[\omega(t-\frac{r}{c})]\frac{\omega d}{2c}\cos\theta\right]\frac{1}{r}(1+\frac{d}{2r}\cos\theta) \\ -\left[\cos[\omega(t-\frac{r}{c})] + \sin[\omega(t-\frac{r}{c})]\frac{\omega d}{2c}\cos\theta\right]\frac{1}{r}(1-\frac{d}{2r}\cos\theta) \end{array} \right\}$$

$$\cong \frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left[-\frac{\omega}{c}\sin[\omega(t-\frac{r}{c}) + \frac{1}{r}\cos[\omega(t-\frac{r}{c})]\right]$$

Griffiths

The Retarded Scalar Potential

Approximation #3: at the radiation zone. $\frac{c}{\omega} << r$

The retarded scalar potential is:

$$V(\mathbf{r},t) \cong \frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left[-\frac{\omega}{c}\sin[\omega(t-\frac{r}{c})\right]$$

Three approximations

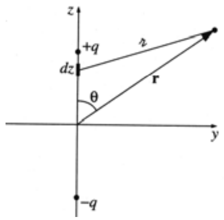
$$d << r \qquad d << \frac{c}{\omega}(=\frac{\lambda}{2\pi}) \qquad \frac{c}{\omega} << r$$

$$\Rightarrow d << \lambda << r$$

Griffiths

The Retarded Vector Potential

The retarded vector potential is determined by the current density.



$$I(t) = \frac{dq}{dt}\hat{\mathbf{z}} = -q_0\omega\sin\omega t\hat{\mathbf{z}}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}d\tau' = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2}\frac{-q\omega\sin[\omega(t-\mathscr{r}/c)]\hat{\mathbf{z}}}{\mathscr{r}}dz$$

$$\cong -\frac{\mu_0 p_0\omega}{4\pi r}\sin[\omega(t-\frac{r}{c})]\hat{\mathbf{z}} \qquad @\, d << \lambda << r$$

Retarded potentials:

$$\begin{cases} V(\mathbf{r},t) = -\frac{p_0\omega}{4\pi\varepsilon_0 c}\left[\frac{\cos\theta}{r}\sin[\omega(t-\frac{r}{c})\right] \\ \mathbf{A}(\mathbf{r},t) = -\frac{\mu_0 p_0\omega}{4\pi r}\sin[\omega(t-\frac{r}{c})]\hat{\mathbf{z}} \end{cases} \qquad \begin{cases} \mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} \\ \mathbf{B} = \nabla\times\mathbf{A} \end{cases}$$

Griffiths

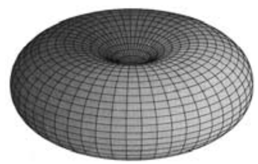
The Electromagnetic Fields and Poynting Vector

$$\begin{cases} \mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} = -\frac{\mu_0 p_0\omega^2}{4\pi\varepsilon_0 c}(\frac{\sin\theta}{r})\cos[\omega(t-\frac{r}{c})]\hat{\boldsymbol{\theta}} \\ \mathbf{B} = \nabla\times\mathbf{A} = -\frac{\mu_0 p_0\omega^2}{4\pi c}(\frac{\sin\theta}{r})\cos[\omega(t-\frac{r}{c})]\hat{\boldsymbol{\varphi}} \end{cases}$$

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}(\frac{\sin\theta}{r})\cos[\omega(t-\frac{r}{c})]\right\}^2\hat{\mathbf{r}}$$

The total power radiated is

$$<P> = \int\langle\mathbf{S}\rangle\cdot d\mathbf{a} = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\int(\frac{\sin\theta}{r})^2 r^2\sin\theta\, d\theta\, d\phi$$



$$= \frac{\mu_0 p_0^2\omega^4}{12\pi c}$$

9.2 Electric Dipole Fields and Radiation

Rewrite (1):

$$\mathbf{A}(\mathbf{x}) = \mu_0 \sum_{l,m} \left\{ \begin{array}{l} \frac{1}{2l+1} Y_{lm}(\theta,\phi) \frac{e^{ikr}}{r^{l+1}}[1+a_1(ikr)+a_2(ikr)^2+\cdots+a_l(ikr)^l] \\ \cdot \int d^3x' \mathbf{J}(\mathbf{x}') r'^l Y^*_{lm}(\theta',\phi') \end{array} \right\} \quad (1)$$

Take the $l=0$ term [$Y_{00}=\frac{1}{\sqrt{4\pi}}$] and denote it by $\mathbf{A}^p(\mathbf{x})$

$$\mathbf{A}^p(\mathbf{x}) = \mathbf{A}(\mathbf{x})^{l=0} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\mathbf{J}(\mathbf{x}')$$

$$= -\frac{i\mu_0\omega}{4\pi}\mathbf{p}\frac{e^{ikr}}{r}, \quad (9.16)$$

where $\mathbf{p}=\int \mathbf{x}'\rho(\mathbf{x}')d^3x'$ (4.8)

(9.16) gives the electric dipole contribution to the solution. It is valid for $kd \ll 1$ and any $\mathbf{x}$ outside the source.

Question: Why is there no monopole term (see p. 410)?

$$\iiint J_x dxdydz$$

$$= \iint dydz\left[\cancel{xJ_x\Big|_{-d}^{d}} - \int x\frac{\partial J_x}{\partial x}dx\right]$$

$$= -\iiint x\Big(\frac{\partial J_x}{\partial x} + \underbrace{\frac{\partial J_y}{\partial y}+\frac{\partial J_z}{\partial z}}\Big)dxdydz$$

give no contribution because $\mathbf{J}$ is localized: $\int \frac{\partial J_y}{\partial y}dy = J_y\Big|_{-d}^{d} = 0$

$$= -\int x\nabla\cdot\mathbf{J}d^3x$$

$$\Rightarrow \int \mathbf{J}d^3x = -\int \mathbf{x}\nabla\cdot\mathbf{J}d^3x$$

$$= -i\omega\underbrace{\int \mathbf{x}\rho(\mathbf{x})d^3x}_{\mathbf{p}} = -i\omega\mathbf{p}$$

$\nabla\cdot\mathbf{J}+\frac{\partial\rho}{\partial t}=0$

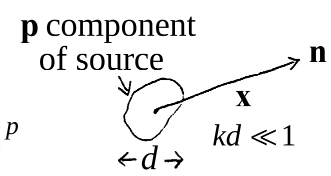
Rewrite (9.16): $\mathbf{A}^p(\mathbf{x}) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}\frac{e^{ikr}}{r}$ (9.16)

From (9.4), $\mathbf{H}^p = \frac{1}{\mu_0}\nabla\times\mathbf{A}^p$ and from (9.5), $\mathbf{E}^p = \frac{iZ_0}{k}\nabla\times\mathbf{H}^p$

$$\Rightarrow \begin{cases} \mathbf{H}^p = \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right) \\ \mathbf{E}^p = \frac{1}{4\pi\varepsilon_0}\left\{k^2(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\frac{e^{ikr}}{r} + [3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\} \end{cases} \quad (9.18)$$

In the far zone ($kr \gg 1$), (9.18) reduces to a spherical wave

$$\begin{cases} \mathbf{H}^p \simeq \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r} \\ \mathbf{E}^p \simeq Z_0\mathbf{H}^p\times\mathbf{n} \end{cases} \quad (9.19)$$

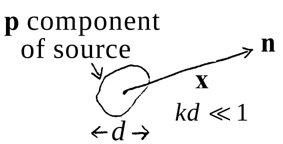


In (9.19), we see that $\mathbf{E}^p$ and $\mathbf{H}^p$ are in phase, and $\mathbf{E}^p$, $\mathbf{H}^p$, and $\mathbf{n}$ are mutually perpendicular. This is a general property of EM waves in unbounded, uniform space. Given any two of these quantities, we can find the third.

$$\begin{cases} \mathbf{H}^p = \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right) \\ \mathbf{E}^p = \frac{1}{4\pi\varepsilon_0}\left\{k^2(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\frac{e^{ikr}}{r} + [3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\} \end{cases} \quad (9.18)$$

In the near zone ($kr \ll 1$), (9.18) reduces to

$$\begin{cases} \mathbf{H}^p \simeq \frac{i\omega}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{1}{r^2} \\ \mathbf{E}^p \simeq \frac{1}{4\pi\varepsilon_0}[3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}]\frac{1}{r^3} \end{cases} \quad (9.20)$$



$\Rightarrow$
- (i) $\mathbf{E}^p$ and $\mathbf{H}^p$ are 90^{o} out of phase $\Rightarrow$ average power $=0$.
- (ii) $\mathbf{E}^p$ has the same spatial pattern as that of the static electric dipole in (4.13), but with $e^{-i\omega t}$ dependence.
- (iii) $\mu_0|H|^2 \sim (kr)^2\varepsilon_0|E|^2 \Rightarrow$ **E**-field energy $\gg$ **B**-field energy.

Questions: (i) Why does $\mathbf{E}^p$ have the static field pattern?

(ii) To obtain (9.20), we have neglected a few terms in (9.18). But some of the neglected terms are still important in the near zone? What are they and in what sense are they important?

$\left\langle\frac{dP}{d\Omega}\right\rangle_t$ = time-averaged power in the far zone/unit solid angle

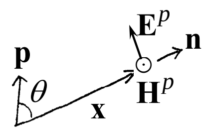
$$= \frac{1}{2}\mathrm{Re}\left[r^2\mathbf{n}\cdot\left(\mathbf{E}^p\times\mathbf{H}^{p*}\right)\right] \quad (9.21)$$

$$\xrightarrow{\boxed{(9.19)}} = \frac{c^2Z_0}{32\pi^2}k^4\,|\underbrace{(\mathbf{n}\times\mathbf{p})\times\mathbf{n}}|^2 \quad (9.22)$$

This vector gives the direction of $\mathbf{E}^p$, i.e. the polarization of the radiation (see figure below.)

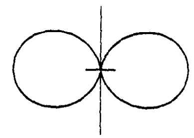
$$\Rightarrow \langle P\rangle_t = \text{total power radiated} = \frac{c^2Z_0k^4}{12\pi}|\mathbf{p}|^2 \quad (9.24)$$

In general, $\mathbf{p} = p_xe^{i\alpha}\mathbf{e}_x + p_ye^{i\beta}\mathbf{e}_y + p_ze^{i\gamma}\mathbf{e}_z$. If $\alpha=\beta=\gamma$, then $\mathbf{p}$ has a fixed direction, $\mathbf{p}=\mathbf{p}_0e^{i\alpha}$ with $\mathbf{p}_0 = p_x\mathbf{e}_x + p_y\mathbf{e}_y + p_z\mathbf{e}_z$, and



$$\left\langle\frac{dP}{d\Omega}\right\rangle_t = \frac{c^2Z_0}{32\pi^2}k^4|\mathbf{p}|^2\sin^2\theta. \quad (9.23)$$

Otherwise, the direction of $\mathbf{p}$ (hence $\left\langle\frac{dP}{d\Omega}\right\rangle_t$) vary with time, but $\langle P\rangle_t$ is still given by (9.24).



dipole radiation pattern

9.3 Magnetic Dipole and Electric Quadrupole Field

Rewrite (1):

$$\mathbf{A}(\mathbf{x}) = \mu_0 \sum_{l,m} \left\{ \begin{array}{l} \frac{1}{2l+1} Y_{lm}(\theta,\phi) \frac{e^{ikr}}{r^{l+1}} [1 + a_1(ikr) + a_2(ikr)^2 + \cdots + a_l(ikr)^l] \\ \cdot \int d^3x' \mathbf{J}(\mathbf{x}') r'^l Y^*_{lm}(\theta',\phi') \end{array} \right\} \quad (1)$$

Take the $l=1$ terms $[a_1 = -1]$

$$\mathbf{A}(\mathbf{x})^{l=1} = \frac{\mu_0}{3} \frac{e^{ikr}}{r^2}(1-ikr) \underbrace{\sum_{m=-1,0,1} Y_{1m}(\theta,\phi) \int d^3x' \mathbf{J}(\mathbf{x}') r' Y^*_{1m}(\theta',\phi')}$$

p. 109

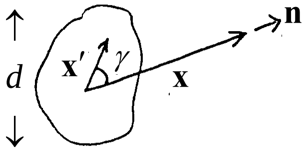
$$\sum_{m=-1,0,1} Y_{1m}(\theta,\phi) Y^*_{1m}(\theta',\phi') = \frac{3}{8\pi} \sin\theta \sin\theta' e^{i(\phi-\phi')}$$

$$+ \frac{3}{4\pi} \cos\theta \cos\theta' + \frac{3}{8\pi} \sin\theta \sin\theta' e^{-i(\phi-\phi')}$$

$$= \frac{3}{4\pi} [\sin\theta \sin\theta' \cos(\phi-\phi') + \cos\theta \cos\theta']$$

$$= \frac{3}{4\pi} \cos\gamma = \frac{3}{4\pi r'} \mathbf{n} \cdot \mathbf{x}'$$

set $l=1$ in (3.68)



Thus,

$$\mathbf{A}(\mathbf{x})^{l=1} = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \left(\frac{1}{r} - ik\right) \int d^3x' \mathbf{J}(\mathbf{x}')(\mathbf{n} \cdot \mathbf{x}') \qquad (9.30)$$

$$= \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \left(\frac{1}{r} - ik\right) \Big\{ \underbrace{\int d^3x' \tfrac{1}{2} [(\mathbf{n} \cdot \mathbf{x}')\mathbf{J} + (\mathbf{n} \cdot \mathbf{J})\mathbf{x}']}_{\text{electric quadrupole radiation}} + \underbrace{\int d^3x' \tfrac{1}{2} (\mathbf{x}' \times \mathbf{J}) \times \mathbf{n}}_{\text{magnetic dipole radiation}} \Big\}$$

$$= \mathbf{A}^Q + \mathbf{A}^m,$$

where $\mathbf{A}^m(\mathbf{x}) = \frac{ik\mu_0}{4\pi} (\mathbf{n} \times \mathbf{m}) \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr}\right) \left[\begin{array}{l} \text{for } kd \ll 1 \text{ and any} \\ \mathbf{x} \text{ outside the source} \end{array}\right] \qquad (9.33)$

with $\mathbf{m} = \frac{1}{2} \int (\mathbf{x} \times \mathbf{J}) d^3x$ [magnetic dipole moment]. $\mathbf{A}^m$ gives the magnetic dipole contribution through (9.4) and (9.5) (see p.15):

$$\left\{ \begin{array}{ll} \mathbf{H}^m = \frac{1}{4\pi} \left\{ k^2 (\mathbf{n} \times \mathbf{m}) \times \mathbf{n} \frac{e^{ikr}}{r} + [3\mathbf{n}(\mathbf{n} \cdot \mathbf{m}) - \mathbf{m}] \left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{ikr} \right\} & (9.35) \\ \mathbf{E}^m = -\frac{Z_0}{4\pi} k^2 (\mathbf{n} \times \mathbf{m}) \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr}\right) & (9.36) \end{array} \right.$$

In the far zone ($kr \gg 1$), we have the spherical wave sloution:

$$\left\{ \begin{array}{l} \mathbf{H}^m \simeq \frac{k^2}{4\pi} (\mathbf{n} \times \mathbf{m}) \times \mathbf{n} \frac{e^{ikr}}{r} \\ \mathbf{E}^m \simeq Z_0 \mathbf{H}^m \times \mathbf{n} \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \left\langle \frac{dP}{d\Omega} \right\rangle_t \simeq \frac{Z_0}{32\pi^2} k^4 |\underbrace{\mathbf{m} \times \mathbf{n}}|^2 \\ \langle P \rangle_t \simeq \frac{Z_0}{12\pi} k^4 |\mathbf{m}|^2 \end{array} \right.$$

$\Rightarrow$ direction of $\mathbf{E}^m$

In the near zone ($kr \ll 1$),

$$\left\{ \begin{array}{l} \mathbf{H}^m \simeq \frac{1}{4\pi} [3\mathbf{n}(\mathbf{n} \cdot \mathbf{m}) - \mathbf{m}] \frac{1}{r^3} \\ \mathbf{E}^m \simeq \frac{Z_0 k}{4\pi i} (\mathbf{n} \times \mathbf{m}) \frac{1}{r^2} \end{array} \right. \Rightarrow$$

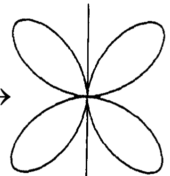
(i) $\mathbf{E}^m$ and $\mathbf{H}^m$ are 90° out of phase $\Rightarrow$ average power $= 0$.

(ii) $\mathbf{H}^m$ has the same spatial pattern as that of the static magnetic dipole in (5.56), but with $e^{-i\omega t}$ dependence.

(iii) **B**-field energy $\gg$ **E**-field energy.

The electric quadrupole radiation, discussed in (9.37)-(9.52), is more complicated. Here, we only illustrate its radiation pattern by the figure to the right.

quadrupole radiation pattern $\rightarrow$



Comparison between Static and Time-dependent Cases

	relations between ρ, **J**, **E**, and **B**	multipole expansion	definition of multipole moments	r-dependence of **E** and **B** (d : dimension of the source)
static case	$\rho(\mathbf{x}) \leftrightarrow \mathbf{E}(\mathbf{x})$ $\mathbf{J}(\mathbf{x}) \leftrightarrow \mathbf{B}(\mathbf{x})$	spherical harmonics expansion [(3.70)] or Taylor series [(4.10)] of $\frac{1}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$	$q = \int \rho(\mathbf{x}') d^3x'$ $\mathbf{p} = \int \mathbf{x}' \rho(\mathbf{x}') d^3x'$ $Q_{ij} = \int (3x'_i x'_j - r'^2 \delta_{ij}) \rho(\mathbf{x}') d^3x'$ $\mathbf{m} = \frac{1}{2} \int \mathbf{x}' \times \mathbf{J}(\mathbf{x}') d^3x'$	**E** or $\mathbf{B} \propto 1/r^{l+2}$ For $r \sim d$, all multipole fields can be significant. For $r \gg d$, multipole fields are dominated by the lowest-order nonvanishing term.
time-dependent case	$\left\{\begin{array}{c}\rho(\mathbf{x}) \\ \updownarrow \\ \mathbf{J}(\mathbf{x})\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\mathbf{E}(\mathbf{x}) \\ \updownarrow \\ \mathbf{B}(\mathbf{x})\end{array}\right\}$ $\Rightarrow$ EM waves	spherical harmonics expansion [(9.98)] of $\frac{e^{ik\lvert\mathbf{x}-\mathbf{x}'\rvert}}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$	There is no time-dependent monopole for an isolated source (see p. 410). **p**, Q_{ij}, and **m** have the same expressions as those of their static counterparts, but with the $e^{-i\omega t}$ time dependence. In time-dependent cases, electric multipoles can generate **B**-fields and magnetic multipoles can generate **E**-fields.	(a) near zone $\lambda \gg r \gg d$ **E** or $\mathbf{B} \propto e^{-i\omega t}/r^{l+2}$ Approx. the same field pattern and r-dependence as for the corresponding static multipole, but with $e^{-i\omega t}$ dependence (hence called quasi-static fields.) (b) far zone $r \gg \lambda \gg d$ $\mathbf{E}, \mathbf{B} \propto e^{ikr - i\omega t}/r$ (spherical EM waves) All multipole fields $\propto 1/r$, relative power levels unchanged with distance.

Induced Electric and Magnetic Dipoles

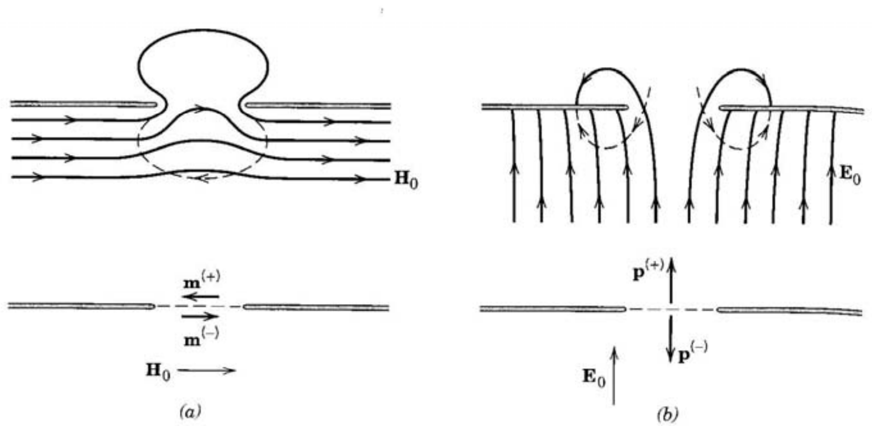
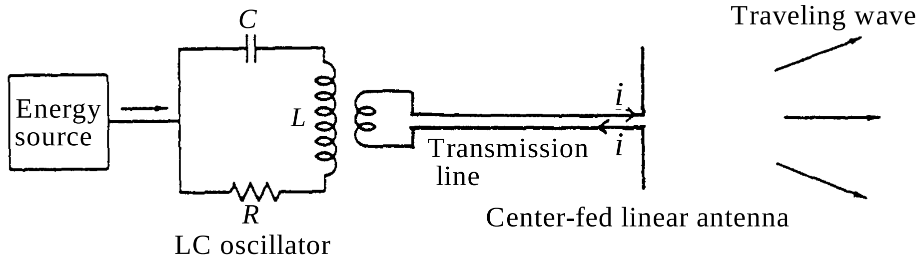
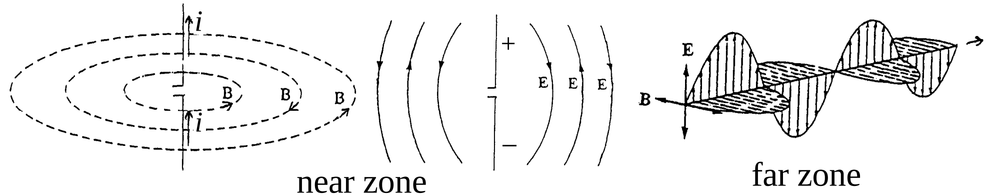


Figure 9.4 Distortion of (a) the tangential magnetic field and (b) the normal electric field by a small aperture in a perfectly conducting surface. The effective dipole moments, as viewed from above and below the surface, are indicated beneath.

9.4 Center-Fed Linear Antenna

A Qualitative Look at the Center - Fed Linear Antenna :





In the near zone, **E** and **B** are principally generated by ρ and **J**, respectively ($\Rightarrow$ largely static field patterns). In the far zone, **E** and **B** are regenerative through $\frac{d}{dt}\mathbf{B}$ and $\frac{d}{dt}\mathbf{E}$ ($\Rightarrow$ EM waves).

Detailed Analysis: The center-fed linear antenna is a case of special interest, because it allows the solution of (9.3) in closed form for any value of *kd*, whereas in Secs. 9.2 and 9.3, we assume *kd*<<1.

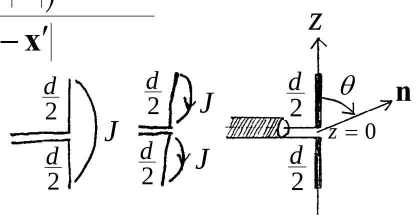
$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}'), \qquad (9.3)$$

where $\mathbf{J}(\mathbf{x}) = I\sin\left(\frac{kd}{2} - k|z|\right)\delta(x)\delta(y)\mathbf{e}_z \qquad (9.53)$

$$\Rightarrow \mathbf{A}(\mathbf{x}) = \mathbf{e}_z \frac{\mu_0 I}{4\pi}\int_{-d/2}^{d/2} dz' \frac{\sin\left(\frac{kd}{2} - k|z'|\right)e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$$

Note: (i) **J** is symmetric about $z = 0$. $\mathbf{J}(z) = \mathbf{J}(-z) \rightarrow$

(ii) I is the peak current only when $kd \geq \pi$.

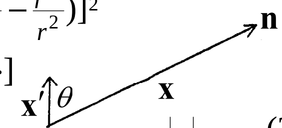


Question: The antenna appears to be an open circuit. How can there be current flowing on it?

$$|\mathbf{x}-\mathbf{x}'| = (r^2 - 2rr'\cos\theta + r'^2)^{\frac{1}{2}} = r[1 - (\tfrac{2\mathbf{n}\cdot\mathbf{x}'}{r} - \tfrac{r'^2}{r^2})]^{\frac{1}{2}}$$

$$= r[1 - \tfrac{1}{2}(\tfrac{2\mathbf{n}\cdot\mathbf{x}'}{r} - \tfrac{r'^2}{r^2}) - \tfrac{1}{8}(\tfrac{2\mathbf{n}\cdot\mathbf{x}'}{r} - \tfrac{r'^2}{r^2})^2 + \cdots]$$

$$= r - \mathbf{n}\cdot\mathbf{x}' + \tfrac{1}{2r}[r'^2 - (\mathbf{n}\cdot\mathbf{x}')^2] + \cdots$$



$r = |\mathbf{x}|$ (2)

$r' = |\mathbf{x}'|$

$\Rightarrow |\mathbf{x}-\mathbf{x}'| \approx r - \mathbf{n}\cdot\mathbf{x}'$ if $r \gg r'$

Hence, if $r \gg d$, we can write $|\mathbf{x}-\mathbf{x}'| \simeq r - z'\cos\theta$.

$$\Rightarrow \mathbf{A}(\mathbf{x}) \underset{\boxed{r \gg d}}{\approx} \mathbf{e}_z \frac{\mu_0 I e^{ikr}}{4\pi}\int_{-d/2}^{d/2} dz' \frac{\sin\left(\frac{kd}{2} - k|z'|\right)e^{-ikz'\cos\theta}}{\underbrace{r - z'\cos\theta}_{\approx r}} \qquad (9.54)$$

$$= \mathbf{e}_z \frac{\mu_0 I e^{ikr}}{2\pi kr}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right] \qquad (9.55)$$

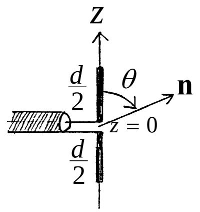
Note: $z'\cos\theta$ in $\frac{1}{r - z'\cos\theta}$ can be neglected if $r \gg d$. But $z'\cos\theta$ in $e^{ik(r - z'\cos\theta)}$ makes an important contribution to the phase angle even at $r \gg d$.

(9.16 & 9.19)

In the far zone, $\boxed{\mathbf{E} = Z_0 \mathbf{H} \times \mathbf{n}}$ $\boxed{\mathbf{H} = \frac{1}{\mu_0}\nabla \times \mathbf{A} = \frac{ik}{\mu_0}\mathbf{n} \times \mathbf{A} \Rightarrow |\mathbf{H}| = \frac{k \sin\theta |\mathbf{A}|}{\mu_0}}$

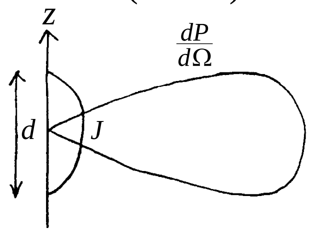
$$\left\langle \frac{dP}{d\Omega} \right\rangle_t = \frac{1}{2}\mathrm{Re}\left[r^2 \mathbf{n} \cdot \mathbf{E} \times \mathbf{H}^* \right] = \frac{Z_0}{2} r^2 |\mathbf{H}|^2 = \frac{Z_0}{2\mu_0^2} k^2 r^2 \sin^2\theta |\mathbf{A}|^2 \quad (3)$$

$$= \frac{Z_0 I^2}{8\pi^2} \left| \frac{\cos(\frac{kd}{2}\cos\theta) - \cos(\frac{kd}{2})}{\sin\theta} \right|^2, \quad \left[\begin{array}{l} \text{for } r \gg d \\ \text{and any } kd \end{array} \right] \quad (9.56)$$

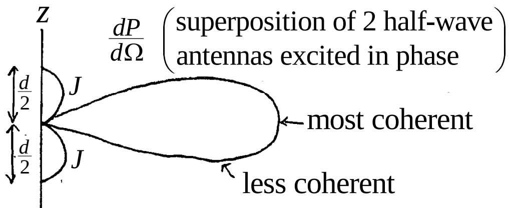


$$= \frac{Z_0 I^2}{8\pi^2} \begin{cases} \cos^2(\frac{\pi}{2}\cos\theta)/\sin^2\theta, & kd = \pi \\ 4\cos^4(\frac{\pi}{2}\cos\theta)/\sin^2\theta, & kd = 2\pi \end{cases} \quad (9.57)$$

half-wave antenna ($kd = \pi$)



full-wave antenna ($kd = 2\pi$)



narrower beam width

Rewrite (9.56)

$$\left\langle \frac{dP}{d\Omega} \right\rangle_t = \frac{Z_0 I^2}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta} \right|^2, \quad \left[\begin{array}{l} \text{for } r \gg d \\ \text{and any } kd \end{array} \right] \quad (9.56)$$

Limiting case (dipole approximation): $kd \ll 1$ (i.e. $\lambda \gg d$)

$$\cos x \simeq 1 - \frac{x^2}{2} \quad (x \ll 1)$$

$$\Rightarrow \begin{cases} \cos\left(\frac{kd}{2}\cos\theta\right) \simeq 1 - \frac{k^2 d^2}{8}\cos^2\theta \\ \cos\left(\frac{kd}{2}\right) \simeq 1 - \frac{k^2 d^2}{8} \end{cases}$$

$$\Rightarrow \left\langle \frac{dP}{d\Omega} \right\rangle_t \approx \frac{Z_0 I^2}{8\pi^2} \left| \frac{1 - \frac{k^2 d^2}{8}\cos^2\theta - 1 + \frac{k^2 d^2}{8}}{\sin\theta} \right|^2$$

$$= \frac{Z_0 I^2}{512\pi^2}(kd)^4 \sin^2\theta \quad [\text{valid for } kd \ll 1] \quad (4)$$

This has the same k and θ depedence as in (9.23, electric dipole), which was derived by assuming $kd \ll 1$.

Radiation Resistance and Equivalent Circuit:

$$\mathbf{J}(\mathbf{x}) = I \sin\left(\frac{kd}{2} - k|z|\right)\delta(x)\delta(y)\mathbf{e}_z \approx \underbrace{\frac{kd}{2} I}_{I_0 \text{ (peak current, } \because |z| \le d)} \left(1 - \frac{2|z|}{d}\right)\delta(x)\delta(y)\mathbf{e}_z$$

$\boxed{kd \ll 1}$

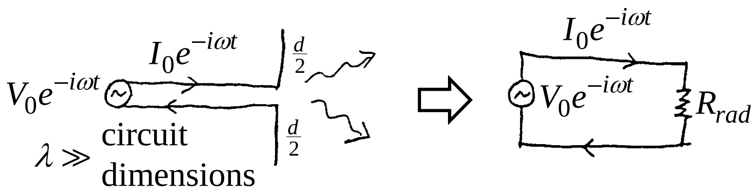
Thus, from (4), $\left\langle \frac{dP}{d\Omega} \right\rangle_t \approx \frac{Z_0 I^2}{512\pi^2}(kd)^4 \sin^2\theta = \frac{Z_0 I_0^2}{128\pi^2}(kd)^2 \sin^2\theta$ (9.28)

$$\Rightarrow \langle P \rangle_t \approx \int \left\langle \frac{dP}{d\Omega} \right\rangle_t d\Omega = \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \left\langle \frac{dP}{d\Omega} \right\rangle_t = \frac{Z_0 I_0^2}{48\pi}(kd)^2 \quad (9.29)$$

$$= \frac{I_0^2}{2} R_{rad},$$

[R_{rad}: radiation resistance.
R_{rad} is part of the field definition of impedance, see 2nd term in (6.137).]

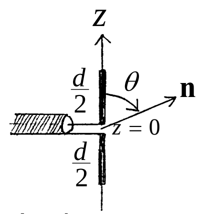
where $R_{rad} \equiv \frac{Z_0}{24\pi}(kd)^2 \approx 5(kd)^2$ ohms [See pp. 412-3.]



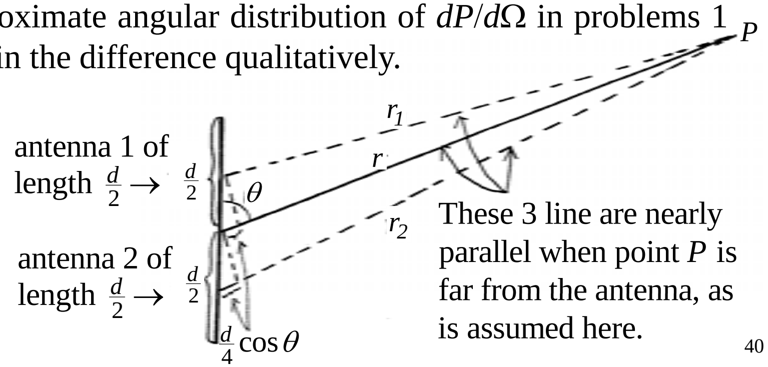
Equivalent circuit for a center-fed antenna

Problems:

1. The full-wave antenna radiation in (9.57) can be thought of as the superposition of two half-wave antennas, one above the other, excited in phase. Demonstrate this by rederiving $dP/d\Omega$ for the full-wave antenna [(9.57), $kd = 2\pi$] by superposing the fields of two half-wave antennas (each of length $d/2$, see figure below).
2. If the two half-wave antennas in problem 1 are excited 180° out of phase, derive $dP/d\Omega$ again by the method of superposition.
3. Plot the approximate angular distribution of $dP/d\Omega$ in problems 1 and 2. Explain the difference qualitatively.



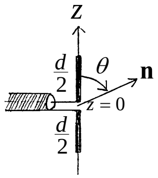
single antenna of length d



Solution to problem 1: Principle of superposition requires that we add the fields (not the powers) of the 2 antennas, each of total length $\frac{d}{2}$.

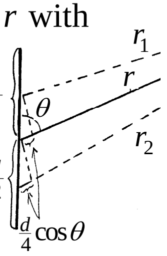
Rewrite (9.55)

$$\mathbf{A}(\mathbf{x}) = \mathbf{e}_z \frac{\mu_0 I e^{ikr}}{2\pi kr}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right] \quad (9.55)$$



(9.55) applies to a single antenna of total length d (see fig. above.) So the field of each of the 2 antennas in this problem can be obtained from (9.55) by replacing d in (9.55) with $\frac{d}{2}$ and expressing r with respect to the center of each antenna (i.e. by r_1 and r_2).

$$\mathbf{A}_{1,2} = \mathbf{e}_z \frac{\mu_0 I e^{ikr_{1,2}}}{2\pi kr_{1,2}}\left[\frac{\cos\left(\frac{kd}{4}\cos\theta\right) - \cos\left(\frac{kd}{4}\right)}{\sin^2\theta}\right], \quad (5)$$



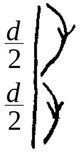
where $r_1 = r - \frac{d}{4}\cos\theta$ and $r_2 = r + \frac{d}{4}\cos\theta$.

We may approximate $r_{1,2}$ in the denominator of (5) by r, but must use the correct $r_{1,2}$ for the phase angles in the exponential terms.

It is assumed that each antenna in this problem is excited in the half-wave pattern, hence we set $k\frac{d}{2} = \pi$ in (5) and the superposed field of the 2 antennas (excited in phase) is given by

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 = \mathbf{e}_z \frac{\mu_0}{2\pi}\frac{I}{kr}e^{ikr}[e^{-i\frac{\pi}{2}\cos\theta} + e^{i\frac{\pi}{2}\cos\theta}]\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} \quad (6)$$

$$= \mathbf{e}_z \frac{\mu_0}{\pi}\frac{I}{kr}e^{ikr}\frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$



From (3), $\left\langle\frac{dP}{d\Omega}\right\rangle_t = \frac{Z_0}{2\mu_0^2}k^2 r^2 \sin^2\theta |\mathbf{A}|^2$

$$= \frac{Z_0 I^2}{2\pi^2}\cos^4\left(\tfrac{\pi}{2}\cos\theta\right)\Big/\sin^2\theta$$

[same as the full wave solution in (9.57)]

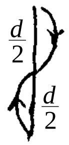
Solution to problem 2:

If the two half-wave antennas in problem 1 are excited 180° out of phase, we simply replace the "+" sign in (6) with a "−" sign.

Thus,

$$\mathbf{A} = \mathbf{A}_1 - \mathbf{A}_2 = \mathbf{e}_z \frac{\mu_0}{2\pi}\frac{I}{kr}e^{ikr}[e^{-i\frac{\pi}{2}\cos\theta} - e^{i\frac{\pi}{2}\cos\theta}]\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$

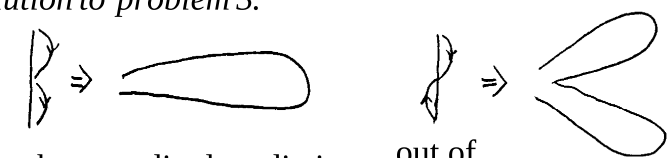
$$= -i\mathbf{e}_z \frac{\mu_0}{\pi}\frac{I}{kr}e^{ikr}\frac{\sin(\frac{\pi}{2}\cos\theta)\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$



From (3), $\left\langle\frac{dP}{d\Omega}\right\rangle_t = \frac{Z_0}{2\mu_0^2}k^2 r^2 \sin^2\theta |\mathbf{A}|^2$

$$= \frac{Z_0 I^2}{2\pi^2}\frac{\sin^2(\frac{\pi}{2}\cos\theta)\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} = \frac{Z_0 I^2}{8\pi^2}\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}$$

Solution to problem 3:



in phase $\Rightarrow$ dipole radiation

out of phase $\Rightarrow$ quadrupole radiation

Question: How does a phased array antenna work?

Homework of Chap. 9

Problems: 3, 6, ~~8, 9,~~ 14, 16, 17, 22, 23