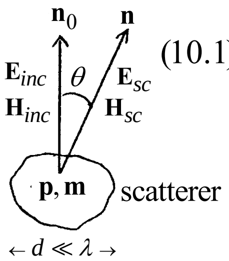
Chapter 10: Scattering and Diffraction

10.1 Scattering at Long Wavelength

Differential Scattering Cross Section : Consider a plane wave

$$\begin{cases} \mathbf{E}_{inc} = \boldsymbol{\varepsilon}_0 E_0 e^{ik\mathbf{n}_0\cdot\mathbf{x}} \\ \mathbf{H}_{inc} = \mathbf{n}_0 \times \mathbf{E}_{inc} / Z_0 \end{cases} \quad \left[\begin{array}{l}\text{Assume free space.} \\ Z_0 = \sqrt{\mu_0/\varepsilon_0}\end{array}\right] \quad (10.1)$$



incident onto an object of dimension $d \ll \lambda$, where $\boldsymbol{\varepsilon}_0$ can be real (linearly polarized) or complex [e.g. for circularly polarized wave, $\boldsymbol{\varepsilon}_{0\pm} = \frac{1}{\sqrt{2}}(\boldsymbol{\varepsilon}_x \pm i\boldsymbol{\varepsilon}_y)$].

$\mathbf{E}_{inc}$ and $\mathbf{H}_{inc}$ will induce multipoles on the object, which in turn generate scattered radiation ($\mathbf{E}_{sc}$, $\mathbf{H}_{sc}$). For $\lambda \gg d$, only the induced **p** and **m** are important. From (9.19) and (9.36), we have

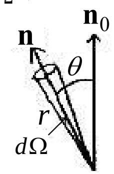
$$\begin{cases} \mathbf{E}_{sc} = \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[(\mathbf{n}\times\mathbf{p})\times\mathbf{n} - \mathbf{n}\times\mathbf{m}/c\right] \\ \mathbf{H}_{sc} = \mathbf{n}\times\mathbf{E}_{sc}/Z_0 \end{cases} \quad \text{[in far zone]} \quad (10.2)$$

$$\begin{cases} \mathbf{H}^p \simeq \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r} \\ \mathbf{E}^p \simeq Z_0\mathbf{H}^p\times\mathbf{n} \end{cases} + \begin{cases} \mathbf{H}^m \simeq \frac{k^2}{4\pi}(\mathbf{n}\times\mathbf{m})\times\mathbf{n}\frac{e^{ikr}}{r} \\ \mathbf{E}^m \simeq Z_0\mathbf{H}^m\times\mathbf{n} \end{cases}$$

Hence, to find $\mathbf{E}_{sc}$ and $\mathbf{H}_{sc}$, we need to find the induced **p** and **m**.

For scattering problems, a useful figure of merit is the scattered power ralative to incident power. Furthermore, it is often important to know the polarization state of the scattered radiation. Thus we define a differential scattering cross section (with dimension m^2) as

$$\frac{d\sigma}{d\Omega}(\mathbf{n},\boldsymbol{\varepsilon};\ \mathbf{n}_0,\boldsymbol{\varepsilon}_0) \equiv \frac{\dfrac{\text{radiated power in } \mathbf{n}\text{-direction with } \boldsymbol{\varepsilon}\text{-polarization}}{\text{unit solid angle}}}{\dfrac{\text{incident power in } \mathbf{n}_0\text{-direction with } \boldsymbol{\varepsilon}_0\text{-polarization}}{\text{unit area}}}$$



$$= \frac{r^2 \frac{1}{2Z_0}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{E}_{sc}\right|^2}{\frac{1}{2Z_0}\left|\boldsymbol{\varepsilon}_0^*\cdot\mathbf{E}_{inc}\right|^2} \quad \left[\begin{array}{l}\text{The meaning of } \sigma \text{ will} \\ \text{become clear in (10.11).}\end{array}\right] \quad (10.3)$$

Note: (i) For a circularly polarized state, $\boldsymbol{\varepsilon}$ an be written $\boldsymbol{\varepsilon}_\pm = \frac{1}{\sqrt{2}}(\boldsymbol{\varepsilon}_1 \pm i\boldsymbol{\varepsilon}_2)$, where $\boldsymbol{\varepsilon}_1 \perp \boldsymbol{\varepsilon}_2$.

(ii) $\boldsymbol{\varepsilon}_0$ and $\boldsymbol{\varepsilon}_0^* \perp \mathbf{n}_0$; $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}^* \perp \mathbf{n}$; $\boldsymbol{\varepsilon}_0\cdot\boldsymbol{\varepsilon}_0^* = 1$; $\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}^* = 1$

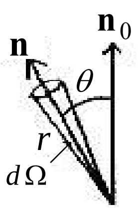
(iii) $\boldsymbol{\varepsilon}$ is not necessarily the direction of $\mathbf{E}_{sc}$. $\boldsymbol{\varepsilon}^*\cdot\mathbf{E}_{sc}$ gives the $\boldsymbol{\varepsilon}$-component of $\mathbf{E}_{sc}$.

Rewrite (10.3): $$\frac{d\sigma}{d\Omega}(\mathbf{n},\boldsymbol{\varepsilon};\ \mathbf{n}_0,\boldsymbol{\varepsilon}_0) = \frac{r^2 \frac{1}{2Z_0}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{E}_{sc}\right|^2}{\frac{1}{2Z_0}\left|\boldsymbol{\varepsilon}_0^*\cdot\mathbf{E}_{inc}\right|^2} \quad (10.3)$$

Sub. $$\begin{cases} \mathbf{E}_{inc} = \boldsymbol{\varepsilon}_0 E_0 e^{ik\mathbf{n}_0\cdot\mathbf{x}} \\ \mathbf{E}_{sc} = \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[(\mathbf{n}\times\mathbf{p})\times\mathbf{n} - \mathbf{n}\times\mathbf{m}/c\right] \end{cases}$$ into (10.3)

$$\frac{d\sigma}{d\Omega}(\mathbf{n},\boldsymbol{\varepsilon};\ \mathbf{n}_0,\boldsymbol{\varepsilon}_0) = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2}\left|\underbrace{\boldsymbol{\varepsilon}^*\cdot[(\mathbf{n}\times\mathbf{p})\times\mathbf{n}]}_{} - \boldsymbol{\varepsilon}^*\cdot\frac{\mathbf{n}\times\mathbf{m}}{c}\right|^2$$

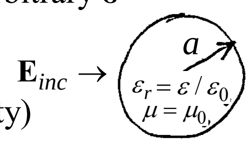
$$\boxed{\begin{aligned} &= \boldsymbol{\varepsilon}^*\cdot[\mathbf{p} - \mathbf{n}(\mathbf{n}\cdot\mathbf{p})] \\ &= \boldsymbol{\varepsilon}^*\cdot\mathbf{p} - \underbrace{(\boldsymbol{\varepsilon}^*\cdot\mathbf{n})}_{0}(\mathbf{n}\cdot\mathbf{p}) \\ &= \boldsymbol{\varepsilon}^*\cdot\mathbf{p} \end{aligned}}$$

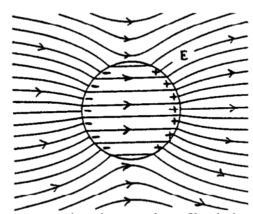


$$= \frac{k^4}{(4\pi\varepsilon_0 E_0)^2}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{p} + \frac{(\mathbf{n}\times\boldsymbol{\varepsilon}^*)\cdot\mathbf{m}}{c}\right|^2 \quad (10.4)$$

Example 1: Scattering by a small ($a \ll \lambda$), uniform dielectric sphere with $\mu=\mu_0$ and arbitrary ε

$$\begin{cases} \mu = \mu_0 \Rightarrow \mathbf{m} = 0 \\ \varepsilon_r = \varepsilon/\varepsilon_0 \text{ (relative permitivity)} \end{cases}$$





total electric field

From (4.56), we obtain the electric dipole moment **p** induced on the scatterer by $\mathbf{E}_{inc}$ $\boxed{= \boldsymbol{\varepsilon}_0 E_0 e^{ik\mathbf{n}_0\cdot\mathbf{x}}}$

$$\mathbf{p} = 4\pi\varepsilon_0\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)a^3\mathbf{E}_{inc} \qquad \boxed{= 0 \text{ by assumption}} \quad (4.56)\ \&\ (10.5)$$

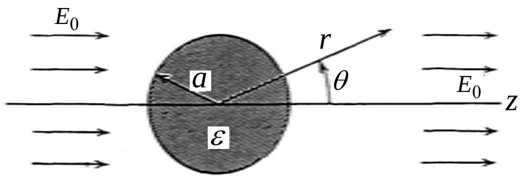
Sub. (10.5) into $$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{p} + (\mathbf{n}\times\boldsymbol{\varepsilon}^*)\cdot\mathbf{m}/c\right|^2 \quad (10.4)$$

$$\frac{d\sigma}{d\Omega}(\mathbf{n},\boldsymbol{\varepsilon};\ \mathbf{n}_0,\boldsymbol{\varepsilon}_0) = k^4a^6\left|\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2\left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\right|^2 \quad (10.6)$$

Question: (4.56) is derived for a dielectric sphere in a static field. Why is it also valid for the time-dependent field here?

Reminder

Example: A dielectric sphere is placed in a uniform electric field. Find ϕ everywhere.



We choose the spherical coordinates and divide the space into two regions: $r < a$ and $r > a$. In both regions, we have $\nabla^2\phi = 0$ with the

solution: $\phi = \begin{Bmatrix} r^l \\ r^{-l-1} \end{Bmatrix} \begin{Bmatrix} P_l^m(\cos\theta) \\ Q_l^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$ [Sec. 3.1 of lecture notes]

$$\text{b.c.} \begin{Bmatrix} \phi \text{ is independent of } \varphi. \\ \phi \text{ is finite at } \cos\theta = \pm 1. \\ \phi_{in} \text{ is finite at } r = 0. \end{Bmatrix} \Rightarrow \begin{cases} \phi_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \\ \phi_{out} = \sum_{l=0}^{\infty} \left[B_l r^l + C_l r^{-l-1} \right] P_l(\cos\theta) \end{cases}$$

Question: If $l > 0$, $\phi_{out} \to \infty$ as $r \to \infty$. Why then keep the $l > 0$ terms in ϕ_{out}?

Reminder

$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}.$

Rewrite: $\phi_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$, $\phi_{out} = \sum_{l=0}^{\infty} \left[B_l r^l + C_l r^{-l-1} \right] P_l(\cos\theta)$

b.c. (i): $\phi_{out}(\infty) = -E_0 z + const. = -E_0 r\cos\theta + const.$

$\Rightarrow B_0 = const.;\ B_1 = -E_0;\ B_l(l > 1) = 0$

$P_1(\cos\theta) = \cos\theta$

b.c. (ii): $\phi_{in}(a) = \phi_{out}(a)$ $[\Rightarrow E_t^{in}(a) = E_t^{out}(a)]$

$$\Rightarrow A_l a^l = B_l a^l + \frac{C_l}{a^{l+1}} \Rightarrow \begin{cases} A_0 = B_0 + C_0/a & (8) \\ A_1 = -E_0 + C_1/a^3 & (9) \\ A_l = C_l/a^{2l+1},\ l > 1 & (10) \end{cases}$$

b.c. (iii): $\varepsilon E_r^{in}(a) = \varepsilon_0 E_r^{out}(a) \Rightarrow -\varepsilon \frac{\partial}{\partial r}\phi_{in}\Big|_{r=a} = -\varepsilon_0 \frac{\partial}{\partial r}\phi_{out}\Big|_{r=a}$

$$\Rightarrow \varepsilon l A_l a^{l-1} = \varepsilon_0 \left[l B_l a^{l-1} - (l+1) C_l / a^{l+2} \right]$$

$$\Rightarrow \begin{cases} 0 = -\varepsilon_0 C_0 / a^2, & l = 0 & (11) \\ \varepsilon A_1 = -\varepsilon_0 [E_0 + 2C_1/a^3], & l = 1 & (12) \\ \varepsilon l A_l = -\varepsilon_0 (l+1) C_l / a^{2l+1}, & l > 1 & (13) \end{cases}$$

Reminder

(7), (11) $\Rightarrow A_0 = B_0 = const.$ (let it be 0.)

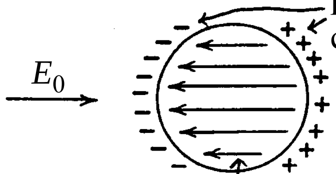
(9), (12) $\Rightarrow A_1 = -\frac{3E_0}{2+\varepsilon/\varepsilon_0};\ C_1 = \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right) a^3 E_0$

(10), (13) $\Rightarrow A_l = C_l = 0$ for $l > 1$

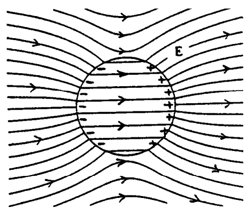
This is the only way (3) & (6) can both be satisfied.

$$\Rightarrow \begin{cases} \phi_{in} = -\frac{3}{2+\varepsilon/\varepsilon_0} E_0 r\cos\theta \\ \phi_{out} = \underbrace{-E_0 r\cos\theta}_{\text{applied field}} + \underbrace{\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} E_0 \frac{a^3}{r^2}\cos\theta}_{\text{dipole field with } p = 4\pi\varepsilon_0 a^3 E_0 \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}} \end{cases} \quad (4.54)$$

[cf. (4.10)]



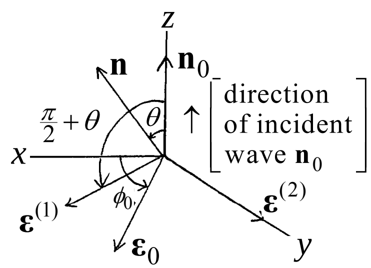
E due to polarization charge



total electric field

We define the $\mathbf{n}$-$\mathbf{n}_0$ plane as the scattering plane. Let $\mathbf{n}_0$ be along the z-axis and $\mathbf{n}$ lie on the x-z plane. The orientations $(\theta,\ \phi)$ of unit vectors $\boldsymbol{\varepsilon}_0$, $\boldsymbol{\varepsilon}^{(1)}$, and $\boldsymbol{\varepsilon}^{(2)}$ are specified accordingly as follows

$$\begin{cases} \boldsymbol{\varepsilon}_0 = (\frac{\pi}{2},\ \phi_0) & \text{[polarization of incident wave]} \\ \boldsymbol{\varepsilon}^{(1)} = (\frac{\pi}{2}+\theta,\ 0) & \text{[polarization state of scattered wave } \parallel \text{ to scattering plane]} \\ \boldsymbol{\varepsilon}^{(2)} = (\frac{\pi}{2},\ \frac{\pi}{2}) & \text{[polarization state of scattered wave } \perp \text{ to scattering plane]} \end{cases}$$



where $\boldsymbol{\varepsilon}_0$ is on the x-y plane making an angle ϕ_0 with the x-axis, $\boldsymbol{\varepsilon}^{(1)}$ is on the x-z (scattering) plane, $\boldsymbol{\varepsilon}^{(2)} (= \mathbf{e}_y)$ is $\perp$ to the scattering plane, and $\mathbf{n}$, $\boldsymbol{\varepsilon}^{(1)}$, and $\boldsymbol{\varepsilon}^{(2)}$ are mutually orthogonal. Polarization vector $(\boldsymbol{\varepsilon}_0)$ of the incident wave and polarization states $[\boldsymbol{\varepsilon}^{(1)}, \boldsymbol{\varepsilon}^{(2)}]$ of the scattered wave are all assumed to be real, representing linear polarization.

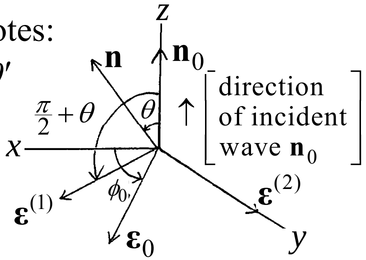
Applying Eq. (1) in Ch. 3 of lecture notes:

$\cos\gamma = \sin\theta\sin\theta'\cos(\phi-\phi') + \cos\theta\cos\theta'$

[γ: angle between $(\theta,\ \phi)$ and $(\theta',\ \phi')$]

to $\boldsymbol{\varepsilon}_0 = (\frac{\pi}{2},\ \phi_0)$, $\boldsymbol{\varepsilon}^{(1)} = (\frac{\pi}{2}+\theta,\ 0)$, and

$\boldsymbol{\varepsilon}^{(2)} = (\frac{\pi}{2},\ \frac{\pi}{2})$, we obtain



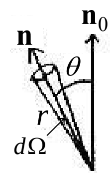
$$\begin{cases} \boldsymbol{\varepsilon}^{(1)}\cdot\boldsymbol{\varepsilon}_0 = \sin\left(\frac{\pi}{2}+\theta\right)\sin\frac{\pi}{2}\cos(0-\phi_0) + \cos\left(\frac{\pi}{2}+\theta\right)\cos\frac{\pi}{2} \\ \qquad\quad = \cos\phi_0\cos\theta \\ \boldsymbol{\varepsilon}^{(2)}\cdot\boldsymbol{\varepsilon}_0 = \sin\frac{\pi}{2}\sin\frac{\pi}{2}\cos\left(\frac{\pi}{2}-\phi_0\right) + \cos\frac{\pi}{2}\cos\frac{\pi}{2} \\ \qquad\quad = \sin\phi_0 \end{cases}$$

Rewrite (10.6): $\frac{d\sigma}{d\Omega}(\mathbf{n},\boldsymbol{\varepsilon};\ \mathbf{n}_0,\boldsymbol{\varepsilon}_0) = k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2 |\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0|^2$

$$\Rightarrow \begin{cases} \frac{d\sigma_\parallel}{d\Omega} = k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2 |\boldsymbol{\varepsilon}^{(1)}\cdot\boldsymbol{\varepsilon}_0|^2 = k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2 \cos^2\phi_0\cos^2\theta \\ \frac{d\sigma_\perp}{d\Omega} = k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2 |\boldsymbol{\varepsilon}^{(2)}\cdot\boldsymbol{\varepsilon}_0|^2 = k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2 \sin^2\phi_0 \end{cases}$$

Assume that the incident radiation has a fixed direction $\mathbf{n}_0$, but is unpolarized (i.e. ϕ_0 is random). We take the average over ϕ_0:

$$\begin{cases} \left\langle\frac{d\sigma_\parallel}{d\Omega}\right\rangle_{\phi_0} = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\sigma_\parallel}{d\Omega}d\phi_0 = \frac{k^4a^6}{2}\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2\cos^2\theta \\ \left\langle\frac{d\sigma_\perp}{d\Omega}\right\rangle_{\phi_0} = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\sigma_\perp}{d\Omega}d\phi_0 = \frac{k^4a^6}{2}\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2 \end{cases} \quad (10.7)$$



$$\Rightarrow\ \Pi(\theta) \equiv \frac{\left\langle\frac{d\sigma_\perp}{d\Omega}\right\rangle_{\phi_0} - \left\langle\frac{d\sigma_\parallel}{d\Omega}\right\rangle_{\phi_0}}{\left\langle\frac{d\sigma_\perp}{d\Omega}\right\rangle_{\phi_0} + \left\langle\frac{d\sigma_\parallel}{d\Omega}\right\rangle_{\phi_0}} = \frac{\sin^2\theta}{1+\cos^2\theta} \quad \left[\begin{array}{l}\Rightarrow 100\%\text{ linearly} \\ \text{polarized at } \theta=\frac{\pi}{2}\end{array}\right] \quad (10.9)$$

where $\Pi(\theta)$ gives the degree of polarization of the scattered radiation.

$$\left\langle\frac{d\sigma}{d\Omega}\right\rangle_{\phi_0} = \left\langle\frac{d\sigma_\perp}{d\Omega}\right\rangle_{\phi_0} + \left\langle\frac{d\sigma_\parallel}{d\Omega}\right\rangle_{\phi_0} = k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2\frac{1}{2}\left(1+\cos^2\theta\right) \quad (10.10)$$

$$\Rightarrow\ \langle\sigma\rangle_{\phi_0} = \int\left\langle\frac{d\sigma}{d\Omega}\right\rangle_{\phi_0}d\Omega = \frac{8\pi}{3}k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2 \ll \pi a^2\ \ [ka\ll 1] \quad (10.11)$$

Question 1: In (10.10), why add powers instead of adding fields?

(10.11) gives $\langle\sigma\rangle_{\phi_0} \ll \pi a^2$, impling that only a small fraction of the radiation incident on the dielectric sphere is scattered. This is true even if the scatterer is a perfectly conducting sphere (with radius $\ll\lambda$). See next example.

Example 2: Scattering by a small perfectly conducting sphere

The incident radiation will induce both electric and magnetic dipole moments (**p** and **m**) on the conductor. **p** and **m** are given by

$\mathbf{p} = 4\pi\varepsilon_0 a^3\mathbf{E}_{inc}$ [See Sec. 3.3 of lecture notes.] (10.12)

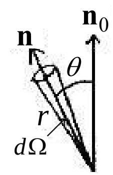
$\mathbf{m} = -2\pi a^3\mathbf{H}_{inc}$ [See next problem.] (10.13)

$$\text{From}\ \begin{cases} \mathbf{E}_{inc} = \boldsymbol{\varepsilon}_0E_0e^{ik\mathbf{n}_0\cdot\mathbf{x}} & (10.1) \\ \mathbf{H}_{inc} = \mathbf{n}_0\times\mathbf{E}_{inc}/Z_0\quad [Z_0\equiv\sqrt{\mu_0/\varepsilon_0}\,] & \\ \frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\varepsilon_0E_0)^2}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{p} + (\mathbf{n}\times\boldsymbol{\varepsilon}^*)\cdot\mathbf{m}/c\right|^2 & (10.4) \end{cases}$$

we obtain $\frac{d\sigma}{d\Omega} = k^4a^6\left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0 - \frac{1}{2}(\mathbf{n}\times\boldsymbol{\varepsilon}^*)\cdot(\mathbf{n}_0\times\boldsymbol{\varepsilon}_0)\right|^2$ (10.14)

As in Example 1, for unploarized incident radiation, (10.14) yields

$$\begin{cases} \left\langle\frac{d\sigma_\parallel}{d\Omega}\right\rangle_{\phi_0} = \frac{k^4a^6}{2}(\cos\theta-\frac{1}{2})^2 \\ \left\langle\frac{d\sigma_\perp}{d\Omega}\right\rangle_{\phi_0} = \frac{k^4a^6}{2}(1-\frac{1}{2}\cos\theta)^2 \end{cases} \quad (10.15)$$



$$\Rightarrow \left\langle\frac{d\sigma}{d\Omega}\right\rangle_{\phi_0} = \left\langle\frac{d\sigma_\parallel}{d\Omega}\right\rangle_{\phi_0} + \left\langle\frac{d\sigma_\perp}{d\Omega}\right\rangle_{\phi_0} = k^4a^6[\tfrac{5}{8}(1+\cos^2\theta)-\cos\theta] \quad (10.16)$$

$$\Pi(\theta) = \frac{3\sin^2\theta}{5(1+\cos^2\theta)-8\cos\theta}\quad [\text{peak at } \theta=60°] \quad (10.17)$$

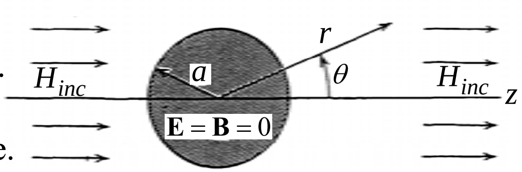
$$\langle\sigma\rangle_{\phi_0} = \int\left\langle\frac{d\sigma}{d\Omega}\right\rangle_{\phi_0}d\Omega = \frac{10}{3}\pi k^4a^6 \ll \pi a^2\quad [ka\ll 1]$$

Again, we find $\langle\sigma\rangle_{\phi_0} \ll \pi a^2$. By geometric optics, the scatterer (a conductor) would be opaque to the incident radiation, and the incident radiation would have been totally blocked [$\langle\sigma\rangle_{\phi_0} = \pi a^2$]. This example demonstrates that geometric optics completely breaks down for $\lambda \gg a$, where we need physical optics, as in scattering/diffraction theory.

Problem: Derive the dipole moment in (10.13): $\mathbf{m} = -2\pi a^3 \mathbf{H}_{inc}$.

Solution: Since $\lambda \gg a$, we may assume $\mathbf{H}_{inc}$ to be uniform.

For a perfect conductor, we have $\mathbf{E} = \mathbf{B} = 0$ inside the sphere.



In Sec. 9.3, we have shown that in the near zone ($r \ll \lambda$), the magnetic dipole radiation has negligible **E**-field. Hence, we assume $\nabla \times \mathbf{B} = -\frac{\partial}{\partial t}\mathbf{E} \approx 0$ outside the sphere and write $\mathbf{B} = \nabla\phi$. Then,

$\nabla \cdot \mathbf{B} = 0 \;\Rightarrow\; \nabla^2\phi = 0$ with the solution: [Sec. 3.1 of lecture notes]

$$\phi = \begin{Bmatrix} r^l \\ r^{-l-1} \end{Bmatrix} \begin{Bmatrix} P_l^m(\cos\theta) \\ Q_l^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$$

This model is valid for $r \ll \lambda$, which is sufficient for us to find the dipole moment of a sphere with radius $\ll \lambda$.

subject to boundary conditions:

$$\begin{cases} \mathbf{B}(r \to \infty) = \mu_0 H_{inc}\mathbf{e}_z \;\Rightarrow\; \phi(r \to \infty) = \mu_0 H_{inc} z = \mu_0 H_{inc} r\cos\theta \\ B_\perp(r = a) = 0 \Rightarrow \frac{\partial}{\partial r}\phi\Big|_{r=a} = 0 \end{cases}$$

Rewrite $\phi = \begin{Bmatrix} r^l \\ r^{-l-1} \end{Bmatrix} \begin{Bmatrix} P_l^m(\cos\theta) \\ Q_l^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$

$\boxed{P_1(\cos\theta) = \cos\theta}$

b.c. $\begin{Bmatrix} \phi \text{ is independent of } \varphi. \\ \phi \text{ is finite at } \cos\theta = \pm 1. \end{Bmatrix} \Rightarrow \phi = \sum_{l=0}^{\infty}\left[A_l r^l + C_l r^{-l-1}\right] P_l(\cos\theta)$

b.c. $\phi(r \to \infty) = \mu_0 H_{inc} r\cos\theta \;\Rightarrow\; A_1 = \mu_0 H_{inc}$ & $A_l = 0$ if $\ell \neq 1$

$$\Rightarrow \phi = \mu_0 H_{inc} r\cos\theta + \sum_{l=0}^{\infty} C_l r^{-l-1} P_l(\cos\theta)$$

b.c. $\frac{\partial}{\partial r}\phi\Big|_{r=a} = 0 \;\Rightarrow\; (\mu_0 H_{inc} - \frac{2}{a^3}C_1)\cos\theta - \sum_{l=2}^{\infty}\frac{l+1}{a^{l+2}} C_l P_l(\cos\theta) = 0$

$\Rightarrow C_1 = \frac{1}{2}\mu_0 a^3 H_{inc}$ & $C_l = 0$ if $\ell \neq 1$

$\Rightarrow \phi = \mu_0 H_{inc} r\cos\theta + \frac{1}{2}\mu_0 a^3 H_{inc}\frac{\cos\theta}{r^2}$

$\Rightarrow \mathbf{B}(\text{due to the sphere}) = \nabla\phi(\text{2nd term}) = -\frac{\mu_0 a^3}{2} H_{inc}\frac{2\cos\theta\mathbf{e}_r + \sin\theta\mathbf{e}_\theta}{r^3}$

Comparing with (5.41), we find that this is a magnetic dipole field produced by a (induced) dipole moment of $\mathbf{m} = -2\pi a^3 \mathbf{H}_{inc}$.

Optional

10.2 Perturbation Theory of Scattering

General Theory: Conside a slightly non-uniform medium with

$$\begin{cases} \varepsilon(\mathbf{x}) = \varepsilon_0 + \delta\varepsilon(\mathbf{x}) \\ \mu(\mathbf{x}) = \mu_0 + \delta\mu(\mathbf{x}) \end{cases}$$

[In Sec. 10.1, ε of the scatterer can be of any value, but the solution is more restricted by the scatterer geometry.]

where ε_0 and μ_0 are independent of **x** and t (ε_0 and μ_0 are not necessarily the free space values.)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \;\Rightarrow\; \nabla\times\nabla\times\varepsilon_0\mathbf{E} + \varepsilon_0\frac{\partial}{\partial t}\nabla\times\mathbf{B} = 0 \qquad (1)$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \;\Rightarrow\; \varepsilon_0\frac{\partial}{\partial t}\nabla\times\mu_0\mathbf{H} = \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\mathbf{D} \qquad (2)$$

$$(1)-(2) \Rightarrow \nabla\times\nabla\times\varepsilon_0\mathbf{E} + \varepsilon_0\frac{\partial}{\partial t}\nabla\times(\mathbf{B} - \mu_0\mathbf{H}) = -\mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\mathbf{D} \qquad (3)$$

$$\nabla\times\nabla\times\mathbf{D} = \nabla\underbrace{(\nabla\cdot\mathbf{D})}_{=\rho_{free}=0} - \nabla^2\mathbf{D} = -\nabla^2\mathbf{D} \qquad (4)$$

The purpose of the above manipulation is to obtain this small quantity, which can be treated as a perturbation.

$(3)-(4) \Rightarrow$

$$\nabla^2\mathbf{D} - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\mathbf{D} = \overbrace{-\nabla\times\nabla\times(\mathbf{D} - \varepsilon_0\mathbf{E}) + \varepsilon_0\frac{\partial}{\partial t}\nabla\times(\mathbf{B} - \mu_0\mathbf{H})} \qquad (10.22)$$

Optional

Assume $\mathbf{D}, \mathbf{E}, \mathbf{B}, \mathbf{H} \sim e^{-i\omega t}$, (10.22) $\Rightarrow$

$$(\nabla^2 + \underbrace{\mu_0\varepsilon_0\omega^2}_{k^2})\mathbf{D} = -\nabla\times\nabla\times(\mathbf{D} - \varepsilon_0\mathbf{E}) - i\varepsilon_0\omega\nabla\times(\mathbf{B} - \mu_0\mathbf{H}) \qquad (10.23)$$

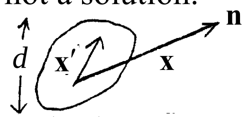
$(\nabla^2 + k^2)G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') \Rightarrow G(\mathbf{x}, \mathbf{x}') = \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$. Hence,

$$\mathbf{D} = \mathbf{D}^{(0)} + \frac{1}{4\pi}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\begin{Bmatrix} \nabla'\times\nabla'\times(\mathbf{D} - \varepsilon_0\mathbf{E}) \\ +i\varepsilon_0\omega\nabla'\times(\mathbf{B} - \mu_0\mathbf{H}) \end{Bmatrix} \qquad (10.24)$$

Note: (i) $\mathbf{D}(0)$ is an incident plane wave which satisfies the homogeneous Helmholtz eq. [i.e. the RHS of (10.23) = 0]

(ii) (10.24) is an integral relation, not a solution.

Let the integrand in (10.24) be of dimension d and $r \gg d$, then $|\mathbf{x} - \mathbf{x}'| \simeq r - \mathbf{n}\cdot\mathbf{x}'$ and we can write **D** as



$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \approx \frac{e^{ik(r-\mathbf{n}\cdot\mathbf{x}')}}{r - \underbrace{\mathbf{n}\cdot\mathbf{x}'}_{\text{neglect}}} \quad \text{for } r \gg d$$

$\mathbf{D} \simeq \mathbf{D}^{(0)} + \mathbf{A}_{sc}\frac{e^{ikr}}{r}$ with

$$\mathbf{A}_{sc} = \frac{1}{4\pi}\int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\begin{Bmatrix} \nabla'\times\nabla'\times(\mathbf{D} - \varepsilon_0\mathbf{E}) \\ +i\varepsilon_0\omega\nabla'\times(\mathbf{B} - \mu_0\mathbf{H}) \end{Bmatrix} \qquad (10.26)$$

$$\int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\nabla'\times\mathbf{a} \qquad [\mathbf{a} \text{ is any vector function of } \mathbf{x}.]$$

integration by parts

$$= \int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left[\mathbf{e}_x\left(\frac{\partial a_z}{\partial y'}-\frac{\partial a_y}{\partial z'}\right)+\mathbf{e}_y\left(\frac{\partial a_x}{\partial z'}-\frac{\partial a_z}{\partial x'}\right)+\mathbf{e}_z\left(\frac{\partial a_y}{\partial x'}-\frac{\partial a_x}{\partial y'}\right)\right]$$

$$= \int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left[i\mathbf{e}_x(k_y a_z - k_z a_y)+\mathbf{e}_y(\cdots)+\mathbf{e}_z(\cdots)\right]$$

$$= \int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'} i(\mathbf{k}\times\mathbf{a}) = \int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'} ik(\mathbf{n}\times\mathbf{a})$$

$\Rightarrow$ The end result is to replace "∇" with "$ik\mathbf{n}$"

$$(10.26) \Rightarrow \mathbf{A}_{sc} = \frac{k^2}{4\pi}\int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left\{\begin{matrix}[\mathbf{n}\times(\mathbf{D}-\varepsilon_0\mathbf{E})]\times\mathbf{n} \\ -\frac{\varepsilon_0\omega}{k}\mathbf{n}\times(\mathbf{B}-\mu_0\mathbf{H})\end{matrix}\right\} \qquad (10.27)$$

From (10.3), we obtain $\dfrac{d\sigma}{d\Omega} = \dfrac{|\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}|^2}{|\mathbf{D}^{(0)}|^2}$ [$\boldsymbol{\varepsilon}$: polarization vector of the scattered wave] (10.28)

Note: (i) $\mathbf{A}_{sc}$ gives the scattered field $\mathbf{D}_{sc} = \mathbf{A}_{sc}e^{ikr}/r$ [hence $\mathbf{H}_{sc}$ through (10.2)]. $\mathbf{A}_{sc}$ is NOT a vector potential.
(ii) (10.27) is an integral equation for $\mathbf{A}_{sc}$, NOT a solution.

Born Approximation: Rewrite (10.27)

$$\mathbf{A}_{sc} = \frac{k^2}{4\pi}\int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left\{[\mathbf{n}\times(\mathbf{D}-\varepsilon_0\mathbf{E})]\times\mathbf{n} - \frac{\varepsilon_0\omega}{k}\mathbf{n}\times(\mathbf{B}-\mu_0\mathbf{H})\right\} \qquad (10.27)$$

For a linear medium,

$$\begin{cases}\mathbf{D}(\mathbf{x}) = [\varepsilon_0 + \delta\varepsilon(\mathbf{x})]\mathbf{E}(\mathbf{x}) \\ \mathbf{B}(\mathbf{x}) = [\mu_0 + \delta\mu(\mathbf{x})]\mathbf{H}(\mathbf{x})\end{cases} \Rightarrow \begin{cases}\mathbf{D}-\varepsilon_0\mathbf{E} = \delta\varepsilon(\mathbf{x})\mathbf{E} \\ \mathbf{B}-\mu_0\mathbf{H} = \delta\mu(\mathbf{x})\mathbf{H}\end{cases} \qquad (10.29)$$

We see from (10.29) that the integrand of (10.27) is composed of small quantities $\delta\varepsilon\mathbf{E}$ and $\delta\mu\mathbf{H}$. To first order in $\delta\varepsilon$ and $\delta\mu$, we only need to use the zero order (or unperturbed) $\mathbf{E}^{(0)}$ and $\mathbf{H}^{(0)}$ for $\mathbf{E}$ and $\mathbf{H}$ in $\delta\varepsilon\mathbf{E}$ and $\delta\mu\mathbf{H}$. Thus, we write

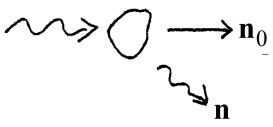
$$\begin{cases}\mathbf{D}-\varepsilon_0\mathbf{E} = \delta\varepsilon(\mathbf{x})\mathbf{E} \approx \dfrac{\delta\varepsilon(\mathbf{x})}{\varepsilon_0}\mathbf{D}^{(0)} \\ \mathbf{B}-\mu_0\mathbf{H} = \delta\mu(\mathbf{x})\mathbf{H} \approx \dfrac{\delta\mu(\mathbf{x})}{\mu_0}\mathbf{B}^{(0)}\end{cases} \qquad (10.30)$$

[This approx., called the Born approx., turns the integral eq. (10.27) into a solution for $\mathbf{A}_{sc}$.]

Let the unperturbed fields be those of a plane wave,

$$\mathbf{D}^{(0)}(\mathbf{x}) = \boldsymbol{\varepsilon}_0 D_0 e^{ik\mathbf{n}_0\cdot\mathbf{x}}, \quad \mathbf{B}^{(0)}(\mathbf{x}) = \sqrt{\frac{\mu_0}{\varepsilon_0}}\mathbf{n}_0\times\mathbf{D}^{(0)}(\mathbf{x})$$

Sub. $\mathbf{D}^{(0)}(\mathbf{x})$ and $\mathbf{B}^{(0)}(\mathbf{x})$ into (10.30), then sub. (10.30) into (10.27), and finally multiply the result by $\boldsymbol{\varepsilon}^*/D_0$



$$\frac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}^{(1)}}{D_0} = \frac{k^2}{4\pi}\int d^3x' e^{i\mathbf{q}\cdot\mathbf{x}'}\left\{\begin{matrix}\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\dfrac{\delta\varepsilon(\mathbf{x}')}{\varepsilon_0} \\ +(\mathbf{n}\times\boldsymbol{\varepsilon}^*)\cdot(\mathbf{n}_0\times\boldsymbol{\varepsilon}_0)\dfrac{\delta\mu(\mathbf{x}')}{\mu_0}\end{matrix}\right\} \qquad (10.31)$$

where $q \equiv k(\mathbf{n}_0 - \mathbf{n})$. The absolute square of (10.31) gives the differential scattering cross section through (10.28).

$$\frac{d\sigma}{d\Omega} = \frac{|\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}|^2}{|\mathbf{D}^{(0)}|^2} \qquad (10.28)$$

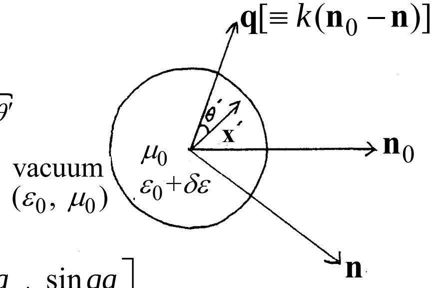
Example: Scattering by a uniform dielectric sphere with $\varepsilon = \varepsilon_0 + \delta\varepsilon$ and $\mu = \mu_0$

$$\int d^3x' e^{i\mathbf{q}\cdot\mathbf{x}'}$$

$$= \int_0^a r'^2 dr' \int_0^{2\pi} d\phi' \int_{-1}^{1} d\overbrace{\cos\theta'}^{y}\, e^{iqr'\overbrace{\cos\theta'}^{y}}$$

$$= 2\pi\int_0^a r'^2 dr'\left[\frac{1}{iqr'}e^{iqr'y}\right]_{y=-1}^{y=1}$$

$$= \frac{4\pi}{q}\int_0^a r'\sin(qr')dr' = 4\pi\left[-\frac{a\cos qa}{q^2} + \frac{\sin qa}{q^3}\right]$$



Thus, from (10.31) (let $\delta\mu = 0$)

$$\frac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}}{D_0} = k^2\frac{\delta\varepsilon}{\varepsilon_0}(\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0)\left[\frac{\sin qa - qa\cos qa}{q^3}\right]$$

$$\xrightarrow{qa\to 0} k^2 a^3\frac{\delta\varepsilon}{3\varepsilon_0}(\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0)$$

$\sin x \approx x - \frac{1}{6}x^3,\ x\to 0$
$\cos x \approx 1 - \frac{1}{2}x^2,\ x\to 0$

Optional

Sub. $\left.\frac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}}{D_0}\right|_{qa\to 0} = k^2a^3\frac{\delta\varepsilon}{3\varepsilon_0}(\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0)$ into $\frac{d\sigma}{d\Omega} = \frac{|\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}|^2}{|\mathbf{D}^{(0)}|^2}$ (10.28)

$$\Rightarrow \lim_{qa\to 0}\left(\frac{d\sigma}{d\Omega}\right)_{\text{Born}} \simeq k^4a^6\left|\frac{\delta\varepsilon}{3\varepsilon_0}\right|^2|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0|^2 \qquad (10.32)$$

in agreement with $\frac{d\sigma}{d\Omega} = k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0|^2$ (10.6) in the limit $\varepsilon_r = \varepsilon/\varepsilon_0 \to 1$.

Question: (10.6) and (10.32) both give the differential scattering cross section ($d\sigma/d\Omega$) of a dielectric sphere with radius much smaller than the wavelength. (10.6) is valid for arbitrary values of ε_r (= $\varepsilon/\varepsilon_0$). It reduces to (10.32) in the limit $\varepsilon_r \to 1$. A physical effect in included in (10.6) [but not in (10.32)] that keeps $d\sigma/d\Omega$ at a finite value in the limit $\varepsilon_r \to \infty$? What is it? Explain why it keeps $d\sigma/d\Omega$ finite.

Blue Sky and Red Sunset: Scattering by gases

$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ (4.34) $\Rightarrow$ $\mathbf{D} = \varepsilon_0\mathbf{E} + N\mathbf{p} = \varepsilon_0\mathbf{E} + N\gamma_{mol}\varepsilon_0\mathbf{E} = \varepsilon\mathbf{E}$

Macroscopically, we have

$$\varepsilon = \varepsilon_0(1+N\gamma_{mol})$$

Microscopically, we may write

$\mathbf{p}$: dipole moment per molecule
$\mathbf{p} = \gamma_{mol}\varepsilon_0\mathbf{E}$
γ_{mol}: molecular polarizability [see (4.72) & (4.73)]
N: no of molecules/unit volume

$$\varepsilon(\mathbf{x}) = \varepsilon_0 + \overbrace{\sum_j \gamma_{mol}\varepsilon_0\delta(\mathbf{x}-\mathbf{x}_j)}^{\ll\,\varepsilon_0\text{, when spreaded over the size of the molecule}} \;\Rightarrow\; \delta\varepsilon(\mathbf{x}) = \varepsilon_0\sum_j\gamma_{mol}\delta(\mathbf{x}-\mathbf{x}_j) \qquad (10.33)$$

Since $\varepsilon(\mathbf{x})$ fluctuates microscopically with a weak variation $\delta\varepsilon(\mathbf{x})$, we may apply the perturbation theory just developed.

Sub. $\delta\varepsilon(\mathbf{x})$ into (10.31), then sub. (10.31) into (10.28), we obtain

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2}|\gamma_{mol}|^2|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0|^2F(\mathbf{q}), \text{ [assume } \delta\mu = 0]$$

$$\text{where } F(\mathbf{q}) = \left|\sum_j e^{i\mathbf{q}\cdot\mathbf{x}_j}\right|^2 = \sum_j\sum_{j'}e^{i\mathbf{q}\cdot(\mathbf{x}_j-\mathbf{x}_{j'})} \overset{\text{for randomly distributed molecules}}{=} \begin{bmatrix}\text{total no of molecules}\\ \text{(incoherent radiation)}\end{bmatrix} \qquad (10.19)$$

We now relate γ_{mol} to the macroscopic quantities ε, n, and N.

$$\varepsilon = \varepsilon_0(1+N\gamma_{mol}) \;\Rightarrow\; \gamma_{mol} = \frac{\frac{\varepsilon}{\varepsilon_0}-1}{N} = \frac{n^2-1}{N} \approx \frac{2(n-1)}{N}$$

index of refraction

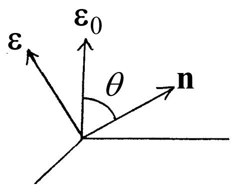
$$n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} \approx 1$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2}|\gamma_{mol}|^2|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0|^2F(\mathbf{q})$$
$$= \frac{k^4}{4\pi^2N^2}|n-1|^2|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0|^2F(\mathbf{q})$$

$\Rightarrow$ Total scattering cross section per molecule is given by

$$\sigma = \frac{1}{F(\mathbf{q})}\int\frac{d\sigma}{d\Omega}d\Omega \quad [F(\mathbf{q})\text{: total number of scatterers}]$$
$$= \frac{k^4}{4\pi^2N^2}|n-1|^2\int_0^{2\pi}d\phi\underbrace{\int_{-1}^{1}d\cos\theta|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0|^2}$$
$$= \frac{2k^4}{3\pi N^2}|n-1|^2 \qquad (10.34)$$

$\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0 = \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$
$\int_{-1}^{1}\sin^2\theta d\cos\theta = \frac{4}{3}$



$\boldsymbol{\varepsilon}$ is on the $\boldsymbol{\varepsilon}_0$-$\mathbf{n}$ plane for dipole scatterer (p.458).

Let I be the intensity (power/unit area) of the incident wave, then

$$\frac{dI}{dx} = -IN\sigma = -I\alpha,$$

(10.34) and (10.35) describe what is known as Rayleigh scattering.

where $\alpha = N\sigma \simeq \frac{2k^4}{3\pi N}|n-1|^2$ [attenuation coefficient] (10.35)

Discussion:

(i) $\alpha \propto k^4 \Rightarrow$ Violet light ($\lambda \simeq 410$ nm) is scattered more than red light ($\lambda \simeq 650$ nm) by a factor of $(\frac{650}{410})^4 \simeq 6.3$.

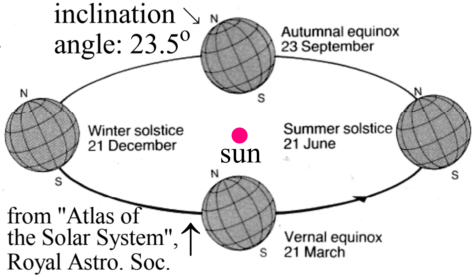
(ii) In (10.35), $n-1 \simeq \frac{1}{2}N\gamma_{mol}$ (see last page). Hence, $\alpha \propto N$ if atoms (or molecules) of the same type are added or taken out.

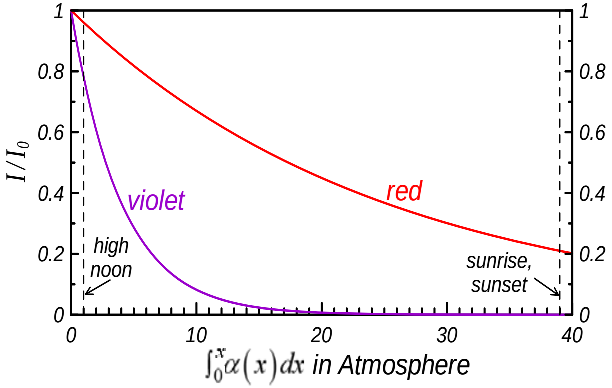
(iii) The atoms in a gas radiate incoherently, but the charges within an atom radiate coherently. Suppose there are 10 electron-ion pairs in each atom and we were able to split all the atoms into a gas of single electron-ion pairs, each with the same p. Then, the macroscopic n remains the same, but the split pairs no longer radiate coherently, resulting in a scattered intensity 10 times weaker. This explains the factor $\frac{1}{N}$ in (10.35) (See p. 468).

In the earth atmosphere, α is a function of x. Then,

$$\frac{dI(x)}{dx} = -I(x)\alpha(x)$$

$$\Rightarrow I(x) = I_0 e^{-\int_0^x \alpha(x)dx}$$





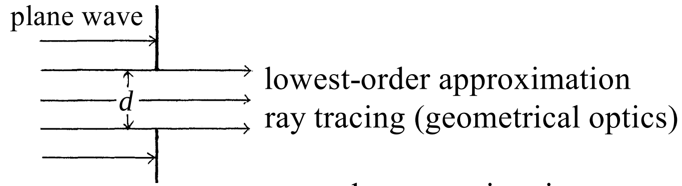
Questions:

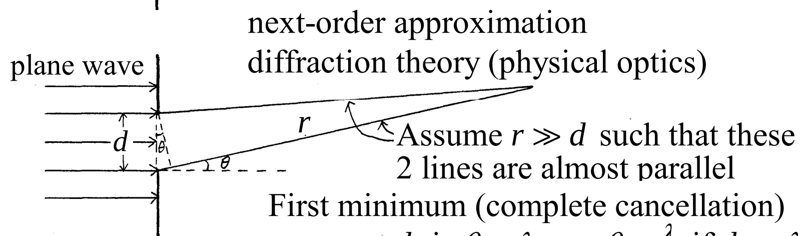
(i) Why is the sky blue instead of violet?

(ii) Why is it more likely to get a sunburn in the summer?

(iii) Hot summer/cold winter results mostly from a different cause than in (ii). What is it?

10.5 Scalar Diffraction Theory





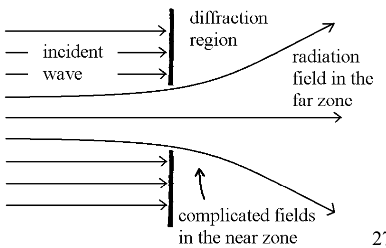
First minimum (complete cancellation) occurs at $d\sin\theta = \lambda \;\Rightarrow\; \theta \approx \frac{\lambda}{d}$ if $d \gg \lambda$.

Nature of the diffraction problem: Physically, the diffraction problem here is not separable from the scattering problem. However, the treatments are different. The scattering problem treated in this chapter assumes $\lambda >> d$. The scalar diffraction theory is most valid when $d >> \lambda$, for which it gives the next-order correction to the geometrical optics (see p. 478).

Justification of the Scalar Diffraction Theory: Physically, electronic responses (**J**, ρ) of the aperture material to the incident wave generate electromagnetic fields in addition to dissipating some of the incident wave. Far from the edges of the aperture, **J** and ρ principally result in reflection of the incident wave, while **J** and ρ near the edges produce fields that pass to the right of the aperture together with the incident wave. The superposed fields form the diffraction pattern. In the far zone of the diffraction region (>a few λ from the aperture), the fields take the form of an EM wave, which obeys

$$\mathbf{E} = Z_0 \mathbf{H} \times \mathbf{n} \quad \text{[see (9.19)]}$$

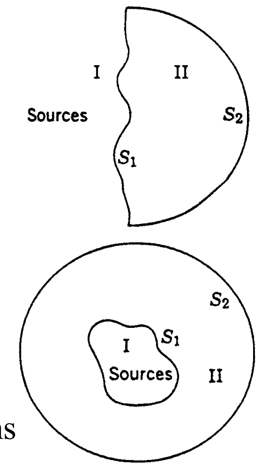
where $Z_0 = (\mu_0/\varepsilon_0)^{1/2}$ is the impedance of vacuum, and **n** is the direction of wave propagation.



Thus, **E**, **H**, and **n** are mutually orthogonal, and the amplitudes of **E** and **H** have a known ratio Z_0. Therefore, one component of the fields gives most of the information (phase and intensity, but not the polarization) about the far fields. This justifies a scalar theory for the diffraction phenomenon and explains why it has been the basis of most of the work on diffraction.

The Kirchhoff Integral Formula: In the scattering problem, we calculate the scattered fields due to **J** and ρ associated with the dipole moments induced by the incident fields. In the diffraction problem, the fields are produced in part by the induced **J** and ρ on the aperture material, but **J** and ρ do not appear explicitly in field equations. They are implicit in the boundary conditions. The Kirchhoff integral formula expresses the diffracted fields in terms of the boundary fields. Determination of the near fields requires accurate handling of the b.c.'s (very few cases can be solved completely). However, the far fields can be fairly accurately determined with crude b.c.'s.

Refer to the figures to the right. S_1 is an opaque surface with aperture(s) on it. The diffraction region (Region II) is the volume enclosed by S_1 and S_2.



Let $\Psi(\mathbf{x},t)=\Psi(\mathbf{x})e^{-i\omega t}$ be a scalar field (a component of $\mathbf{E}$ or $\mathbf{B}$), then

$$\left(\nabla^2+k^2\right)\Psi(\mathbf{x})=0, \quad k=\omega/c \qquad (10.73)$$

Note: Ψ gives the phase and intensity, but not the polarization, of the fields.

Below, we will express Ψ in Region II in terms of Ψ and $\frac{\partial\Psi}{\partial n}$ on the boundary surfaces by making use of Green's thm.

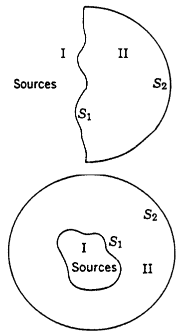
$$\int_V(\phi\nabla^2\psi-\psi\nabla^2\phi)d^3x=\oint_S\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)da \qquad (1.35)$$

Rewrite $\int_V(\phi\nabla^2\psi-\psi\nabla^2\phi)d^3x=\oint_S\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)da \qquad (1.35)$

Introduce a Green's function $G(\mathbf{x},\mathbf{x}')$ satisfying

$$(\nabla^2+k^2)G(\mathbf{x},\mathbf{x}')=-\delta(\mathbf{x}-\mathbf{x}') \qquad (10.74)$$

Apply (1.35) to the volume enclosed by S_1 and S_2 (Region II) and let $\psi=\Psi$ and $\phi=G$.



$$\int_V d^3x'[G(\mathbf{x},\mathbf{x}')\overbrace{\nabla'^2\Psi(\mathbf{x}')}^{-k^2\Psi(\mathbf{x}')}-\Psi(\mathbf{x}')\overbrace{\nabla'^2G(\mathbf{x},\mathbf{x}')}^{-k^2G(\mathbf{x},\mathbf{x}')-\delta(\mathbf{x}-\mathbf{x}')}]$$

$$=-\oint_{S_1+S_2}da'[G(\mathbf{x},\mathbf{x}')\mathbf{n}'\cdot\nabla'\Psi(\mathbf{x}')-\Psi(\mathbf{x}')\mathbf{n}'\cdot\nabla'G(\mathbf{x},\mathbf{x}')]$$

For an observation point $\mathbf{x}$ inside region II,

$$\Psi(\mathbf{x})=\oint_{S_1+S_2}da'[\Psi(\mathbf{x}')\mathbf{n}'\cdot\nabla'G(\mathbf{x},\mathbf{x}')-G(\mathbf{x},\mathbf{x}')\mathbf{n}'\cdot\nabla'\Psi(\mathbf{x}')] \qquad (10.75)$$

Note: $\mathbf{n}'$ is inwardly directed into the volume instead of outwardly directed as in (1.35).

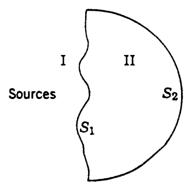
Is this a good choice?

Solution of (10.74): $G(\mathbf{x},\mathbf{x}')=\frac{e^{ikR}}{4\pi R}$ with $\mathbf{R}=\mathbf{x}-\mathbf{x}'$. $\qquad (10.76)$

Green function with outgoing wave b.c.

(note: $\nabla'R=-\nabla R$)

$$\Rightarrow \nabla'G(\mathbf{x},\mathbf{x}')=\underbrace{\left(\frac{d}{dR}G\right)}_{ik\frac{e^{ikR}}{4\pi R}-\frac{e^{ikR}}{4\pi R^2}}\cdot\overbrace{\nabla'R}^{\frac{-\mathbf{R}}{R}}=\frac{-e^{ikR}}{4\pi R}ik\left(1+\frac{i}{kR}\right)\frac{\mathbf{R}}{R}$$



Hence,

$$\Psi(\mathbf{x})=-\frac{1}{4\pi}\oint_{S_1+S_2}da'\frac{e^{ikR}}{R}\mathbf{n}'\cdot\left[\nabla'\Psi(\mathbf{x}')+ik\left(1+\frac{i}{kR}\right)\frac{\mathbf{R}}{R}\Psi(\mathbf{x}')\right] \qquad (10.77)$$

We assume that Ψ on S_2 is transmitted through S_1. Then, $\Psi|_{S_2}\propto\frac{1}{r}$ and the contribution to the integral in (10.77) from S_2 vanishes as the inverse of the radius of the sphere. Assume further that the radius goes to infinity and hence neglect the contribution from S_2. (10.77) then gives the Kirchhoff integral formula

$$\Rightarrow \Psi(\mathbf{x})=-\frac{1}{4\pi}\int_{S_1}da'\frac{e^{ikR}}{R}\mathbf{n}'\cdot\left[\nabla'\Psi(\mathbf{x}')+ik\left(1+\frac{i}{kR}\right)\frac{\mathbf{R}}{R}\Psi(\mathbf{x}')\right] \qquad (10.79)$$

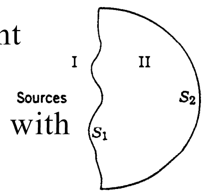
Ψ in Region II is now expressed in terms of Ψ and $\frac{\partial\Psi}{\partial n}$ on S_1.

Kirchhoff Approximation: Rewrite (10.79),

$$\Psi(\mathbf{x})=-\frac{1}{4\pi}\int_{S_1}da'\frac{e^{ikR}}{R}\mathbf{n}'\cdot\left[\nabla'\Psi(\mathbf{x}')+ik\left(1+\frac{i}{kR}\right)\frac{\mathbf{R}}{R}\Psi(\mathbf{x}')\right] \qquad (10.79)$$

(10.79) is an integral equation for Ψ. It becomes a solution for Ψ under the Kirchhoff approximation, which consists of

1. Ψ and $\frac{\partial\Psi}{\partial n}$ vanish everywhere on S_1 except in the openings.
2. Ψ and $\frac{\partial\Psi}{\partial n}$ in the openings are those of the incident wave in the absence of any obstacles.



There are, however, **mathematical inconsistencies** with the Kirchhoff approximation:

1. If Ψ and $\frac{\partial\Psi}{\partial n}$ vanish on any finite surface, then $\Psi=0$ everywhere (true for both Laplace and Helmholtz equations).
2. (10.79) does not yield on S_1 the assumed values of Ψ and $\frac{\partial\Psi}{\partial n}$.

Approximations made here work best for $\lambda\ll d$, and fail badly for $\lambda\sim d$ or $\lambda>d$ (d: size of the aperture or obstacle). See p.478.

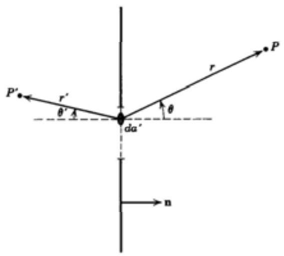
Remove the mathematical inconsistencies in the Kirchhoff Approximation by the choice of a proper Green function.

If Ψ is known on the surface S_1, a Dirichet Green function $G_D(\mathbf{x},\mathbf{x}')$, satisfying $G_D(\mathbf{x},\mathbf{x}')=0$ for $\mathbf{x}'$ on S is required.

A generalized Kirchhoff integral:

$$\Psi(\mathbf{x}) = \int_{S_1} da'[\Psi(\mathbf{x}')\mathbf{n}'\cdot\nabla' G(\mathbf{x},\mathbf{x}')] \qquad (10.81)$$

Consider a plane screen with aperature (s). The method of images can be used to give the Dirichlet Green functions explicit form:



$$G_D(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi}\left(\frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'}\right) \qquad (10.84)$$

where $\begin{cases} \mathbf{R} = \mathbf{x}-\mathbf{x}' = (x-x', y-y', z-z') \\ \mathbf{R}' = \mathbf{x}-\mathbf{x}'' = (x-x', y-y', z+z') \end{cases}$

$$\Psi(\mathbf{x}) = \frac{k}{2\pi i}\int_{S_1} \frac{e^{ikR}}{R}\left(1+\frac{i}{kR}\right)\frac{\mathbf{n}'\cdot\mathbf{R}}{R}\Psi(\mathbf{x}')da' \qquad (10.85)$$

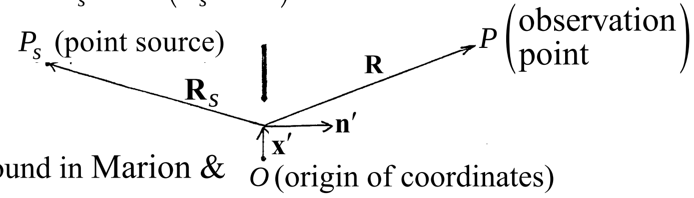
A Special Case*: Diffraction of spherical waves originating from a point source at P_s.

$$\Psi(\mathbf{x}') = \frac{e^{ikR_s}}{R_s} \quad \text{(by Kirchhoff approximation)} \qquad (5)$$

$$\Rightarrow \nabla'\Psi(\mathbf{x}') = -\frac{e^{ikR_s}}{R_s} ik\left(1+\frac{i}{kR_s}\right)\frac{\mathbf{R}_s}{R_s} \qquad \boxed{G(\mathbf{x},\mathbf{x}') = \frac{e^{ikR}}{4\pi R}} \qquad (6)$$

Sub. (5), (6) into (10.79), assume kR & $kR_s \gg 1$ and hence neglect $O\left(\frac{1}{kR}\right)$ and $O\left(\frac{1}{kR_s}\right)$ terms, we obtain

$$\Psi(P) = \frac{ik}{4\pi}\int_{S_1} da' \frac{e^{ik(R+R_s)}}{RR_s}\mathbf{n}'\cdot\left(\frac{\mathbf{R}_s}{R_s} - \frac{\mathbf{R}}{R}\right) \qquad (7)$$



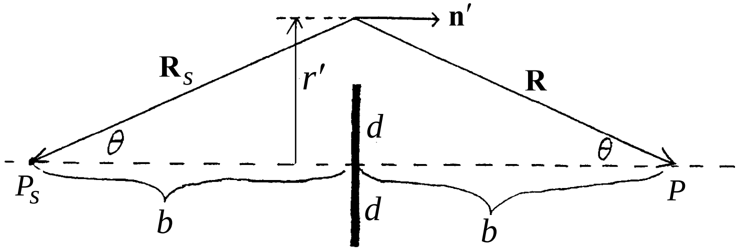
* More cases can be found in Marion & Heald, "Classical Electromagnetic Radiation," following Eq. (12.14).

As we will see from the following example, the scalar diffraction theory agrees with observations, although it is highly artificial.

Example: Diffraction by a circular disk. For simplicity, we assume

(i) P_s and P are on the axis of the disk.

(ii) P_s and P are at equal distance from the disk.



$$\left.\begin{array}{l} R_s = R \\ da' = 2\pi r'dr' \left(\begin{array}{l} R^2 = r'^2 + b^2 \Rightarrow r'dr' = RdR \\ \text{Hence, } da' = 2\pi RdR \end{array}\right) \\ \mathbf{n}'\cdot\frac{\mathbf{R}_s}{R_s} = -\cos\theta = -\frac{b}{R}, \quad \mathbf{n}'\cdot\frac{\mathbf{R}}{R} = \cos\theta = \frac{b}{R} \end{array}\right\} \qquad (8)$$

Sub. (8) into $\Psi(P) = \frac{ik}{4\pi}\int_{S_1} da' \frac{e^{ik(R+R_s)}}{RR_s}\mathbf{n}'\cdot\left(\frac{\mathbf{R}_s}{R_s} - \frac{\mathbf{R}}{R}\right)$ (7)

$$\Rightarrow \Psi(P) = -ikb\int_{\sqrt{d^2+b^2}}^{\infty} \frac{e^{2ikR}}{R^2}dR \qquad (9)$$

Integrating by parts $[\int_{a_1}^{a_2} udv = uv\Big|_{a_1}^{a_2} - \int_{a_1}^{a_2} vdu,\ u = \frac{1}{R^2},\ dv = e^{2ikR}dR]$

$$\Psi(P) = -ikb\left[\frac{e^{2ikR}}{2ikR^2}\bigg|_{\sqrt{d^2+b^2}}^{\infty} + \frac{1}{2ik}\int_{\sqrt{d^2+b^2}}^{\infty}\frac{e^{2ikR}}{R^3}dR\right]$$

(integrating by parts again)

$$= -ikb\left[\frac{e^{2ikR}}{2ikR^2}\bigg|_{\sqrt{d^2+b^2}}^{\infty} - \underbrace{\frac{e^{2ikR}}{4k^2R^3}\bigg|_{\sqrt{d^2+b^2}}^{\infty}}_{\substack{\text{negligible,} \\ \text{since } kR \gg 1}} + \cdots\right] \simeq \frac{be^{2ik\sqrt{d^2+b^2}}}{2\left(d^2+b^2\right)} \qquad (10)$$

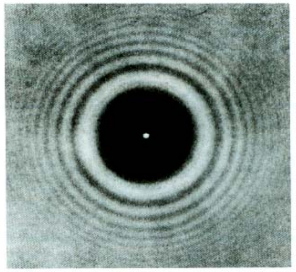
Questions:

(i) Intensity at P: $I(P)\propto|\Psi(P)|^2=b^2/[4(d^2+b^2)^2]$ (11)

Since $I(P)>0$ for all b, there is always a bright spot (Fresnel bright spot) at any point on the axis. What is the physical reason?

(ii) $\lim_{d\to 0}\Psi(P)=\frac{e^{2ikR}}{2b}$ (12)

In the limit of no obsticle ($d\to 0$), $\Psi(P)$ reduces to the exact solution for a point source at P_s, i.e. the approximate solution in (10) becomes the exact solution in (12). What is the mathematical reason?



← The diffraction pattern of a disk (from Halliday, Resnick, and Walker). Note the Fresnel bright spot at the center of the pattern. The concentric diffraction rings are not predictable by (11), which applies only to fields on the axis.

A historical anecdote about the Fresnel bright spot: (The following paragraphs are taken from Halliday, Resnick, and Walker.)

"Diffraction finds a ready explanation in the wave theory of light. However, this theory, originally advanced by Huygens and used 123 years later by Young to explain double-slit interference, was very slow in being adopted, largely because it ran counter to Newton's theory that light was a stream of particles.

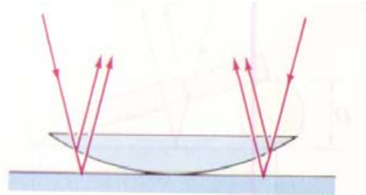
Newton's view was the prevailing view in French scientific circles of the early nineteenth century, when Augustin Fresnel was a young military engineer. Fresnel, who believed in the wave theory of light, submitted a paper to the French Academy of Sciences describing his experiments and his wave-theory explanations of them.

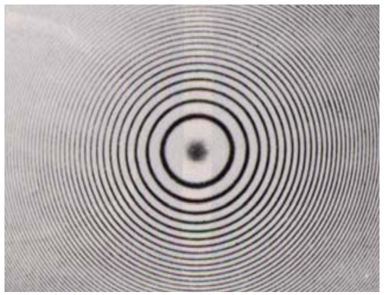
In 1819, the Academy, dominated by supporters of Newton and thinking to challenge the wave point of view, organized a prize competition for an essay on the subject of diffraction. Fresnel won. The Newtonians, however, were neither converted nor silenced. One of them, S. D. Poisson, pointed out the "strange result" that if Fresnel's theories were correct, then light waves should flare into the shadow region of a sphere as they pass the edge of the sphere, producing a bright spot at the center of the shadow. The prize committee arranged a test of the famous mathematician's prediction and discovered that the predicted Fresnel bright spot, as we call it today, was indeed there! Nothing builds confidence in a theory so much as having one of its unexpected and counterintuitive predictions verified by experiment."

Benson

Newton's Ring

When a lens with a large radius of curvature is place on a flat plate, as in Fig. 37.19, a **thin film of air** is formed. When Newton is illuminated with **mono-chronomatic** light, **circular fringes**, called **Newton's Rings**, can be seen.



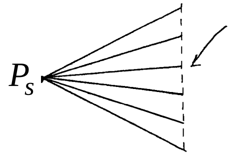


Why the center spot is dark unlike Fresnel bright spot?

This is the wave nature.

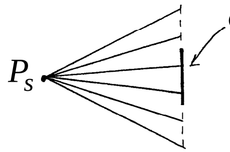
10.8 Babinet's Principle

Rewrite $\Psi(\mathbf{x})=-\frac{1}{4\pi}\int_{S_1}da'\frac{e^{ikR}}{R}\mathbf{n}'\cdot\left[\nabla'\Psi(\mathbf{x}')+ik\left(1+\frac{i}{kR}\right)\frac{\mathbf{R}}{R}\Psi(\mathbf{x}')\right]$ (10.79)



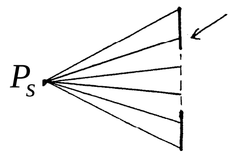
no diffraction screen, imaginary surface

$\cdot\ \Psi(P)=-\frac{1}{4\pi}\int_{\text{dashed surface}}(\cdots)$



diffraction screen

$\cdot\ \Psi_a(P)=-\frac{1}{4\pi}\int_{\text{dashed surface}}(\cdots)$



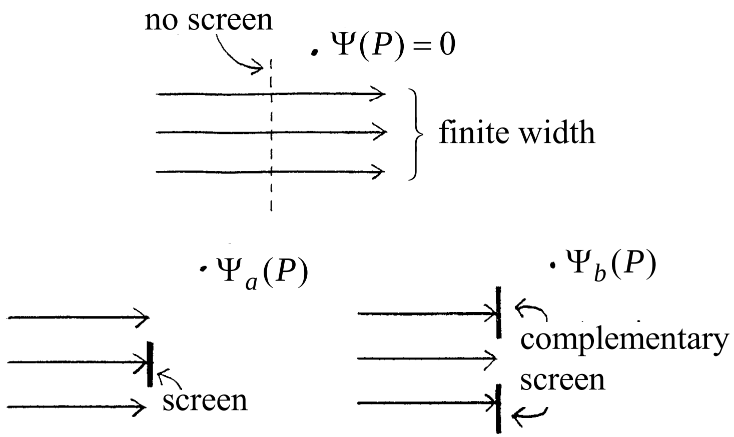
complementary diffraction screen

$\cdot\ \Psi_b(P)=-\frac{1}{4\pi}\int_{\text{dashed surface}}(\cdots)$

By Kirchoff's approx.: $\begin{cases}\text{on the obatacle}: \Psi \text{ and } \frac{\partial\Psi}{\partial n}=0\\ \text{elsewhere}: \Psi \text{ and } \frac{\partial\Psi}{\partial n}=\text{those of the source}\end{cases}$,

we have $\Psi(P)=\Psi_a(P)+\Psi_b(P)$ [Babinet's principle]

Example: a light beam of finite width

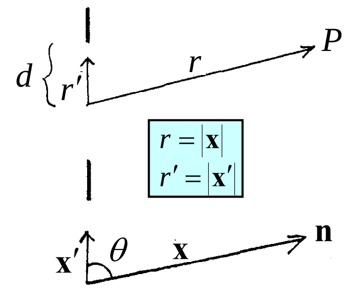


Babinet's principle $\Rightarrow\ \Psi(P)=\Psi_a(P)+\Psi_b(P)=0$

$$\Rightarrow\ \Psi_a(P)=-\Psi_b(P)$$

Fresnel and Fraunhofer Diffraction: (see p.491)

There is a clear diffraction pattern only when $r \gg d$. So, In integrals such as (10.77), $R(=|\mathbf{x}-\mathbf{x}'|)$ can be approximated by $r(=|\mathbf{x}|)$ everywhere except in e^{ikR}, where the phase angle kR must be evaluated more accurately.



Consider three length scales: r, d, and λ.

$$R=|\mathbf{x}-\mathbf{x}'|=(r^2-2rr'\cos\theta+r'^2)^{1/2}$$

$$=r\left[1-\left(\frac{2\mathbf{n}\cdot\mathbf{x}'}{r}-\frac{r'^2}{r^2}\right)\right]^{1/2}=r\left[1-\tfrac{1}{2}\left(\frac{2\mathbf{n}\cdot\mathbf{x}'}{r}-\frac{r'^2}{r^2}\right)-\tfrac{1}{8}\left(\frac{2\mathbf{n}\cdot\mathbf{x}'}{r}-\frac{r'^2}{r^2}\right)^2+\cdots\right]$$

$$=r\left[1-\frac{\mathbf{n}\cdot\mathbf{x}'}{r}+\tfrac{1}{2}\left(\frac{r'^2}{r^2}-\frac{(\mathbf{n}\cdot\mathbf{x}')^2}{r^2}\right)+\cdots\right]=r-\mathbf{n}\cdot\mathbf{x}'+\frac{1}{2r}\left[r'^2-(\mathbf{n}\cdot\mathbf{x}')^2+\cdots\right]$$

$$\Rightarrow\ kR=O(kr)+O(kd)+O\left(\frac{kd^2}{r}\right)+\cdots$$

If the 3rd and higher terms are neglected, we have the Fraunhofer diffraction (far field). If the 3rd term is kept, but higher order terms are neglected, we have the Fresnel diffraction (near field).

Homework of Chap. 10

Problems: 2, 3, 7, ~~12, 14~~