清大物理系 “電動力學(二)” 任課老師:張存續
“Electrodynamics (II)” (PHYS 532000)

Spring Semester, 2012

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一百學年度第二學期 PHYS532000 電動力學二（3 學分）教師: 張存續

教學目標：鼓勵學生思考、整合、創造。

課本：

J. D. Jackson, *Classical Electrodynamics*, Third Edition

參考書：

D. J. Griffiths, *Introduction to Electrodynamics*.

D. K. Cheng, *Field and Wave Electromagnetics* or D. M. Pozar, *Microwave Engineering*.

R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics II*.

講義下載：

ED: http://www.phys.nthu.edu.tw/~thschang/ED.htm

全部講義: http://www.phys.nthu.edu.tw/~hf5/

上課時間與地點：

(表訂)每星期二(10:10 – 11:00 am)與星期四(10:10 – 12:00 am)在物理館 019。

上課進度：如附表。

學期成績：期中考 40%，期末考 40%，4 次小考 20%

給用心且認真同學加分：點名 6%+課程參與度 4%。

學期成績會視考試狀況稍作調整，期望能正確的反應大家的學習效果。

習題：有 60-75% 題目會類似勾選的習題（確保有努力就有收穫）。

另外 25-40%組合觀念題（讓程度好的同學可以發揮）。

考試：

以英文命題。

考試於原上課時間舉行，考試地點為原上課教室，除非另有宣佈。

考試時，除非學生有充分理由，不得請假。未事先請假者，該次考試以 0 分計。

演習課：每週 1 次，每星期二第四節(11:10 – 12:00 am) 暫定。

Schedule (depending on students' learning condition)

週次	時間	上 課 內 容
一	02/21 (二)	導論(課程簡介、評分規定) 複習 Maxwell Equations
	02/23 (四)	Chap.8 ***Waveguides, Resonant Cavities, and Optical Fibers***
二	02/28 (二)	和平紀念日
	03/01 (四)	Chap.8+ **Quiz #1**
三	03/06 (二)	Chap.8
	03/08 (四)	參加國際會議
四	03/13 (二)	Chap.8
	03/15 (四)	Chap.9 ***Radiating System, Multiple Fields and Radiation***
五	03/20 (二)	Chap.9
	03/22 (四)	Chap.9
六	03/27 (二)	Chap.9
	03/29 (四)	Chap.9+ **Quiz #2**
七	04/03 (二)	校際活動週
	04/05 (四)	Chap.10 ***Scattering and Diffraction***
八	04/10 (二)	Chap.10
	04/12 (四)	Chap.10
九	04/17 (二)	Chap.10
	04/19 (四)	Chap.10
十	04/24 (二)	Chap.11 ***Special Theory of Relativity***
	04/26 (四)	**Midterm Chs. 8, 9, and 10**
十一	05/01 (二)	Chap.11
	05/03 (四)	Chap.11
十二	05/08 (二)	Chap.11+ **Quiz #3**
	05/10 (四)	Chap.11
十三	05/15 (二)	Chap.11
	05/17 (四)	Chap.14 ***Radiation by Moving Charges***
十四	05/22 (二)	Chap.14
	05/24 (四)	Chap.14
十五	05/29 (二)	Chap.14+ **Quiz #4**
	05/31 (四)	Chap.14
十六	06/05 (二)	Chap.14
	06/07 (四)	Chap.14
十七	06/12 (二)	Chap.14
	06/14 (四)	**Final Chs. 10, 11 and 14**
十八	06/19 (二)	
	06/21 (四)	**Makeup exam if needed.**

* 此進度表僅供參考，實際情形視學習狀況調整。

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1. **Textbook and Contents of the Course**:

   J. D. Jackson, “Classical Electrodynamics”, 3rd edition, Chapters 8-11, 14.

   Other books will be referenced in the lecture notes when needed.

2. **Conduct of Class :**

   Lecture notes will be projected sequentially on the screen during the class. Physical concepts will be emphasized, while algebraic details in the lecture notes will often be skipped. *Questions are encouraged*. It is assumed that students have at least gone through the algebra in the lecture notes before attending classes (*important*!).

3. **Grading Policy:** Midterm (40%); Final (40%); Quiz x 4 (20%) and extra points (10%). The overall score will be normalized to reflect an average consistent with other courses.

4. **Lecture Notes:** Starting from basic equations, the lecture notes follow Jackson closely with algebraic details filled in.

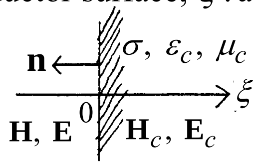
   Equations numbered in the format of (8.7), (8.9)... refer to Jackson. Supplementary equations derived in lecture notes, which will later be referenced, are numbered (1), (2)... [restarting from (1) in each chapter.] Equations in Appendices A, B…of each chapter are numbered (A.1), (A.2)…and (B.1), (B.2)…

   Page numbers cited in the text (e.g. p. 395) refer to Jackson. Section numbers (e.g. Sec. 8.1) refer to Jackson (except for sections in Ch. 11). Main topics within each section are highlighted by **boldfaced** characters. Some words are typed in *italicized* characters for attention. Technical terms which are introduced for the first time are underlined.

Chapter 8: Waveguides, Resonant Cavities, and Optical Fibers

8.1 Fields at the Surface of and Within a Good Conductor*

Notations: $\mathbf{H}$, $\mathbf{E}$: fields outside the conductor; $\mathbf{H}_C$, $\mathbf{E}_C$: fields inside the conductor; $\mathbf{n}$: a unit vector $\perp$ to conductor surface; ξ : a normal coordinate into the conductor.



Assume: (i) fields $\sim e^{-i\omega t}$

(ii) good but not perfect conductor, i.e.

$\sigma \neq \infty$, but $\frac{\sigma}{\omega\varepsilon_b} \gg 1$ [See Ch. 7 of lecture notes, Eq. (24)].

(iii) $\mathbf{H}_{\parallel}(\xi = 0)$ is known.

Find: $\mathbf{E}_c(\xi)$, $\mathbf{H}_c(\xi)$, and power loss, etc. in terms of $\mathbf{H}_{\parallel}(\xi = 0)$

*The main results in Sec. 8.1 [(8.9), (8.10), (8.12), (8.14), and (8.15)] have been derived with a much simpler method in Ch. 7 of lecture notes. [See contents following Eq. (26)]. So, we will not cover this section in classes.

Calculation of $\mathbf{E}_c(\xi)$, $\mathbf{H}_c(\xi)$: In the conductor, we have

good conductor assumption

$$\begin{cases} \nabla\times\mathbf{E}_c = -\frac{\partial}{\partial t}\mathbf{B}_c = i\omega\mu_c\mathbf{H}_c & (1) \\ \nabla\times\mathbf{H}_c = \mathbf{J}+\frac{\partial}{\partial t}\mathbf{D}_c = \sigma\mathbf{E}_c - i\omega\varepsilon_b\mathbf{E}_c \simeq \sigma\mathbf{E}_c & (2) \end{cases}$$

$\nabla \simeq -\mathbf{n}\frac{\partial}{\partial\xi}$ [In a good conductor, fields vary rapidly along the normal to the surface, see Ch. 7 of lecture notes.] (3)

$$(1),(2),(3) \Rightarrow \begin{cases} \mathbf{E}_c \approx -\frac{1}{\sigma}\mathbf{n}\times\frac{\partial}{\partial\xi}\mathbf{H}_c & (4) \\ \mathbf{H}_c \approx \frac{i}{\mu_c\omega}\mathbf{n}\times\frac{\partial}{\partial\xi}\mathbf{E}_c & (5) \end{cases} \qquad \boxed{\delta \equiv \sqrt{\frac{2}{\mu_c\omega\sigma}} = \text{skin depth}}$$

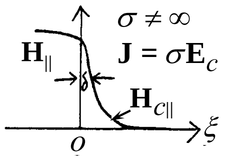
Sub. (4) into (5): $\frac{\partial^2}{\partial\xi^2}(\mathbf{n}\times\mathbf{H}_c)+\frac{2i}{\delta^2}(\mathbf{n}\times\mathbf{H}_c)\approx 0$ (8.7)

$\Rightarrow \mathbf{n}\times\mathbf{H}_c(\xi)\approx\mathbf{n}\times\mathbf{H}_c(0)e^{-\frac{\xi}{\delta}}e^{\frac{i\xi}{\delta}}$ $\boxed{\text{b.c. at } \xi=0\text{: } \mathbf{H}_{\parallel}(0)=\mathbf{H}_{c\parallel}(0)}$

$\Rightarrow \mathbf{H}_{c\parallel}(\xi)\approx\mathbf{H}_{c\parallel}(0)e^{-\frac{\xi}{\delta}}e^{\frac{i\xi}{\delta}} = \mathbf{H}_{\parallel}(0)e^{-\frac{\xi}{\delta}}e^{\frac{i\xi}{\delta}}$ (6)

$\mathbf{n}\cdot(5)\Rightarrow \mathbf{n}\cdot\mathbf{H}_c(\xi)\approx 0 \Rightarrow \mathbf{H}_{c\parallel}(\xi)\approx\mathbf{H}_c(\xi)$ (7)

Sub. (7) into (6) $\Rightarrow \mathbf{H}_c(\xi)\approx\mathbf{H}_{\parallel}(0)e^{-\frac{\xi}{\delta}}e^{\frac{i\xi}{\delta}}$ (8.9)



Sub. $\mathbf{H}_c(\xi)\approx\mathbf{H}_{\parallel}(0)e^{-\frac{\xi}{\delta}}e^{\frac{i\xi}{\delta}}$ into $\mathbf{E}_c(\xi)\approx-\frac{1}{\sigma}\mathbf{n}\times\frac{\partial}{\partial\xi}\mathbf{H}_c(\xi)$

$$\Rightarrow \mathbf{E}_c(\xi)\approx\sqrt{\frac{\mu_C\omega}{2\sigma}}(1-i)\left[\mathbf{n}\times\mathbf{H}_{\parallel}(\xi=0)\right]e^{-\frac{\xi}{\delta}}e^{\frac{i\xi}{\delta}} \qquad (8.10)$$

$$\mathbf{E}_{\parallel}(\xi=0)=\mathbf{E}_{c\parallel}(\xi=0)\approx\mathbf{E}_c(\xi=0)\approx\sqrt{\frac{\mu_C\omega}{2\sigma}}(1-i)\left[\mathbf{n}\times\mathbf{H}_{\parallel}(\xi=0)\right] \qquad (8.11)$$

$\boxed{\text{b.c. at } \xi=0}$ $\boxed{\mathbf{n}\cdot(4)\Rightarrow\mathbf{n}\cdot\mathbf{E}_C\simeq 0\Rightarrow\mathbf{E}_{C\parallel}\simeq\mathbf{E}_C}$

Power Loss Per Unit Area:

$$\begin{aligned} \frac{dP_{loss}}{da} &= \text{time averaged power into conductor per unit area} \\ &= -\tfrac{1}{2}\mathrm{Re}\left[\mathbf{n}\cdot\mathbf{E}(\xi=0)\times\mathbf{H}^*(\xi=0)\right] \\ &= -\tfrac{1}{2}\mathrm{Re}\left[\mathbf{n}\cdot\mathbf{E}_{\parallel}(\xi=0)\times\mathbf{H}_{\parallel}^*(\xi=0)\right] \\ \boxed{(8.11)}\rightarrow &= \tfrac{1}{4}\mu_C\omega\delta\left|\mathbf{H}_{\parallel}(\xi=0)\right|^2 = \tfrac{1}{2\sigma\delta}\left|\mathbf{H}_{\parallel}(\xi=0)\right|^2 \qquad (8.12) \\ &\propto \mu_C^{\frac{1}{2}}\omega^{\frac{1}{2}}\sigma^{-\frac{1}{2}}\left|\mathbf{H}_{\parallel}(\xi=0)\right|^2 \end{aligned}$$

Alternative method to derive (8.12):

$$(8.10)\Rightarrow\mathbf{J}(\xi)=\sigma\mathbf{E}_c(\xi)\approx\tfrac{1}{\delta}(1-i)\left[\mathbf{n}\times\mathbf{H}_{\parallel}(\xi=0)\right]e^{-\frac{\xi(1-i)}{\delta}} \qquad (8.13)$$

$$\begin{bmatrix}\text{time averaged power}\\ \text{loss in conductor per}\\ \text{unit volume}\end{bmatrix} = \tfrac{1}{2}\mathrm{Re}\left[\mathbf{J}(\xi)\cdot\mathbf{E}_c^*(\xi)\right]=\tfrac{1}{2\sigma}\left|\mathbf{J}(\xi)\right|^2$$

$\boxed{(8.13)}$

$$\frac{dP_{loss}}{da}=\frac{1}{2\sigma}\int_0^\infty d\xi\left|\mathbf{J}(\xi)\right|^2 = \frac{1}{\sigma\delta^2}\left|\mathbf{H}_{\parallel}(\xi=0)\right|^2\int_0^\infty e^{-\frac{2\xi}{\delta}}d\xi$$

$$=\frac{1}{2\sigma\delta}\left|\mathbf{H}_{\parallel}(\xi=0)\right|^2, \quad \text{same as (8.12)}$$

Effective surface current $\mathbf{K}_{eff}$:

$$\mathbf{K}_{eff} = \int_0^\infty\mathbf{J}(\xi)d\xi = \tfrac{1}{\delta}(1-i)\left[\mathbf{n}\times\mathbf{H}_{\parallel}(\xi=0)\right]\int_0^\infty e^{-\frac{\xi(1-i)}{\delta}}d\xi$$

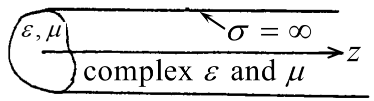
$$=\mathbf{n}\times\mathbf{H}_{\parallel}(\xi=0) \quad \boxed{(8.13)} \qquad (8.14)$$

$$(8.12)\ \&\ (8.14)\Rightarrow \frac{dP_{loss}}{da} == \frac{1}{2\sigma\delta}\left|\mathbf{K}_{eff}\right|^2 \qquad (8.15)$$

8.2-8.4 Modes in a Waveguide

Consider a hollow conductor of infinite length and uniform cross section of arbitrary shape (see figure). We assume that the filling medium is uniform, linear, and isotropic ($\mathbf{B}=\mu\mathbf{H}$; $\mathbf{D}=\varepsilon\mathbf{E}$, where ε and μ are in general complex numbers). This is a structure commonly used to guide EM waves as well as a rare case where exact solutions are possible (for some simple cross sections.) Maxwell equations can be written

$$\begin{cases} \nabla\times\mathbf{E}=-\frac{\partial}{\partial t}\mathbf{B} & (8) \\ \nabla\times\mathbf{B}=\mu\varepsilon\frac{\partial}{\partial t}\mathbf{E} & (9) \\ \nabla\cdot\mathbf{E}=0 & (10) \\ \nabla\cdot\mathbf{B}=0 & (11) \end{cases}$$



$$\nabla\times(8)\Rightarrow\nabla\times\nabla\times\mathbf{E}=-\frac{\partial}{\partial t}\nabla\times\mathbf{B}\Rightarrow\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}=-\frac{\partial}{\partial t}\left(\mu\varepsilon\frac{\partial}{\partial t}\mathbf{E}\right)$$

$$\Rightarrow\nabla^2\mathbf{E}-\mu\varepsilon\frac{\partial^2}{\partial t^2}\mathbf{E}=0 \qquad (12)$$

Similarly, $\nabla\times(9)\Rightarrow\nabla^2\mathbf{B}-\mu\varepsilon\frac{\partial^2}{\partial t^2}\mathbf{B}=0$ (13)

Let $\begin{cases} \mathbf{E}(\mathbf{x},t)=\mathbf{E}(\mathbf{x}_t)e^{\pm ik_z z-i\omega t} \\ \mathbf{B}(\mathbf{x},t)=\mathbf{B}(\mathbf{x}_t)e^{\pm ik_z z-i\omega t} \end{cases}$,

> $\mathbf{x}_t$: coordinates transverse to z, e.g. (x,y) or (r,θ)
> k_z here $\leftrightarrow$ k in Jackson

where, in general, ω and k_z are complex constants. To be specific, we assume that the real parts of ω and k_z are both positive. Then, $e^{ik_z z-i\omega t}$ and $e^{-ik_z z-i\omega t}$ have forward and backward phase velocities, respectively. As will be seen in (31), $e^{ik_z z-i\omega t}$ and $e^{-ik_z z-i\omega t}$ also have forward and backward group velocities, respectively. Hence, we call $e^{ik_z z-i\omega t}$ a <u>forward wave</u> and $e^{-ik_z z-i\omega t}$ a <u>backward wave</u>.

With the assumed z and t dependences, we have

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\to -\omega^2 \\ \dfrac{\partial^2}{\partial z^2}\to -k_z^2 \\ \nabla^2=\nabla_t^2+\dfrac{\partial^2}{\partial z^2}=\nabla_t^2-k_z^2 \end{cases}$$

$$\boxed{\nabla_t^2=\begin{cases} \dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}, & \text{Cartesian} \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial}{\partial r}\right)+\dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}, & \text{cylindrical} \end{cases}}$$

Thus,

$$\begin{cases} \nabla^2\mathbf{E}-\mu\varepsilon\dfrac{\partial^2}{\partial t^2}\mathbf{E}=0 \\ \nabla^2\mathbf{B}-\mu\varepsilon\dfrac{\partial^2}{\partial t^2}\mathbf{B}=0 \end{cases} \Rightarrow \left(\nabla_t^2+\mu\varepsilon\omega^2-k_z^2\right)\begin{Bmatrix}\mathbf{E}(\mathbf{x}_t)\\ \mathbf{B}(\mathbf{x}_t)\end{Bmatrix}=0 \qquad (8.19)$$

$$\Rightarrow \left(\nabla_t^2+\mu\varepsilon\omega^2-k_z^2\right)\begin{Bmatrix}E_z(\mathbf{x}_t)\\ B_z(\mathbf{x}_t)\end{Bmatrix}=0 \qquad (14)$$

It is in general not possible to obtain from (8.19). So our strategy here is to solve (14) for $E_z(\mathbf{x}_t)$ and $B_z(\mathbf{x}_t)$, and then express the other components of the fields $[\mathbf{E}_t(\mathbf{x}_t)$ and $\mathbf{B}_t(\mathbf{x}_t)]$ in terms of $E_z(\mathbf{x}_t)$ and $B_z(\mathbf{x}_t)$ through Eqs. (17) and (18).

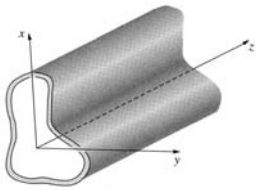
Exercise: Writing $\mathbf{E}(\mathbf{x}_t)=E_r\mathbf{e}_r+E_\theta\mathbf{e}_\theta+E_z\mathbf{e}_z$ and using the cylindrical coordinate system, derive the equations for E_r and E_θ from (8.19).
(hint: $\frac{\partial}{\partial\theta}\mathbf{e}_r=\mathbf{e}_\theta$, $\frac{\partial}{\partial\theta}\mathbf{e}_\theta=-\mathbf{e}_r$)

Griffiths

9.5 Guided Waves
9.5.1 Wave Guides

Can the electromagnetic waves propagate in a hollow metal pipe?
Yes, wave guide.

Waveguides generally made of good conductor, so that **E**=0 and **B**=0 inside the material.



The boundary conditions at the inner wall are: $\mathbf{E}^{//}=0$ and $B^{\perp}=0$ …

The generic form of the monochromatic waves:

$$\begin{cases} \tilde{\mathbf{E}}(x,y,z,t)=\tilde{\mathbf{E}}_0(x,y)e^{i(\tilde{k}z-\omega t)}=(\tilde{E}_x\hat{\mathbf{x}}+\tilde{E}_y\hat{\mathbf{y}}+\tilde{E}_z\hat{\mathbf{z}})e^{i(\tilde{k}z-\omega t)} \\ \tilde{\mathbf{B}}(x,y,z,t)=\tilde{\mathbf{B}}_0(x,y)e^{i(\tilde{k}z-\omega t)}=(\tilde{B}_x\hat{\mathbf{x}}+\tilde{B}_y\hat{\mathbf{y}}+\tilde{B}_z\hat{\mathbf{z}})e^{i(\tilde{k}z-\omega t)} \end{cases}$$

Griffiths

General Properties of Wave Guides

In the interior of the wave guide, the waves satisfy Maxwell's equations:

$$\nabla\cdot\mathbf{E}=0 \quad \nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$$

Why $\rho_f=0$ and $J_f=0$?

$$\nabla\cdot\mathbf{B}=0 \quad \nabla\times\mathbf{B}=\frac{1}{v^2}\frac{\partial\mathbf{E}}{\partial t} \quad \text{where } v=\frac{1}{\sqrt{\varepsilon\mu}}$$

We obtain

$$\text{(i)}\ \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=i\omega B_z \qquad \text{(iv)}\ \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=-\frac{i\omega}{c^2}E_z$$

$$\text{(ii)}\ \frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=i\omega B_x \qquad \text{(v)}\ \frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}=-\frac{i\omega}{c^2}E_x$$

$$\text{(iii)}\ \frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=i\omega B_y \qquad \text{(vi)}\ \frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}=-\frac{i\omega}{c^2}E_y$$

Griffiths

TE, TM, and TEM Waves

Determining the longitudinal components E_z and B_z, we could quickly calculate all the others.

$$E_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$$

$$E_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)$$

$$B_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$$

$$B_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$$

Try to derive these relations by yourself.

We obtain

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{v^2} - k^2\right]E_z = 0$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{v^2} - k^2\right]B_z = 0$$

If $E_z = 0 \Rightarrow$ TE (transverse electric) waves;
If $B_z = 0 \Rightarrow$ TM (transverse magnetic) waves;
If $E_z = 0$ and $B_z = 0 \Rightarrow$ TEM waves.

General Approach

Let $\begin{cases}\mathbf{E} = \mathbf{E}_t + E_z\mathbf{e}_z \\ \mathbf{B} = \mathbf{B}_t + B_z\mathbf{e}_z \\ \nabla = \nabla_t + \mathbf{e}_z\frac{\partial}{\partial z} = \nabla_t \pm ik_z\mathbf{e}_z\end{cases}$

$$\boxed{\nabla_t = \begin{cases}\mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y}, & \text{Cartesian} \\ \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}, & \text{cylindrical}\end{cases}}$$

$$\nabla\times\mathbf{E} = -\tfrac{\partial}{\partial t}\mathbf{B} \;\Rightarrow (\nabla_t \pm ik_z\mathbf{e}_z)\times(\mathbf{E}_t + E_z\mathbf{e}_z) = i\omega(\mathbf{B}_t + B_z\mathbf{e}_z) \quad (15)$$

$$\nabla\times\mathbf{B} = \mu\varepsilon\tfrac{\partial}{\partial t}\mathbf{E} \Rightarrow (\nabla_t \pm ik_z\mathbf{e}_z)\times(\mathbf{B}_t + B_z\mathbf{e}_z) = -i\mu\varepsilon\omega(\mathbf{E}_t + E_z\mathbf{e}_z) \quad (16)$$

Using the relations: $\left\{\begin{matrix}(\nabla_t\times\mathbf{E}_t)\parallel\mathbf{e}_z \\ (\nabla_t\times E_z\mathbf{e}_z)\perp\mathbf{e}_z\end{matrix}\right\}$, we obtain from the transverse components of (15) and (16):

$$\nabla_t\times E_z\mathbf{e}_z \pm ik_z\mathbf{e}_z\times\mathbf{E}_t = i\omega\mathbf{B}_t \quad (17)$$

$$\nabla_t\times B_z\mathbf{e}_z \pm ik_z\mathbf{e}_z\times\mathbf{B}_t = -i\mu\varepsilon\omega\mathbf{E}_t \quad (18)$$

In (15)-(18), the $\left\{\begin{matrix}\text{upper}\\\text{lower}\end{matrix}\right\}$ sign applies to the $\left\{\begin{matrix}\text{forward}\\\text{backward}\end{matrix}\right\}$ wave.

Rewrite (17) and (18)

$$\nabla_t\times E_z\mathbf{e}_z \pm ik_z\mathbf{e}_z\times\mathbf{E}_t = i\omega\mathbf{B}_t \quad (17)$$

$$\nabla_t\times B_z\mathbf{e}_z \pm ik_z\mathbf{e}_z\times\mathbf{B}_t = -i\mu\varepsilon\omega\mathbf{E}_t \quad (18)$$

Since E_z and B_z have already been solved from (14), (17) and (18) are algebraic (rather than differential) equations. We now manipulate (17) and (18) to eliminate $\mathbf{B}_t$ and thus express $\mathbf{E}_t$ in terms of E_z and B_z.

$$\mathbf{e}_z\times(17)\Rightarrow\; \underbrace{\mathbf{e}_z\times(\nabla_t\times E_z\mathbf{e}_z)}_{\nabla_t E_z\times\mathbf{e}_z + E_z\underbrace{\nabla_t\times\mathbf{e}_z}_{0}} \pm ik_z\overbrace{\mathbf{e}_z\times(\mathbf{e}_z\times\mathbf{E}_t)}^{-\mathbf{E}_t} = i\omega\mathbf{e}_z\times\mathbf{B}_t$$

$$\boxed{\begin{array}{l}\nabla\times\psi\mathbf{a} = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a} \\ \text{If } \psi,\ \mathbf{a} \text{ are both independent of } z\text{, then} \\ \nabla_t\times\psi\mathbf{a} = \nabla_t\psi\times\mathbf{a} + \psi\nabla_t\times\mathbf{a}\end{array}}$$

$$\Rightarrow\; i\omega\mathbf{e}_z\times\mathbf{B}_t = \nabla_t E_z \mp ik_z\mathbf{E}_t \quad (19)$$

Sub. (19) into (18)

$$\underbrace{\nabla_t\times B_z\mathbf{e}_z}_{\nabla_t B_z\times\mathbf{e}_z} \pm ik_z\frac{1}{i\omega}(\nabla_t E_z \mp ik_z\mathbf{E}_t) = -i\mu\varepsilon\omega\mathbf{E}_t \quad (20)$$

Multiply (20) by $i\omega$: $i\omega\nabla_t B_z\times\mathbf{e}_z \pm ik_z\nabla_t E_z + k_z^2\mathbf{E}_t = \mu\varepsilon\omega^2\mathbf{E}_t$

$$\Rightarrow\; (\mu\varepsilon\omega^2 - k_z^2)\mathbf{E}_t = i(\omega\nabla_t B_z\times\mathbf{e}_z \pm k_z\nabla_t E_z)$$

$$\Rightarrow\; \mathbf{E}_t = \frac{i}{\mu\varepsilon\omega^2 - k_z^2}\left[\pm k_z\nabla_t E_z - \omega\mathbf{e}_z\times\nabla_t B_z\right] \quad (8.26a)$$

Similarly,

$$\mathbf{B}_t = \frac{i}{\mu\varepsilon\omega^2 - k_z^2}\left[\pm k_z\nabla_t B_z + \mu\varepsilon\omega\mathbf{e}_z\times\nabla_t E_z\right] \quad (8.26b)$$

Thus, once E_z and B_z have been solved from (14), the solutions for $\mathbf{E}_t$ and $\mathbf{B}_t$ are given by (8.26a) and (8.26b).

Discussion:

(i) $\mathbf{E}_t$, $\mathbf{B}_t$, E_z, B_z in (8.26a) and (8.26b) are functions of $\mathbf{x}_t$ only.

(ii) ε and μ can be complex. Im(ε) or Im(μ) implies dissipation.

(iii) By letting $B_z = 0$, we may obtain a set of solutions for E_z, $\mathbf{E}_t$, and $\mathbf{B}_t$ from (14), (8.26a), and (8.26b), respectively. It can be shown that if the boundary condition on E_z is satisfied, then boundary conditions on $\mathbf{E}_t$ and $\mathbf{B}_t$ are also satisfied. Hence, this gives a set of valid solutions called the TM (transverse magnetic) modes. Similarly, by letting $E_z = 0$, we may obtain a set of valid solutions called the TE (transverse electric) modes.

(iv) E_z is the generating function for the TM mode and B_z is the generating function for the TE mode. The generating function is denoted by Ψ in Jackson.

TM Mode of a Waveguide ($B_z = 0$): (see pp. 359-360)

$$\begin{cases} (\nabla_t^2 + \gamma^2)E_z = 0 \text{ with boundary condition } E_z|_s = 0 & (21) \\ \mathbf{E}_t = \pm \dfrac{ik_z}{\gamma^2}\nabla_t E_z & (21a) \\ \mathbf{H}_t = \pm \dfrac{\varepsilon\omega}{k_z}\mathbf{e}_z \times \mathbf{E}_t = \pm \dfrac{1}{Z_e}\mathbf{e}_z \times \mathbf{E}_t & (21b) \\ \gamma^2 = \mu\varepsilon\omega^2 - k_z^2 & (21c) \end{cases}$$

Assume perfectly conducting wall.

$Z_e \equiv k_z/\varepsilon\omega$, wave impedance of TM modes

TE Mode of a Waveguide ($E_z = 0$): (see pp. 359-360)

$$\begin{cases} (\nabla_t^2 + \gamma^2)H_z = 0 \text{ with boundary condition } \frac{\partial}{\partial n}H_z|_s = 0 & (22) \\ \mathbf{H}_t = \pm \dfrac{ik_z}{\gamma^2}\nabla_t H_z & (22a) \\ \mathbf{E}_t = \mp \dfrac{\mu\omega}{k_z}\mathbf{e}_z \times \mathbf{H}_t = \mp Z_h \mathbf{e}_z \times \mathbf{H}_t & (22b) \\ \gamma^2 = \mu\varepsilon\omega^2 - k_z^2 & (22c) \end{cases}$$

$Z_h \equiv \mu\omega/k_z$, wave impedance of TE modes

s $\mathbf{n}$ $\rightarrow \mathbf{e}_z$

b.c. $\mathbf{n}\cdot\mathbf{H}|_s = 0$

$\mathbf{n} \perp \mathbf{e}_z \Rightarrow \mathbf{n}\cdot\mathbf{H}_t|_s = 0$

(22a) $\Rightarrow \mathbf{n}\cdot\nabla_t H_z|_s = 0$

$\Rightarrow \frac{\partial}{\partial n}H_z|_s = 0$

Discussion:

(i) Either (21) or (22) constitutes an eigenvalue problem (see lecture notes, Ch. 3, Appendix A). The eigenvalue γ^2 will be an infinite set of discrete values fixed by the boundary condition, each representing an eigenmode of the waveguide (An example will be provided below.)

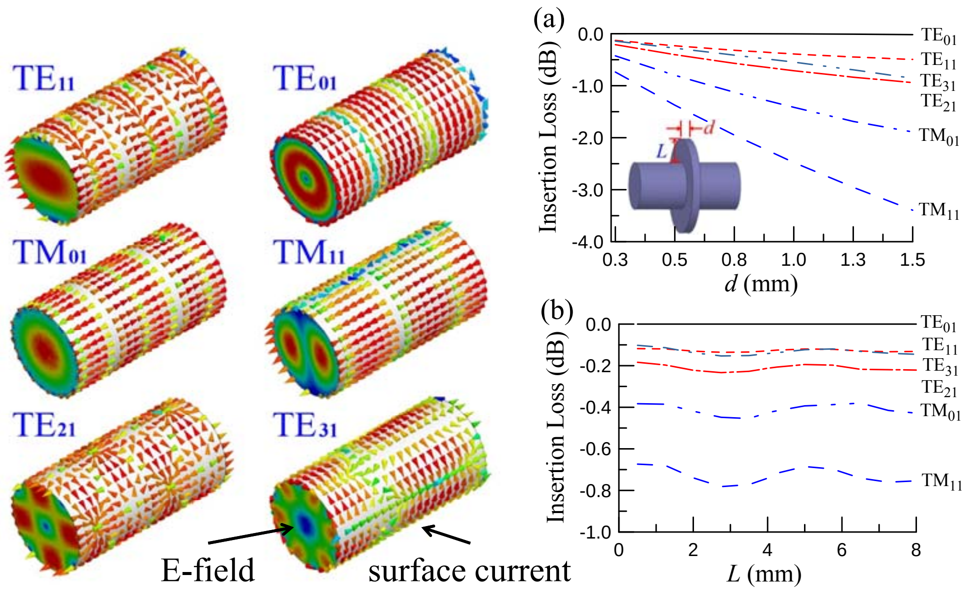
(ii) (21b) and (22b) show that $\mathbf{E}_t$ is perpendicular to $\mathbf{B}_t$ (also true in a cavity).

(iii) (21b) and (22b) show that $\mathbf{E}_t$ and $\mathbf{B}_t$ are *in phase* if μ, ε, ω, k_z are all real (not true in a cavity).

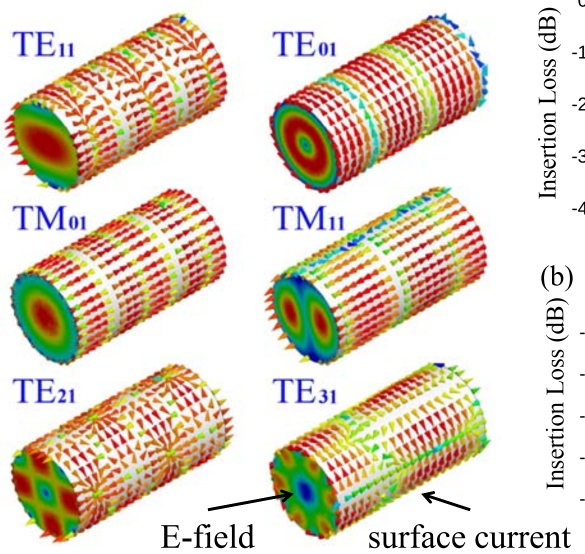
(iv) (21c) [or (22c)] is the dispersion relation, which relates ω and k_z for a given mode.

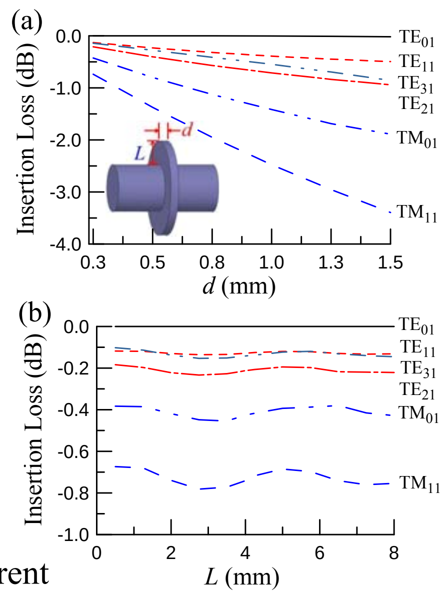
(v) The wave impedance, Z_e or Z_h, gives the ratio of E_t to H_t in the waveguide.

Field Patterns of Circular Waveguide Modes

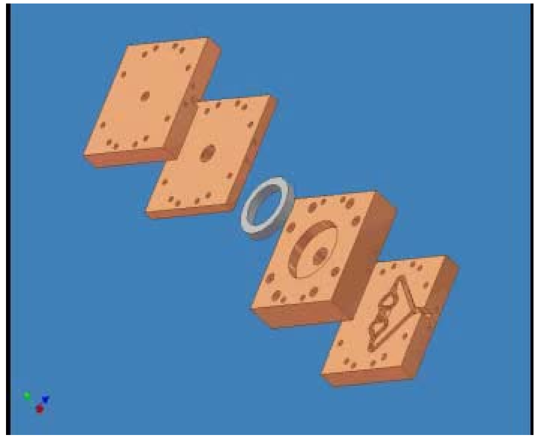


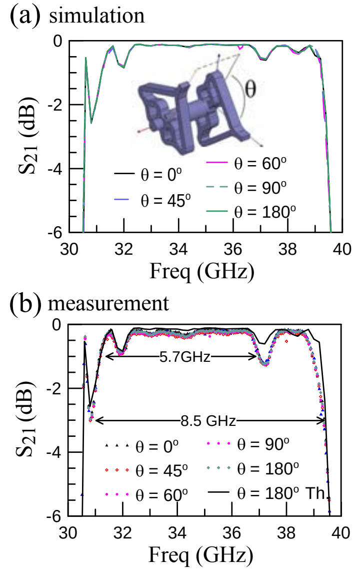
Field Patterns of Circular Waveguide Modes





Characterization of Circularly Symmetric TE01 Mode





T. H. Chang and B. R. Yu, "High-Power Millimeter-Wave Rotary Joint", Rev. Sci. Instrum. 80, 034701 (2009).

TEM Mode of Coaxial and Parallel-Wire Transmission Lines

($E_z = B_z = 0$)**:** (see Jackson p. 341)

$$\text{Rewrite} \begin{cases} \mathbf{E}_t = \dfrac{i}{\mu\varepsilon\omega^2 - k_z^2}\left[\pm k_z \nabla_t E_z - \omega \mathbf{e}_z \times \nabla_t B_z\right] & \text{(8.26a)} \\ \mathbf{B}_t = \dfrac{i}{\mu\varepsilon\omega^2 - k_z^2}\left[\pm k_z \nabla_t B_z + \mu\varepsilon\omega \mathbf{e}_z \times \nabla_t E_z\right] & \text{(8.26b)} \end{cases}$$

These 2 equations fail for a different class of modes, called the TEM (transverse electromagnetic) mode, for which $E_z = B_z = 0$. However, they give the condition for the existence of this mode:

$$\boxed{\omega^2 = k_z^2 / \mu\varepsilon}. \quad \left[\begin{array}{l}\text{Equations in rectangular boxes are} \\ \text{basic equations for the TEM mode.}\end{array}\right] \qquad (8.27)$$

(8.27) is also the dispersion relation in infinite space. This makes the TEM mode very useful because it can propagate at any frequency.

To calulate $\mathbf{E}_t$ and $\mathbf{B}_t$, we need to go back to Maxwell equations.

Let $E_z = B_z = 0$ and $\begin{Bmatrix}\mathbf{E}_t \\ \mathbf{B}_t\end{Bmatrix} = \begin{Bmatrix}\mathbf{E}_{\text{TEM}}(\mathbf{x}_t) \\ \mathbf{B}_{\text{TEM}}(\mathbf{x}_t)\end{Bmatrix} e^{\pm i k_z z - i\omega t}$,

then, because $B_z = 0$, the z-component of $\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$ gives

$$\nabla_t \times \mathbf{E}_{\text{TEM}} = 0 \;\Rightarrow\; \boxed{\mathbf{E}_{\text{TEM}} = -\nabla_t \Phi_{\text{TEM}}(\mathbf{x}_t)},$$

$$\boxed{\begin{array}{c}\nabla_t \times \mathbf{A}_t(\mathbf{x}_t) = 0 \\ \Updownarrow \\ \mathbf{A}_t(\mathbf{x}_t) = -\nabla_t \Phi(\mathbf{x}_t)\end{array}}$$

and, because $E_z = 0$, $\nabla \cdot \mathbf{E} = 0$ gives

$$\nabla_t \cdot \mathbf{E}_{\text{TEM}} = 0 \;\Rightarrow\; \boxed{\nabla_t^2 \Phi_{\text{TEM}}(\mathbf{x}_t) = 0},$$

where Φ_{TEM} is the generating function for the TEM modes. Because $\mathbf{E}_{\tan} = 0$ on the surface of a perfect conductor, Φ_{TEM} is subject to the boundary condition $\Phi_{\text{TEM}} = const.$ on the conductor. This gives $\Phi_{\text{TEM}} = const.$ or $\mathbf{E}_{\text{TEM}} = 0$ everywhere, if there is only one conductor. So, TEM modes exist only in 2-conductor configurations, such as coaxial and parallel-wire transmission lines. Finally, $\mathbf{B}_{\text{TEM}}$ is given by the transverse components of $\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$: $\boxed{\mathbf{B}_{\text{TEM}} = \pm \frac{k_z}{\omega} \mathbf{e}_z \times \mathbf{E}_{\text{TEM}}}$.

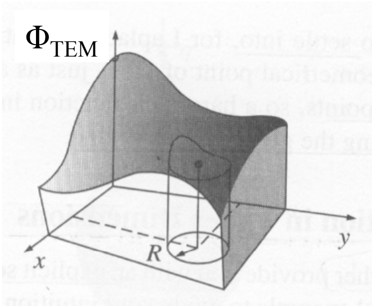
Why single conductor cannot support TEM waves? (I)

Let's consider the property of 2D Laplace equation.

Suppose Φ_{TEM} depends on two variables.

$$\frac{\partial^2\Phi_{\text{TEM}}}{\partial x^2}+\frac{\partial^2\Phi_{\text{TEM}}}{\partial y^2}=0 \quad \begin{cases}\text{a partial differential equation (PDE);}\\ \text{not a ordinary differential equation (ODE).}\end{cases}$$

Harmonic functions in two dimensions have the same properties as we noted in one dimension:



Φ_{TEM} has no local maxima or minima. All extrema occur at the boundaries. (The surface may not be an equalpotential.)

If Φ_{TEM}=const., $\mathbf{E}_{\text{TEM}}$=0 & $\mathbf{B}_{\text{TEM}}$=0 (Not a flawless argument)

Why single conductor cannot support TEM waves? (II)

David Cheng's explanation. Chap. 10, p.525.

1. The magnetic flux lines always close upon themselves. For a TEM wave, the magnetic field line would form closed loops in a transverse plane.
2. The generalized Ampere's law requires that the line integral of the magnetic field around any closed loop in a transverse plane must equal the sum of the longitudinal conduction and displace conduction current inside the waveguide.
3. There is no longitudinal conduction current inside the waveguide and no longitudinal displace current (E_z=0).
4. There can be no closed loops of magnetic field lines in any transverse plane. (weak conclusion)

The TEM wave cannot exist in a single-conductor hollow waveguide of any shape. (Again, not a perfect argument)

In summary, the TEM modes are governed by the following set of equtions:

$$\begin{cases}\nabla_t^2\Phi_{\text{TEM}}(\mathbf{x}_t)=0 & (23)\\ \mathbf{E}_{\text{TEM}}=-\nabla_t\Phi_{\text{TEM}}(\mathbf{x}_t) & (23a)\\ \mathbf{B}_{\text{TEM}}=\pm\frac{k_z}{\omega}\mathbf{e}_z\times\mathbf{E}_{\text{TEM}} & (23b)\\ \left(\text{or } \mathbf{H}_{\text{TEM}}=\pm\frac{k_z}{\omega\mu}\mathbf{e}_z\times\mathbf{E}_{\text{TEM}}=\pm\sqrt{\frac{\varepsilon}{\mu}}\mathbf{e}_z\times\mathbf{E}_{\text{TEM}}=\pm Y\mathbf{e}_z\times\mathbf{E}_{\text{TEM}}\right) & \\ \omega^2=\frac{k_z^2}{\mu\varepsilon} & (23c)\end{cases}$$

where $Y\ (=\sqrt{\varepsilon/\mu})$ is the (intrinsic) admittance of the filling medium defined in Ch. 7 of lecture notes (the last page of Sec. II).

Discussion:

(i) For the TEM modes, we solve a 2-D equation $\nabla_t^2\Phi_{\text{TEM}}(\mathbf{x}_t)=0$ for $\Phi_{\text{TEM}}(\mathbf{x}_t)$. But this is not a 2-D problem because Φ_{TEM} is not the full solution. The full solution is $\begin{Bmatrix}\mathbf{E}_t(\mathbf{x},t)\\ \mathbf{B}_t(\mathbf{x},t)\end{Bmatrix}=\begin{Bmatrix}\mathbf{E}_{\text{TEM}}(\mathbf{x}_t)\\ \mathbf{B}_{\text{TEM}}(\mathbf{x}_t)\end{Bmatrix}e^{\pm ik_z z-i\omega t}$ with $\mathbf{E}_{\text{TEM}}=-\nabla_t\Phi_{\text{TEM}}(\mathbf{x}_t)$ and $\mathbf{B}_{\text{TEM}}=\pm\frac{k_z}{\omega}\mathbf{e}_z\times\mathbf{E}_{\text{TEM}}$.

For an actual 2-D electrostatic problem $[\Phi(\mathbf{x})=\Phi(\mathbf{x}_t)]$, we have $\nabla_t^2\Phi(\mathbf{x}_t)=0,$ which gives the full solution $\mathbf{E}_t(\mathbf{x}_t)=-\nabla_t\Phi(\mathbf{x}_t)$.

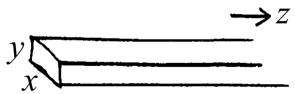
(ii) Note the difference between the scalar potentials discussed here and in Ch. 1 and Ch. 6.

$$\begin{cases}\mathbf{E}_{\text{TEM}}=-\nabla_t\Phi_{\text{TEM}}(\mathbf{x}_t) & \text{regard } \Phi_{\text{TEM}} \text{ as a mathematical tool.}\\ \mathbf{E}(\mathbf{x})=-\nabla\Phi(\mathbf{x}) & \text{regard } \Phi \text{ as a physical quantity.}\\ \mathbf{E}(\mathbf{x},t)=-\nabla\Phi(\mathbf{x},t)-\frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t) & \text{regard } \Phi \text{ and } \mathbf{A} \text{ as mathematical tools.}\end{cases}$$

Example 1: TE mode of a rectangular waveguide

Rewrite the basic equations for the TE mode:

$$\begin{cases} (\nabla_t^2+\gamma^2)H_z=0 \text{ with boundary condition } \left.\frac{\partial}{\partial n}H_z\right|_s=0 & (22)\\ \mathbf{H}_t=\pm\dfrac{ik_z}{\gamma^2}\nabla_t H_z & (22a)\\ \mathbf{E}_t=\mp\dfrac{\mu\omega}{k_z}\mathbf{e}_z\times\mathbf{H}_t=\mp Z_h\mathbf{e}_z\times\mathbf{H}_t & (22b)\\ \gamma^2=\mu\varepsilon\omega^2-k_z^2 & (22c)\end{cases}$$



Rectangular geometry $\Rightarrow$ Cartesian system $\Rightarrow \nabla_t^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$

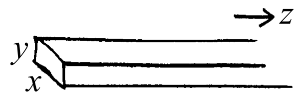
Hence, the wave equation in (22) becomes:

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\mu\varepsilon\omega^2-k_z^2\right]H_z=0 \qquad (24)$$

Rewrite (24): $\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\mu\varepsilon\omega^2-k_z^2\right]H_z=0 \qquad (24)$

Assuming $e^{ik_x x+ik_y y}$ dependence for H_z, we obtain

$$\left[\mu\varepsilon\omega^2-k_x^2-k_y^2-k_z^2\right]H_z=0$$



In order for $H_z\neq 0$, we must have

$$\mu\varepsilon\omega^2-k_x^2-k_y^2-k_z^2=0,$$

which is satisfied for $\pm k_x,\ \pm k_y,\ \pm k_z$. Since $(e^{ik_x x}, e^{-ik_x x})$, $(e^{ik_y y}, e^{-ik_y y})$, and $(e^{ik_z z}, e^{-ik_z z})$ are all linearly independent pairs, the complete solution for H_z is

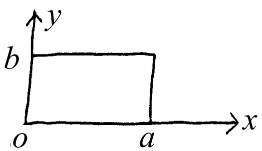
$$H_z=e^{-i\omega t}\left[A_1e^{ik_x x}+A_2e^{-ik_x x}\right]\left[B_1e^{ik_y y}+B_2e^{-ik_y y}\right]\cdot\left[C_+e^{ik_z z}+C_-e^{-ik_z z}\right] \qquad (25)$$

$$\boxed{\mathbf{H}_t=\pm\frac{ik_z}{\gamma^2}\nabla_t H_z}$$

Applying boundary conditions [see (22)] to (25):

$$H_z=e^{-i\omega t}\left[A_1e^{ik_x x}+A_2e^{-ik_x x}\right]\left[B_1e^{ik_y y}+B_2e^{-ik_y y}\right]\left[C_+e^{ik_z z}+C_-e^{-ik_z z}\right]$$

$$\begin{cases} B_x\propto\left.\frac{\partial}{\partial x}B_z\right|_{x=0}=0\Rightarrow ik_xA_1-ik_xA_2=0\Rightarrow A_1=A_2\\ B_y\propto\left.\frac{\partial}{\partial y}B_z\right|_{y=0}=0\Rightarrow ik_yB_1-ik_yB_2=0\Rightarrow B_1=B_2\end{cases}$$



$$\Rightarrow H_z=\cos k_x x\cos k_y y\left[C_+e^{-i\omega t+ik_z z}+C_-e^{-i\omega t-ik_z z}\right]$$

$$\begin{cases} B_x\propto\left.\frac{\partial}{\partial x}B_z\right|_{x=a}=0\Rightarrow \sin k_x a=0\Rightarrow k_x=m\pi/a,\ m=0,1,2,\ldots\\ B_y\propto\left.\frac{\partial}{\partial y}B_z\right|_{y=b}=0\Rightarrow \sin k_y b=0\Rightarrow k_y=n\pi/b,\ n=0,1,2,\ldots\end{cases}$$

$$\Rightarrow H_z=\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\Big[\underbrace{C_+e^{ik_z z-i\omega t}}_{\text{forward wave}}+\underbrace{C_-e^{-ik_z z-i\omega t}}_{\text{backward wave}}\Big] \qquad (26)$$

Sub. $k_x=\frac{m\pi}{a},\ k_y=\frac{n\pi}{b}$ into $\mu\varepsilon\omega^2-k_x^2-k_y^2-k_z^2=0$, we obtain

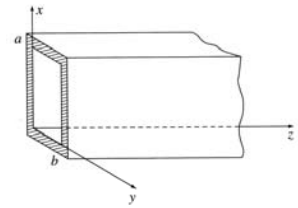
$$\mu\varepsilon\omega^2-k_z^2-\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)=0,\ m,n=0,1,2,\ldots \qquad (27)$$

Griffiths

9.5.2 TE Waves in a Rectangular Wave Guide

$E_z=0$, and $B_z(x,y)=X(x)Y(y)\leftarrow$ separation of variables

$$\frac{1}{X}\frac{\partial^2X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2Y}{\partial y^2}+\left(\frac{\omega^2}{v^2}-k^2\right)=0$$



$$\frac{1}{X}\frac{\partial^2X}{\partial x^2}=-k_x^2 \text{ and } \frac{1}{Y}\frac{\partial^2Y}{\partial y^2}=-k_y^2$$

with $\frac{\omega^2}{v^2}=k^2+k_x^2+k_y^2$

*Griffiths' derivation uses different boundary condition --- $\mathbf{E}_t=0$.

$$X(x)=A\sin k_x x+B\cos k_x x$$
$$Y(y)=C\sin k_y y+D\cos k_y y$$

Griffiths

TE Waves in a Rectangular Wave Guide (II)

$$E_x \propto \frac{\partial B_z}{\partial y} \propto C\cos k_y y - D\sin k_y y \qquad \boxed{\mathbf{E}_t = \mp\frac{i\mu\omega}{\gamma^2}\mathbf{e}_z\times\nabla_t H_z}$$

$$E_x(@\, y=0)=0 \Rightarrow C=0$$

$$E_x(@\, y=b)=0 \Rightarrow \sin k_y b = 0,\ k_y = \frac{n\pi}{b}\ (n=0,1,2,\ldots)$$

$$E_y \propto \frac{\partial B_z}{\partial x} \propto A\cos k_x x - B\sin k_x x$$

$$E_y(@\, x=0)=0 \Rightarrow A=0$$

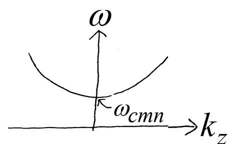
$$E_y(@\, x=a)=0 \Rightarrow \sin k_x a = 0,\ k_x = \frac{m\pi}{a}\ (m=0,1,2,\ldots)$$

$B_z(x,y)=B_0\cos(m\pi x/a)\cos(n\pi y/b) \leftarrow$ the TE_{mn} mode

$$k=\sqrt{(\omega/v)^2-\pi^2[(m/a)^2+(n/b)^2]}$$

Rewrite (27) as $\mu\varepsilon\omega^2 - k_z^2 - \mu\varepsilon\omega_{cmn}^2 = 0,$ (28)

where $\omega_{cmn} = \frac{\pi}{\sqrt{\mu\varepsilon}}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{1/2},\ \ m,n=0,1,2,\ldots$ (29)



Each pair of $(m,\ n)$ gives a <u>normal mode</u> (TE_{mn} mode) of the waveguide. m and n cannot both be 0, because that will creat a $\frac{0}{0}$ situation on (8.26) or (22a), making $\mathbf{H}_t$ and $\mathbf{E}_t$ indeterminable.

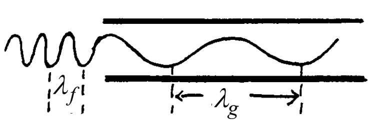
ω_{cmn} is the <u>cutoff frequency</u> (the frequency at which $k_z=0$) of the waveduide for the TE_{mn} mode. Waves with $\omega<\omega_{cmn}$ cannot propagate as a TE_{mn} mode because k_z becomes purely imaginary.

(28) is the TE_{mn} mode dispersion relation of a waveguide filled with a dielectric medium with constant (in general complex) ε and μ.

For the usual case of an unfilled waveguide, we have $\varepsilon=\varepsilon_0$ and $\mu=\mu_0$ ($\Rightarrow \mu\varepsilon=\mu_0\varepsilon_0=\frac{1}{c^2}$), and (28) (29) can be written

$$\omega^2-k_z^2c^2-\omega_{cmn}^2=0 \text{ with } \omega_{cmn}=\pi c\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{1/2}\ \left[\begin{array}{l}\text{for unfilled}\\ \text{waveguide}\end{array}\right] \quad (30)$$

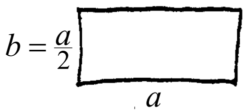
$$\omega^2-k_z^2c^2-\omega_{cmn}^2=0,\ \ \omega_{cmn}=\pi c\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{\frac{1}{2}}$$

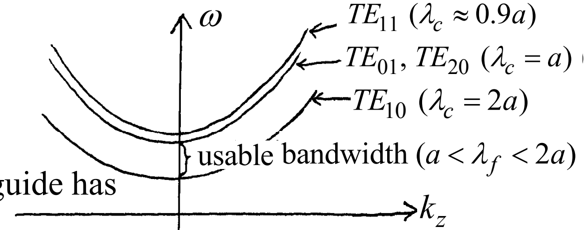


$$\begin{cases}\lambda_g = \underline{\text{guide wavelength}} \equiv 2\pi/k_z\\ \lambda_c = \underline{\text{cutoff wavelength}} \equiv 2\pi c/\omega_{cmn}\\ \lambda_f = \underline{\text{free space wavelength}} \equiv 2\pi c/\omega\end{cases}$$

depend on →	mode & geometry	wave freq.
λ_g	yes	yes
λ_c	yes	no
λ_f	no	yes

$$\begin{cases}\omega>\omega_{cmn} \Rightarrow k_z = \text{real} \Rightarrow \underline{\text{propagating waves}}\\ \omega=\omega_{cmn} \Rightarrow k_z=0 \Rightarrow \lambda_g=\infty\\ \omega<\omega_{cmn} \Rightarrow k_z=\text{imaginary} \Rightarrow \underline{\text{evanescent fields}}\end{cases}$$





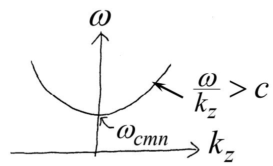
Question 1: A typical waveguide has $a=2b$. Why?

Question 2: Can we use a waveguide to transport waves at 60 Hz?

Other quantities of interest:

(1) Differentiating $\omega^2-k_z^2c^2-\omega_{cmn}^2=0$ with respect to k_z

$$2\omega\frac{d\omega}{dk_z}-2k_zc^2=0$$



$$\Rightarrow v_g=\frac{d\omega}{dk_z}=\frac{k_zc^2}{\omega}\ \text{[group velocity in unfilled waveguide]}$$

$$\Rightarrow \begin{cases}v_g<c\\ v_g\to 0 \text{ as } \omega\to\omega_{\text{cmn}}\end{cases}$$

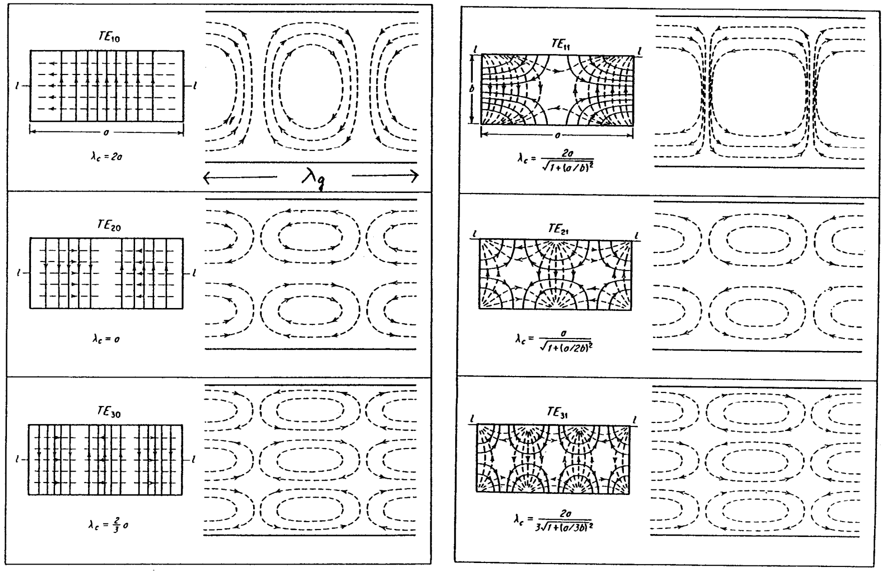
(2) The remaining field components (E_x, E_y, H_x, and H_y) can be obtained from H_z through

$$\mathbf{H}_t=\pm\frac{ik_z}{\gamma^2}\nabla_t H_z \qquad \left[\begin{array}{l}\gamma^2=\mu\varepsilon\omega^2-k_z^2=\frac{\omega_{cmn}^2}{c^2}\\ \text{see (22c) and (30).}\end{array}\right] \quad (22a)$$

$$\mathbf{E}_t=\mp\frac{\mu\omega}{k_z}\mathbf{e}_z\times\mathbf{H}_t \quad (22b)$$

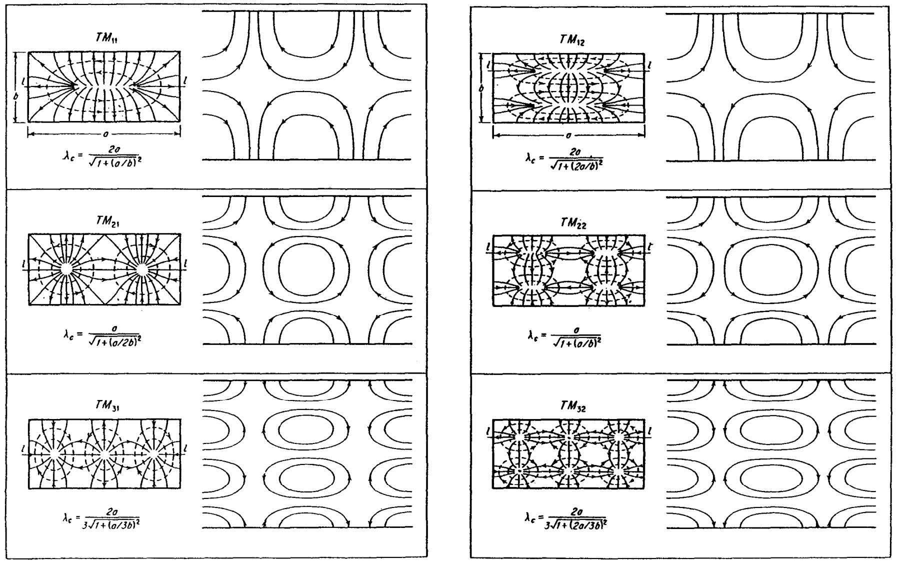
where the $\begin{Bmatrix}\text{upper}\\ \text{lower}\end{Bmatrix}$ sign applies to the $\begin{Bmatrix}\text{forward}\\ \text{backward}\end{Bmatrix}$ wave.

TE mode field patterns of rectangular waveguide



from E. L. Ginzton, "Microwave measurements". λ_c : cutoff frequency
solid curve: **E**-field lines; dashed curves: **B**-field lines

TM mode field patterns of rectangular waveguide

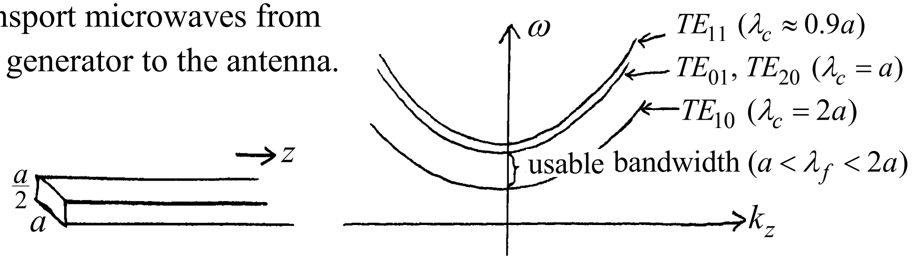


from E. L. Ginzton, "Microwave measurements". λ_c : cutoff frequency
solid curve: **E**-field lines; dashed curves: **B**-field lines

Discussion: *Waveguide and microwaves*

A typical waveguide has $a = 2b$ to maximize the usable bandwidth ($a < \lambda_f < 2a$) over which only the TE_{10} mode can propagate and hence mode purity is maintained. Waves are normally transported by the TE_{10} mode over this frequency range. Waveguides come in different sizes. Usable bandwidths of waveguides of practical dimensions (0.1 cm $< a <$ 100 cm) cover the entire microwave band (300 MHz to 300 GHz).

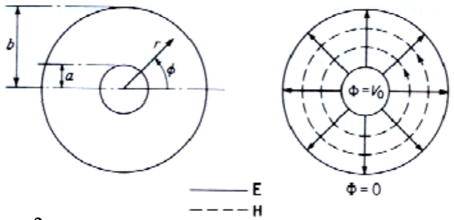
Compared with coaxial transmission lines, the waveguide is capable of handling much higher power. Hence, it is commonly used in high-power microwave systems. In a radar system, for example, it is used to transport microwaves from the generator to the antenna.



Example 2: TEM modes of a coaxial transmission line

TEM modes are governed by the following set of equtions:

$$\begin{cases} \nabla_t^2 \Phi_{\text{TEM}}(\mathbf{x}_t) = 0 & (23) \\ \mathbf{E}_{\text{TEM}} = -\nabla_t \Phi_{\text{TEM}}(\mathbf{x}_t) & (23a) \\ \mathbf{H}_{\text{TEM}} = \pm Y \mathbf{e}_z \times \mathbf{E}_{\text{TEM}} & (23b) \\ \omega^2 = \dfrac{k_z^2}{\mu\varepsilon} & (23c) \end{cases}$$



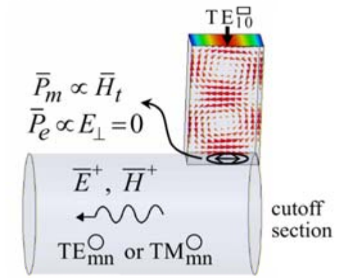
(23) gives $\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \Phi_{\text{TEM}}}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 \Phi_{\text{TEM}}}{\partial \varphi^2} = 0.$

Neglect the $\frac{\partial \Phi}{\partial \varphi} \neq 0$ modes $\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \Phi_{\text{TEM}}}{\partial r}) = 0 \Rightarrow \Phi_{\text{TEM}} = C_1 In(r) + C_2.$

Apply b.c. $\begin{cases} \Phi_{\text{TEM}}(r=a) = V_0 \\ \Phi_{\text{TEM}}(r=b) = 0 \end{cases} \Rightarrow \begin{cases} C_1 = V_0 / In(a/b) \\ C_2 = -C_1 In(b) \end{cases} \Rightarrow \Phi_{\text{TEM}} = V_0 \frac{In(r/b)}{In(a/b)}.$

(23a, b) then give
$$\begin{cases} \mathbf{E}_{\text{TEM}}(\mathbf{x},t) = \dfrac{V_0}{In(b/a)}\dfrac{1}{r} e^{\pm i k_z z - i\omega t}\mathbf{e}_r \\ \mathbf{H}_{\text{TEM}}(\mathbf{x},t) = \pm\dfrac{YV_0}{In(b/a)}\dfrac{1}{r} e^{\pm i k_z z - i\omega t}\mathbf{e}_\varphi \end{cases} \quad (31)$$

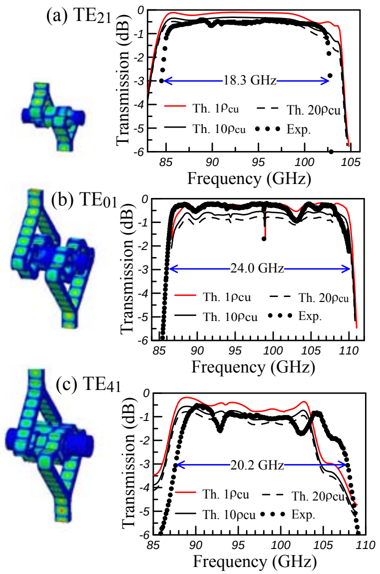
Exciting a Specific Mode



$$\bar{M} = j\omega\mu_0\bar{P}_m$$

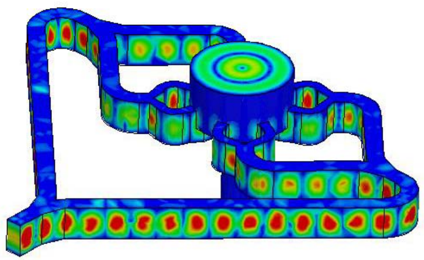
$$\bar{E}_2^+ = \sum_n A_n^+ \left(\bar{e}_n + \hat{z}e_{zn}\right)\cdot e^{-j\beta_n z}$$

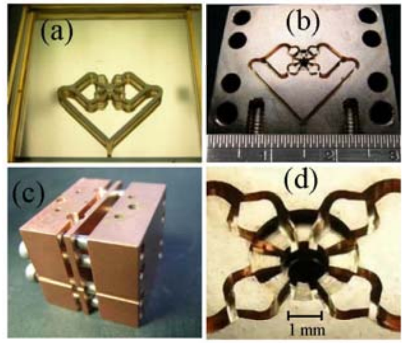
$$\frac{P_{01}^{total}}{P_{41}^{total}} = \frac{\beta_{41}p_{01}'^{\,4}\varepsilon_{04}}{\beta_{01}p_{41}'^{\,4}\varepsilon_{00}}\frac{(p_{01}'^{\,2}-0^2)}{(p_{41}'^{\,2}-4^2)}\frac{J_0^4(p_{01}')}{J_4^4(p_{41}')} = -18 \text{ dB}$$

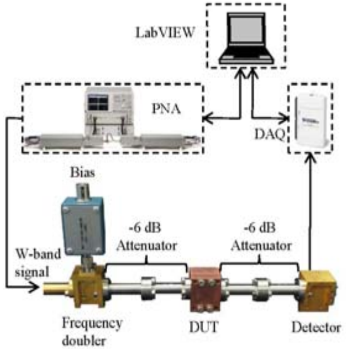


T. H. Chang, C. S. Lee, C. N. Wu, and C. F. Yu, "Exciting circular TEmn modes at low terahertz region", Appl. Phys. Lett. 93, 111503 (2008).

Difficulties of Exciting a Higher-Order Mode: Take TE_{02} as an Example



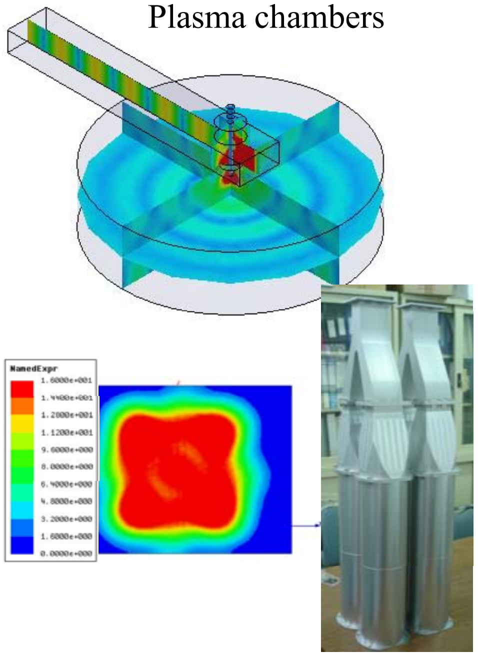




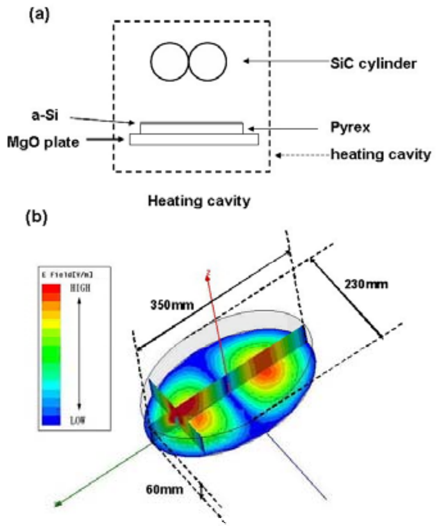
Desired mode	TE_{02}	
Coupling structure	octa-feed	
Waveguide radius	1.86 mm	
Parasitic modes	$TE_{11,A}$, $TE_{11,B}$	$TE_{21,A}$, $TE_{21,B}$
	TE_{01}	
	$TE_{31,A}$, $TE_{31,B}$	$TE_{12,A}$, $TE_{12,B}$
	TM_{01}	
	$TM_{11,A}$, $TM_{11,B}$	$TM_{21,A}$, $TM_{21,B}$
	$TE_{41,A}$, $TE_{41,B}$	$TE_{12,A}$, $TE_{12,B}$
	TM_{02}	
	$TM_{31,A}$, $TM_{31,B}$	$TE_{51,A}$, $TE_{51,B}$
	$TE_{22,A}$, $TE_{22,B}$	$TM_{12,A}$, $TM_{12,B}$

Applications of Waveguide Modes (I)

Plasma chambers



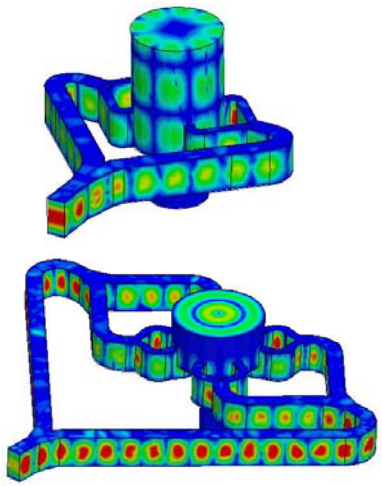
Material processing



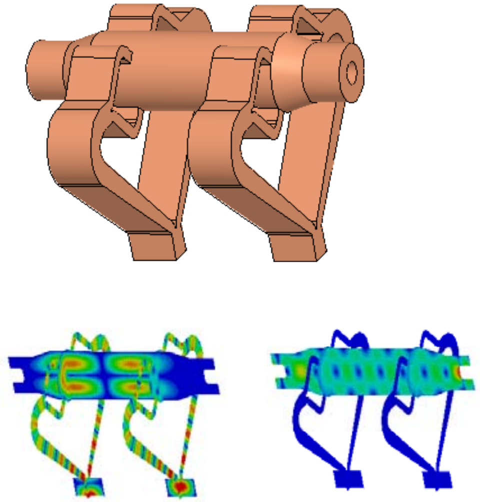
Appl. Phys. Lett. 94, 102104 (2009)

Applications of Waveguide Modes (II)

Mode converters



Rotary joints



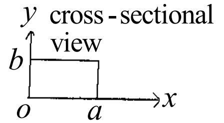
THz waveguide, circulator, isolator, power divider, antenna…

8.7 Modes in Cavities

We consider the example of a rectangular cavity (i.e. a rectangular waveguide with two ends closed by conductors), for which we have two additional boundary conditions at the ends.

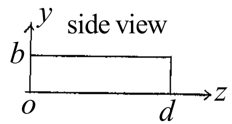
Rewrite (27): $H_z = \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\left[C_+ e^{ik_z z - i\omega t} + C_- e^{-ik_z z - i\omega t}\right]$

b.c. (i): $H_z(z=0)=0 \;\Rightarrow\; C_+ = -C_-$



$\Rightarrow H_z = H_{z0} e^{-i\omega t}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin k_z z$

b.c. (ii): $H_z(z=d)=0$



$\Rightarrow \sin k_z d = 0 \;\Rightarrow\; k_z = \frac{l\pi}{d},\; l = 1, 2, \ldots$ (32)

$\Rightarrow H_z = H_{z0} e^{-i\omega t}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{l\pi z}{d}$, $\left[\begin{matrix} m, n = 0, 1, 2, \ldots \\ l = 1, 2, \ldots \end{matrix}\right]$ (33)

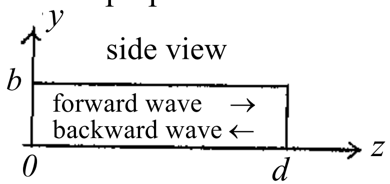
Sub. (32) into $\omega^2 - k_z^2 c^2 - \omega_{cmn}^2 = 0$, where $\omega_{cmn} = \pi c(\frac{m^2}{a^2} + \frac{n^2}{b^2})^{\frac{1}{2}}$

$\Rightarrow \omega = \omega_{mnl} = \pi c(\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{l^2}{d^2})^{1/2}$ $\left[\begin{matrix} \omega_{mnl}\text{ : resonant frequency} \\ \text{of the TE}_{mnl}\text{ mode} \end{matrix}\right]$ (34)

$\boxed{C_+ = -C_-}$

From (26): $H_z = \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\left[C_+ e^{ik_z z - i\omega t} + C_- e^{-ik_z z - i\omega t}\right]$,

we see that a cavity mode is formed of a forward wave and a backward wave of equal amplitude. The forward wave is reflected at the right end to become a backward wave, and turns into a forward wave again at the the left end. The forward and backward waves superpose into a standing wave [see (33)]. Thus, we may obtain the other components of the cavity field by superposing the other components of the two traveling waves, as in (26).



Comparison with vibrational modes of a string:

	dependent variable(s)	independent variables	mode index
string	x (oscillation amp.)	z, t	l
cavity	E_x, E_y, B_x, B_y, E_z (or B_z)	x, y, z, t	m, n, l

8.5 Energy Flow and Attenuation in Waveguides

Power in a Lossless Waveguide: Consider a TM mode ($\mathbf{E} = \mathbf{E}_t + E_z\mathbf{e}_z$, $\mathbf{H} = \mathbf{H}_t$) in a medium with real ε, μ (hence real ω, k_z).

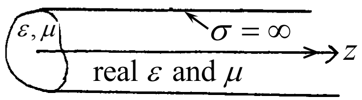
$\mathbf{S}_{\text{TM}} = \frac{1}{2}\mathbf{E}\times\mathbf{H}^* = \frac{1}{2}[\mathbf{E}_t\times\mathbf{H}_t^* + E_z\mathbf{e}_z\times\mathbf{H}_t^*]$ [complex Poynting vector]

$= \frac{1}{2}\frac{\varepsilon\omega}{k_z}[\underbrace{\mathbf{E}_t\times(\mathbf{e}_z\times\mathbf{E}_t^*)}_{\mathbf{e}_z|\mathbf{E}_t|^2} + E_z\underbrace{\mathbf{e}_z\times(\mathbf{e}_z\times\mathbf{E}_t^*)}_{-\mathbf{E}_t^*}]$ [for TM modes]

$\boxed{(21b)}$

$= \frac{\varepsilon\omega}{2k_z}[\mathbf{e}_z\frac{k_z^2}{\gamma^4}|\nabla_t E_z|^2 + \frac{ik_z}{\gamma^2}E_z\nabla_t E_z^*]$

$\boxed{(21a)}$



$= \frac{\omega k_z\varepsilon}{2\gamma^4}[\mathbf{e}_z|\nabla_t E_z|^2 + \frac{i\gamma^2}{k_z}E_z\nabla_t E_z^*]$

P_{TM} = time averaged power in the z-direction

$= \int_A \mathbf{e}_z\cdot[\text{Re}\,\mathbf{S}_{\text{TM}}]da$ [A: crossectional area]

$= \frac{\omega k_z\varepsilon}{2\gamma^4}\int_A(\nabla_t E_z^*\cdot\nabla_t E_z)da$ (35)

Green's first identity: $\int_V(\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)d^3x = \oint_S\phi\frac{\partial\psi}{\partial n}da$ (1.34)

Let ϕ and ψ be independent of z and apply (1.34) to a slab of end surface area A (on the x-y plane) and infinistesimal thickness Δz in z,

$\Delta z\int_A(\phi\nabla_t^2\psi + \nabla_t\phi\cdot\nabla_t\psi)da = \Delta z\oint_C\phi\frac{\partial\psi}{\partial n}dl + \left[\begin{matrix}\text{surface integrals on} \\ \text{two ends of the} \\ \text{slab, which vanish.}\end{matrix}\right]$

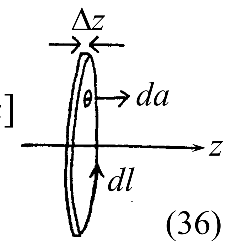
$\Rightarrow \int_A(\phi\nabla_t^2\psi + \nabla_t\phi\cdot\nabla_t\psi)da = \oint_C\phi\frac{\partial\psi}{\partial n}dl$

Let $\phi = E_z^*$ and $\psi = E_z$, then

$\int_A(\nabla_t E_z^*\cdot\nabla_t E_z)da = [\oint_C\underbrace{E_z^*}_{=0}\frac{\partial}{\partial n}E_z dl - \int_A E_z^*\underbrace{\nabla_t^2 E_z}_{-\gamma^2 E_z}da]$

by boundary condition   by (14)

$= \gamma^2\int_A|E_z|^2 da.$ (36)



Sub. (36) into (35): $P_{\text{TM}} = \frac{\omega k_z\varepsilon}{2\gamma^4}\int_A(\nabla_t E_z^*\cdot\nabla_t E_z)da$, we obtain

$P_{\text{TM}} = \frac{\omega k_z\varepsilon}{2\gamma^2}\int_A|E_z|^2 da$, [where $\gamma^2 = \mu\varepsilon\omega^2 - k_z^2$] (37)

$$\gamma^2 = \mu\varepsilon\omega^2 - k_z^2 \;\Rightarrow \omega_c = \frac{\gamma}{\sqrt{\mu\varepsilon}} \quad \left[\begin{array}{l}\omega_c\,(\text{i.e. } \omega \text{ at } k_z = 0) \text{ is the} \\ \text{cutoff freq. of the mode.}\end{array}\right] \tag{38}$$

$$\Rightarrow\; k_z = (\mu\varepsilon\omega^2 - \gamma^2)^{\frac{1}{2}} = \sqrt{\mu\varepsilon}\,\omega\Big(1 - \frac{\omega_c^2}{\omega^2}\Big)^{\frac{1}{2}} \tag{39}$$

Sub. (38) and (39) into (37)

$$P_{\text{TM}} = \tfrac{1}{2}\sqrt{\tfrac{\varepsilon}{\mu}}\Big(\frac{\omega}{\omega_c}\Big)^2\Big(1 - \frac{\omega_c^2}{\omega^2}\Big)^{\frac{1}{2}}\int_A |E_z|^2\, da \quad [\text{cf. } (8.51)] \tag{40}$$

Similarly, for the TE mode and real μ, ε, ω, and k_z, we obtain from (22), (22a), and (22b),

$$\mathbf{S}_{\text{TE}} = \frac{\omega k_z \mu}{2\gamma^4}\Big[\mathbf{e}_z |\nabla_t H_z|^2 - \frac{i\gamma^2}{k_z} H_z^* \nabla_t H_z\Big] \tag{41}$$

$$P_{\text{TE}} = \int_A \mathbf{e}_z \cdot [\text{Re}\,\mathbf{S}_{\text{TE}}]\, da = \frac{\omega k_z \mu}{2\gamma^2}\int_A |H_z|^2\, da$$

$$= \tfrac{1}{2}\sqrt{\tfrac{\mu}{\varepsilon}}\Big(\frac{\omega}{\omega_c}\Big)^2\Big(1 - \frac{\omega_c^2}{\omega^2}\Big)^{\frac{1}{2}}\int_A |H_z|^2\, da \quad [\text{cf. } (8.51)] \tag{42}$$

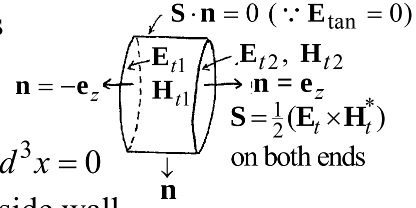
Note: P_{TM} and P_{TE} are expressed in terms of the generating function.

Energy in a Lossless Waveguide:

$$\oint_S \mathbf{S}\cdot\mathbf{n}\,da + \tfrac{1}{2}\int_V \mathbf{J}^*\cdot\mathbf{E}\,d^3x + 2i\omega\int_V (w_e - w_m)\, d^3x = 0 \tag{6.134}$$

$$\begin{cases} w_e = \frac{1}{4}\mathbf{E}\cdot\mathbf{D}^* = \frac{1}{4}\varepsilon|E|^2 \\ w_m = \frac{1}{4}\mathbf{B}\cdot\mathbf{H}^* = \frac{1}{4\mu}|B|^2 \end{cases} \quad \left[\begin{array}{l}\text{if } \varepsilon,\ \mu \text{ are real, } w_e \text{ and } w_m \text{ are} \\ \text{also real and represent time} \\ \text{averaged field energy densities.}\end{array}\right] \tag{6.133}$$

Apply (6.134) to a section of a lossless waveguide [i.e. μ, ε are real and the wall conductivity $\sigma = \infty$].



$\sigma = 0$ (inside volume) $\Rightarrow \mathbf{J} = 0 \Rightarrow \int_V \mathbf{J}^*\cdot\mathbf{E}\,d^3x = 0$

$\mathbf{E}_t = 0$ on the side wall $\Rightarrow \mathbf{S}\cdot\mathbf{n} = 0$ on the side wall

μ, ε (hence ω, k_z) are real $\Rightarrow \mathbf{E}_t$ and $\mathbf{H}_t$ are in phase [by (21b)&(22b)]

$\Rightarrow \mathbf{E}_t \times \mathbf{H}_t^*$ is real $\Rightarrow \mathbf{S}$ is real on both ends $\Rightarrow \oint_S \mathbf{S}\cdot\mathbf{n}\,da$ is real

$$\begin{cases} \text{Re}[(6.134)] \Rightarrow \oint_S \mathbf{S}\cdot\mathbf{n}\,da = 0 \;\; \text{(no net power into or out of volume)} \\ \text{Im}[(6.134)] \Rightarrow \int_V w_e\, d^3x = \int_V w_m\, d^3x \;\; \text{(B-field energy = E-field energy)} \end{cases}$$

For the TM mode ($H_z = 0$):

U_{TM} = field energy per unit length

$$= \int_A (w_e + w_m)\, da = 2\int_A w_m\, da = \frac{\mu}{2}\int_A |\mathbf{H}_t|^2\, da \overset{\boxed{(21b)}}{=} \frac{\mu}{2}\frac{\varepsilon^2\omega^2}{k_z^2}\int_A |\mathbf{E}_t|^2\, da$$

$$\underset{\boxed{(21a)}}{=} \frac{\mu}{2}\frac{\varepsilon^2\omega^2}{k_z^2}\frac{k_z^2}{\gamma^4}\underbrace{\int_A |\nabla_t E_z|^2\, da}_{\substack{\gamma^2\int_A |E_z|^2 da \\ \text{by (36)}}} = \frac{\mu\varepsilon^2\omega^2}{2\gamma^2}\int_A |E_z|^2\, da \underset{\boxed{\gamma^2 = \mu\varepsilon\omega_c^2}}{=} \frac{\varepsilon}{2}\Big(\frac{\omega}{\omega_c}\Big)^2\int_A |E_z|^2\, da \tag{43}$$

Similarly, for the TE mode ($E_z = 0$):

$$U_{\text{TE}} = 2\int_A W_e\, da = \frac{\varepsilon}{2}\int_A |\mathbf{E}_t|^2\, da \underset{\boxed{\text{Use (22a,b) and Green's 1st identity}}}{=} \frac{\mu}{2}\Big(\frac{\omega}{\omega_c}\Big)^2\int_A |H_z|^2\, da \tag{44}$$

From (40), (42), (43), and (44)

$$\frac{P_{\text{TM}}}{U_{\text{TM}}} = \frac{P_{\text{TE}}}{U_{\text{TE}}} = \frac{1}{\sqrt{\mu\varepsilon}}\Big(1 - \frac{\omega_c^2}{\omega^2}\Big)^{\frac{1}{2}} \underset{\boxed{(39)}}{=} \frac{k_z}{\mu\varepsilon\omega} \underset{\boxed{\frac{d}{dk_z}(21.c)}}{=} v_g \tag{8.53}$$

$v_p = \omega / k_z$

$$\Rightarrow v_p v_g = 1/\mu\varepsilon \tag{8.54}$$

Attenuation in Waveguides Due to Ohmic Loss on the Wall:

We express k_z for a lossless ($\sigma = \infty$) and lossy ($\sigma \neq \infty$) waveguide

as
$$k_z = \begin{cases} k_z^{(0)}, & \sigma = \infty \\ k_z^{(0)} + \alpha + i\beta, & \sigma \neq \infty \end{cases} \tag{8.55}$$

where $k_z^{(0)}$ is the solution of the dispersion relation for $\sigma = \infty$, i.e.

$$\mu\varepsilon\omega^2 - k_z^2 - \mu\varepsilon\omega_c^2 = 0 \quad [\text{derived in } (28)] \tag{45}$$

The expression for $\sigma \neq \infty$ in (8.55) assumes that the wall loss modifies $k_z^{(0)}$ by a small real part α and a small imaginary part β, where α and β are to be dertermined.

Physical reason for α: Effective waveguide radius increases by by an amount ~ skin depth δ. A larger waveguide has a smaller ω_c. Hence, $\alpha > 0$.

Physical reason for β: Power dissipation on the wall.

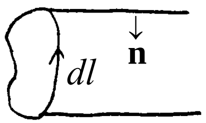
In $k_z = k_z^{(0)} + \alpha + i\beta$, α is not of primary interest because it modifies the guide wavelength slightly. However, β results in attentuation, which can be very significant over a long distance. We outline below how β can be evaluated.

$$P = \text{power flow} \left(\propto \text{Re}[\mathbf{E}_t \times \mathbf{H}_t^*] \propto e^{ik_z z} \cdot e^{-ik_z^* z} = e^{-2k_{zi}z} = e^{-2\beta z} \right)$$

$$= P_0 e^{-2\beta z} \quad \boxed{\text{power dissipation/unit length}} \tag{8.56}$$

$$\Rightarrow \quad \beta = -\frac{1}{2P}\frac{dP}{dz} = \text{field } \underline{\text{attenuation constant}} \tag{8.57}$$

$$(8.15) \Rightarrow \frac{dP}{dz} = -\frac{1}{2\sigma\delta}\oint_c \left|\mathbf{K}_{eff}\right|^2 dl \tag{46}$$



$$(8.14) \Rightarrow \mathbf{K}_{eff} = \mathbf{n} \times \mathbf{H} \tag{47}$$

$$(46)(47) \Rightarrow \frac{dP}{dz} = -\frac{1}{2\sigma\delta}\oint_c \left|\mathbf{n} \times \mathbf{H}\right|^2 dl \tag{8.58}$$

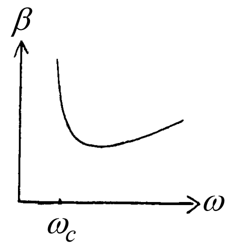
Since the wall loss can be regarded as a small perturbation, we may use the zero-order $\mathbf{H}$ derived for $\sigma = \infty$ in Sec.8.1 to calculate $\frac{dP}{dz}$.

Specifically, we calculate the zero-order **E** and **H,** and use the zero-order **E** and **H** to calculate P from (8.51) and dP/dz from (8.58). β is then found from (8.57).

Formulae for β for rectangular and cylindrical waveguides are tabulated in many microwave textbooks, e.g. R. E. Collin, "Foundation of Microwave Engineering" (2nd Ed.) p. 189 & p.197 (where the attenuation constant is denoted by α instead of β).

Note:

(i) β has been calculated by a perturbation method. The method is invalid near the cutoff frequency, at which there is a large "perturbation". Sec 8.6 gives a method which calculates both α and β (due to wall loss) valid for all frequencies.



(ii) Other types of losses (e.g. lossy filling medium or complex ε) can also contribute to α and β.

(iii) Note there are two definitions of the attenuation constant.

In Ch. 8 of Jackson, the attentuation constant for the waveguide is denoted by β and it is defined as

$$\beta = -\frac{1}{2P}\frac{dP}{dz}, \tag{8.57}$$

This is the *field* attentuation constant, i.e.

$$\mathbf{E}, \mathbf{B} \propto e^{-\beta z}.$$

In Ch. 7 of Jackson, the attentuation constant for a uniform medium is denoted by α [see (7.53)] and it is defined as

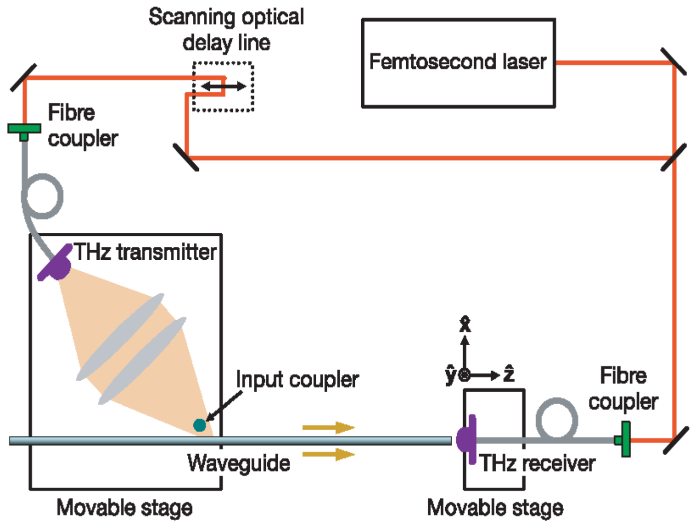
$$\alpha = -\frac{1}{P}\frac{dP}{dz}$$

This is the *power* attentuation constant, i.e.

$$P \propto e^{-\alpha z}$$

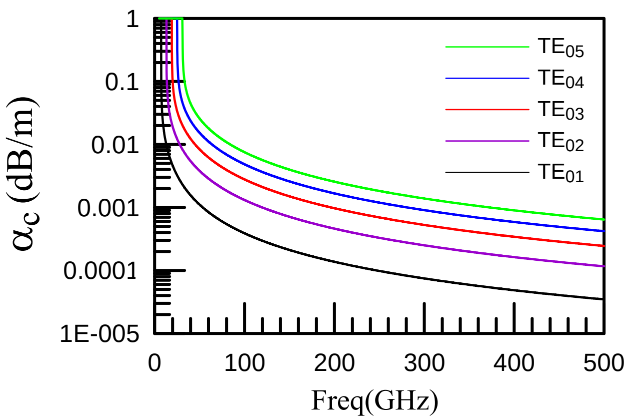
Obviously, the power attentuation constant is twice the value of the field attentuation constant.

Terahertz Waveguide (I)



K. Wang and D. M. Mittleman, "Metal wires for terahertz wave guiding", Nature, vol.432, No. 18, p.376, 2004.

Terahertz Waveguide (II): Using The Lowest Lossy TE01 Mode



References
1. Pozar, p.161.
2. Collin, p.197.

Q: How to excite the TE01 mode and fabricate it at the terahertz region?
A possible solution: X-ray micro-fabrication (LIGA).

8.8 Cavity Power Loss and *Q*

Definition of Q : We have so far assumed a real ω for EM waves in infinite space or a waveguide. Since fields are *stored* in a cavity, it damps in time if there are losses, represented by a complex ω. Thus, fields at any point in the cavity have the time dependence given by

$$E(t)=\begin{cases} E_0 e^{-i\omega_0 t}, & \sigma=\infty \\ E_0 e^{-i(\omega_0+\Delta\omega+i\frac{\omega_0}{2Q})t}=E_0 e^{-i(\omega_0+\Delta\omega)t-\frac{\omega_0}{2Q}t}, & \sigma\neq\infty \end{cases} \qquad (8.88)$$

where ω_0 is the resonant frequency [e.g. (34)] without the wall loss.

(8.88) assumes that the wall loss modifies ω_0 by a small real part $\Delta\omega$ and a small imaginary part $\frac{\omega_0}{2Q}$, where $\Delta\omega$ and Q are to be dertermined.

Physical reason for $\Delta\omega$: Effective cavity size increases by an amount ~ skin depth δ. A larger cavity has a lower frequency. Hence, $\Delta\omega < 0$.

Physical reason for Q : power dissipation on the wall

U = stored energy in the cavity $[\propto |E|^2 \propto e^{-i\omega t}\cdot e^{i\omega^* t}=e^{2\omega_i t}=e^{-\frac{\omega_0 t}{Q}}]$

$$=U_0 e^{-\frac{\omega_0 t}{Q}}$$

$$\boxed{\begin{aligned}&\mathbf{E}(t)=E_0 e^{-i(\omega_0+\Delta\omega)t-\frac{\omega_0}{2Q}t}\\ &\Rightarrow \omega_i=-\frac{\omega_0}{2Q}\end{aligned}}$$

$$\Rightarrow \frac{dU}{dt}=-\frac{\omega_0}{Q}U \text{ (power loss)} \qquad (8.87)$$

$$\Rightarrow Q=\omega_0\frac{\text{stored energy}}{\text{power loss}} \quad (\underline{\text{time-space definition of } Q}) \qquad (8.86)$$

(8.88) represents a damped oscillation which does not have a single frequency. To exame the frequency of $E(t)$. we write

$$E(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}E(\omega)e^{-i\omega t}d\omega,$$

where

Use (8.88), assume $E(t) = 0$ for $t < 0$

$$E(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}E(t)e^{i\omega t}dt=\frac{1}{\sqrt{2\pi}}E_0\int_0^{\infty}e^{-\frac{\omega_0}{2Q}t+i(\omega-\omega_0-\Delta\omega)t}dt$$

$$=\frac{1}{\sqrt{2\pi}}\frac{E_0}{-i(\omega-\omega_0-\Delta\omega)+\frac{\omega_0}{2Q}}$$

The frequency spectrum is best seen form the field energy distribution in ω-space

$$|E(\omega)|^2\propto\frac{1}{(\omega-\omega_0-\Delta\omega)^2+\left(\frac{\omega_0}{2Q}\right)^2}=\begin{cases}\max, & \omega=\omega_0+\Delta\omega\\ \frac{1}{2}\max, & \omega=\omega_0+\Delta\omega\pm\frac{\omega_0}{2Q}\end{cases} \qquad (8.90)$$

$$\Rightarrow \delta\omega=\begin{bmatrix}\text{full width at}\\ \text{half-maximum points}\end{bmatrix}=\frac{\omega_0}{Q}$$

$$\Rightarrow Q=\frac{\omega_0}{\delta\omega} \quad (\underline{\text{frequency-space definition of } Q}) \qquad (8.91)$$

Note: ω_0 is the resonant frequency of the cavity in the absence of any loss. $\omega_0+\Delta\omega$ is the resonant frequency in the presence of losses. In most cases, the difference is insignificant.

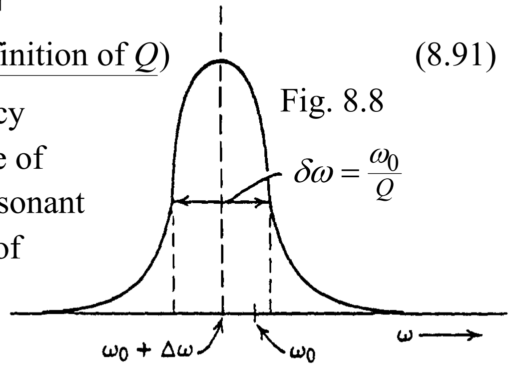


Fig. 8.8

Physical Interpretation of Q :

(i) Use the time-space definition: $Q = \omega_0 \dfrac{\text{stored energy}}{\text{power loss}}$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{\tau_0}$$

wave period

$$\frac{\text{stored energy}}{\text{power loss}} \approx \tau_d$$

decay time of stored energy

$$\Rightarrow Q = \omega_0 \frac{\text{stored energy}}{\text{power loss}} \approx 2\pi \frac{\tau_d}{\tau_0} \quad (48)$$

(48) shows that Q, which results from the power loss, is approximately 2π times the number of oscillations during the decay time. A larger Q value implies that the field energy can be stored in the cavity for a longer time. Hence, Q is often referred to as the quality factor.

(ii) Use the frequency-space definition: $Q = \dfrac{\omega_0}{\delta\omega}$ (see Fig. 8.8)

For a lossy cavity, a resonant mode can be excited not just at one frequency (as is the case with a lossless cavity) but at a range of frequencies ($\delta\omega$). The resonant frequency ($\omega_0+\Delta\omega$, see Fig. 8.8) of a lossy cavity is the frequency at which the cavity can be excited with the largest inside-field amplitude, given the same source power. The resonant width $\delta\omega$ of a mode is equal to the resonant frequency divided by the Q value of that mode (see Fig. 8.8). Note that each mode has a different Q value.

Figure 8.8 can be easily generated in experiment to measure the Q value.

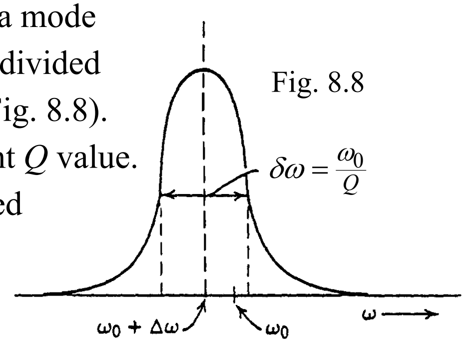


Fig. 8.8

$$Q = \omega_0 \frac{\text{stored energy}}{\text{power loss}}$$

Using the results of Sec. 8.1, we can calculate Q (but not $\Delta\omega$) due to the ohmic loss. We first calculate the zero order **E** and **H** of a specific cavity assuming $\sigma = \infty$, then use the zero order **E** and **H** to calculate U and power loss,

$$\text{stored energy} = \int_V (w_e + w_m) d^3x = \begin{cases} 2\int_V w_e d^3x = \frac{\varepsilon}{2}\int_V |\mathbf{E}|^2 d^3x \\ 2\int_V w_m d^3x = \frac{\mu}{2}\int_V |\mathbf{H}|^2 d^3x \end{cases}$$

(6.133)

(8.15)

$$\text{power loss} = \frac{1}{2\sigma\delta}\oint_S |\mathbf{K}_{eff}|^2 da$$

$$= \frac{1}{2\sigma\delta}\oint_S |\mathbf{n}\times\mathbf{H}|^2 da$$

(8.14)

Formulae for Q (due to ohmic loss) for rectangular and cylindrical cavities can be found in, for example, R. E. Collin, "Foundation of Microwave Engineering", p. 503 and p. 506.

Q due to other types of losses : If there are several types of power losses in a cavity (e.g. due to Imε and coupling losses), Q can be expressed as follows:

$$Q = \omega_0 \frac{\text{stored energy}}{\sum_n (\text{power loss})_n} \quad (49)$$

n-th type of power loss

$$\Rightarrow \frac{1}{Q} = \sum_n \frac{1}{Q_n} \quad (50)$$

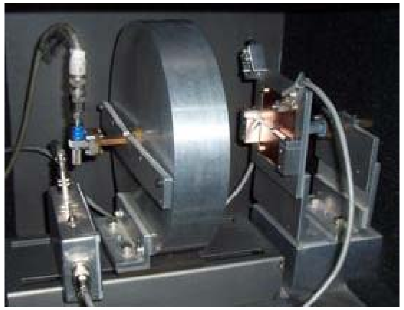
where Q_n (Q due to the n-th type of power loss) is given by

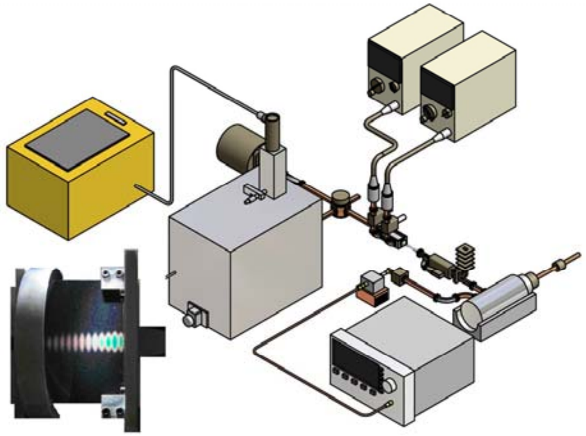
$$Q_n = \omega_0 \frac{\text{stored energy}}{(\text{power loss})_n}$$

A Comparison between Waveguides and Cavities

	Waveguide	Cavity
Function	transport EM energy	store EM energy
Characteri-zation	dispersion relation and attenuation constant	resonant frequency and Q
Examples of applications (mostly for microwaves, 0.3-300 GHz)	transport of high power microwaves (such as multi-kW waves for long-range radars and communi-cations)	(1) particle acceleration (2) frequency measurement

High-Q Microwave/Material Applicator





$$Q = \omega_0 \frac{\text{stored energy}}{\sum_n (\text{power loss})_n}$$

Conductor loss, dielectric loss, radiation loss, diffraction loss…

Homework of Chap. 8

Problems: 2, 3, 4, 5, 6,
18, 19, 20

Chapter 9: Radiating Systems, Multipole Fields and Radiation

An Overview of Chapters on EM Waves : (covered in this course)

	source term in wave equation	boundary
Ch. 7	none	plane wave in ∞ space or in two semi-∞ spaces separated by the x - y plane
Ch. 8	none	conducting walls
Ch. 9	$\mathbf{J}, \rho \sim e^{-i\omega t}$ prescribed, as in an antenna	outgoing wave to ∞
Ch. 10	$\mathbf{J}, \rho \sim e^{-i\omega t}$ induced by incident EM waves, as in the case of scattering of a plane wave by a dielectric object.	outgoing wave to ∞
Ch. 14	moving charges, such as electrons in a synchrotron	outgoing wave to ∞

9.6 Spherical Wave Solutions of the Scalar Wave Equation

Spherical Bessel Functions and Hankel functions : Although this chapter deals with radiating systems, here we first solve the scalar source-free wave equation in the spherical coordinate syatem. The purpose is to obtain a complete set of spherical Bessel funtions and Hankel functions, with which we will expand the fields produced by the sources.

The scalar source-free wave equation is [see (6.32)]

$$\nabla^2\psi(\mathbf{x},t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\mathbf{x},t)=0 \qquad (9.77)$$

Let $\psi(\mathbf{x},t)=\int_{-\infty}^{\infty}\psi(\mathbf{x},\omega)e^{-i\omega t}d\omega \qquad (9.78)$

$\Rightarrow$ Each Fourier component satisfies the Helmholtz wave eq.

$$(\nabla^2+k^2)\psi(\mathbf{x},\omega)=0, \qquad (9.79)$$

where $k\equiv\frac{\omega}{c}$

In spherical coordinates, $(\nabla^2+k^2)\psi=0$ is written

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}+k^2\psi=0$$

Let $\psi=U(r)P(\theta)Q(\varphi)$, we obtain

$$PQ\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dU}{dr}\right)+UQ\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+UP\frac{1}{r^2\sin^2\theta}\frac{d^2Q}{d\varphi^2}+k^2UPQ=0$$

Multiply by $\dfrac{r^2\sin^2\theta}{UPQ}$

The only term with φ-dependence, so this term must be a constant. Let it be $-m^2$.

$$\sin^2\theta[\underbrace{\frac{1}{U}\frac{d}{dr}(r^2\frac{dU}{dr})+k^2r^2}_{=l(l+1)}+\frac{1}{P\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dP}{d\theta})]+\underbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}_{=-m^2}=0$$

Dividing all terms by $\sin^2\theta$, we see that this is the only term with r-dependence. So it must be a constant. Let it be $l(l+1)$.

$$\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\varphi}$$

Thus, as in Sec. 3.1 of lecture notes,

$P=P_\ell^m(\cos\theta),\ Q_\ell^m(\cos\theta);\ Q=e^{im\varphi},e^{-im\varphi}\Rightarrow\ PQ=Y_{lm}(\theta,\varphi)$

rejected because of divergence at $\theta=\pm\pi$

$U(r)$ is governed by $\frac{d}{dr}(r^2\frac{dU}{dr})+k^2r^2U=l(l+1)U$. Rewrite U as $f_l(r)$. Then, $\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+k^2-\frac{l(l+1)}{r^2}\right]f_l(r)=0 \qquad (9.81)$

Let $f_l(r)=\frac{1}{r^{1/2}}u_l(r)\Rightarrow\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+k^2-\frac{(l+1/2)^2}{r^2}\right]u_l(r)=0 \qquad (9.83)$

$\Rightarrow\ u_l(r)=J_{l+\frac{1}{2}}(kr),\ N_{l+\frac{1}{2}}(kr)$ [Bessel functions of fractional order]

$\Rightarrow\ f_l(r)=\frac{1}{r^{1/2}}J_{l+\frac{1}{2}}(kr),\ \frac{1}{r^{1/2}}N_{l+\frac{1}{2}}(kr)$

Define $\begin{cases} j_l(kr)=\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}}J_{l+\frac{1}{2}}(kr) \\ n_l(kr)=\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}}N_{l+\frac{1}{2}}(kr)\end{cases}$ and $\begin{cases} h_l^{(1)}(kr)=j_l(kr)+in_l(kr) \\ h_l^{(2)}(kr)=j_l(kr)-in_l(kr)\end{cases}$

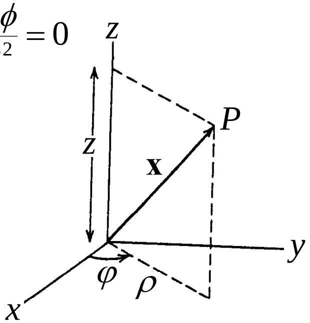
spherical Bessel functions — Hankel functions

$$\Rightarrow\psi(\mathbf{x},\omega)=\sum_{lm}\left[A_{lm}^{(1)}h_l^{(1)}(kr)+A_{lm}^{(2)}h_l^{(2)}(kr)\right]Y_{lm}(\theta,\phi)\quad [k=\tfrac{\omega}{c}] \qquad (9.92)$$

Review

3.7 Laplace Equation in Cylindrical Coordinates; Bessel Functions

$$\nabla^2\phi(\mathbf{x})=0 \Rightarrow \frac{\partial^2\phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\phi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2}+\frac{\partial^2\phi}{\partial z^2}=0$$



Let $\phi(\mathrm{x})=R(\rho)Q(\varphi)Z(z)$

$$\Rightarrow \begin{cases} \dfrac{\partial^2 Z}{\partial z^2}-k^2 Z=0 \Rightarrow Z=e^{\pm kz} \\ \dfrac{\partial^2 Q}{\partial\varphi^2}+\nu^2 Q=0 \Rightarrow Q=e^{\pm i\nu\varphi} \\ \dfrac{\partial^2 R}{\partial\rho^2}+\dfrac{1}{\rho}\dfrac{\partial R}{\partial\rho}+\left(k^2-\dfrac{\nu^2}{\rho^2}\right)R=0 \Rightarrow R=J_\nu(k\rho),\ N_\nu(k\rho) \end{cases}$$

where J_ν and N_ν are <u>Bessel functions</u> of the first and second kind, respectively (see following pages).

$$\Rightarrow \phi=\begin{Bmatrix} J_\nu(k\rho) \\ N_\nu(k\rho) \end{Bmatrix}\begin{Bmatrix} e^{i\nu\varphi} \\ e^{-i\nu\varphi} \end{Bmatrix}\begin{Bmatrix} e^{kz} \\ e^{-kz} \end{Bmatrix} \qquad (3)$$

Review

Bessel Functions: If we let $x=k\rho$, the equation for R takes the standard form of the Bessel equation,

$$\frac{d^2R}{dx^2}+\frac{1}{x}\frac{dR}{dx}+\left(1-\frac{\nu^2}{x^2}\right)R=0 \qquad (3.77)$$

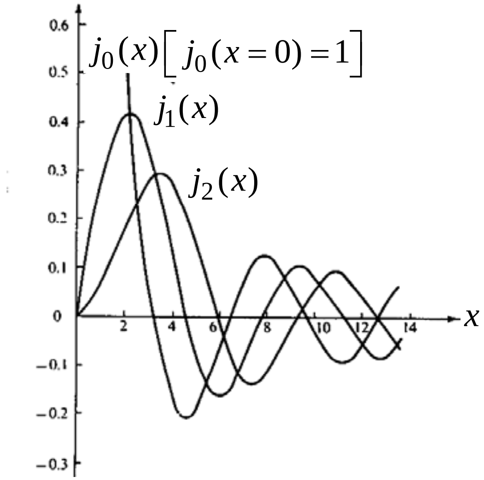
with solutions $J_\nu(x)$ and $N_\nu(x)$, from which we define the <u>Hankel functions</u>:

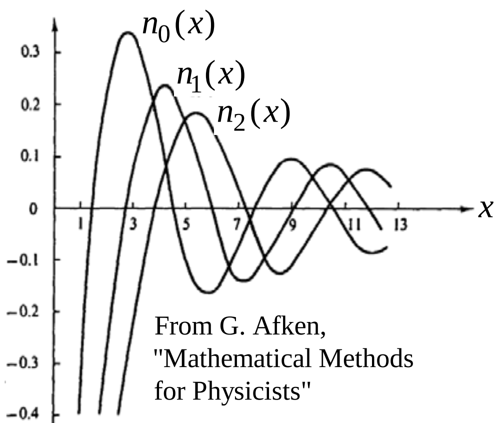
$$\begin{cases} H_\nu^{(1)}(x)=J_\nu(x)+iN_\nu(x) \\ H_\nu^{(2)}(x)=J_\nu(x)-iN_\nu(x) \end{cases} \qquad (3.86)$$

and the <u>modified Bessel functions</u> (Bessel functions of imaginary argument)

$$\begin{cases} I_\nu(x)=i^{-\nu}J_\nu(ix) & (3.100) \\ K_\nu(x)=\frac{\pi}{2}i^{\nu+1}H_\nu^{(1)}(ix) & (3.101) \end{cases}$$

See Jackson pp. 112-116, Gradshteyn & Ryzhik, and Abramowitz & Stegun for properties of these special functions.





From G. Afken, "Mathematical Methods for Physicists"

$$j_l(x)\xrightarrow{x<<1,\ l}\frac{x^l}{(2l+1)!!}\left[1-\frac{x^2}{2(2l+3)}+\cdots\right]$$

$$n_l(x)\xrightarrow{x<<1,\ l}-\frac{(2l-1)!!}{x^{l+1}}\left[1-\frac{x^2}{2(1-2l)}+\cdots\right]$$

$$j_l(x)\xrightarrow{x>>l}\tfrac{1}{x}\sin\left(x-\tfrac{l\pi}{2}\right)$$

$$n_l(x)\xrightarrow{x>>l}-\tfrac{1}{x}\cos\left(x-\tfrac{l\pi}{2}\right)$$

$h_l^{(1)}(x)\xrightarrow{x>>l}(-i)^{l+1}\frac{e^{ix}}{x}$ [$\Rightarrow$ spatial dependence of spherical waves.]

See Jackson pp. 426-427 for further properties of j_l, n_l, $h_l^{(1)}$, and $h_l^{(2)}$.

Expansion of the Green function: Solution of the Green equation

$$(\nabla^2+k^2)G(\mathbf{x},\mathbf{x}')=-4\pi\delta(\mathbf{x}-\mathbf{x}') \qquad (6.36)$$

is given by (derived in Sec. 6.4.)

$$G(\mathbf{x},\mathbf{x}')=\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\ \left[\begin{array}{l}\text{in infinite space and for outgoing-} \\ \text{wave boundary condition.}\end{array}\right] \qquad (6.40)$$

We may solve (6.36) in the same way as in Sec. 3.9, i.e. write

$$G(\mathbf{x},\mathbf{x}')=\sum_{lm} g_l(r,r')Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi),$$

solve for $g_l(r,r')$ for $r>r'$ and $r<r'$ [where $\delta(\mathbf{x}-\mathbf{x}')=0$], and then apply boundary conditions at $r=0$, $r=\infty$, and $r=r'$. The result is

$$G(\mathbf{x},\mathbf{x}')=4\pi ik\sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>)\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

Equating the two expressions above for $G(\mathbf{x},\mathbf{x}')$, we obtain

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}=4\pi ik\sum_{l=0}^{\infty} j_l(kr_<)h_l^{(1)}(kr_>)\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi), \qquad (9.98)$$

where $r_<$ and $r_>$ are, respectively, the smaller and larger of r and r'.

Part I

Summary of Differential Equations and Solutions:

Source-free D.E.	Laplace eq. $\nabla^2\phi = 0$	Helmholtz eq. $\left(\nabla^2 + k^2\right)\psi = 0$
Solutions: Cartesian, cylindrical, spherical	$e^{i\alpha x}$, $e^{i\beta y}$, $e^{\sqrt{\alpha^2+\beta^2}z}$, etc. (Sec. 2.9); $J_m(kr)$, $e^{im\theta}$, e^{kz}, etc. (Sec. 3.7); $Y_{lm}(\theta,\phi)$, r^l, etc. (Secs. 3.1, 3.2)	$e^{ik_x x}$, $e^{ik_y y}$, $e^{ik_z z}$, etc. (Sec. 8.4); $J_m\left(\sqrt{\frac{\omega^2}{c^2}-k_z^2}\; r\right)$, $e^{im\theta}$, $e^{ik_z z}$, etc. (Sec. 8.7); $Y_{lm}(\theta,\phi)$, $j_l(kr)$, $n_l(kr)$, etc. (Sec. 9.6)
D.E. with a point source	$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ b.c.: $G(\infty)=0$	$(\nabla^2+k^2)G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ b.c.: outgoing wave
Solutions (Green functions)	$G = \frac{1}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$	$G = \frac{e^{ik\lvert\mathbf{x}-\mathbf{x}'\rvert}}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$ [Eq. (6.40)]
Series expansin of Green function	Eqs. (3.70), (3.148), (3.168)	Eq. (9.98)

Part II

Summary of Differential Equations and Solutions:

Source-free D.E.	Helmholtz eq. $\left(\nabla^2 + k^2\right)\psi = 0$	Wave Eq. $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = 0$
Solutions: Cartesian, cylindrical, spherical	$e^{ik_x x}$, $e^{ik_y y}$, $e^{ik_z z}$, etc. (Sec. 8.4); $J_m\left(\sqrt{\frac{\omega^2}{c^2}-k_z^2}\; r\right)$, $e^{im\theta}$, $e^{ik_z z}$, etc. (Sec. 8.7); $Y_{lm}(\theta,\phi)$, $j_l(kr)$, $n_l(kr)$, etc. (Sec. 9.6)	$\begin{Bmatrix}\mathbf{A}(\mathbf{x},t)\\ \Phi(\mathbf{x},t)\end{Bmatrix} = \iint d^3x'dt' \frac{\delta\left[t'-\left(t-\frac{\lvert\mathbf{x}-\mathbf{x}'\rvert}{c}\right)\right]}{4\pi\lvert\mathbf{x}-\mathbf{x}'\rvert}\begin{Bmatrix}\mu_0\mathbf{J}(\mathbf{x}',t')\\ \rho(\mathbf{x}',t')/\varepsilon_0\end{Bmatrix}$
D.E. with a point source	$(\nabla^2+k^2)G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ b.c.: outgoing wave	$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})G^+(\mathbf{x},t,\mathbf{x}',t') = -4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$ b.c.: outgoing wave
Solutions (Green functions)	$G = \frac{e^{ik\lvert\mathbf{x}-\mathbf{x}'\rvert}}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$ [Eq. (6.40)]	$G^+(\mathbf{x},t,\mathbf{x}',t') = \frac{\delta\left[t'-\left(t-\frac{\lvert\mathbf{x}-\mathbf{x}'\rvert}{c}\right)\right]}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$ [Eq. (6.44)]
Series expansin of Green function	Eq. (9.98)	

9.1 Radiation of a Localized Oscillating Source

Review of Inhomogeneous Wave Equations and Solutions:

$$\begin{cases}\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi = -\rho/\varepsilon_0 & (6.15)\\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\mu_0\mathbf{J} & (6.16)\end{cases}$$

[in free space, Φ and $\mathbf{A}$ satisfy Lorenz gauge.]

Basic structure of the inhomogenous wave equation:

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = -4\pi f(\mathbf{x},t) \qquad (6.32)$$

Solution of (6.32) with outgoing-wave b.c.:

$$\psi(\mathbf{x},t) = \psi_{in}(\mathbf{x},t) + \int d^3x' \int dt' G^+(\mathbf{x},t,\mathbf{x}',t') f(\mathbf{x}',t') \qquad (6.45)$$

homogeneous solution

$f(\mathbf{x}',t')$ in (6.45) is evaluated at the retarded time.

where $G^+(\mathbf{x},t,\mathbf{x}',t') = \frac{\delta\left[t'-\left(t-\frac{\lvert\mathbf{x}-\mathbf{x}'\rvert}{c}\right)\right]}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$ (6.44)

is the solution of

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})G^+(\mathbf{x},t,\mathbf{x}',t') = -4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \qquad (6.41)$$

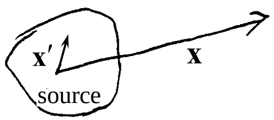
with outgoing wave b.c.

Using (6.45) (assume $\psi_{in}=0$) on (6.15) & (6.16), we obtain the gereral solutions for $\mathbf{A}$ and Φ, which are valid for arbitrary $\mathbf{J}$ and ρ.

$$\begin{Bmatrix}\mathbf{A}(\mathbf{x},t)\\ \Phi(\mathbf{x},t)\end{Bmatrix} = \frac{1}{4\pi}\int d^3x' \int dt' \frac{\delta\left[t'-\left(t-\frac{\lvert\mathbf{x}-\mathbf{x}'\rvert}{c}\right)\right]}{\lvert\mathbf{x}-\mathbf{x}'\rvert}\begin{Bmatrix}\mu_0\mathbf{J}(\mathbf{x}',t')\\ \rho(\mathbf{x}',t')/\varepsilon_0\end{Bmatrix} \qquad (6.48), (9.2)$$

In general, the sources, $\mathbf{J}(\mathbf{x}',t')$ and $\rho(\mathbf{x}',t')$, contain a static part and a time dependent part. For static $\mathbf{J}(\mathbf{x})$ and $\rho(\mathbf{x})$, (9.2) gives the static $\mathbf{A}$ and Φ in Ch. 5 and Ch. 1, respecticely.

$$\mathbf{A}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{x}')}{\lvert\mathbf{x}-\mathbf{x}'\rvert} \qquad (5.32)$$



$$\Phi(\mathbf{x}) = \Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho(\mathbf{x}')}{\lvert\mathbf{x}-\mathbf{x}'\rvert} \qquad (1.17)$$

Question: It is stated on p. 408 that (9.2) is valid provided no boundary surfaces are present. Why? [See discussion below (6.47) in Ch. 6 of lectures notes.]

Fields by Harmonic Sources : Only time-dependent sources can radiate. Radiation from moving charges are treated in Ch. 13 and Ch. 14. Here, specialize to sources of the form (as in an antenna):

$$\begin{aligned}\rho(\mathbf{x},t) &= \rho(\mathbf{x})e^{-i\omega t}\\ \mathbf{J}(\mathbf{x},t) &= \mathbf{J}(\mathbf{x})e^{-i\omega t}\end{aligned} \qquad (9.1)$$

Sub. (9.1) into (9.2) and carry out the t'-integration, we obtain

$$\mathbf{A}(\mathbf{x},t) = \mathbf{A}(\mathbf{x})e^{-i\omega t} \text{ with } \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}'), \qquad (9.3)$$

where $k \equiv \frac{\omega}{c}$.

We shall assume that $\mathbf{J}(\mathbf{x})$ is independent of $\mathbf{A}(\mathbf{x})$, i.e. the source will not be affected by the fields they radiate. Otherwise, (9.3) is an integral equation for $\mathbf{A}(\mathbf{x})$.

A simpler derivation of (9.3): We specialize to harmonic sources from the outset. Then, only (6.16) is required.

$$\nabla^2\mathbf{A}(\mathbf{x},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}(\mathbf{x},t) = -\mu_0\mathbf{J}(\mathbf{x},t) \qquad (6.16)$$

Let $\mathbf{J}(\mathbf{x},t) = \mathbf{J}(\mathbf{x})e^{-i\omega t}$ and $\mathbf{A}(\mathbf{x},t) = \mathbf{A}(\mathbf{x})e^{-i\omega t}$

$\Rightarrow \left(\nabla^2 + k^2\right)\mathbf{A}(\mathbf{x}) = -\mu_0\mathbf{J}(\mathbf{x})$ [inhomogeneous Helmholtz wave eq.]

The Green equation for the above equation is

$$\left(\nabla^2 + k^2\right)G_k(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \qquad (6.36)$$

Solution of (6.36) with outgoing wave b.c.

$$G_k(\mathbf{x},\mathbf{x}') = \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \qquad (6.40)$$

$$\Rightarrow \mathbf{A}(\mathbf{x}) = \int d^3x' G_k(\mathbf{x},\mathbf{x}')\frac{\mu_0}{4\pi}\mathbf{J}(\mathbf{x}') = \frac{\mu_0}{4\pi}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}'),$$

which is (9.3).

Rewrite (9.3), $\quad \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}'), \qquad (9.3)$

Maxwell eqs. give

$$\begin{cases}\mathbf{H} = \frac{1}{\mu_0}\nabla\times\mathbf{A} & \text{(everywhere)} \qquad (9.4)\\ \mathbf{E} = \frac{iZ_0}{k}\nabla\times\mathbf{H} & \text{(outside the source)} \qquad (9.5)\end{cases}$$

where $Z_0 = \sqrt{\mu_0/\varepsilon_0} = 377\ \Omega$ (impedance of free space, p. 297).

Thus, given the source function $\mathbf{J}(\mathbf{x})$, we may in principle evaluate $\mathbf{A}(\mathbf{x})$ from (9.3) and then obtain the fields $\mathbf{H}$ and $\mathbf{E}$ from (9.4) and (9.5).

Note that $e^{-i\omega t}$ dependence has been assumed for $\mathbf{J}$, hence for all other quantities which are expressed in terms of $\mathbf{J}$.

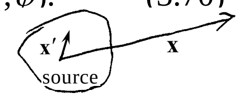
Note: The charge distribution ρ and scalar potential Φ are not required for the determination of $\mathbf{H}$ and $\mathbf{E}$? (why?)

Near-Field Expansion of $\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}') \qquad (9.3)$

Before going into algebraic details, we may readily observe some general properties of $\mathbf{A}(\mathbf{x})$ near the source ($r \ll \lambda$).

For $\mathbf{x}$ outside the source and $r \ll \lambda$ (or $kr \ll 1$), we let $e^{ik|\mathbf{x}-\mathbf{x}'|} \approx 1$

and use $\dfrac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi). \qquad (3.70)$



Since $r > r'$, we have $r_> = r$ and $r_< = r'$.

$$\Rightarrow \mathbf{A}(\mathbf{x}) \underset{\boxed{kr \ll 1}}{\approx} \mu_0\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{1}{r^{l+1}}Y_{lm}(\theta,\varphi)\int d^3x'\mathbf{J}(\mathbf{x}')r'^l Y_{lm}^*(\theta',\varphi') \qquad (9.6)$$

The integral in (9.6) yields multipole coefficients as in (4.2). Thus, (9.6) shows that, for $kr \ll 1$, $\mathbf{A}(\mathbf{x})$ can be decomposed into multipole fields, which fall off as $r^{-(l+1)}$ just as the static multipole fields, but with the $e^{-i\omega t}$ dependence. However, we will show later that, far from the source ($kr \gg 1$), $\mathbf{A}(\mathbf{x})$ behaves as an outgoing spherical wave.

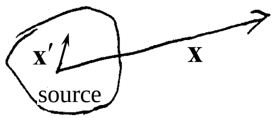
Full Expansion of $\mathbf{A}(\mathbf{x})$: We may in fact expand $\mathbf{A}(\mathbf{x})$, without approximations, by using (9.98). For $\mathbf{x}$ outside the source, we have $r_> = |\mathbf{x}| = r$, $r_< = |\mathbf{x}'| = r'$. Hence, (9.98) can be written

$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} = 4\pi ik \sum_{l=0}^{\infty} j_l(kr')h_l^{(1)}(kr) \sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

Sub. this equation into $\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}')$, we obtain

$$\mathbf{A}(\mathbf{x}) = \mu_0 ik \sum_{l,m} h_l^{(1)}(kr)Y_{lm}(\theta,\phi)\int d^3x'\mathbf{J}(\mathbf{x}')\, j_l(kr')Y_{lm}^*(\theta',\phi'), \quad (9.11)$$

where $h_l^{(1)}(kr) = \frac{e^{ikr}(2l-1)!!}{i(kr)^{l+1}} \sum_{n=0}^{l} a_n(ikr)^n$

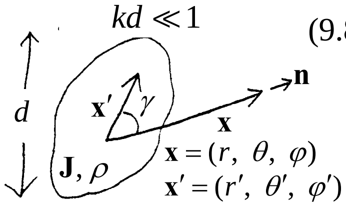


with $a_n = \frac{(-1)^n(2l-n)!}{(2l-1)!!(2l-2n)!!n!}$ $(a_0 = 1,\ a_1 = -1, \cdots)$

(See Abramowitz & Stegun, "Handbook of Mathematical Functions," p. 439.)

(9.11) is an exact expression for $\mathbf{A}(\mathbf{x})$. We now assume $kd \ll 1$ (i.e. source dimension $\ll$ wavelength). Then, $kr' \ll 1$ and $j_l(kr')$ reduces to

$$j_l(kr')\big|_{kr'\ll 1} = \frac{(kr')^l}{(2l+1)!!} \quad (9.88)$$



Sub. $h_l^{(1)}(kr) = \frac{e^{ikr}(2l-1)!!}{i(kr)^{l+1}} \sum_{n=0}^{l} a_n(ikr)^n$

and (9.88) into (9.11), we obtain

$$\mathbf{A}(\mathbf{x}) = \mu_0 \sum_{l,m} \left\{ \begin{array}{l} \frac{1}{2l+1} Y_{lm}(\theta,\phi)\frac{e^{ikr}}{r^{l+1}}[1 + a_1(ikr) + a_2(ikr)^2 + \cdots + a_l(ikr)^l] \\ \cdot\int d^3x'\mathbf{J}(\mathbf{x}')r'^l Y_{lm}^*(\theta',\phi') \end{array} \right\} \quad (1)$$

(1) is the combination of (9.6) and (9.12) in Jackson. It is valid for $kd \ll 1$ and any $\mathbf{x}$ outside the source. The region outside the source is commonly divided into 3 zones (by their different physical characters):

The near (static) zone: $d \ll r \ll \lambda \quad (\Rightarrow kr \ll 1)$

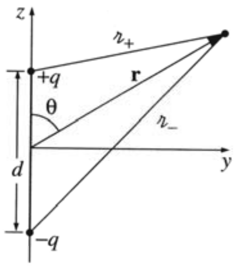
The intermediate (induction) zone: $d \ll r \sim \lambda \quad (\Rightarrow kr \sim 1)$

The far (radiation) zone: $d \ll \lambda \ll r \quad (\Rightarrow kr \gg 1)$

Griffiths

11.1.2 Electric Dipole Radiation

Consider two point charges of $+q$ and $-q$ separating by a distance $d(t)$. Assume $d(t)$ can be expressed in sinusoidal form.



The result is an oscillating electric dipole:

$$\mathbf{p}(t) = qd(t)\hat{\mathbf{z}} = qd\cos(\omega t)\hat{\mathbf{z}} = p_0\cos(\omega t)\hat{\mathbf{z}}, \quad \text{where } p_0 \equiv qd.$$

The retarded potential is:

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\left\{ \frac{q_0\cos[\omega(t-\mathscr{r}_+/c)]}{\mathscr{r}_+} - \frac{q_0\cos[\omega(t-\mathscr{r}_-/c)]}{\mathscr{r}_-} \right\}$$

Griffiths

Electric Dipole Radiation: Approximations

Approximation #1: Make this physical dipole into a perfect dipole.

$d << r$

Estimate the spearation distances by the law of cosines.

$$\mathscr{r}_\pm = \sqrt{r^2 \mp rd\cos\theta + (d/2)^2} \cong r(1 \mp \frac{d}{2r}\cos\theta)$$

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}(1 \pm \frac{d}{2r}\cos\theta)$$

$$\cos[\omega(t-\mathscr{r}_\pm/c)] \cong \cos[\omega(t-\frac{r}{c}) \pm \frac{\omega d}{2c}\cos\theta]$$

$$= \cos[\omega(t-\frac{r}{c})]\cos(\frac{\omega d}{2c}\cos\theta) \mp \sin[\omega(t-\frac{r}{c})]\sin(\frac{\omega d}{2c}\cos\theta)$$

Approximation #2: The wavelength is much longer than the dipole size.

$$d << \frac{c}{\omega} = \frac{\lambda}{2\pi}$$

Griffiths

The Retarded Scalar Potential

$$\cos[\omega(t-\mathscr{r}_\pm/c)]\cong\cos[\omega(t-\frac{r}{c})]\underbrace{\cos(\frac{\omega d}{2c}\cos\theta)}_{=1}\mp\sin[\omega(t-\frac{r}{c})]\underbrace{\sin(\frac{\omega d}{2c}\cos\theta)}_{\frac{\omega d}{2c}\cos\theta}$$

$$=\cos[\omega(t-\frac{r}{c})]\mp\sin[\omega(t-\frac{r}{c})]\frac{\omega d}{2c}\cos\theta$$

The retarded scalar potential is:

$$V(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\left\{\begin{array}{l}\left[\cos[\omega(t-\frac{r}{c})]-\sin[\omega(t-\frac{r}{c})]\frac{\omega d}{2c}\cos\theta\right]\frac{1}{r}(1+\frac{d}{2r}\cos\theta)\\ -\left[\cos[\omega(t-\frac{r}{c})]+\sin[\omega(t-\frac{r}{c})]\frac{\omega d}{2c}\cos\theta\right]\frac{1}{r}(1-\frac{d}{2r}\cos\theta)\end{array}\right\}$$

$$\cong\frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left[-\frac{\omega}{c}\sin[\omega(t-\frac{r}{c})+\frac{1}{r}\cos[\omega(t-\frac{r}{c})]\right]$$

Griffiths

The Retarded Scalar Potential

Approximation #3: at the radiation zone. $\frac{c}{\omega}<<r$

The retarded scalar potential is:

$$V(\mathbf{r},t)\cong\frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left[-\frac{\omega}{c}\sin[\omega(t-\frac{r}{c})\right]$$

Three approximations

$$d<<r \qquad d<<\frac{c}{\omega}(=\frac{\lambda}{2\pi}) \qquad \frac{c}{\omega}<<r$$

$$\Rightarrow d<<\lambda<<r$$

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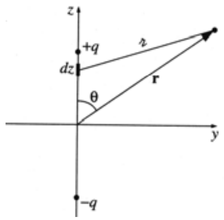
The Retarded Vector Potential

The retarded vector potential is determined by the current density.



$$I(t)=\frac{dq}{dt}\hat{\mathbf{z}}=-q_0\omega\sin\omega t\hat{\mathbf{z}}$$

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}d\tau'=\frac{\mu_0}{4\pi}\int_{-d/2}^{d/2}\frac{-q\omega\sin[\omega(t-\mathscr{r}/c)]\hat{\mathbf{z}}}{\mathscr{r}}dz$$

$$\cong-\frac{\mu_0 p_0\omega}{4\pi r}\sin[\omega(t-\frac{r}{c})]\hat{\mathbf{z}} \qquad @\,d<<\lambda<<r$$

Retarded potentials:

$$\begin{cases}V(\mathbf{r},t)=-\frac{p_0\omega}{4\pi\varepsilon_0 c}\left[\frac{\cos\theta}{r}\sin[\omega(t-\frac{r}{c})\right]\\ \mathbf{A}(\mathbf{r},t)=-\frac{\mu_0 p_0\omega}{4\pi r}\sin[\omega(t-\frac{r}{c})]\hat{\mathbf{z}}\end{cases} \qquad \begin{cases}\mathbf{E}=-\nabla V-\frac{\partial\mathbf{A}}{\partial t}\\ \mathbf{B}=\nabla\times\mathbf{A}\end{cases}$$

Griffiths

The Electromagnetic Fields and Poynting Vector

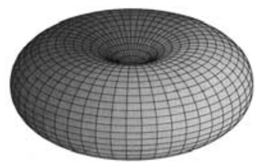
$$\begin{cases}\mathbf{E}=-\nabla V-\frac{\partial\mathbf{A}}{\partial t}=-\frac{\mu_0 p_0\omega^2}{4\pi\varepsilon_0 c}(\frac{\sin\theta}{r})\cos[\omega(t-\frac{r}{c})]\hat{\boldsymbol{\theta}}\\ \mathbf{B}=\nabla\times\mathbf{A}=-\frac{\mu_0 p_0\omega^2}{4\pi c}(\frac{\sin\theta}{r})\cos[\omega(t-\frac{r}{c})]\hat{\boldsymbol{\varphi}}\end{cases}$$

$$\mathbf{S}=\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})=\frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}(\frac{\sin\theta}{r})\cos[\omega(t-\frac{r}{c})]\right\}^2\hat{\mathbf{r}}$$

The total power radiated is

$$<P>=\int\langle\mathbf{S}\rangle\cdot d\mathbf{a}=\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\int(\frac{\sin\theta}{r})^2 r^2\sin\theta d\theta d\phi$$

$$=\frac{\mu_0 p_0^2\omega^4}{12\pi c}$$



9.2 Electric Dipole Fields and Radiation

Rewrite (1):

$$\mathbf{A}(\mathbf{x})=\mu_0\sum_{l,m}\left\{\begin{array}{l}\frac{1}{2l+1}Y_{lm}(\theta,\phi)\frac{e^{ikr}}{r^{l+1}}[1+a_1(ikr)+a_2(ikr)^2+\cdots+a_l(ikr)^l]\\ \cdot\int d^3x'\mathbf{J}(\mathbf{x}')r'^lY^*_{lm}(\theta',\phi')\end{array}\right\}\quad(1)$$

Take the $l=0$ term $[Y_{00}=\frac{1}{\sqrt{4\pi}}]$ and denote it by $\mathbf{A}^p(\mathbf{x})$

$$\mathbf{A}^p(\mathbf{x})=\mathbf{A}(\mathbf{x})^{l=0}=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\mathbf{J}(\mathbf{x}')$$

$$=-\frac{i\mu_0\omega}{4\pi}\mathbf{p}\frac{e^{ikr}}{r},\quad(9.16)$$

where $\mathbf{p}=\int\mathbf{x}'\rho(\mathbf{x}')d^3x'\quad(4.8)$

(9.16) gives the electric dipole contribution to the solution. It is valid for $kd\ll1$ and any $\mathbf{x}$ outside the source.

Question: Why is there no monopole term (see p. 410)?

$$\iiint J_x dxdydz$$

$$=\iint dydz\left[\cancel{xJ_x\Big|_{-d}^{d}}-\int x\frac{\partial J_x}{\partial x}dx\right]$$

$$=-\iiint x(\frac{\partial J_x}{\partial x}+\underbrace{\frac{\partial J_y}{\partial y}+\frac{\partial J_z}{\partial z}})dxdydz$$

give no contribution because $\mathbf{J}$ is localized: $\int\frac{\partial J_y}{\partial y}dy=J_y\Big|_{-d}^{d}=0$

$$=-\int x\nabla\cdot\mathbf{J}d^3x$$

$$\Rightarrow\int\mathbf{J}d^3x=-\int\mathbf{x}\nabla\cdot\mathbf{J}d^3x$$

$$=-i\omega\underbrace{\int\mathbf{x}\rho(\mathbf{x})d^3x}_{\mathbf{p}}=-i\omega\mathbf{p}$$

$$\nabla\cdot\mathbf{J}+\frac{\partial\rho}{\partial t}=0$$

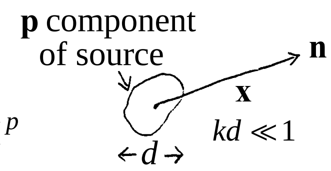
Rewrite (9.16): $\mathbf{A}^p(\mathbf{x})=-\frac{i\mu_0\omega}{4\pi}\mathbf{p}\frac{e^{ikr}}{r}\quad(9.16)$

From (9.4), $\mathbf{H}^p=\frac{1}{\mu_0}\nabla\times\mathbf{A}^p$ and from (9.5), $\mathbf{E}^p=\frac{iZ_0}{k}\nabla\times\mathbf{H}^p$

$$\Rightarrow\begin{cases}\mathbf{H}^p=\frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\\ \mathbf{E}^p=\frac{1}{4\pi\varepsilon_0}\left\{k^2(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\frac{e^{ikr}}{r}+[3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}\end{cases}\quad(9.18)$$

In the far zone ($kr\gg1$), (9.18) reduces to a spherical wave

$$\begin{cases}\mathbf{H}^p\simeq\frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r}\\ \mathbf{E}^p\simeq Z_0\mathbf{H}^p\times\mathbf{n}\end{cases}\quad(9.19)$$

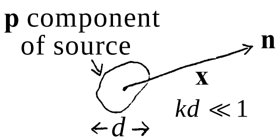


In (9.19), we see that $\mathbf{E}^p$ and $\mathbf{H}^p$ are in phase, and $\mathbf{E}^p$, $\mathbf{H}^p$, and $\mathbf{n}$ are mutually perpendicular. This is a general property of EM waves in unbounded, uniform space. Given any two of these quantities, we can find the third.

$$\begin{cases}\mathbf{H}^p=\frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\\ \mathbf{E}^p=\frac{1}{4\pi\varepsilon_0}\left\{k^2(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\frac{e^{ikr}}{r}+[3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}\end{cases}\quad(9.18)$$

In the near zone ($kr\ll1$), (9.18) reduces to

$$\begin{cases}\mathbf{H}^p\simeq\frac{i\omega}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{1}{r^2}\\ \mathbf{E}^p\simeq\frac{1}{4\pi\varepsilon_0}[3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}]\frac{1}{r^3}\end{cases}\quad(9.20)$$



$\Rightarrow$
(i) $\mathbf{E}^p$ and $\mathbf{H}^p$ are 90° out of phase $\Rightarrow$ average power = 0.
(ii) $\mathbf{E}^p$ has the same spatial pattern as that of the static electric dipole in (4.13), but with $e^{-i\omega t}$ dependence.
(iii) $\mu_0|H|^2\sim(kr)^2\varepsilon_0|E|^2\Rightarrow$ **E**-field energy $\gg$ **B**-field energy.

Questions: (i) Why does $\mathbf{E}^p$ have the static field pattern?

(ii) To obtain (9.20), we have neglected a few terms in (9.18). But some of the neglected terms are still important in the near zone? What are they and in what sense are they important?

$\left\langle\frac{dP}{d\Omega}\right\rangle_t$ = time-averaged power in the far zone/unit solid angle

$$=\frac{1}{2}\mathrm{Re}\left[r^2\mathbf{n}\cdot\left(\mathbf{E}^p\times\mathbf{H}^{p*}\right)\right]\quad(9.21)$$

$$\boxed{(9.19)}\rightarrow\ =\frac{c^2Z_0}{32\pi^2}k^4\,|\underbrace{(\mathbf{n}\times\mathbf{p})\times\mathbf{n}}|^2\quad(9.22)$$

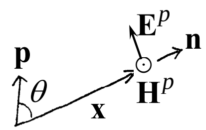
This vector gives the direction of $\mathbf{E}^p$, i.e. the polarization of the radiation (see figure below.)

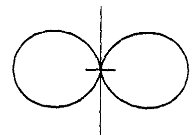
$$\Rightarrow\langle P\rangle_t=\text{total power radiated}=\frac{c^2Z_0k^4}{12\pi}|\mathbf{p}|^2\quad(9.24)$$

In general, $\mathbf{p}=p_xe^{i\alpha}\mathbf{e}_x+p_ye^{i\beta}\mathbf{e}_y+p_ze^{i\gamma}\mathbf{e}_z$. If $\alpha=\beta=\gamma$, then $\mathbf{p}$ has a fixed direction, $\mathbf{p}=\mathbf{p}_0e^{i\alpha}$ with $\mathbf{p}_0=p_x\mathbf{e}_x+p_y\mathbf{e}_y+p_z\mathbf{e}_z$, and

$$\left\langle\frac{dP}{d\Omega}\right\rangle_t=\frac{c^2Z_0}{32\pi^2}k^4|\mathbf{p}|^2\sin^2\theta.\quad(9.23)$$

Otherwise, the direction of $\mathbf{p}$ (hence $\left\langle\frac{dP}{d\Omega}\right\rangle_t$) vary with time, but $\langle P\rangle_t$ is still given by (9.24).





dipole radiation pattern

9.3 Magnetic Dipole and Electric Quadrupole Field

Rewrite (1):

$$\mathbf{A}(\mathbf{x}) = \mu_0 \sum_{l,m} \left\{ \begin{array}{l} \frac{1}{2l+1} Y_{lm}(\theta,\phi) \frac{e^{ikr}}{r^{l+1}} [1 + a_1(ikr) + a_2(ikr)^2 + \cdots + a_l(ikr)^l] \\ \cdot \int d^3x' \mathbf{J}(\mathbf{x}') r'^l Y_{lm}^*(\theta',\phi') \end{array} \right\} \quad (1)$$

Take the $l = 1$ terms $[a_1 = -1]$

$$\mathbf{A}(\mathbf{x})^{l=1} = \frac{\mu_0}{3} \frac{e^{ikr}}{r^2}(1 - ikr) \underbrace{\sum_{m=-1,0,1} Y_{1m}(\theta,\phi) \int d^3x' \mathbf{J}(\mathbf{x}') r' Y_{1m}^*(\theta',\phi')}$$

p. 109

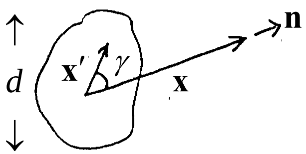
$$\sum_{m=-1,0,1} Y_{1m}(\theta,\phi) Y_{1m}^*(\theta',\phi') = \frac{3}{8\pi} \sin\theta \sin\theta' e^{i(\phi-\phi')}$$

$$+ \frac{3}{4\pi} \cos\theta \cos\theta' + \frac{3}{8\pi} \sin\theta \sin\theta' e^{-i(\phi-\phi')}$$

$$= \frac{3}{4\pi} [\sin\theta \sin\theta' \cos(\phi - \phi') + \cos\theta \cos\theta']$$

$$= \frac{3}{4\pi} \cos\gamma = \frac{3}{4\pi r'} \mathbf{n} \cdot \mathbf{x}'$$

set $l = 1$ in (3.68)



Thus,

$$\mathbf{A}(\mathbf{x})^{l=1} = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} (\tfrac{1}{r} - ik) \int d^3x' \mathbf{J}(\mathbf{x}')(\mathbf{n} \cdot \mathbf{x}') \quad (9.30)$$

$$= \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} (\tfrac{1}{r} - ik) \{ \underbrace{\int d^3x' \tfrac{1}{2}[(\mathbf{n} \cdot \mathbf{x}')\mathbf{J} + (\mathbf{n} \cdot \mathbf{J})\mathbf{x}']}_{\text{electric quadrupole radiation}} + \underbrace{\int d^3x' \tfrac{1}{2}(\mathbf{x}' \times \mathbf{J}) \times \mathbf{n}}_{\text{magnetic dipole radiation}} \}$$

$$= \mathbf{A}^Q + \mathbf{A}^m,$$

where $\mathbf{A}^m(\mathbf{x}) = \frac{ik\mu_0}{4\pi} (\mathbf{n} \times \mathbf{m}) \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr}\right) \left[\begin{array}{l} \text{for } kd \ll 1 \text{ and any} \\ \mathbf{x} \text{ outside the source} \end{array} \right] \quad (9.33)$

with $\mathbf{m} = \frac{1}{2} \int (\mathbf{x} \times \mathbf{J}) d^3x$ [magnetic dipole moment]. $\mathbf{A}^m$ gives the magnetic dipole contribution through (9.4) and (9.5) (see p.15):

$$\left\{ \begin{array}{ll} \mathbf{H}^m = \frac{1}{4\pi} \left\{ k^2 (\mathbf{n} \times \mathbf{m}) \times \mathbf{n} \frac{e^{ikr}}{r} + [3\mathbf{n}(\mathbf{n} \cdot \mathbf{m}) - \mathbf{m}] \left(\frac{1}{r^3} - \frac{ik}{r^2} \right) e^{ikr} \right\} & (9.35) \\ \mathbf{E}^m = -\frac{Z_0}{4\pi} k^2 (\mathbf{n} \times \mathbf{m}) \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr}\right) & (9.36) \end{array} \right.$$

In the far zone ($kr \gg 1$), we have the spherical wave sloution:

$$\left\{ \begin{array}{l} \mathbf{H}^m \simeq \frac{k^2}{4\pi} (\mathbf{n} \times \mathbf{m}) \times \mathbf{n} \frac{e^{ikr}}{r} \\ \mathbf{E}^m \simeq Z_0 \mathbf{H}^m \times \mathbf{n} \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \left\langle \frac{dP}{d\Omega} \right\rangle_t \simeq \frac{Z_0}{32\pi^2} k^4 |\underbrace{\mathbf{m} \times \mathbf{n}}|^2 \\ \langle P \rangle_t \simeq \frac{Z_0}{12\pi} k^4 |\mathbf{m}|^2 \end{array} \right.$$

$\Rightarrow$ direction of $\mathbf{E}^m$

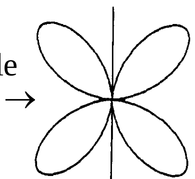
In the near zone ($kr \ll 1$),

$$\left\{ \begin{array}{l} \mathbf{H}^m \simeq \frac{1}{4\pi} [3\mathbf{n}(\mathbf{n} \cdot \mathbf{m}) - \mathbf{m}] \frac{1}{r^3} \\ \mathbf{E}^m \simeq \frac{Z_0 k}{4\pi i} (\mathbf{n} \times \mathbf{m}) \frac{1}{r^2} \end{array} \right. \Rightarrow$$

(i) $\mathbf{E}^m$ and $\mathbf{H}^m$ are 90° out of phase
$\Rightarrow$ average power = 0.
(ii) $\mathbf{H}^m$ has the same spatial pattern as that of the static magnetic dipole in (5.56), but with $e^{-i\omega t}$ dependence.
(iii) **B**-field energy $\gg$ **E**-field energy.

The electric quadrupole radiation, discussed in (9.37)-(9.52), is more complicated. Here, we only illustrate its radiation pattern by the figure to the right.

quadrupole radiation pattern $\rightarrow$



Comparison between Static and Time-dependent Cases

	relations between ρ, $\mathbf{J}$, $\mathbf{E}$, and $\mathbf{B}$	multipole expansion	definition of multipole moments	r-dependence of $\mathbf{E}$ and $\mathbf{B}$ (d : dimension of the source)
static case	$\rho(\mathbf{x}) \leftrightarrow \mathbf{E}(\mathbf{x})$ $\mathbf{J}(\mathbf{x}) \leftrightarrow \mathbf{B}(\mathbf{x})$	spherical harmonics expansion [(3.70)] or Taylor series [(4.10)] of $\frac{1}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$	$q = \int \rho(\mathbf{x}') d^3x'$ $\mathbf{p} = \int \mathbf{x}' \rho(\mathbf{x}') d^3x'$ $Q_{ij} = \int (3x_i' x_j' - r'^2 \delta_{ij}) \rho(\mathbf{x}') d^3x'$ $\mathbf{m} = \frac{1}{2} \int \mathbf{x}' \times \mathbf{J}(\mathbf{x}') d^3x'$	$\mathbf{E}$ or $\mathbf{B} \propto 1/r^{l+2}$ For $r \sim d$, all multipole fields can be significant. For $r \gg d$, multipole fields are dominated by the lowest-order nonvanishing term.
time-dependent case	$\left\{ \begin{array}{c} \rho(\mathbf{x}) \\ \updownarrow \\ \mathbf{J}(\mathbf{x}) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \mathbf{E}(\mathbf{x}) \\ \updownarrow \\ \mathbf{B}(\mathbf{x}) \end{array} \right\}$ $\Rightarrow$ EM waves	spherical harmonics expansion [(9.98)] of $\frac{e^{ik\lvert\mathbf{x}-\mathbf{x}'\rvert}}{\lvert\mathbf{x}-\mathbf{x}'\rvert}$	There is no time-dependent monopole for an isolated source (see p. 410). $\mathbf{p}$, Q_{ij}, and $\mathbf{m}$ have the same expressions as those of their static counterparts, but with the $e^{-i\omega t}$ time dependence. In time-dependent cases, electric multipoles can generate **B**-fields and magnetic multipoles can generate **E**-fields.	(a) near zone $\lambda \gg r \gg d$ $\mathbf{E}$ or $\mathbf{B} \propto e^{-i\omega t}/r^{l+2}$ Approx. the same field pattern and r-dependence as for the corresponding static multipole, but with $e^{-i\omega t}$ dependence (hence called quasi-static fields.) (b) far zone $r \gg \lambda \gg d$ $\mathbf{E}, \mathbf{B} \propto e^{ikr-i\omega t}/r$ (spherical EM waves) All multipole fields $\propto 1/r$, relative power levels unchanged with distance.

Induced Electric and Magnetic Dipoles

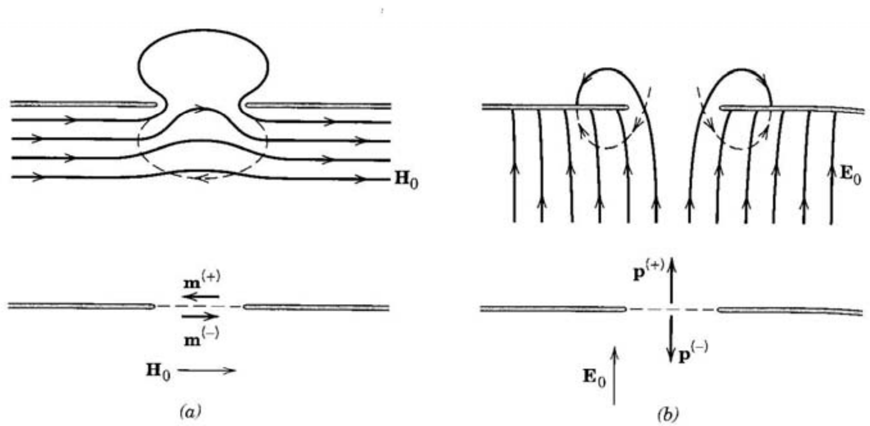
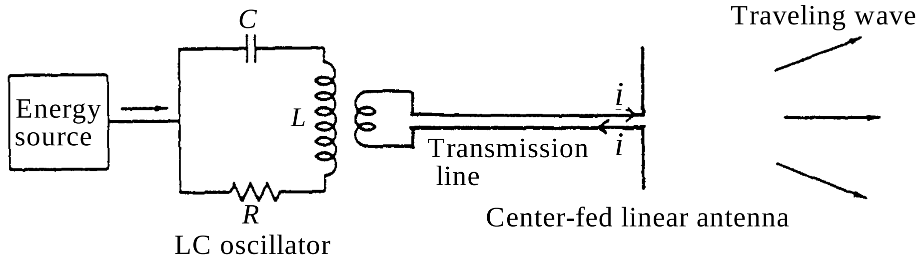
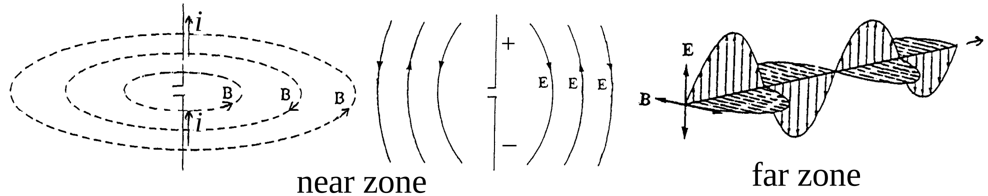


Figure 9.4 Distortion of (a) the tangential magnetic field and (b) the normal electric field by a small aperture in a perfectly conducting surface. The effective dipole moments, as viewed from above and below the surface, are indicated beneath.

9.4 Center-Fed Linear Antenna

A Qualitative Look at the Center - Fed Linear Antenna :





In the near zone, **E** and **B** are principally generated by ρ and **J**, respectively ($\Rightarrow$ largely static field patterns). In the far zone, **E** and **B** are regenerative through $\frac{d}{dt}\mathbf{B}$ and $\frac{d}{dt}\mathbf{E}$ ($\Rightarrow$ EM waves).

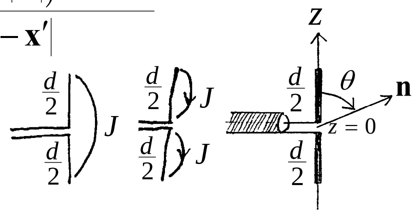
Detailed Analysis: The center-fed linear antenna is a case of special interest, because it allows the solution of (9.3) in closed form for any value of kd, whereas in Secs. 9.2 and 9.3, we assume $kd \ll 1$.

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}'), \qquad (9.3)$$

where $\mathbf{J}(\mathbf{x}) = I\sin\left(\frac{kd}{2} - k|z|\right)\delta(x)\delta(y)\mathbf{e}_z \qquad (9.53)$

$$\Rightarrow \mathbf{A}(\mathbf{x}) = \mathbf{e}_z \frac{\mu_0 I}{4\pi}\int_{-d/2}^{d/2} dz' \frac{\sin\left(\frac{kd}{2} - k|z'|\right)e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$$

Note: (i) **J** is symmetric about $z = 0$. $\mathbf{J}(z) = \mathbf{J}(-z) \rightarrow$
(ii) I is the peak current only when $kd \geq \pi$.

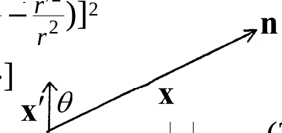


Question: The antenna appears to be an open circuit. How can there be current flowing on it?

$$|\mathbf{x}-\mathbf{x}'| = (r^2 - 2rr'\cos\theta + r'^2)^{\frac{1}{2}} = r[1 - (\tfrac{2\mathbf{n}\cdot\mathbf{x}'}{r} - \tfrac{r'^2}{r^2})]^{\frac{1}{2}}$$

$$= r[1 - \tfrac{1}{2}(\tfrac{2\mathbf{n}\cdot\mathbf{x}'}{r} - \tfrac{r'^2}{r^2}) - \tfrac{1}{8}(\tfrac{2\mathbf{n}\cdot\mathbf{x}'}{r} - \tfrac{r'^2}{r^2})^2 + \cdots]$$

$$= r - \mathbf{n}\cdot\mathbf{x}' + \tfrac{1}{2r}[r'^2 - (\mathbf{n}\cdot\mathbf{x}')^2] + \cdots \qquad (2)$$



$r = |\mathbf{x}|$, $r' = |\mathbf{x}'|$

$\Rightarrow |\mathbf{x}-\mathbf{x}'| \approx r - \mathbf{n}\cdot\mathbf{x}'$ if $r \gg r'$

Hence, if $r \gg d$, we can write $|\mathbf{x}-\mathbf{x}'| \simeq r - z'\cos\theta$.

$$\Rightarrow \underset{\boxed{r \gg d}}{\mathbf{A}(\mathbf{x}) \approx} \mathbf{e}_z \frac{\mu_0 I e^{ikr}}{4\pi}\int_{-d/2}^{d/2} dz' \frac{\sin\left(\frac{kd}{2} - k|z'|\right)e^{-ikz'\cos\theta}}{\underbrace{r - z'\cos\theta}_{\approx r}} \qquad (9.54)$$

$$= \mathbf{e}_z \frac{\mu_0 I e^{ikr}}{2\pi kr}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right] \qquad (9.55)$$

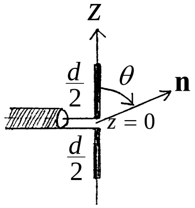
Note: $z'\cos\theta$ in $\frac{1}{r - z'\cos\theta}$ can be neglected if $r \gg d$. But $z'\cos\theta$ in $e^{ik(r - z'\cos\theta)}$ makes an important contribution to the phase angle even at $r \gg d$.

In the far zone, $\boxed{\mathbf{E} = Z_0 \mathbf{H} \times \mathbf{n}}$ $\boxed{\mathbf{H} = \frac{1}{\mu_0}\nabla \times \mathbf{A} = \frac{ik}{\mu_0}\mathbf{n}\times\mathbf{A} \Rightarrow |\mathbf{H}| = \frac{k\sin\theta|\mathbf{A}|}{\mu_0}}$ (9.16 & 9.19)

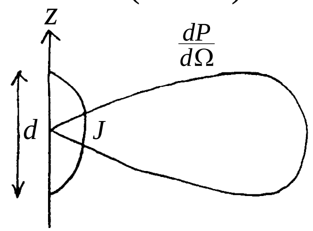
$$\left\langle \frac{dP}{d\Omega}\right\rangle_t = \frac{1}{2}\mathrm{Re}\left[r^2 \mathbf{n}\cdot\mathbf{E}\times\mathbf{H}^*\right] = \frac{Z_0}{2}r^2|\mathbf{H}|^2 = \frac{Z_0}{2\mu_0^2}k^2r^2\sin^2\theta|\mathbf{A}|^2 \quad (3)$$

$$= \frac{Z_0 I^2}{8\pi^2}\left|\frac{\cos(\frac{kd}{2}\cos\theta) - \cos(\frac{kd}{2})}{\sin\theta}\right|^2, \quad \left[\begin{array}{l}\text{for } r \gg d \\ \text{and any } kd\end{array}\right] \quad (9.56)$$

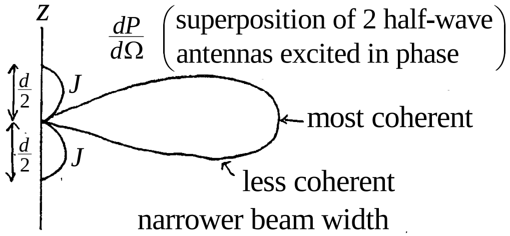
$$= \frac{Z_0 I^2}{8\pi^2}\begin{cases}\cos^2(\frac{\pi}{2}\cos\theta)/\sin^2\theta, & kd = \pi \\ 4\cos^4(\frac{\pi}{2}\cos\theta)/\sin^2\theta, & kd = 2\pi\end{cases} \quad (9.57)$$



half-wave antenna ($kd = \pi$)



full-wave antenna ($kd = 2\pi$)



narrower beam width

Rewrite (9.56)

$$\left\langle \frac{dP}{d\Omega}\right\rangle_t = \frac{Z_0 I^2}{8\pi^2}\left|\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta}\right|^2, \quad \left[\begin{array}{l}\text{for } r \gg d \\ \text{and any } kd\end{array}\right] \quad (9.56)$$

Limiting case (dipole approximation): $kd \ll 1$ (i.e. $\lambda \gg d$)

$$\cos x \simeq 1 - \frac{x^2}{2} \quad (x \ll 1)$$

$$\Rightarrow \begin{cases}\cos\left(\frac{kd}{2}\cos\theta\right) \simeq 1 - \frac{k^2d^2}{8}\cos^2\theta \\ \cos\left(\frac{kd}{2}\right) \simeq 1 - \frac{k^2d^2}{8}\end{cases}$$

$$\Rightarrow \left\langle \frac{dP}{d\Omega}\right\rangle_t \approx \frac{Z_0 I^2}{8\pi^2}\left|\frac{1 - \frac{k^2d^2}{8}\cos^2\theta - 1 + \frac{k^2d^2}{8}}{\sin\theta}\right|^2$$

$$= \frac{Z_0 I^2}{512\pi^2}(kd)^4\sin^2\theta \quad [\text{valid for } kd \ll 1] \quad (4)$$

This has the same k and θ depedence as in (9.23, electric dipole), which was derived by assuming $kd \ll 1$.

Radiation Resistance and Equivalent Circuit:

$$\mathbf{J}(\mathbf{x}) = I\sin\left(\frac{kd}{2} - k|z|\right)\delta(x)\delta(y)\mathbf{e}_z \approx \underbrace{\frac{kd}{2}I}_{I_0}\left(1 - \frac{2|z|}{d}\right)\delta(x)\delta(y)\mathbf{e}_z$$

$\boxed{kd \ll 1}$ I_0 (peak current, $\because |z| \le d$)

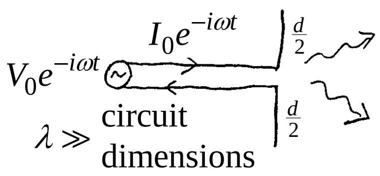
Thus, from (4), $\left\langle \frac{dP}{d\Omega}\right\rangle_t \approx \frac{Z_0 I^2}{512\pi^2}(kd)^4\sin^2\theta = \frac{Z_0 I_0^2}{128\pi^2}(kd)^2\sin^2\theta$ (9.28)

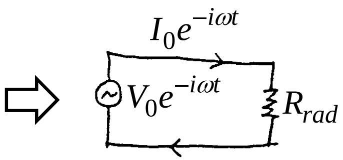
$$\Rightarrow \langle P\rangle_t \approx \int\left\langle \frac{dP}{d\Omega}\right\rangle_t d\Omega = \int_0^{2\pi}d\phi\int_{-1}^{1}d\cos\theta\left\langle \frac{dP}{d\Omega}\right\rangle_t = \frac{Z_0 I_0^2}{48\pi}(kd)^2 \quad (9.29)$$

$$= \frac{I_0^2}{2}R_{rad},$$

[R_{rad}: radiation resistance.
R_{rad} is part of the field definition of impedance, see 2nd term in (6.137).]

where $R_{rad} \equiv \frac{Z_0}{24\pi}(kd)^2 \approx 5(kd)^2$ ohms [See pp. 412-3.]

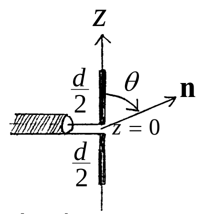




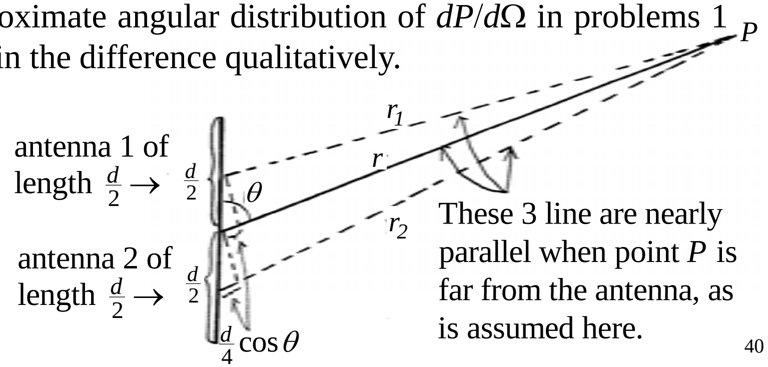
Equivalent circuit for a center-fed antenna

Problems:

1. The full-wave antenna radiation in (9.57) can be thought of as the superposition of two half-wave antennas, one above the other, excited in phase. Demonstrate this by rederiving $dP/d\Omega$ for the full-wave antenna [(9.57), $kd = 2\pi$] by superposing the fields of two half-wave antennas (each of length $d/2$, see figure below).
2. If the two half-wave antennas in problem 1 are excited 180° out of phase, derive $dP/d\Omega$ again by the method of superposition.
3. Plot the approximate angular distribution of $dP/d\Omega$ in problems 1 and 2. Explain the difference qualitatively.

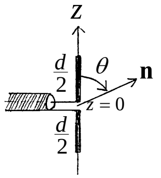


single antenna of length d



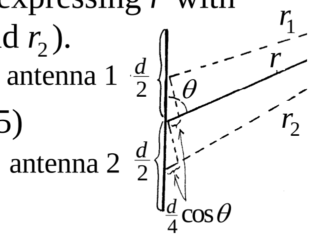
Solution to problem 1: Principle of superposition requires that we add the fields (not the powers) of the 2 antennas, each of total length $\frac{d}{2}$.

Rewrite (9.55)



$$\mathbf{A}(\mathbf{x}) = \mathbf{e}_z \frac{\mu_0 I e^{ikr}}{2\pi kr}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right] \quad (9.55)$$

(9.55) applies to a single antenna of total length d (see fig. above.) So the field of each of the 2 antennas in this problem can be obtained from (9.55) by replacing d in (9.55) with $\frac{d}{2}$ and expressing r with respect to the center of each antenna (i.e. by r_1 and r_2).



$$\mathbf{A}_{1,2} = \mathbf{e}_z \frac{\mu_0 I e^{ikr_{1,2}}}{2\pi k r_{1,2}}\left[\frac{\cos\left(\frac{kd}{4}\cos\theta\right)-\cos\left(\frac{kd}{4}\right)}{\sin^2\theta}\right], \quad (5)$$

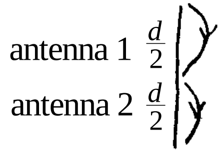
where $r_1 = r - \frac{d}{4}\cos\theta$ and $r_2 = r + \frac{d}{4}\cos\theta$.

We may approximate $r_{1,2}$ in the denominator of (5) by r, but must use the correct $r_{1,2}$ for the phase angles in the exponential terms.

It is assumed that each antenna in this problem is excited in the half-wave pattern, hence we set $k\frac{d}{2} = \pi$ in (5) and the superposed field of the 2 antennas (excited in phase) is given by

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 = \mathbf{e}_z \frac{\mu_0}{2\pi}\frac{I}{kr} e^{ikr}\left[e^{-i\frac{\pi}{2}\cos\theta} + e^{i\frac{\pi}{2}\cos\theta}\right]\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} \quad (6)$$

$$= \mathbf{e}_z \frac{\mu_0}{\pi}\frac{I}{kr} e^{ikr}\frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$



From (3), $\left\langle \frac{dP}{d\Omega}\right\rangle_t = \frac{Z_0}{2\mu_0^2}k^2 r^2 \sin^2\theta |\mathbf{A}|^2$

$$= \frac{Z_0 I^2}{2\pi^2}\cos^4\left(\frac{\pi}{2}\cos\theta\right)\Big/\sin^2\theta \quad \left[\begin{array}{l}\text{same as the full wave} \\ \text{solution in (9.57)}\end{array}\right]$$

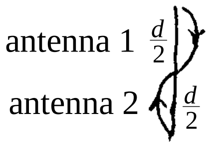
Solution to problem 2:

If the two half-wave antennas in problem 1 are excited 180° out of phase, we simply replace the "+" sign in (6) with a "−" sign.

Thus,

$$\mathbf{A} = \mathbf{A}_1 - \mathbf{A}_2 = \mathbf{e}_z \frac{\mu_0}{2\pi}\frac{I}{kr} e^{ikr}\left[e^{-i\frac{\pi}{2}\cos\theta} - e^{i\frac{\pi}{2}\cos\theta}\right]\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$

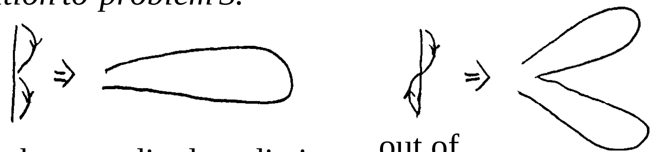
$$= -i\mathbf{e}_z \frac{\mu_0}{\pi}\frac{I}{kr} e^{ikr}\frac{\sin(\frac{\pi}{2}\cos\theta)\cos(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$



From (3), $\left\langle \frac{dP}{d\Omega}\right\rangle_t = \frac{Z_0}{2\mu_0^2}k^2 r^2 \sin^2\theta |\mathbf{A}|^2$

$$= \frac{Z_0 I^2}{2\pi^2}\frac{\sin^2(\frac{\pi}{2}\cos\theta)\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} = \frac{Z_0 I^2}{8\pi^2}\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}$$

Solution to problem 3:



in phase $\Rightarrow$ dipole radiation   out of phase $\Rightarrow$ quadrupole radiation

Question: How does a phased array antenna work?

Homework of Chap. 9

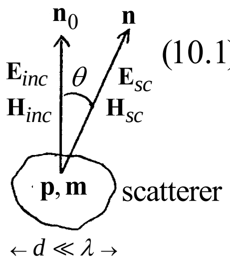
Problems: 3, 6, ~~8, 9,~~ 14, 16, 17, 22, 23

Chapter 10: Scattering and Diffraction

10.1 Scattering at Long Wavelength

Differential Scattering Cross Section : Consider a plane wave

$$\begin{cases} \mathbf{E}_{inc} = \boldsymbol{\varepsilon}_0 E_0 e^{ik\mathbf{n}_0\cdot\mathbf{x}} \\ \mathbf{H}_{inc} = \mathbf{n}_0 \times \mathbf{E}_{inc}/Z_0 \end{cases} \quad \begin{bmatrix} \text{Assume free space.} \\ Z_0 = \sqrt{\mu_0/\varepsilon_0} \end{bmatrix} \qquad (10.1)$$



incident onto an object of dimension $d \ll \lambda$, where $\boldsymbol{\varepsilon}_0$ can be real (linearly polarized) or complex [e.g. for circularly polarized wave, $\boldsymbol{\varepsilon}_{0\pm} = \frac{1}{\sqrt{2}}(\boldsymbol{\varepsilon}_x \pm i\boldsymbol{\varepsilon}_y)$].

$\mathbf{E}_{inc}$ and $\mathbf{H}_{inc}$ will induce multipoles on the object, which in turn generate scattered radiation ($\mathbf{E}_{sc}$, $\mathbf{H}_{sc}$). For $\lambda \gg d$, only the induced $\mathbf{p}$ and $\mathbf{m}$ are important. From (9.19) and (9.36), we have

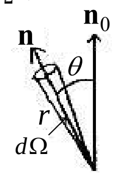
$$\begin{cases} \mathbf{E}_{sc} = \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[(\mathbf{n}\times\mathbf{p})\times\mathbf{n} - \mathbf{n}\times\mathbf{m}/c\right] \\ \mathbf{H}_{sc} = \mathbf{n}\times\mathbf{E}_{sc}/Z_0 \end{cases} \quad \text{[in far zone]} \qquad (10.2)$$

$$\begin{cases} \mathbf{H}^p \simeq \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r} \\ \mathbf{E}^p \simeq Z_0 \mathbf{H}^p \times \mathbf{n} \end{cases} + \begin{cases} \mathbf{H}^m \simeq \frac{k^2}{4\pi}(\mathbf{n}\times\mathbf{m})\times\mathbf{n}\frac{e^{ikr}}{r} \\ \mathbf{E}^m \simeq Z_0 \mathbf{H}^m \times \mathbf{n} \end{cases}$$

Hence, to find $\mathbf{E}_{sc}$ and $\mathbf{H}_{sc}$, we need to find the induced $\mathbf{p}$ and $\mathbf{m}$.

For scattering problems, a useful figure of merit is the scattered power ralative to incident power. Furthermore, it is often important to know the polarization state of the scattered radiation. Thus we define a differential scattering cross section (with dimension m^2) as

$$\frac{d\sigma}{d\Omega}(\mathbf{n}, \boldsymbol{\varepsilon};\ \mathbf{n}_0, \boldsymbol{\varepsilon}_0) \equiv \frac{\dfrac{\text{radiated power in } \mathbf{n}\text{-direction with } \boldsymbol{\varepsilon}\text{-polarization}}{\text{unit solid angle}}}{\dfrac{\text{incident power in } \mathbf{n}_0\text{-direction with } \boldsymbol{\varepsilon}_0\text{-polarization}}{\text{unit area}}}$$



$$= \frac{r^2 \frac{1}{2Z_0}|\boldsymbol{\varepsilon}^*\cdot\mathbf{E}_{sc}|^2}{\frac{1}{2Z_0}|\boldsymbol{\varepsilon}_0^*\cdot\mathbf{E}_{inc}|^2} \quad \begin{bmatrix}\text{The meaning of } \sigma \text{ will} \\ \text{become clear in (10.11).}\end{bmatrix} \qquad (10.3)$$

Note: (i) For a circularly polarized state, $\boldsymbol{\varepsilon}$ an be written
$\boldsymbol{\varepsilon}_\pm = \frac{1}{\sqrt{2}}(\boldsymbol{\varepsilon}_1 \pm i\boldsymbol{\varepsilon}_2)$, where $\boldsymbol{\varepsilon}_1 \perp \boldsymbol{\varepsilon}_2$.

(ii) $\boldsymbol{\varepsilon}_0$ and $\boldsymbol{\varepsilon}_0^* \perp \mathbf{n}_0$; $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}^* \perp \mathbf{n}$; $\boldsymbol{\varepsilon}_0\cdot\boldsymbol{\varepsilon}_0^* = 1$; $\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}^* = 1$

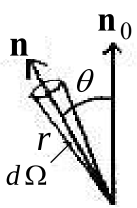
(iii) $\boldsymbol{\varepsilon}$ is not necessarily the direction of $\mathbf{E}_{sc}$. $\boldsymbol{\varepsilon}^*\cdot\mathbf{E}_{sc}$ gives the $\boldsymbol{\varepsilon}$-component of $\mathbf{E}_{sc}$.

Rewrite (10.3): $$\frac{d\sigma}{d\Omega}(\mathbf{n}, \boldsymbol{\varepsilon};\ \mathbf{n}_0, \boldsymbol{\varepsilon}_0) = \frac{r^2 \frac{1}{2Z_0}|\boldsymbol{\varepsilon}^*\cdot\mathbf{E}_{sc}|^2}{\frac{1}{2Z_0}|\boldsymbol{\varepsilon}_0^*\cdot\mathbf{E}_{inc}|^2} \qquad (10.3)$$

Sub. $$\begin{cases} \mathbf{E}_{inc} = \boldsymbol{\varepsilon}_0 E_0 e^{ik\mathbf{n}_0\cdot\mathbf{x}} \\ \mathbf{E}_{sc} = \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[(\mathbf{n}\times\mathbf{p})\times\mathbf{n} - \mathbf{n}\times\mathbf{m}/c\right] \end{cases}$$ into (10.3)

$$\frac{d\sigma}{d\Omega}(\mathbf{n}, \boldsymbol{\varepsilon};\ \mathbf{n}_0, \boldsymbol{\varepsilon}_0) = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2}\left|\underbrace{\boldsymbol{\varepsilon}^*\cdot[(\mathbf{n}\times\mathbf{p})\times\mathbf{n}]}_{} - \boldsymbol{\varepsilon}^*\cdot\frac{\mathbf{n}\times\mathbf{m}}{c}\right|^2$$

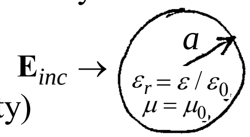
$$\boxed{\begin{aligned} &= \boldsymbol{\varepsilon}^*\cdot[\mathbf{p} - \mathbf{n}(\mathbf{n}\cdot\mathbf{p})] \\ &= \boldsymbol{\varepsilon}^*\cdot\mathbf{p} - \underbrace{(\boldsymbol{\varepsilon}^*\cdot\mathbf{n})}_{0}(\mathbf{n}\cdot\mathbf{p}) \\ &= \boldsymbol{\varepsilon}^*\cdot\mathbf{p} \end{aligned}}$$

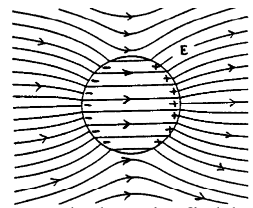


$$= \frac{k^4}{(4\pi\varepsilon_0 E_0)^2}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{p} + \frac{(\mathbf{n}\times\boldsymbol{\varepsilon}^*)\cdot\mathbf{m}}{c}\right|^2 \qquad (10.4)$$

Example 1: Scattering by a small ($a << \lambda$), uniform dielectric sphere with $\mu = \mu_0$ and arbitrary ε

$$\begin{cases} \mu = \mu_0 \Rightarrow \mathbf{m} = 0 \\ \varepsilon_r = \varepsilon/\varepsilon_0 \text{ (relative permitivity)} \end{cases}$$





total electric field

From (4.56), we obtain the electric dipole moment $\mathbf{p}$ induced on the scatterer by $\mathbf{E}_{inc}$ $\boxed{= \boldsymbol{\varepsilon}_0 E_0 e^{ik\mathbf{n}_0\cdot\mathbf{x}}}$

$$\mathbf{p} = 4\pi\varepsilon_0\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)a^3\mathbf{E}_{inc} \qquad \boxed{= 0 \text{ by assumption}} \quad (4.56) \;\&\; (10.5)$$

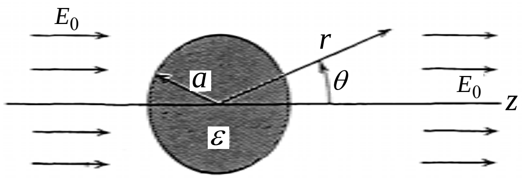
Sub. (10.5) into $$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{p} + (\mathbf{n}\times\boldsymbol{\varepsilon}^*)\cdot\mathbf{m}/c\right|^2 \qquad (10.4)$$

$$\frac{d\sigma}{d\Omega}(\mathbf{n}, \boldsymbol{\varepsilon};\ \mathbf{n}_0, \boldsymbol{\varepsilon}_0) = k^4 a^6\left|\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0|^2 \qquad (10.6)$$

Question: (4.56) is derived for a dielectric sphere in a static field. Why is it also valid for the time-dependent field here?

Reminder

Example: A dielectric sphere is placed in a uniform electric field. Find ϕ everywhere.



We choose the spherical coordinates and divide the space into two regions: $r < a$ and $r > a$. In both regions, we have $\nabla^2\phi = 0$ with the

solution: $\phi = \begin{Bmatrix} r^l \\ r^{-l-1} \end{Bmatrix} \begin{Bmatrix} P_l^m(\cos\theta) \\ Q_l^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$ [Sec. 3.1 of lecture notes]

$$\text{b.c.}\begin{cases} \phi \text{ is independent of } \varphi. \\ \phi \text{ is finite at } \cos\theta = \pm 1. \\ \phi_{in} \text{ is finite at } r = 0. \end{cases} \Rightarrow \begin{cases} \phi_{in} = \sum\limits_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \\ \phi_{out} = \sum\limits_{l=0}^{\infty} \left[B_l r^l + C_l r^{-l-1} \right] P_l(\cos\theta) \end{cases}$$

Question: If $l > 0$, $\phi_{out} \to \infty$ as $r \to \infty$. Why then keep the $l > 0$ terms in ϕ_{out}?

Reminder

$\boxed{\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}.}$

Rewrite: $\phi_{in} = \sum\limits_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$, $\phi_{out} = \sum\limits_{l=0}^{\infty} \left[B_l r^l + C_l r^{-l-1} \right] P_l(\cos\theta)$

b.c. (i): $\phi_{out}(\infty) = -E_0 z + const. = -E_0 r\cos\theta + const.$

$\Rightarrow B_0 = const.;\ B_1 = -E_0;\ B_l(l > 1) = 0$ $\quad \boxed{P_1(\cos\theta) = \cos\theta}$

b.c. (ii): $\phi_{in}(a) = \phi_{out}(a)$ $[\Rightarrow E_t^{in}(a) = E_t^{out}(a)]$

$$\Rightarrow A_l a^l = B_l a^l + \frac{C_l}{a^{l+1}} \Rightarrow \begin{cases} A_0 = B_0 + C_0/a & (8) \\ A_1 = -E_0 + C_1/a^3 & (9) \\ A_l = C_l/a^{2l+1},\ l > 1 & (10) \end{cases}$$

b.c. (iii): $\varepsilon E_r^{in}(a) = \varepsilon_0 E_r^{out}(a) \Rightarrow -\varepsilon \frac{\partial}{\partial r}\phi_{in}\Big|_{r=a} = -\varepsilon_0 \frac{\partial}{\partial r}\phi_{out}\Big|_{r=a}$

$$\Rightarrow \varepsilon l A_l a^{l-1} = \varepsilon_0 \left[l B_l a^{l-1} - (l+1) C_l / a^{l+2} \right]$$

$$\Rightarrow \begin{cases} 0 = -\varepsilon_0 C_0 / a^2, & l = 0 & (11) \\ \varepsilon A_1 = -\varepsilon_0 [E_0 + 2C_1/a^3], & l = 1 & (12) \\ \varepsilon l A_l = -\varepsilon_0 (l+1) C_l / a^{2l+1}, & l > 1 & (13) \end{cases}$$

Reminder

(7), (11) $\Rightarrow A_0 = B_0 = const.$ (let it be 0.)

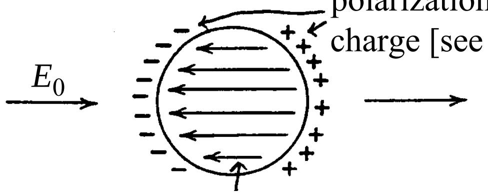
(9), (12) $\Rightarrow A_1 = -\dfrac{3E_0}{2+\varepsilon/\varepsilon_0}$; $C_1 = \left(\dfrac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right) a^3 E_0$

(10), (13) $\Rightarrow A_l = C_l = 0$ for $l > 1$ $\quad$ This is the only way (3) & (6) can both be satisfied.

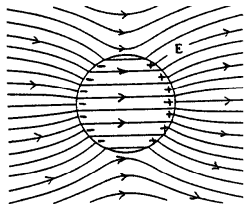
$$\Rightarrow \begin{cases} \phi_{in} = -\dfrac{3}{2+\varepsilon/\varepsilon_0} E_0 r\cos\theta \\ \phi_{out} = \underbrace{-E_0 r\cos\theta}_{\text{applied field}} + \underbrace{\dfrac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} E_0 \dfrac{a^3}{r^2}\cos\theta}_{\text{dipole field with } p = 4\pi\varepsilon_0 a^3 E_0 \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}} \end{cases} \quad (4.54)$$

[cf. (4.10)]

polarization charge [see (4.58)]



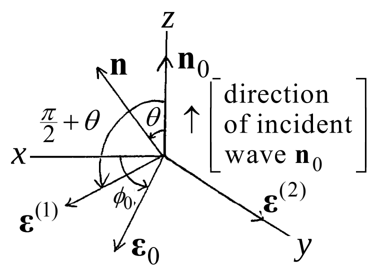
E due to polarization charge



total electric field

We define the $\mathbf{n}$-$\mathbf{n}_0$ plane as the scattering plane. Let $\mathbf{n}_0$ be along the z-axis and $\mathbf{n}$ lie on the x-z plane. The orientations (θ, ϕ) of unit vectors $\boldsymbol{\varepsilon}_0$, $\boldsymbol{\varepsilon}^{(1)}$, and $\boldsymbol{\varepsilon}^{(2)}$ are specified accordingly as follows

$$\begin{cases} \boldsymbol{\varepsilon}_0 = (\frac{\pi}{2},\ \phi_0) & \left[\begin{array}{l}\text{polarization of} \\ \text{incident wave}\end{array}\right] \\ \boldsymbol{\varepsilon}^{(1)} = (\frac{\pi}{2}+\theta,\ 0) & \left[\begin{array}{l}\text{polarization state} \\ \text{of scattered wave} \\ \parallel \text{ to scattering plane}\end{array}\right] \\ \boldsymbol{\varepsilon}^{(2)} = (\frac{\pi}{2},\ \frac{\pi}{2}) & \left[\begin{array}{l}\text{polarization state} \\ \text{of scattered wave} \\ \perp \text{ to scattering plane}\end{array}\right] \end{cases}$$



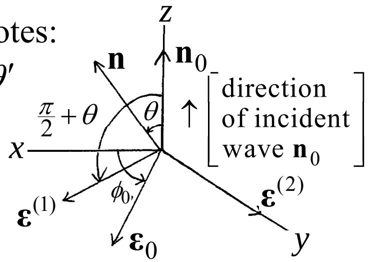
where $\boldsymbol{\varepsilon}_0$ is on the x-y plane making an angle ϕ_0 with the x-axis, $\boldsymbol{\varepsilon}^{(1)}$ is on the x-z (scattering) plane, $\boldsymbol{\varepsilon}^{(2)} (= \mathbf{e}_y)$ is $\perp$ to the scattering plane, and $\mathbf{n}$, $\boldsymbol{\varepsilon}^{(1)}$, and $\boldsymbol{\varepsilon}^{(2)}$ are mutually orthogonal. Polarization vector $(\boldsymbol{\varepsilon}_0)$ of the incident wave and polarization states $[\boldsymbol{\varepsilon}^{(1)}, \boldsymbol{\varepsilon}^{(2)}]$ of the scattered wave are all assumed to be real, representing linear polarization.

Applying Eq. (1) in Ch. 3 of lecture notes:

$\cos\gamma = \sin\theta\sin\theta'\cos(\phi-\phi') + \cos\theta\cos\theta'$

[γ: angle between $(\theta,\ \phi)$ and $(\theta',\ \phi')$]

to $\boldsymbol{\varepsilon}_0 = (\frac{\pi}{2},\ \phi_0)$, $\boldsymbol{\varepsilon}^{(1)} = (\frac{\pi}{2}+\theta,\ 0)$, and $\boldsymbol{\varepsilon}^{(2)} = (\frac{\pi}{2},\ \frac{\pi}{2})$, we obtain



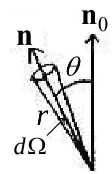
$$\begin{cases} \boldsymbol{\varepsilon}^{(1)}\cdot\boldsymbol{\varepsilon}_0 = \sin\left(\frac{\pi}{2}+\theta\right)\sin\frac{\pi}{2}\cos(0-\phi_0) + \cos\left(\frac{\pi}{2}+\theta\right)\cos\frac{\pi}{2} \\ \qquad = \cos\phi_0\cos\theta \\ \boldsymbol{\varepsilon}^{(2)}\cdot\boldsymbol{\varepsilon}_0 = \sin\frac{\pi}{2}\sin\frac{\pi}{2}\cos\left(\frac{\pi}{2}-\phi_0\right) + \cos\frac{\pi}{2}\cos\frac{\pi}{2} \\ \qquad = \sin\phi_0 \end{cases}$$

Rewrite (10.6): $\frac{d\sigma}{d\Omega}(\mathbf{n},\boldsymbol{\varepsilon};\ \mathbf{n}_0,\boldsymbol{\varepsilon}_0) = k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2 |\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0|^2$

$$\Rightarrow \begin{cases} \frac{d\sigma_\parallel}{d\Omega} = k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2 \left|\boldsymbol{\varepsilon}^{(1)}\cdot\boldsymbol{\varepsilon}_0\right|^2 = k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2\cos^2\phi_0\cos^2\theta \\ \frac{d\sigma_\perp}{d\Omega} = k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2 \left|\boldsymbol{\varepsilon}^{(2)}\cdot\boldsymbol{\varepsilon}_0\right|^2 = k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2\sin^2\phi_0 \end{cases}$$

Assume that the incident radiation has a fixed direction $\mathbf{n}_0$, but is unpolarized (i.e. ϕ_0 is random). We take the average over ϕ_0 :

$$\begin{cases} \left\langle\frac{d\sigma_\parallel}{d\Omega}\right\rangle_{\phi_0} = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\sigma_\parallel}{d\Omega}d\phi_0 = \frac{k^4a^6}{2}\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2\cos^2\theta \\ \left\langle\frac{d\sigma_\perp}{d\Omega}\right\rangle_{\phi_0} = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\sigma_\perp}{d\Omega}d\phi_0 = \frac{k^4a^6}{2}\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2 \end{cases} \qquad (10.7)$$



$$\Rightarrow \Pi(\theta) \equiv \frac{\left\langle\frac{d\sigma_\perp}{d\Omega}\right\rangle_{\phi_0} - \left\langle\frac{d\sigma_\parallel}{d\Omega}\right\rangle_{\phi_0}}{\left\langle\frac{d\sigma_\perp}{d\Omega}\right\rangle_{\phi_0} + \left\langle\frac{d\sigma_\parallel}{d\Omega}\right\rangle_{\phi_0}} = \frac{\sin^2\theta}{1+\cos^2\theta} \quad \left[\begin{array}{l}\Rightarrow 100\% \text{ linearly} \\ \text{polarized at } \theta=\frac{\pi}{2}\end{array}\right] \qquad (10.9)$$

where $\Pi(\theta)$ gives the degree of polarization of the scattered radiation.

$$\left\langle\frac{d\sigma}{d\Omega}\right\rangle_{\phi_0} = \left\langle\frac{d\sigma_\perp}{d\Omega}\right\rangle_{\phi_0} + \left\langle\frac{d\sigma_\parallel}{d\Omega}\right\rangle_{\phi_0} = k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2\frac{1}{2}\left(1+\cos^2\theta\right) \qquad (10.10)$$

$$\Rightarrow \langle\sigma\rangle_{\phi_0} = \int\left\langle\frac{d\sigma}{d\Omega}\right\rangle_{\phi_0}d\Omega = \frac{8\pi}{3}k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2 \ll \pi a^2 \quad [ka \ll 1] \qquad (10.11)$$

Question 1: In (10.10), why add powers instead of adding fields?

(10.11) gives $\langle\sigma\rangle_{\phi_0} \ll \pi a^2$, impling that only a small fraction of the radiation incident on the dielectric sphere is scattered. This is true even if the scatterer is a perfectly conducting sphere (with radius $\ll \lambda$). See next example.

Example 2: Scattering by a small perfectly conducting sphere

The incident radiation will induce both electric and magnetic dipole moments ($\mathbf{p}$ and $\mathbf{m}$) on the conductor. $\mathbf{p}$ and $\mathbf{m}$ are given by

$\mathbf{p} = 4\pi\varepsilon_0 a^3\mathbf{E}_{inc}$ [See Sec. 3.3 of lecture notes.] (10.12)

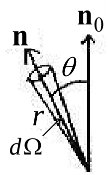
$\mathbf{m} = -2\pi a^3\mathbf{H}_{inc}$ [See next problem.] (10.13)

$$\text{From} \begin{cases} \mathbf{E}_{inc} = \boldsymbol{\varepsilon}_0 E_0 e^{ik\mathbf{n}_0\cdot\mathbf{x}} \\ \mathbf{H}_{inc} = \mathbf{n}_0\times\mathbf{E}_{inc}/Z_0 \quad [Z_0 \equiv \sqrt{\mu_0/\varepsilon_0}] \\ \frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2}\left|\boldsymbol{\varepsilon}^*\cdot\mathbf{p} + (\mathbf{n}\times\boldsymbol{\varepsilon}^*)\cdot\mathbf{m}/c\right|^2 \end{cases} \qquad \begin{array}{r}(10.1)\\ \\ (10.4)\end{array}$$

we obtain $\frac{d\sigma}{d\Omega} = k^4a^6\left|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0 - \frac{1}{2}(\mathbf{n}\times\boldsymbol{\varepsilon}^*)\cdot(\mathbf{n}_0\times\boldsymbol{\varepsilon}_0)\right|^2$ (10.14)

As in Example 1, for unploarized incident radiation, (10.14) yields

$$\begin{cases} \left\langle\frac{d\sigma_\parallel}{d\Omega}\right\rangle_{\phi_0} = \frac{k^4a^6}{2}(\cos\theta - \frac{1}{2})^2 \\ \left\langle\frac{d\sigma_\perp}{d\Omega}\right\rangle_{\phi_0} = \frac{k^4a^6}{2}(1-\frac{1}{2}\cos\theta)^2 \end{cases} \qquad (10.15)$$



$$\Rightarrow \left\langle\frac{d\sigma}{d\Omega}\right\rangle_{\phi_0} = \left\langle\frac{d\sigma_\parallel}{d\Omega}\right\rangle_{\phi_0} + \left\langle\frac{d\sigma_\perp}{d\Omega}\right\rangle_{\phi_0} = k^4a^6[\tfrac{5}{8}(1+\cos^2\theta) - \cos\theta] \qquad (10.16)$$

$$\Pi(\theta) = \frac{3\sin^2\theta}{5(1+\cos^2\theta) - 8\cos\theta} \quad [\text{peak at } \theta = 60°] \qquad (10.17)$$

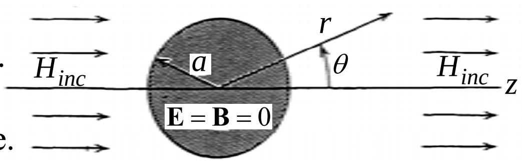
$$\langle\sigma\rangle_{\phi_0} = \int\left\langle\frac{d\sigma}{d\Omega}\right\rangle_{\phi_0}d\Omega = \frac{10}{3}\pi k^4a^6 \ll \pi a^2 \quad [ka \ll 1]$$

Again, we find $\langle\sigma\rangle_{\phi_0} \ll \pi a^2$. By geometric optics, the scatterer (a conductor) would be opaque to the incident radiation, and the incident radiation would have been totally blocked [$\langle\sigma\rangle_{\phi_0} = \pi a^2$]. This example demonstrates that geometric optics completely breaks down for $\lambda \gg a$, where we need physical optics, as in scattering/diffraction theory.

Problem: Derive the dipole moment in (10.13): $\mathbf{m} = -2\pi a^3 \mathbf{H}_{inc}$.

Solution: Since $\lambda \gg a$, we may assume $\mathbf{H}_{inc}$ to be uniform.

For a perfect conductor, we have $\mathbf{E} = \mathbf{B} = 0$ inside the sphere.



In Sec. 9.3, we have shown that in the near zone ($r \ll \lambda$), the magnetic dipole radiation has negligible **E**-field. Hence, we assume $\nabla \times \mathbf{B} = -\frac{\partial}{\partial t}\mathbf{E} \approx 0$ outside the sphere and write $\mathbf{B} = \nabla\phi$. Then,

$\nabla \cdot \mathbf{B} = 0 \;\Rightarrow\; \nabla^2\phi = 0$ with the solution: [Sec. 3.1 of lecture notes]

$$\phi = \begin{Bmatrix} r^l \\ r^{-l-1} \end{Bmatrix} \begin{Bmatrix} P_l^m(\cos\theta) \\ Q_l^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$$

This model is valid for $r \ll \lambda$, which is sufficient for us to find the dipole moment of a sphere with radius $\ll \lambda$.

subject to boundary conditions:

$$\begin{cases} \mathbf{B}(r \to \infty) = \mu_0 H_{inc}\mathbf{e}_z \;\Rightarrow\; \phi(r \to \infty) = \mu_0 H_{inc} z = \mu_0 H_{inc} r\cos\theta \\ B_\perp(r = a) = 0 \Rightarrow \frac{\partial}{\partial r}\phi\big|_{r=a} = 0 \end{cases}$$

Rewrite $\phi = \begin{Bmatrix} r^l \\ r^{-l-1} \end{Bmatrix} \begin{Bmatrix} P_l^m(\cos\theta) \\ Q_l^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} e^{im\varphi} \\ e^{-im\varphi} \end{Bmatrix}$

$\boxed{P_1(\cos\theta) = \cos\theta}$

b.c. $\begin{Bmatrix} \phi \text{ is independent of } \varphi. \\ \phi \text{ is finite at } \cos\theta = \pm 1. \end{Bmatrix} \Rightarrow \phi = \sum_{l=0}^{\infty}\left[A_l r^l + C_l r^{-l-1}\right] P_l(\cos\theta)$

b.c. $\phi(r \to \infty) = \mu_0 H_{inc} r\cos\theta \;\Rightarrow\; A_1 = \mu_0 H_{inc}$ & $A_l = 0$ if $\ell \neq 1$

$$\Rightarrow \phi = \mu_0 H_{inc} r\cos\theta + \sum_{l=0}^{\infty} C_l r^{-l-1} P_l(\cos\theta)$$

b.c. $\frac{\partial}{\partial r}\phi\big|_{r=a} = 0 \;\Rightarrow\; (\mu_0 H_{inc} - \frac{2}{a^3}C_1)\cos\theta - \sum_{l=2}^{\infty}\frac{l+1}{a^{l+2}} C_l P_l(\cos\theta) = 0$

$\Rightarrow C_1 = \frac{1}{2}\mu_0 a^3 H_{inc}$ & $C_l = 0$ if $\ell \neq 1$

$\Rightarrow \phi = \mu_0 H_{inc} r\cos\theta + \frac{1}{2}\mu_0 a^3 H_{inc}\frac{\cos\theta}{r^2}$

$\Rightarrow \mathbf{B}(\text{due to the sphere}) = \nabla\phi(\text{2nd term}) = -\frac{\mu_0 a^3}{2} H_{inc}\frac{2\cos\theta\mathbf{e}_r + \sin\theta\mathbf{e}_\theta}{r^3}$

Comparing with (5.41), we find that this is a magnetic dipole field produced by a (induced) dipole moment of $\mathbf{m} = -2\pi a^3 \mathbf{H}_{inc}$.

Optional

10.2 Perturbation Theory of Scattering

General Theory: Conside a slightly non-uniform medium with

$$\begin{cases} \varepsilon(\mathbf{x}) = \varepsilon_0 + \delta\varepsilon(\mathbf{x}) \\ \mu(\mathbf{x}) = \mu_0 + \delta\mu(\mathbf{x}) \end{cases}$$

[In Sec. 10.1, ε of the scatterer can be of any value, but the solution is more restricted by the scatterer geometry.]

where ε_0 and μ_0 are independent of $\mathbf{x}$ and t (ε_0 and μ_0 are not necessarily the free space values.)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \;\Rightarrow\; \nabla\times\nabla\times\varepsilon_0\mathbf{E} + \varepsilon_0\frac{\partial}{\partial t}\nabla\times\mathbf{B} = 0 \quad (1)$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \;\Rightarrow\; \varepsilon_0\frac{\partial}{\partial t}\nabla\times\mu_0\mathbf{H} = \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\mathbf{D} \quad (2)$$

$$(1)-(2) \Rightarrow \nabla\times\nabla\times\varepsilon_0\mathbf{E} + \varepsilon_0\frac{\partial}{\partial t}\nabla\times(\mathbf{B} - \mu_0\mathbf{H}) = -\mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\mathbf{D} \quad (3)$$

$$\nabla\times\nabla\times\mathbf{D} = \nabla(\underbrace{\nabla\cdot\mathbf{D}}_{=\rho_{free}=0}) - \nabla^2\mathbf{D} = -\nabla^2\mathbf{D} \quad (4)$$

The purpose of the above manipulation is to obtain this small quantity, which can be treated as a perturbation.

$(3)-(4) \Rightarrow$

$$\nabla^2\mathbf{D} - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\mathbf{D} = \overbrace{-\nabla\times\nabla\times(\mathbf{D} - \varepsilon_0\mathbf{E}) + \varepsilon_0\frac{\partial}{\partial t}\nabla\times(\mathbf{B} - \mu_0\mathbf{H})} \quad (10.22)$$

Optional

Assume $\mathbf{D}, \mathbf{E}, \mathbf{B}, \mathbf{H} \sim e^{-i\omega t}$, (10.22) $\Rightarrow$

$$(\nabla^2 + \underbrace{\mu_0\varepsilon_0\omega^2}_{k^2})\mathbf{D} = -\nabla\times\nabla\times(\mathbf{D} - \varepsilon_0\mathbf{E}) - i\varepsilon_0\omega\nabla\times(\mathbf{B} - \mu_0\mathbf{H}) \quad (10.23)$$

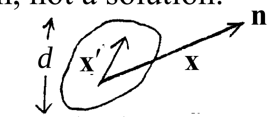
$(\nabla^2 + k^2)G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') \Rightarrow G(\mathbf{x}, \mathbf{x}') = \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$. Hence,

$$\mathbf{D} = \mathbf{D}^{(0)} + \frac{1}{4\pi}\int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\begin{Bmatrix} \nabla'\times\nabla'\times(\mathbf{D} - \varepsilon_0\mathbf{E}) \\ +i\varepsilon_0\omega\nabla'\times(\mathbf{B} - \mu_0\mathbf{H}) \end{Bmatrix} \quad (10.24)$$

Note: (i) $\mathbf{D}(0)$ is an incident plane wave which satisfies the homogeneous Helmholtz eq. [i.e. the RHS of (10.23) = 0]

(ii) (10.24) is an integral relation, not a solution.

Let the integrand in (10.24) be of dimension d and $r \gg d$, then $|\mathbf{x} - \mathbf{x}'| \simeq r - \mathbf{n}\cdot\mathbf{x}'$ and we can write $\mathbf{D}$ as



$$\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \approx \frac{e^{ik(r-\mathbf{n}\cdot\mathbf{x}')}}{r - \underbrace{\mathbf{n}\cdot\mathbf{x}'}_{\text{neglect}}} \quad \text{for } r \gg d$$

$\mathbf{D} \simeq \mathbf{D}^{(0)} + \mathbf{A}_{sc}\frac{e^{ikr}}{r}$ with

$$\mathbf{A}_{sc} = \frac{1}{4\pi}\int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\begin{Bmatrix} \nabla'\times\nabla'\times(\mathbf{D} - \varepsilon_0\mathbf{E}) \\ +i\varepsilon_0\omega\nabla'\times(\mathbf{B} - \mu_0\mathbf{H}) \end{Bmatrix} \quad (10.26)$$

$$\int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\nabla'\times\mathbf{a} \qquad [\mathbf{a} \text{ is any vector function of } \mathbf{x}.]$$

integration by parts

$$= \int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left[\mathbf{e}_x\left(\frac{\partial a_z}{\partial y'} - \frac{\partial a_y}{\partial z'}\right) + \mathbf{e}_y\left(\frac{\partial a_x}{\partial z'} - \frac{\partial a_z}{\partial x'}\right) + \mathbf{e}_z\left(\frac{\partial a_y}{\partial x'} - \frac{\partial a_x}{\partial y'}\right)\right]$$

$$= \int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left[i\mathbf{e}_x(k_y a_z - k_z a_y) + \mathbf{e}_y(\cdots) + \mathbf{e}_z(\cdots)\right]$$

$$= \int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'} i(\mathbf{k}\times\mathbf{a}) = \int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'} ik(\mathbf{n}\times\mathbf{a})$$

$\Rightarrow$ The end result is to replace "∇" with "$ik\mathbf{n}$"

$$(10.26) \Rightarrow \mathbf{A}_{sc} = \tfrac{k^2}{4\pi}\int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\begin{Bmatrix}[\mathbf{n}\times(\mathbf{D}-\varepsilon_0\mathbf{E})]\times\mathbf{n} \\ -\frac{\varepsilon_0\omega}{k}\mathbf{n}\times(\mathbf{B}-\mu_0\mathbf{H})\end{Bmatrix} \qquad (10.27)$$

From (10.3), we obtain
$$\frac{d\sigma}{d\Omega} = \frac{|\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}|^2}{|\mathbf{D}^{(0)}|^2} \qquad (10.28)$$
[$\boldsymbol{\varepsilon}$: polarization vector of the scattered wave]

Note: (i) $\mathbf{A}_{sc}$ gives the scattered field $\mathbf{D}_{sc} = \mathbf{A}_{sc}e^{ikr}/r$ [hence $\mathbf{H}_{sc}$ through (10.2)]. $\mathbf{A}_{sc}$ is NOT a vector potential.
(ii) (10.27) is an integral equation for $\mathbf{A}_{sc}$, NOT a solution.

Born Approximation: Rewrite (10.27)

$$\mathbf{A}_{sc} = \tfrac{k^2}{4\pi}\int d^3x' e^{-ik\mathbf{n}\cdot\mathbf{x}'}\left\{[\mathbf{n}\times(\mathbf{D}-\varepsilon_0\mathbf{E})]\times\mathbf{n} - \tfrac{\varepsilon_0\omega}{k}\mathbf{n}\times(\mathbf{B}-\mu_0\mathbf{H})\right\} \quad (10.27)$$

For a linear medium,

$$\begin{cases}\mathbf{D}(\mathbf{x}) = [\varepsilon_0 + \delta\varepsilon(\mathbf{x})]\mathbf{E}(\mathbf{x}) \\ \mathbf{B}(\mathbf{x}) = [\mu_0 + \delta\mu(\mathbf{x})]\mathbf{H}(\mathbf{x})\end{cases} \Rightarrow \begin{cases}\mathbf{D}-\varepsilon_0\mathbf{E} = \delta\varepsilon(\mathbf{x})\mathbf{E} \\ \mathbf{B}-\mu_0\mathbf{H} = \delta\mu(\mathbf{x})\mathbf{H}\end{cases} \qquad (10.29)$$

We see from (10.29) that the integrand of (10.27) is composed of small quantities $\delta\varepsilon\mathbf{E}$ and $\delta\mu\mathbf{H}$. To first order in $\delta\varepsilon$ and $\delta\mu$, we only need to use the zero order (or unperturbed) $\mathbf{E}^{(0)}$ and $\mathbf{H}^{(0)}$ for $\mathbf{E}$ and $\mathbf{H}$ in $\delta\varepsilon\mathbf{E}$ and $\delta\mu\mathbf{H}$. Thus, we write

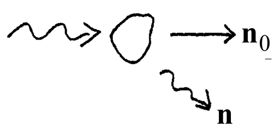
$$\begin{cases}\mathbf{D}-\varepsilon_0\mathbf{E} = \delta\varepsilon(\mathbf{x})\mathbf{E} \approx \dfrac{\delta\varepsilon(\mathbf{x})}{\varepsilon_0}\mathbf{D}^{(0)} \\ \mathbf{B}-\mu_0\mathbf{H} = \delta\mu(\mathbf{x})\mathbf{H} \approx \dfrac{\delta\mu(\mathbf{x})}{\mu_0}\mathbf{B}^{(0)}\end{cases} \qquad (10.30)$$

[This approx., called the <u>Born approx.</u>, turns the integral eq. (10.27) into a solution for $\mathbf{A}_{sc}$.]

Let the unperturbed fields be those of a plane wave,

$$\mathbf{D}^{(0)}(\mathbf{x}) = \boldsymbol{\varepsilon}_0 D_0 e^{ik\mathbf{n}_0\cdot\mathbf{x}},\ \mathbf{B}^{(0)}(\mathbf{x}) = \sqrt{\tfrac{\mu_0}{\varepsilon_0}}\mathbf{n}_0\times\mathbf{D}^{(0)}(\mathbf{x})$$

Sub. $\mathbf{D}^{(0)}(\mathbf{x})$ and $\mathbf{B}^{(0)}(\mathbf{x})$ into (10.30), then sub. (10.30) into (10.27), and finally multiply the result by $\boldsymbol{\varepsilon}^*/D_0$



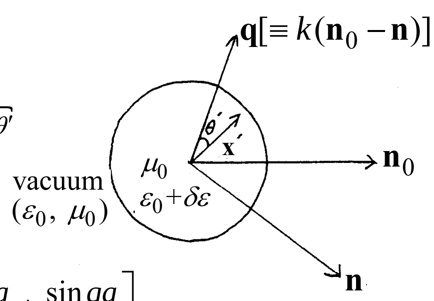
$$\frac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}^{(1)}}{D_0} = \frac{k^2}{4\pi}\int d^3x' e^{i\mathbf{q}\cdot\mathbf{x}'}\begin{Bmatrix}\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0\dfrac{\delta\varepsilon(\mathbf{x}')}{\varepsilon_0} \\ +(\mathbf{n}\times\boldsymbol{\varepsilon}^*)\cdot(\mathbf{n}_0\times\boldsymbol{\varepsilon}_0)\dfrac{\delta\mu(\mathbf{x}')}{\mu_0}\end{Bmatrix} \qquad (10.31)$$

where $q \equiv k(\mathbf{n}_0 - \mathbf{n})$. The absolute square of (10.31) gives the differential scattering cross section through (10.28).

$$\frac{d\sigma}{d\Omega} = \frac{|\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}|^2}{|\mathbf{D}^{(0)}|^2} \qquad (10.28)$$

Example: Scattering by a uniform dielectric sphere with
$\varepsilon = \varepsilon_0 + \delta\varepsilon$ and $\mu = \mu_0$

$$\int d^3x' e^{i\mathbf{q}\cdot\mathbf{x}'}$$
$$= \int_0^a r'^2 dr' \int_0^{2\pi} d\phi' \int_{-1}^{1} d\overbrace{\cos\theta'}^{y}\, e^{iqr'\overbrace{\cos\theta'}^{y}}$$
$$= 2\pi\int_0^a r'^2 dr'\left[\tfrac{1}{iqr'}e^{iqr'y}\right]_{y=-1}^{y=1}$$
$$= \tfrac{4\pi}{q}\int_0^a r'\sin(qr')dr' = 4\pi\left[-\frac{a\cos qa}{q^2} + \frac{\sin qa}{q^3}\right]$$



Thus, from (10.31) (let $\delta\mu = 0$)

$$\frac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}}{D_0} = k^2\tfrac{\delta\varepsilon}{\varepsilon_0}(\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0)\left[\frac{\sin qa - qa\cos qa}{q^3}\right]$$
$$\xrightarrow{qa\to 0} k^2a^3\tfrac{\delta\varepsilon}{3\varepsilon_0}(\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0)$$

$$\sin x \approx x - \tfrac{1}{6}x^3,\ x\to 0$$
$$\cos x \approx 1 - \tfrac{1}{2}x^2,\ x\to 0$$

Optional

Sub. $\left.\frac{\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}}{D_0}\right|_{qa\to 0} = k^2a^3\frac{\delta\varepsilon}{3\varepsilon_0}(\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0)$ into $\frac{d\sigma}{d\Omega}=\frac{|\boldsymbol{\varepsilon}^*\cdot\mathbf{A}_{sc}|^2}{|\mathbf{D}^{(0)}|^2}$ (10.28)

$$\Rightarrow \lim_{qa\to 0}\left(\frac{d\sigma}{d\Omega}\right)_{\text{Born}} \simeq k^4a^6\left|\frac{\delta\varepsilon}{3\varepsilon_0}\right|^2|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0|^2 \quad (10.32)$$

in agreement with $\frac{d\sigma}{d\Omega}=k^4a^6\left|\frac{\varepsilon_r-1}{\varepsilon_r+2}\right|^2|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0|^2$ (10.6) in the limit $\varepsilon_r=\varepsilon/\varepsilon_0\to 1$.

Question: (10.6) and (10.32) both give the differential scattering cross section ($d\sigma/d\Omega$) of a dielectric sphere with radius much smaller than the wavelength. (10.6) is valid for arbitrary values of ε_r ($=\varepsilon/\varepsilon_0$). It reduces to (10.32) in the limit $\varepsilon_r\to 1$. A physical effect in included in (10.6) [but not in (10.32)] that keeps $d\sigma/d\Omega$ at a finite value in the limit $\varepsilon_r\to\infty$? What is it? Explain why it keeps $d\sigma/d\Omega$ finite.

Blue Sky and Red Sunset: Scattering by gases

$\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P}$ (4.34) $\Rightarrow$ $\mathbf{D}=\varepsilon_0\mathbf{E}+N\mathbf{p}=\varepsilon_0\mathbf{E}+N\gamma_{mol}\varepsilon_0\mathbf{E}=\varepsilon\mathbf{E}$

Macroscopically, we have

$$\varepsilon=\varepsilon_0(1+N\gamma_{mol})$$

p: dipole moment per molecule
$\mathbf{p}=\gamma_{mol}\varepsilon_0\mathbf{E}$
γ_{mol}: molecular polarizability [see (4.72) & (4.73)]
N: no of molecules/unit volume

Microscopically, we may write

$$\varepsilon(\mathbf{x})=\varepsilon_0+\underbrace{\sum_j\gamma_{mol}\varepsilon_0\delta(\mathbf{x}-\mathbf{x}_j)}_{\ll\varepsilon_0\text{, when spreaded over the size of the molecule}} \Rightarrow \delta\varepsilon(\mathbf{x})=\varepsilon_0\sum_j\gamma_{mol}\delta(\mathbf{x}-\mathbf{x}_j) \quad (10.33)$$

Since $\varepsilon(\mathbf{x})$ fluctuates microscopically with a weak variation $\delta\varepsilon(\mathbf{x})$, we may apply the perturbation theory just developed.

Sub. $\delta\varepsilon(\mathbf{x})$ into (10.31), then sub. (10.31) into (10.28), we obtain

$$\frac{d\sigma}{d\Omega}=\frac{k^4}{16\pi^2}|\gamma_{mol}|^2|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0|^2F(\mathbf{q}),\ [\text{assume } \delta\mu=0]$$

where $F(\mathbf{q})=\left|\sum_j e^{i\mathbf{q}\cdot\mathbf{x}_j}\right|^2=\sum_j\sum_{j'}e^{i\mathbf{q}\cdot(\mathbf{x}_j-\mathbf{x}_{j'})}\overset{\text{for randomly distributed molecules}}{=}\begin{bmatrix}\text{total no of molecules}\\ \text{(incoherent radiation)}\end{bmatrix}$ (10.19)

We now relate γ_{mol} to the macroscopic quantities ε, n, and N.

$$\varepsilon=\varepsilon_0(1+N\gamma_{mol})\ \Rightarrow\ \gamma_{mol}=\frac{\frac{\varepsilon}{\varepsilon_0}-1}{N}=\frac{n^2-1}{N}\approx\frac{2(n-1)}{N}$$

n: index of refraction

$$n=\sqrt{\frac{\varepsilon}{\varepsilon_0}}\approx 1$$

$$\Rightarrow\frac{d\sigma}{d\Omega}=\frac{k^4}{16\pi^2}|\gamma_{mol}|^2|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0|^2F(\mathbf{q})$$

$$=\frac{k^4}{4\pi^2N^2}|n-1|^2|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0|^2F(\mathbf{q})$$

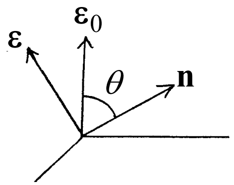
$\Rightarrow$ Total scattering cross section per molecule is given by

$$\sigma=\frac{1}{F(\mathbf{q})}\int\frac{d\sigma}{d\Omega}d\Omega\quad [F(\mathbf{q}):\text{total number of scatterers}]$$

$$=\frac{k^4}{4\pi^2N^2}|n-1|^2\int_0^{2\pi}d\phi\underbrace{\int_{-1}^{1}d\cos\theta|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0|^2}$$

$$=\frac{2k^4}{3\pi N^2}|n-1|^2 \quad (10.34)$$

$\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}_0=\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$
$\int_{-1}^{1}\sin^2\theta d\cos\theta=\frac{4}{3}$



ε is on the $\boldsymbol{\varepsilon}_0$-**n** plane for dipole scatterer (p.458).

Let I be the intensity (power/unit area) of the incident wave, then

$$\frac{dI}{dx}=-IN\sigma=-I\alpha,$$

(10.34) and (10.35) describe what is known as Rayleigh scattering.

where $\alpha=N\sigma\simeq\frac{2k^4}{3\pi N}|n-1|^2$ [attenuation coefficient] (10.35)

Discussion:

(i) $\alpha\propto k^4\Rightarrow\begin{cases}\text{Violet light } (\lambda\simeq 410\text{ nm}) \text{ is scattered more than}\\ \text{red light } (\lambda\simeq 650\text{ nm}) \text{ by a factor of } (\frac{650}{410})^4\simeq 6.3.\end{cases}$

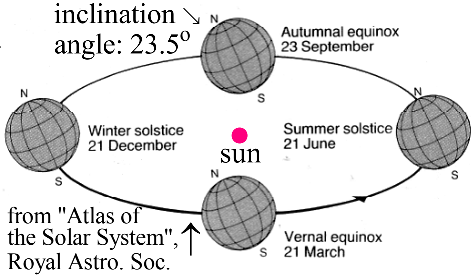
(ii) In (10.35), $n-1\simeq\frac{1}{2}N\gamma_{mol}$ (see last page). Hence, $\alpha\propto N$ if atoms (or molecules) of the same type are added or taken out.

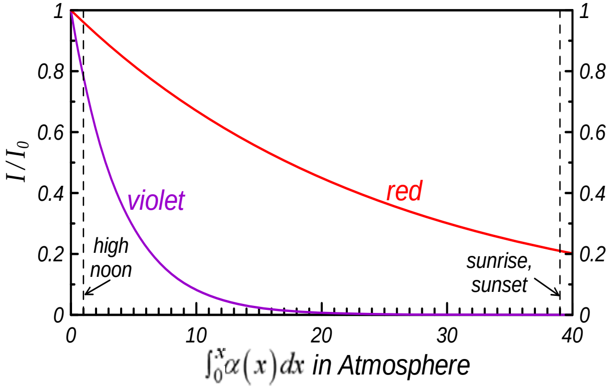
(iii) The atoms in a gas radiate incoherently, but the charges within an atom radiate coherently. Suppose there are 10 electron-ion pairs in each atom and we were able to split all the atoms into a gas of single electron-ion pairs, each with the same p. Then, the macroscopic n remains the same, but the split pairs no longer radiate coherently, resulting in a scattered intensity 10 times weaker. This explains the factor $\frac{1}{N}$ in (10.35) (See p. 468).

In the earth atmosphere, α is a function of x. Then,

$$\frac{dI(x)}{dx} = -I(x)\alpha(x)$$

$$\Rightarrow I(x) = I_0 e^{-\int_0^x \alpha(x)dx}$$





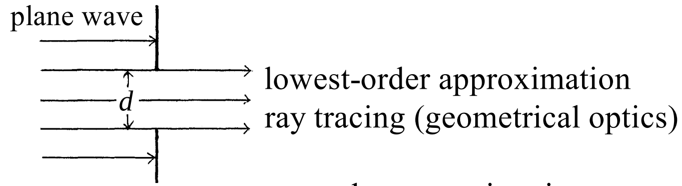
Questions:

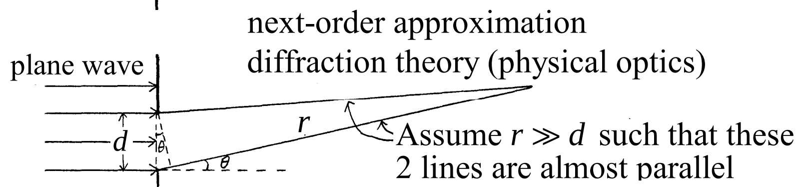
(i) Why is the sky blue instead of violet?

(ii) Why is it more likely to get a sunburn in the summer?

(iii) Hot summer/cold winter results mostly from a different cause than in (ii). What is it?

10.5 Scalar Diffraction Theory





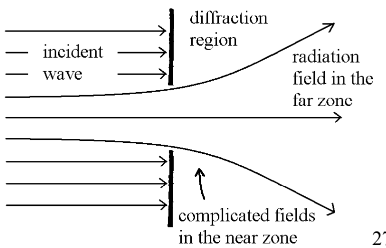
First minimum (complete cancellation) occurs at $d\sin\theta = \lambda \;\Rightarrow\; \theta \approx \frac{\lambda}{d}$ if $d \gg \lambda$.

Nature of the diffraction problem: Physically, the diffraction problem here is not separable from the scattering problem. However, the treatments are different. The scattering problem treated in this chapter assumes $\lambda >> d$. The scalar diffraction theory is most valid when $d >> \lambda$, for which it gives the next-order correction to the geometrical optics (see p. 478).

Justification of the Scalar Diffraction Theory: Physically, electronic responses ($\mathbf{J}$, ρ) of the aperture material to the incident wave generate electromagnetic fields in addition to dissipating some of the incident wave. Far from the edges of the aperture, $\mathbf{J}$ and ρ principally result in reflection of the incident wave, while $\mathbf{J}$ and ρ near the edges produce fields that pass to the right of the aperture together with the incident wave. The superposed fields form the diffraction pattern. In the far zone of the diffraction region (>a few λ from the aperture), the fields take the form of an EM wave, which obeys

$$\mathbf{E} = Z_0 \mathbf{H} \times \mathbf{n} \qquad \text{[see (9.19)]}$$

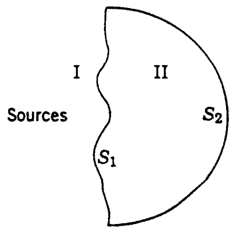
where $Z_0 = (\mu_0/\varepsilon_0)^{1/2}$ is the impedance of vacuum, and $\mathbf{n}$ is the direction of wave propagation.

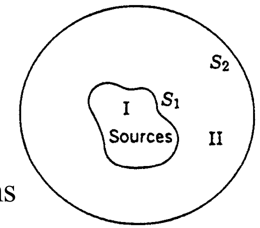


Thus, $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{n}$ are mutually orthogonal, and the amplitudes of $\mathbf{E}$ and $\mathbf{H}$ have a known ratio Z_0. Therefore, one component of the fields gives most of the information (phase and intensity, but not the polarization) about the far fields. This justifies a scalar theory for the diffraction phenomenon and explains why it has been the basis of most of the work on diffraction.

The Kirchhoff Integral Formula: In the scattering problem, we calculate the scattered fields due to $\mathbf{J}$ and ρ associated with the dipole moments induced by the incident fields. In the diffraction problem, the fields are produced in part by the induced $\mathbf{J}$ and ρ on the aperture material, but $\mathbf{J}$ and ρ do not appear explicitly in field equations. They are implicit in the boundary conditions. The Kirchhoff integral formula expresses the diffracted fields in terms of the boundary fields. Determination of the near fields requires accurate handling of the b.c.'s (very few cases can be solved completely). However, the far fields can be fairly accurately determined with crude b.c.'s.

Refer to the figures to the right. S_1 is an opaque surface with aperture(s) on it. The diffraction region (Region II) is the volume enclosed by S_1 and S_2.





Let $\Psi(\mathbf{x},t)=\Psi(\mathbf{x})e^{-i\omega t}$ be a scalar field (a component of **E** or **B**), then

$$\left(\nabla^2+k^2\right)\Psi(\mathbf{x})=0,\quad k=\omega/c \qquad (10.73)$$

Note: Ψ gives the phase and intensity, but not the polarization, of the fields.

Below, we will express Ψ in Region II in terms of Ψ and $\frac{\partial\Psi}{\partial n}$ on the boundary surfaces by making use of Green's thm.

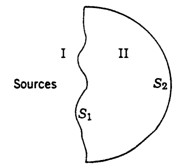
$$\int_V(\phi\nabla^2\psi-\psi\nabla^2\phi)d^3x=\oint_S(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n})da \qquad (1.35)$$

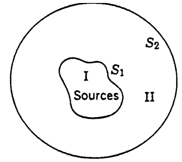
Rewrite $\int_V(\phi\nabla^2\psi-\psi\nabla^2\phi)d^3x=\oint_S(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n})da \qquad (1.35)$

Introduce a Green's function $G(\mathbf{x},\mathbf{x}')$ satisfying

$$(\nabla^2+k^2)G(\mathbf{x},\mathbf{x}')=-\delta(\mathbf{x}-\mathbf{x}') \qquad (10.74)$$

Apply (1.35) to the volume enclosed by S_1 and S_2 (Region II) and let $\psi=\Psi$ and $\phi=G$.





$$\int_V d^3x'[G(\mathbf{x},\mathbf{x}')\overbrace{\nabla'^2\Psi(\mathbf{x}')}^{-k^2\Psi(\mathbf{x}')}-\Psi(\mathbf{x}')\overbrace{\nabla'^2G(\mathbf{x},\mathbf{x}')}^{-k^2G(\mathbf{x},\mathbf{x}')-\delta(\mathbf{x}-\mathbf{x}')}]$$

$$=-\oint_{S_1+S_2}da'[G(\mathbf{x},\mathbf{x}')\mathbf{n}'\cdot\nabla'\Psi(\mathbf{x}')-\Psi(\mathbf{x}')\mathbf{n}'\cdot\nabla'G(\mathbf{x},\mathbf{x}')]$$

For an observation point $\mathbf{x}$ inside region II,

$$\Psi(\mathbf{x})=\oint_{S_1+S_2}da'[\Psi(\mathbf{x}')\mathbf{n}'\cdot\nabla'G(\mathbf{x},\mathbf{x}')-G(\mathbf{x},\mathbf{x}')\mathbf{n}'\cdot\nabla'\Psi(\mathbf{x}')] \qquad (10.75)$$

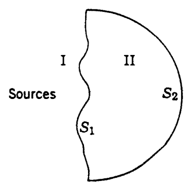
Note: $\mathbf{n}'$ is inwardly directed into the volume instead of outwardly directed as in (1.35).

Is this a good choice?

Solution of (10.74): $G(\mathbf{x},\mathbf{x}')=\frac{e^{ikR}}{4\pi R}$ with $\mathbf{R}=\mathbf{x}-\mathbf{x}'$. (10.76)

Green function with outgoing wave b.c.

$$\Rightarrow \nabla'G(\mathbf{x},\mathbf{x}')=\underbrace{\left(\frac{d}{dR}G\right)}_{ik\frac{e^{ikR}}{4\pi R}-\frac{e^{ikR}}{4\pi R^2}}\cdot\overbrace{\nabla'R}^{\frac{-\mathbf{R}}{R}\text{ (note: }\nabla'R=-\nabla R)}=\frac{-e^{ikR}}{4\pi R}ik\left(1+\frac{i}{kR}\right)\frac{\mathbf{R}}{R}$$



Hence,

$$\Psi(\mathbf{x})=-\frac{1}{4\pi}\oint_{S_1+S_2}da'\frac{e^{ikR}}{R}\mathbf{n}'\cdot\left[\nabla'\Psi(\mathbf{x}')+ik\left(1+\frac{i}{kR}\right)\frac{\mathbf{R}}{R}\Psi(\mathbf{x}')\right] \qquad (10.77)$$

We assume that Ψ on S_2 is transmitted through S_1. Then, $\Psi|_{S_2}\propto\frac{1}{r}$ and the contribution to the integral in (10.77) from S_2 vanishes as the inverse of the radius of the sphere. Assume further that the radius goes to infinity and hence neglect the contribution from S_2. (10.77) then gives the Kirchhoff integral formula

$$\Rightarrow\ \Psi(\mathbf{x})=-\frac{1}{4\pi}\int_{S_1}da'\frac{e^{ikR}}{R}\mathbf{n}'\cdot\left[\nabla'\Psi(\mathbf{x}')+ik\left(1+\frac{i}{kR}\right)\frac{\mathbf{R}}{R}\Psi(\mathbf{x}')\right] \qquad (10.79)$$

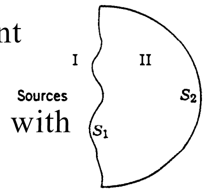
Ψ in Region II is now expressed in terms of Ψ and $\frac{\partial\Psi}{\partial n}$ on S_1.

Kirchhoff Approximation: Rewrite (10.79),

$$\Psi(\mathbf{x})=-\frac{1}{4\pi}\int_{S_1}da'\frac{e^{ikR}}{R}\mathbf{n}'\cdot\left[\nabla'\Psi(\mathbf{x}')+ik\left(1+\frac{i}{kR}\right)\frac{\mathbf{R}}{R}\Psi(\mathbf{x}')\right] \qquad (10.79)$$

(10.79) is an integral equation for Ψ. It becomes a solution for Ψ under the Kirchhoff approximation, which consists of

1. Ψ and $\frac{\partial\Psi}{\partial n}$ vanish everywhere on S_1 except in the openings.
2. Ψ and $\frac{\partial\Psi}{\partial n}$ in the openings are those of the incident wave in the absence of any obstacles.



There are, however, **mathematical inconsistencies** with the Kirchhoff approximation:

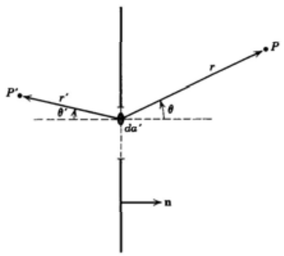
1. If Ψ and $\frac{\partial\Psi}{\partial n}$ vanish on any finite surface, then $\Psi=0$ everywhere (true for both Laplace and Helmholtz equations).
2. (10.79) does not yield on S_1 the assumed values of Ψ and $\frac{\partial\Psi}{\partial n}$.

Approximations made here work best for $\lambda\ll d$, and fail badly for $\lambda\sim d$ or $\lambda>d$ (d: size of the aperture or obstacle). See p.478.

Remove the mathematical inconsistencies in the Kirchhoff Approximation by the choice of a proper Green function.
If Ψ is known on the surface S_1, a Dirichet Green function $G_D(\mathbf{x},\mathbf{x}')$, satisfying $G_D(\mathbf{x},\mathbf{x}')=0$ for $\mathbf{x}'$ on S is required.
A generalized Kirchhoff integral:

$$\Psi(\mathbf{x}) = \int_{S_1} da'[\Psi(\mathbf{x}')\mathbf{n}'\cdot\nabla' G(\mathbf{x},\mathbf{x}')] \qquad (10.81)$$

Consider a plane screen with aperature (s). The method of images can be used to give the Dirichlet Green functions explicit form:



$$G_D(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi}\left(\frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'}\right) \qquad (10.84)$$

where $\begin{cases} \mathbf{R} = \mathbf{x}-\mathbf{x}' = (x-x', y-y', z-z') \\ \mathbf{R}' = \mathbf{x}-\mathbf{x}'' = (x-x', y-y', z+z') \end{cases}$

$$\Psi(\mathbf{x}) = \frac{k}{2\pi i}\int_{S_1}\frac{e^{ikR}}{R}\left(1+\frac{i}{kR}\right)\frac{\mathbf{n}'\cdot\mathbf{R}}{R}\Psi(\mathbf{x}')da' \qquad (10.85)$$

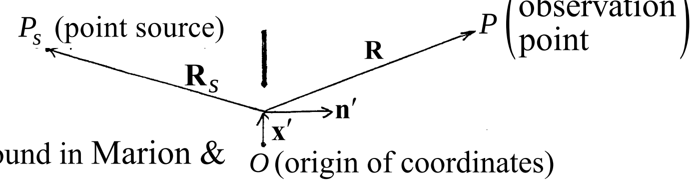
A Special Case*: Diffraction of spherical waves originating from a point source at P_s.

$$\Psi(\mathbf{x}') = \frac{e^{ikR_s}}{R_s} \quad \text{(by Kirchhoff approximation)} \qquad (5)$$

$$\Rightarrow \nabla'\Psi(\mathbf{x}') = -\frac{e^{ikR_s}}{R_s}ik\left(1+\frac{i}{kR_s}\right)\frac{\mathbf{R}_s}{R_s} \qquad \boxed{G(\mathbf{x},\mathbf{x}') = \frac{e^{ikR}}{4\pi R}} \qquad (6)$$

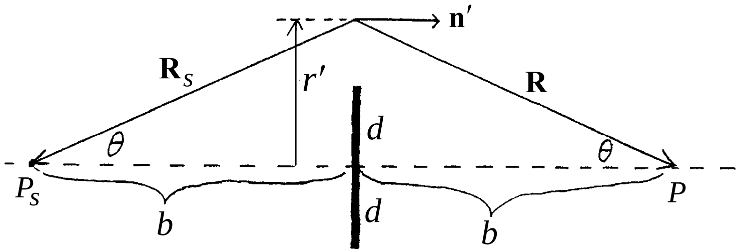
Sub. (5), (6) into (10.79), assume kR & $kR_s \gg 1$ and hence neglect $O\left(\frac{1}{kR}\right)$ and $O\left(\frac{1}{kR_s}\right)$ terms, we obtain

$$\Psi(P) = \frac{ik}{4\pi}\int_{S_1} da'\frac{e^{ik(R+R_s)}}{RR_s}\mathbf{n}'\cdot\left(\frac{\mathbf{R}_s}{R_s}-\frac{\mathbf{R}}{R}\right) \qquad (7)$$



* More cases can be found in Marion & Heald, "Classical Electromagnetic Radiation," following Eq. (12.14).

As we will see from the following example, the scalar diffraction theory agrees with observations, although it is highly artificial.
Example: Diffraction by a circular disk. For simplicity, we assume
(i) P_s and P are on the axis of the disk.
(ii) P_s and P are at equal distance from the disk.



$$\left.\begin{array}{l} R_s = R \\ da' = 2\pi r'dr' \left(\begin{array}{l} R^2 = r'^2+b^2 \Rightarrow r'dr' = RdR \\ \text{Hence, } da' = 2\pi RdR \end{array}\right) \\ \mathbf{n}'\cdot\frac{\mathbf{R}_s}{R_s} = -\cos\theta = -\frac{b}{R}, \quad \mathbf{n}'\cdot\frac{\mathbf{R}}{R} = \cos\theta = \frac{b}{R} \end{array}\right\} \qquad (8)$$

Sub. (8) into $\Psi(P) = \frac{ik}{4\pi}\int_{S_1} da'\frac{e^{ik(R+R_s)}}{RR_s}\mathbf{n}'\cdot\left(\frac{\mathbf{R}_s}{R_s}-\frac{\mathbf{R}}{R}\right) \qquad (7)$

$$\Rightarrow \Psi(P) = -ikb\int_{\sqrt{d^2+b^2}}^{\infty}\frac{e^{2ikR}}{R^2}dR \qquad (9)$$

Integrating by parts $[\int_{a_1}^{a_2} udv = uv|_{a_1}^{a_2} - \int_{a_1}^{a_2} vdu,\ u = \frac{1}{R^2},\ dv = e^{2ikR}dR]$

$$\Psi(P) = -ikb\left[\left.\frac{e^{2ikR}}{2ikR^2}\right|_{\sqrt{d^2+b^2}}^{\infty} + \frac{1}{2ik}\int_{\sqrt{d^2+b^2}}^{\infty}\frac{e^{2ikR}}{R^3}dR\right]$$

(integrating by parts again)

$$= -ikb\left[\left.\frac{e^{2ikR}}{2ikR^2}\right|_{\sqrt{d^2+b^2}}^{\infty} - \underbrace{\left.\frac{e^{2ikR}}{4k^2R^3}\right|_{\sqrt{d^2+b^2}}^{\infty}}_{\text{negligible, since } kR\gg 1} + \cdots\right] \simeq \frac{be^{2ik\sqrt{d^2+b^2}}}{2(d^2+b^2)} \qquad (10)$$

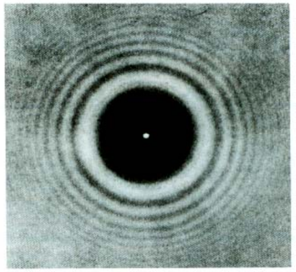
Questions:

(i) Intensity at P: $I(P) \propto |\Psi(P)|^2 = b^2/[4(d^2+b^2)^2]$ (11)

Since $I(P) > 0$ for all b, there is always a bright spot (Fresnel bright spot) at any point on the axis. What is the physical reason?

(ii) $\lim_{d\to 0} \Psi(P) = \frac{e^{2ikR}}{2b}$ (12)

In the limit of no obsticle ($d \to 0$), $\Psi(P)$ reduces to the exact solution for a point source at P_s, i.e. the approximate solution in (10) becomes the exact solution in (12). What is the mathematical reason?



← The diffraction pattern of a disk (from Halliday, Resnick, and Walker). Note the Fresnel bright spot at the center of the pattern. The concentric diffraction rings are not predictable by (11), which applies only to fields on the axis.

A historical anecdote about the Fresnel bright spot: (The following paragraphs are taken from Halliday, Resnick, and Walker.)

"Diffraction finds a ready explanation in the wave theory of light. However, this theory, originally advanced by Huygens and used 123 years later by Young to explain double-slit interference, was very slow in being adopted, largely because it ran counter to Newton's theory that light was a stream of particles.

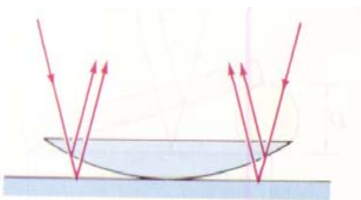
Newton's view was the prevailing view in French scientific circles of the early nineteenth century, when Augustin Fresnel was a young military engineer. Fresnel, who believed in the wave theory of light, submitted a paper to the French Academy of Sciences describing his experiments and his wave-theory explanations of them.

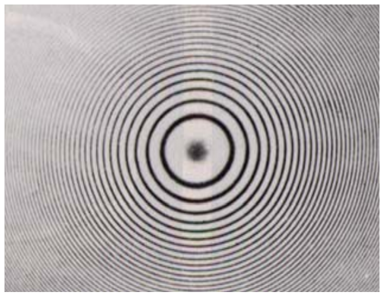
In 1819, the Academy, dominated by supporters of Newton and thinking to challenge the wave point of view, organized a prize competition for an essay on the subject of diffraction. Fresnel won. The Newtonians, however, were neither converted nor silenced. One of them, S. D. Poisson, pointed out the "strange result" that if Fresnel's theories were correct, then light waves should flare into the shadow region of a sphere as they pass the edge of the sphere, producing a bright spot at the center of the shadow. The prize committee arranged a test of the famous mathematician's prediction and discovered that the predicted Fresnel bright spot, as we call it today, was indeed there! Nothing builds confidence in a theory so much as having one of its unexpected and counterintuitive predictions verified by experiment."

Benson

Newton's Ring

When a lens with a large radius of curvature is place on a flat plate, as in Fig. 37.19, a **thin film of air** is formed. When Newton is illuminated with **mono-chronomatic** light, **circular fringes**, called **Newton's Rings**, can be seen.



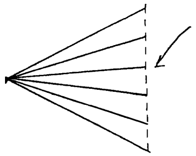


Why the center spot is dark unlike Fresnel bright spot?

This is the wave nature.

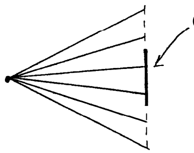
10.8 Babinet's Principle

Rewrite $\Psi(\mathbf{x}) = -\frac{1}{4\pi}\int_{S_1} da' \frac{e^{ikR}}{R} \mathbf{n}' \cdot \left[\nabla'\Psi(\mathbf{x}') + ik\left(1+\frac{i}{kR}\right)\frac{\mathbf{R}}{R}\Psi(\mathbf{x}')\right]$ (10.79)



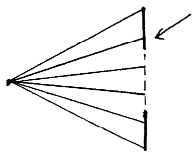
no diffraction screen, imaginary surface

$\cdot\ \Psi(P) = -\frac{1}{4\pi}\int_{\text{dashed surface}}(\cdots)$



diffraction screen

$\cdot\ \Psi_a(P) = -\frac{1}{4\pi}\int_{\text{dashed surface}}(\cdots)$



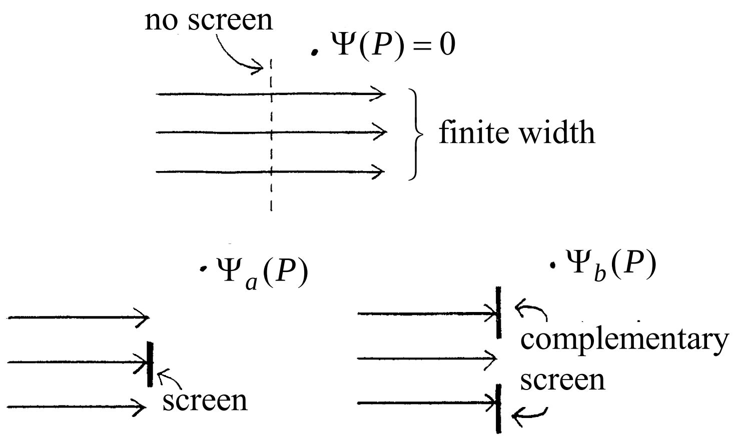
complementary diffraction screen

$\cdot\ \Psi_b(P) = -\frac{1}{4\pi}\int_{\text{dashed surface}}(\cdots)$

By Kirchoff's approx.: $\begin{cases} \text{on the obatacle}: \Psi \text{ and } \frac{\partial\Psi}{\partial n} = 0 \\ \text{elsewhere}: \Psi \text{ and } \frac{\partial\Psi}{\partial n} = \text{those of the source} \end{cases}$,

we have $\Psi(P) = \Psi_a(P) + \Psi_b(P)$ [Babinet's principle]

Example: a light beam of finite width

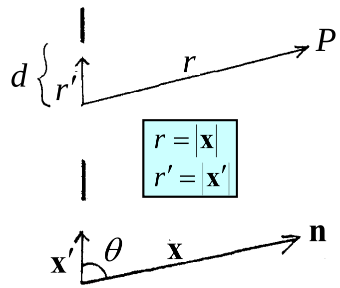


Babinet's principle $\Rightarrow$ $\Psi(P)=\Psi_a(P)+\Psi_b(P)=0$

$$\Rightarrow \Psi_a(P)=-\Psi_b(P)$$

Fresnel and Fraunhofer Diffraction: (see p.491)

There is a clear diffraction pattern only when $r \gg d$. So, In integrals such as (10.77), $R(=|\mathbf{x}-\mathbf{x}'|)$ can be approximated by $r(=|\mathbf{x}|)$ everywhere except in e^{ikR}, where the phase angle kR must be evaluated more accurately.



Consider three length scales: r, d, and λ.

$$R=|\mathbf{x}-\mathbf{x}'|=(r^2-2rr'\cos\theta+r'^2)^{1/2}$$

$$=r\left[1-\left(\frac{2\mathbf{n}\cdot\mathbf{x}'}{r}-\frac{r'^2}{r^2}\right)\right]^{1/2}=r\left[1-\frac{1}{2}\left(\frac{2\mathbf{n}\cdot\mathbf{x}'}{r}-\frac{r'^2}{r^2}\right)-\frac{1}{8}\left(\frac{2\mathbf{n}\cdot\mathbf{x}'}{r}-\frac{r'^2}{r^2}\right)^2+\cdots\right]$$

$$=r\left[1-\frac{\mathbf{n}\cdot\mathbf{x}'}{r}+\frac{1}{2}\left(\frac{r'^2}{r^2}-\frac{(\mathbf{n}\cdot\mathbf{x}')^2}{r^2}\right)+\cdots\right]=r-\mathbf{n}\cdot\mathbf{x}'+\frac{1}{2r}\left[r'^2-(\mathbf{n}\cdot\mathbf{x}')^2+\cdots\right]$$

$$\Rightarrow \; kR=O(kr)+O(kd)+O\left(\frac{kd^2}{r}\right)+\cdots$$

If the 3rd and higher terms are neglected, we have the Fraunhofer diffraction (far field). If the 3rd term is kept, but higher order terms are neglected, we have the Fresnel diffraction (near field).

Homework of Chap. 10

Problems: 2, 3, 7, ~~12, 14~~

Chapter 11: Special Theory of Relativity

(Ref.: Marion & Heald, “Classical Electromagnetic Radiation,” 3rd ed., Ch. 14)

Einstein’s special theory of relativity is based on two postulates:

1. Laws of physics are invariant in form in all Lorentz frames (In relativity, we often call the inertial frame a Lorentz frame.)
2. The speed of light in vacuum has the same value c in all Lorentz frames, independent of the motion of the source.

The basics of the theory are covered in Appendix A on an elementary level with an emphasis on the Lorentz transformation and relativistic momentum/energy. Here, we examine relativity in the four-dimensional space of $\mathbf{x}$ and t, which provides the framework for us to examine the laws of mechanics and electromagnetism. The contents of the lecture notes depart considerably from Ch.11 of Jackson. Instead, we follow Ch. 14 of Marion.

In the lecture notes, section numbers do not follow Jackson.

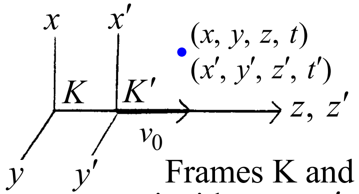
Section 1: Definitions and Operation Rules of Tensors of Different Ranks in the 4-Dimensional Space

The Lorentz Transformation:

Consider two Lorentz frames, K and K'. Frame K' moves along the common z-axis with constant speed v_0 relative to frame K.

Assume that at $t = t' = 0$, coordinate axes of frames K and K' overlap. Postulate 2 leads to the following Lorentz transformation for space and time coordinates. [derived in Appendix A, Eq. (A.15), where the relative motion is assumed to be along the x-axis.]

$$\begin{cases} x' = x \\ y' = y \\ z' = \gamma_0 (z - v_0 t) \\ t' = \gamma_0 (t - \frac{v_0}{c^2} z) \end{cases} \tag{1}$$



Frames K and K′ coincide at $t = t' = 0$.

where $\gamma_0 \equiv (1 - \frac{v_0^2}{c^2})^{-\frac{1}{2}}$ is the Lorentz factor for the transformation.

A note about notation: In many books, the relative speed between two frames is denoted by v and the particle velocity in a given frame is denoted by $\mathbf{u}$. This eventually leads to two definitions for the same notation γ:

$$\gamma \equiv (1 - \frac{v^2}{c^2})^{-\frac{1}{2}} \quad \begin{bmatrix}\text{Lorentz factor for the transformation,} \\ \text{Jackson (11.17)}\end{bmatrix}$$

$$\gamma \equiv (1 - \frac{u^2}{c^2})^{-\frac{1}{2}} \quad \begin{bmatrix}\text{Lorentz factor of a particle in a given frame,} \\ \text{Jackson (11.46) and (11.51)}\end{bmatrix}.$$

To avoid confusion with the notation γ (e.g. when we perform a Lorentz transformation of the Lorentz factor of a particle), we will denote the relative speed between two frames by v_0 and the particle velocity by v throughout this chapter, and thus define

$$\gamma_0 \equiv (1 - \frac{v_0^2}{c^2})^{-\frac{1}{2}} \quad \text{[Lorentz factor for the transformation]}$$

$$\gamma \equiv (1 - \frac{v^2}{c^2})^{-\frac{1}{2}} \quad \text{[Lorentz factor of a particle in a given frame].}$$

Four-Dimension Space Quantities and Operation Rules:

Define a position vector in the 4-dimensional space of $\mathbf{x}$ and t as

$$\underbrace{\mathbf{x}}_{\text{4-vector}} \equiv (x, y, z, ict) = (\underbrace{\mathbf{x}}_{\text{spatial vector}}, ict)$$

and a 4-D matrix as

$$a_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_0 & i\gamma_0\beta_0 \\ 0 & 0 & -i\gamma_0\beta_0 & \gamma_0 \end{bmatrix}, \quad \beta_0 = v_0 / c$$

$\mu = 1-4$, row number

$\nu = 1-4$, column number

then, the Lorentz transformation in (1) can be written

$$\begin{bmatrix} x' \\ y' \\ z' \\ ict' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_0 & i\gamma_0\beta_0 \\ 0 & 0 & -i\gamma_0\beta_0 & \gamma_0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ ict \end{bmatrix} \quad \text{or} \quad x'_\mu = \sum_{\nu=1}^{4} a_{\mu\nu} x_\nu \tag{2}$$

and the inverse Lorentz transformation is: $x_\nu = \sum_{\mu=1}^{4} a_{\mu\nu} x'_\mu$. (3)

The $a_{\mu\nu}$ matrix in (2) shows that the Lorentz transformation is an orthogonal transformation because it satisfies

$$\sum_\mu a_{\mu\nu}a_{\mu\lambda} = \delta_{\nu\lambda} \quad \begin{bmatrix}\text{definition of orthogonal}\\ \text{transformation*}\end{bmatrix} \qquad (4)$$

*See H. Goldstein, "Classical Mechanics," 2nd edition, p.134.

$$\text{Thus, } \sum_\mu x'^2_\mu = \sum_\mu \overbrace{\sum_\nu a_{\mu\nu}x_\nu}^{x'_\mu \text{ by (2)}}\ \overbrace{\sum_\lambda a_{\mu\lambda}x_\lambda}^{x'_\mu \text{ by (2)}} = \sum_{\nu,\lambda}\overbrace{\sum_\mu a_{\mu\nu}a_{\mu\lambda}}^{\delta_{\nu\lambda}\text{ by (4)}}\, x_\nu x_\lambda = \sum_\lambda x_\lambda^2$$

$$\Rightarrow x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2,$$

which is a statement of postulate 2 [see Eqs. (B.1) and (B.2) in Appendix B.]

Just as the 3-dimensional vectors (and tensors in general) are defined by their transformation properties in the **x**-space, we may define 4-vectors (and 4-tensors in general) by their transformation properties in the $(\mathbf{x}, t)$ space and find rules for their operation.

1. Any set of 4 quantities A_μ $(\mu = 1-4)$ or $\mathbf{A} = (A_1, A_2, A_3, A_4)$, which transform in the same way as x_μ, i.e.

$$A'_\mu = \sum_\nu a_{\mu\nu}A_\nu, \qquad (5)$$

is called a 4-vector (or 4-tensor of the first rank).

The position vector $\mathbf{x}\,[\equiv (x,\ y,\ z,\ ict)]$ of a point in the 4-D space is obviously a 4-vector. As another example, the momentum vector of a particle in the 4-D space, defined as

$$\mathbf{p} \equiv (p_x,\ p_y,\ p_z,\ \tfrac{iE}{c}) = (\mathbf{p},\ \tfrac{iE}{c}),$$

is a 4-vector because it transforms as [see Eq. (A.28), Appendix A.]

$$\begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ \frac{iE'}{c}\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_0 & i\gamma_0\beta_0 \\ 0 & 0 & -i\gamma_0\beta_0 & \gamma_0 \end{bmatrix}\begin{bmatrix} p_x \\ p_y \\ p_z \\ \frac{iE}{c}\end{bmatrix} \quad \text{or } p'_\mu = \sum_{\nu=1}^{4} a_{\mu\nu}p_\nu$$

2. If a quantity Φ is unchanged under the Lorentz transformation, it is called a Lorentz scalar (or 4-vector of the zeroth rank). The Lorentz scalar is also called a Lorentz invariant.

   The Lorentz scalar is in general a function of the components of a 4-vector. For example, we have just shown that

$$\sum_\mu x'^2_\mu = \sum_\lambda x^2_\lambda$$

   Hence, $\sum_\lambda x^2_\lambda$ is a Lorentz scalar.

3. Define the 4-D operator, $\Box \equiv [\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z},\ \frac{\partial}{\partial (ict)}]$, as the counterpart of the operator ∇ in the **x**-space. Then, the 4-gradient of a Lorentz scalar, $\Box\Phi \equiv [\frac{\partial \Phi}{\partial x},\ \frac{\partial \Phi}{\partial y},\ \frac{\partial \Phi}{\partial z},\ \frac{\partial \Phi}{\partial (ict)}]$, is a 4-vector.

Proof :
$$(\Box'\Phi)_\mu = \frac{\partial\Phi}{\partial x'_\mu} = \sum_\nu \frac{\partial\Phi}{\partial x_\nu}\underbrace{\frac{\partial x_\nu}{\partial x'_\mu}}_{=a_{\mu\nu}\text{ by (3)}} = \sum_\nu a_{\mu\nu}\frac{\partial\Phi}{\partial x_\nu} = \underbrace{\sum_\nu a_{\mu\nu}(\Box\Phi)_\nu}_{\text{Transforms as a 4-vector.}} \qquad (6)$$

4. The 4-divergence of a 4-vector, $\Box\cdot\mathbf{A} \equiv \sum_\mu \frac{\partial A_\mu}{\partial x_\mu}$, is a Lorentz scalar.

Proof :

A: 4-vector
A_μ: component of **A**

$$\Box'\cdot\mathbf{A}' = \sum_\nu \frac{\partial A'_\nu}{\partial x'_\nu} = \sum_\nu\sum_\mu \underbrace{\frac{\partial x_\mu}{\partial x'_\nu}}_{=a_{\nu\mu}\text{ by (3)}}\ \overbrace{\frac{\partial A'_\nu}{\partial x_\mu}}^{=\frac{\partial}{\partial x_\mu}\sum_\lambda a_{\nu\lambda}A_\lambda \text{ by (5)}} = \sum_{\mu\lambda}\underbrace{\sum_\nu a_{\nu\mu}a_{\nu\lambda}}_{=\delta_{\mu\lambda}\text{ by (4)}}\frac{\partial A_\lambda}{\partial x_\mu} = \sum_\mu \frac{\partial A_\mu}{\partial x_\mu} = \Box\cdot\mathbf{A} \qquad (7)$$

$\Rightarrow$ $\Box\cdot\mathbf{A}$ is unchanged under the Lorentz transformation

5. The 4-Laplacian operator, $\Box^2 \equiv \Box\cdot\Box = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$, (8)

is a Lorentz scalar operator, i.e. $\Box'^2\Phi = \Box^2\Phi$ [Φ : a Lorentz scalar].

Proof : We have shown in item 4 that the divergence of a 4-vector is a Lorentz scalar, i.e. $\Box'\cdot\mathbf{A}' = \Box\cdot\mathbf{A}$. Let Φ be a Lorentz scalar, then $\mathbf{A}' = \Box'\Phi$ and $\mathbf{A} = \Box\Phi$ are both 4-vectors (see item 3). Hence,

$$\Box'\cdot\mathbf{A}' = \Box\cdot\mathbf{A} \Rightarrow \Box'\cdot\Box'\Phi = \Box\cdot\Box\Phi \;\Rightarrow\; \Box'^2\Phi = \Box^2\Phi.$$

6. The dot product of two 4-vectors, $\mathbf{A}\cdot\mathbf{B} \equiv \sum_\mu A_\mu B_\mu$, is a Lorentz scalar.

Proof :

$$\mathbf{A}'\cdot\mathbf{B}' = \sum_\sigma A'_\sigma B'_\sigma = \sum_\sigma \overbrace{\sum_\nu a_{\sigma\nu}A_\nu}^{A'_\sigma} \overbrace{\sum_\lambda a_{\sigma\lambda}B_\lambda}^{B'_\sigma} = \sum_{\nu\lambda} \overbrace{\sum_\sigma a_{\sigma\nu}a_{\sigma\lambda}}^{=\delta_{\nu\lambda} \text{ by (4)}} A_\nu B_\lambda$$
$$= \sum_\lambda A_\lambda B_\lambda = \mathbf{A}\cdot\mathbf{B} \qquad (9)$$

$$\Rightarrow \mathbf{A}'\cdot\mathbf{A}' = \mathbf{A}\cdot\mathbf{A} \Rightarrow \sum_\mu A'^2_\mu = \sum_\mu A^2_\mu \quad \left[\begin{array}{l}\text{a useful property of the}\\ \text{orthogonal transformation}\end{array}\right]$$

Example: In frame K, a particle's position changes by $d\mathbf{x}$ in a time interval dt. Then, $d\mathbf{x} = (dx, dy, dz, icdt)$ is a 4-vector. Hence, $d\mathbf{x}\cdot d\mathbf{x}$ $(= \sum_\mu dx_\mu dx_\mu)$ is a Lorentz invariant, i.e. in frame K', $d\mathbf{x}'\cdot d\mathbf{x}'$ $(= \sum_\mu dx'_\mu dx'_\mu)$ is given by $d\mathbf{x}\cdot d\mathbf{x}$.

Special case: The particle is at rest in frame K' (the rest frame of the particle). Hence, $d\mathbf{x}' = 0$ and $d\mathbf{x}' = (0, 0, 0, icd\tau)$, where we have denoted the differential time in frame K' by $d\tau$ instead of dt', because frame K' is a unique frame. $d\tau$ is called the proper time of the particle.

$$\sum_\mu dx'_\mu dx'_\mu = \sum_\mu dx_\mu dx_\mu \Rightarrow -c^2 d\tau^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$$

$$\Rightarrow\; d\tau = dt\sqrt{1 - \frac{v^2}{c^2}} = \frac{dt}{\gamma} \quad \text{[a Lorentz invariant]} \qquad (10)$$

where $\mathbf{v} = \frac{dx}{dt}\mathbf{e}_x + \frac{dy}{dt}\mathbf{e}_y + \frac{dz}{dt}\mathbf{e}_z$ is the velocity of the particle in frame K.

Discussion: (i) For the special case that K' is the rest frame of the particle, v is also the relative velocity of the 2 frames. Hence, $\gamma = \gamma_0$.

(ii) The Lorentz transformation applies only to inertial frames. If the particle has an acceleration, $d\tau (= \frac{dt}{\gamma})$ in (10) is the differential time in the *instantaneous* rest frame of the particle, in which the particle has zero velocity but a finite acceleration. In general, the speed of the rest frame (hence γ) is a function of time, i. e. $d\tau = \frac{dt}{\gamma(t)}$ [Jackson, (11.26)].

(iii) Consider a special case of *constant* particle velocity. The muon has a lifetime of 2.2 μsec in its rest frame between birth and decay. If the lifetime is measured in a Lorentz frame in which the muon has a constant γ, then by (10), the rest-frame lifetime (τ_d) and the measured lifetime t_d are related by

$$\int_{\tau_{birth}}^{\tau_{decay}} d\tau = \int_{t_{birth}}^{t_{decay}} \frac{dt}{\gamma} = \frac{1}{\gamma}\int_{t_{birth}}^{t_{decay}} dt \Rightarrow \quad \tau_d = \frac{t_d}{\gamma}.$$

This expresses the phenomenon of time dilation; namely, when the time interval of a clock's rest time (e.g. τ_d above) is observed in a moving frame, it is greater by a factor of γ. The invariance of τ_d $(= \frac{t_d}{\gamma})$ means that $\frac{t_d}{\gamma}$ will have the same value in all Lorentz frames.

7. A 4-tensor of the second rank ($\overleftrightarrow{\mathbf{T}}$) is a set of 16 quantities, $T_{\mu\nu}$ $(\mu, \nu = 1-4)$, which transform according to

$$T'_{\mu\nu} = \sum_{\lambda,\sigma} a_{\mu\lambda}a_{\nu\sigma}T_{\lambda\sigma} \qquad (11)$$

8. The dot product of a 4-tensor of the second rank and a 4-vector, $(\overleftrightarrow{\mathbf{T}}\cdot\mathbf{A})_\mu \equiv \sum_\nu T_{\mu\nu}A_\nu$, is a 4-vector.

Proof :

$$(\overleftrightarrow{\mathbf{T}}'\cdot\mathbf{A}')_\mu = \sum_\nu \overset{\boxed{\sum_{\lambda\sigma} a_{\mu\lambda}a_{\nu\sigma}T_{\lambda\sigma}}}{T'_{\mu\nu}} \overset{\boxed{\sum_\alpha a_{\nu\alpha}A_\alpha}}{A'_\nu} = \sum_{\lambda,\sigma,\alpha} a_{\mu\lambda} \overbrace{\sum_\nu a_{\nu\sigma}a_{\nu\alpha}}^{\delta_{\sigma\alpha}} T_{\lambda\sigma}A_\alpha$$
$$= \sum_\lambda a_{\mu\lambda}\sum_\sigma T_{\lambda\sigma}A_\sigma = \underbrace{\sum_\lambda a_{\mu\lambda}(\overleftrightarrow{\mathbf{T}}\cdot\mathbf{A})_\lambda}_{\text{Transform as a 4-vector.}} \qquad (12)$$

9. The 4-divergence of a 4-tensor of the second rank, $(\Box\cdot\overleftrightarrow{\mathbf{T}})_\mu \equiv \sum_\nu \frac{\partial T_{\mu\nu}}{\partial x_\nu}$, is a 4-vector.

Proof :

$$(\Box'\cdot\overleftrightarrow{\mathbf{T}}')_\mu = \sum_\nu \frac{\partial T'_{\mu\nu}}{\partial x'_\nu} = \sum_\nu \frac{\partial}{\partial x'_\nu}\sum_{\lambda,\sigma} a_{\mu\lambda}a_{\nu\sigma}T_{\lambda\sigma} = \sum_\nu\sum_\alpha \underbrace{\frac{\partial x_\alpha}{\partial x'_\nu}}_{a_{\nu\alpha}} \frac{\partial}{\partial x_\alpha}\sum_{\lambda\sigma} a_{\mu\lambda}a_{\nu\sigma}T_{\lambda\sigma}$$

$$= \sum_{\lambda,\sigma,\alpha} a_{\mu\lambda}\overbrace{\sum_\nu a_{\nu\alpha}a_{\nu\sigma}}^{\delta_{\alpha\sigma}}\frac{\partial T_{\lambda\sigma}}{\partial x_\alpha} = \sum_\lambda a_{\mu\lambda}\sum_\sigma\frac{\partial T_{\lambda\sigma}}{\partial x_\sigma} = \underbrace{\sum_\lambda a_{\mu\lambda}(\Box\cdot\overleftrightarrow{\mathbf{T}})_\lambda}_{\text{Transform as a 4-vector.}} \quad (13)$$

10. A 4-tensor of the third rank is a set of 64 quantities, $G_{\lambda\mu\nu}(\lambda,\mu,\nu=1\text{-}4)$, which transform according to

$$G'_{\lambda\mu\nu} = \sum_{ijk} a_{\lambda i}a_{\mu j}a_{\nu k}G_{ijk} \quad (14)$$

Problem 1: If $F_{\mu\nu}$ is a 4-tensor of the second rank, show that

$\frac{\partial F_{\mu\nu}}{\partial x_\lambda}(\lambda,\mu,\nu=1\text{-}4)$ is a 4-tensor of the third rank.

Solution: $F'_{\mu\nu} = \sum_{jk} a_{\mu j}a_{\nu k}F_{jk}$

$$\Rightarrow \frac{\partial F'_{\mu\nu}}{\partial x'_\lambda} = \sum_{jk} a_{\mu j}a_{\nu k}\sum_i \frac{\partial F_{jk}}{\partial x_i}\overbrace{\frac{\partial x_i}{\partial x'_\lambda}}^{a_{\lambda i}} = \underbrace{\sum_{ijk} a_{\lambda i}a_{\mu j}a_{\nu k}\frac{\partial F_{jk}}{\partial x_i}}_{\text{Transform as a 4-tensor of the third rank.}} \quad (15)$$

Problem 2: Show that the set of equations,

$$\frac{\partial F_{\mu\nu}}{\partial x_\lambda}+\frac{\partial F_{\lambda\mu}}{\partial x_\nu}+\frac{\partial F_{\nu\lambda}}{\partial x_\mu}=0\ (\lambda,\mu,\nu=1\text{-}4) \quad (16)$$

is invariant in form under the Lorentz transformation.

Solution: Rewrite (15): $\frac{\partial F'_{\mu\nu}}{\partial x'_\lambda} = \sum_{ijk} a_{\lambda i}a_{\mu j}a_{\nu k}\frac{\partial F_{jk}}{\partial x_i}$

Change indecies in (15) as follows: $\begin{cases}\lambda\to\nu,\ \mu\to\lambda,\ \nu\to\mu\\ i\to k,\ \ k\to j,\ j\to i\end{cases}$

$$\Rightarrow \frac{\partial F'_{\lambda\mu}}{\partial x'_\nu} = \sum_{ijk} a_{\lambda i}a_{\mu j}a_{\nu k}\frac{\partial F_{ij}}{\partial x_k} \quad (17)$$

Change indecies in (17) as follows: $\begin{cases}\nu\to\mu,\ \lambda\to\nu,\ \mu\to\lambda\\ k\to j,\ i\to k,\ \ j\to i\end{cases}$

$$\Rightarrow \frac{\partial F'_{\nu\lambda}}{\partial x'_\mu} = \sum_{ijk} a_{\lambda i}a_{\mu j}a_{\nu k}\frac{\partial F_{ki}}{\partial x_j} \quad (18)$$

Combine (15), (17), and (18),

$$\Rightarrow \frac{\partial F'_{\mu\nu}}{\partial x'_\lambda}+\frac{\partial F'_{\lambda\mu}}{\partial x'_\nu}+\frac{\partial F'_{\nu\lambda}}{\partial x'_\mu} = \sum_{ijk} a_{\lambda i}a_{\mu j}a_{\nu k}\overbrace{\left(\frac{\partial F_{jk}}{\partial x_i}+\frac{\partial F_{ij}}{\partial x_k}+\frac{\partial F_{ki}}{\partial x_j}\right)}^{=0\text{ by (16)}} = 0$$

11. If a physical law can be expressed as a relation between 4-tensors of the same rank, then it's form is invariant in all Lorentz frames.

Example 1: If the physical law in frame K is of the form $\mathbf{A}=\mathbf{B}$, then, $A'_\nu = \sum_\mu a_{\mu\nu}\underbrace{A_\mu}_{B_\mu} = \sum_\mu a_{\mu\nu}B_\mu = B'_\nu$, i.e. $\mathbf{A}=\mathbf{B}\Rightarrow\mathbf{A}'=\mathbf{B}'$. (19)

Example 2: If the physical law in frame K is of the form $\overleftrightarrow{\mathbf{T}}=\overleftrightarrow{\mathbf{F}}$, then, $T'_{\mu\nu} = \sum_{\lambda\sigma} a_{\mu\lambda}a_{\nu\sigma}\underbrace{T_{\lambda\sigma}}_{F_{\mu\nu}} = \sum_{\lambda\sigma} a_{\mu\lambda}a_{\nu\sigma}F_{\mu\nu} = F'_{\mu\nu}$, i.e.

$$\overleftrightarrow{\mathbf{T}}=\overleftrightarrow{\mathbf{F}}\Rightarrow\overleftrightarrow{\mathbf{T}}'=\overleftrightarrow{\mathbf{F}}' \quad \text{[i.e. invariant in form]} \quad (20)$$

In the following section, we examine relativistic mechanics in 4-vector formalism. In Sec. 3, we will demonstrate that laws of electromagnetism are in variant under the Lorentz transformation by expressing them as relations between tensors of the same rank. From the Lorentz transformation of these tensors, we also obtain the transformation equations for various electromagnetic quantities.

Section 2: Relativistic Mechanics

We begin with a note on the terms "conservation", "invariance", and "covariance".

The conservation of a quantity means that it remains unchanged in time in a given Lorentz frame. For example, the relativistic momentum and energy of an isolated system of particles are both conserved after a collision. This is a fundamental law to be discussed in this Section.

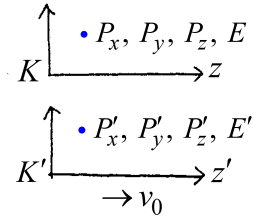
The invariance of a quantity means that it is invariant in value under a Lorentz transformation. Such a quantity is called a Lorentz invariant or Lorentz scalar. For example, the dot product of two 4-vectors is a Lorentz invariant. However, it may or may not be a conserved quantity. An example will be provided in this section.

The term covariance refers to physical laws. A physical law is "covariant" if it is "invariant in form under the Lorentz transformation." As will be shown, the new laws of relativistic mechanics and existing laws of electromagnetism are all covariant.

The 4-Momentum (p) of a Single Particle:

As shown in (A.28), if we define the momentum of a particle as $\mathbf{p} \equiv \gamma m \mathbf{v}$ and energy as $E \equiv \gamma mc^2$ (m is called the rest mass*), then the 4-momentum, $\mathsf{p} \equiv (p_x, p_y, p_z, \frac{iE}{c})$, is a 4-vector, which transforms as

$$\begin{cases} p_x' = p_x & (21.1) \\ p_y' = p_y & (21.2) \\ p_z' = \gamma_0 (p_z - \frac{v_0}{c^2} E) & (21.3) \\ E' = \gamma_0 (E - v_0 p_z) & (21.4) \end{cases}$$



*Throughout this chapter, m and M denote the rest mass.

Discussion: In Appendix A, we first define $\mathbf{p} = \gamma m \mathbf{v}$ and $E = \gamma mc^2$, then show that the law of conservation of momentum and energy is covariant. Conversely, from the requirement of the covariance of this conservation law, we can deduce the definitions of $\mathbf{p} = \gamma m \mathbf{v}$ and $E = \gamma mc^2$ (see Jackson Sec. 11.5).

The dot product of two 4-vectors is a Lorentz scalar, hence

$$\mathsf{p} \cdot \mathsf{p} = \mathsf{p}' \cdot \mathsf{p}' \Rightarrow p^2 - \frac{E^2}{c^2} = p'^2 - \frac{E'^2}{c^2} \qquad (22)$$

i.e. $E^2 - p^2c^2$ is a Lorentz scalar (invariant).

If frame K' is the rest frame of the particle (i.e. $p' = 0$, $E' = mc^2$) then $\mathsf{p}' = (0, 0, 0, imc)$ and $\mathsf{p} \cdot \mathsf{p} = \mathsf{p}' \cdot \mathsf{p}'$ gives $p^2 - \frac{E^2}{c^2} = -m^2c^2$, or

$$E^2 - p^2c^2 = m^2c^4 \qquad (23)$$

Since $E^2 - p^2c^2$ is a Lorentz invariant, (23) shows that the rest mass m is a Lorentz invariant. This has in fact been assumed in Sec. 2 of Appendix A, where we derive the Lorentz transformation equations for $\mathbf{p}$ $(= \gamma m \mathbf{v})$ and E $(= \gamma mc^2)$. (23) is a useful formula for it relates the particle's total energy (E) to its momentum (p). (Momentum in particle physics is often expressed in unit of GeV/c.)

For a relativistic particle, we can still speak of its (macroscopic) kinetic energy K, defined as: $K = E - mc^2 = (\gamma - 1)mc^2$. (24)

The 4-Momentum (P) of a System of Particles

Consider a system of particles, each with the 4-momentum

$$\mathsf{p}_j = (p_{xj}, p_{yj}, p_{zj}, iE_j/c) = (\mathbf{p}_j, iE_j/c), \; j = 1, 2, 3, \cdots$$

Since the Lorentz transformation is a linear transformation, the sum of any number of 4-vectors also obeys the Lorentz tansformation. Thus, $\mathsf{P} = \sum_j \mathsf{p}_j$ is a 4-vector and its components transform as

$$\begin{cases} \sum_j p_{xj}' = \sum_j p_{xj} & (25.1) \\ \sum_j p_{yj}' = \sum_j p_{yj} & (25.2) \\ \sum_j p_{zj}' = \gamma_0 (\sum_j p_{zj} - \frac{v_0}{c^2} \sum_j E_j) & (25.3) \\ \sum_j E_j' = \gamma_0 (\sum_j E_j - v_0 \sum_j p_{zj}) & (25.4) \end{cases}$$

and $\mathsf{P} \cdot \mathsf{P} = (\sum_j \mathsf{p}_j) \cdot (\sum_j \mathsf{p}_j) = (\sum_j \mathbf{p}_j) \cdot (\sum_j \mathbf{p}_j) - (\sum_j E_j/c)^2 \qquad (26)$

is a Lorentz invariant.

Rewrite (25):
$$\begin{cases} \sum_j p'_{xj} = \sum_j p_{xj} \\ \sum_j p'_{yj} = \sum_j p_{yj} \\ \sum_j p'_{zj} = \gamma_0 (\sum_j p_{zj} - \frac{v_0}{c^2} \sum_j E_j) \\ \sum_j E'_j = \gamma_0 (\sum_j E_j - v_0 \sum_j p_{zj}) \end{cases}$$

We see from (25) that only when all the components of **P** (i.e. the three components of total momentum plus the total energy) are each conserved in frame K will all the components of $\mathbf{P}'$ be conserved. If one component of **P** is not conserved, a rotation of the spatial coordinate system can make any component of $\mathbf{P}'$ (momentum or energy) unconserved in the new spatial coordinate system. Thus, the relativistic law of conservation must take the form as described below in order for it to be a covariant law.

Law of Conservation of Momentum and Energy:

For reasons just discussed, in relativity, the conservation of momentum and energy comes in one law rather than separate laws for the momentum and energy as in nonrelativistic mechanics. The law states that, for an *isolated* system of particles,

$$\mathbf{P}(\text{before collision}) = \mathbf{P}(\text{after collision}), \qquad (27)$$

which implies that $\sum_j p_{xj}$, $\sum_j p_{yj}$, $\sum_j p_{zj}$, and $\sum_j E_j$ are each conserved, i.e.

$$\sum_j \mathbf{p}_j(\text{before collision}) = \sum_j \mathbf{p}_j(\text{after collision}) \qquad (28)$$

$$\sum_j E_j(\text{before collision}) = \sum_j E_j(\text{after collision}) \qquad (29)$$

Since the law in (27) is expressed as a 4-vector relation, it has the same form in all Lorentz frames [see (19)]. Thus, in frame K', we have $\mathbf{P}'(\text{before collision}) = \mathbf{P}'(\text{after collision})$.

If **P** is conserved, the dot product $\mathbf{P}\cdot\mathbf{P}$ must also be conserved. Thus,

$$\underbrace{(\sum_j \mathbf{p}_j)\cdot(\sum_j \mathbf{p}_j) - (\sum_j \frac{E_j}{c})^2}_{\text{before collision}} = \underbrace{(\sum_j \mathbf{p}_j)\cdot(\sum_j \mathbf{p}_j) - (\sum_j \frac{E_j}{c})^2}_{\text{after collision}} \qquad (30)$$

Discussion:

(i) $\mathbf{P}\cdot\mathbf{P}$ for an *isolated* system is both a Lorentz invariant [see (26)] and a conserved quantity [see (30)]. If the system is not isolated, it is still a Lorentz invariant, but no longer a conserved quantity.

(ii) $\mathbf{P}(\text{before collision}) = \mathbf{P}(\text{after collision})$ in (27) is a fundamental law (rather than a derived relation), in which the nonrelativistic law of conservation of momentum has been extended to include the energy, $E = \gamma mc^2$. A very important aspect of this law is that it applies to all processes in an isolated system, such as elastic and inelastic collisions, nuclear reactions, and particle decays. As a result, the total rest mass of the system may not be conserved, as is illustrated in the following two problems.

Problem 1: Two identical particles of rest mass m and equal and opposite velocities $\pm\mathbf{v}$ collide head-on inelastically to form a single particle. Find the mass and velocity of the new particle.

Solution:

$$\boxed{\begin{array}{ll} m,\ \gamma\ \bullet \rightarrow \mathbf{v}\quad \mathbf{v} \leftarrow \bullet\ m,\ \gamma & \text{(before)} \\ \qquad M_{cm}\ \bullet & \text{(after)} \end{array}}$$

The total momentum before the collision is $\gamma m\mathbf{v} - \gamma m\mathbf{v} = 0$.

So the collision occurs in the center-of-momentum (CM) frame, i.e. the frame in which the sum of the momentum of all particles vanishes. For later comparison with the result in problem 2, we denote the mass of the new particle by M_{cm} to indicate that it is created in the CM frame.

$$\begin{cases} \text{Conservation of momentum} \Rightarrow \text{The new particle is stationary.} \\ \text{Conservation of energy} \Rightarrow \gamma m + \gamma m = M_{cm} \Rightarrow M_{cm} = 2\gamma m \end{cases}$$

$m,\ \gamma\ \bullet\!\rightarrow \mathbf{v}\quad \mathbf{v}\leftarrow\!\bullet\ m,\ \gamma$ (before)
$M_{cm}\ \bullet$ (after)

Discussion: In this problem, we find $M_{cm} = 2\gamma m > 2m$, i.e. rest mass has been created from the kinetic energy [$(\gamma$-$1)mc^2$] of the colliding particles. There is no need to know what's inside the new particle. We only need to know its rest mass and hence the energy associated with it. A hot object has a rest mass greater than when it's cold. The difference in rest mass due to an increase in temperature can in principle be measured by its acceleration under a known force, and we know that at least some of the added mass is in the form of thermal energy. In many other cases, it's not possible to know what's inside.

Nuclear fusion and fission reactions are examples of non-conservation of rest mass. The total rest mass is reduced after the reaction and the mass deficit appears as kinetic energies and radiation. In fact, all reactions (chemical or nuclear) in which energy is absorbed (e.g. photosynthesis) or released (e.g. digestion of food) involve a corresponding change of the reactants' total rest mass.

Problem 2: A particle of rest mass m and velocity $\mathbf{v}$ collides with a stationary particle of the same rest mass and is absorbed by it. Find the rest mass and velocity of the new particle.

Solution: The collision occurs in the stationary-target (ST) frame. So, we denote the new particle mass by M_{st}, velocity by $\mathbf{V}_{st}$, and Lorentz factor by γ_{st} $[=(1-V_{st}^2/c^2)^{-1/2}]$. $(m, \gamma, \mathbf{v}$ are also ST frame quantities.)

$$\begin{cases} \text{Conservation of momentum} \Rightarrow \ \gamma m\mathbf{v} = \gamma_{st} M_{st} \mathbf{V}_{st} & (31) \\ \text{Conservation of energy} \quad \Rightarrow \ (\gamma+1)m = \gamma_{st} M_{st} & (32) \end{cases}$$

$$\frac{(31)}{(32)} \Rightarrow \mathbf{V}_{st} = \frac{\gamma}{\gamma+1}\mathbf{v}$$

$$(32) \Rightarrow M_{st} = \frac{\gamma+1}{\gamma_{st}} m$$

$m,\ \gamma\ \bullet\!\longrightarrow \mathbf{v}\quad \bullet\ m$ (before)
$M_{st},\ \gamma_{st}\ \bullet\!\longrightarrow \mathbf{V}_{st}$ (after)

$$\Rightarrow M_{st}^2 = m^2\frac{(\gamma+1)^2}{\gamma_{st}^2} = m^2(\gamma+1)^2\left(1-\frac{V_{st}^2}{c^2}\right)^2 = m^2(\gamma+1)^2\left[1-\frac{\gamma^2 v^2}{c^2(\gamma+1)^2}\right]$$

$$= m^2\left(\gamma^2+2\gamma+1-\gamma^2\frac{v^2}{c^2}\right) = m^2\left[\gamma^2\left(1-\frac{v^2}{c^2}\right)^2+2\gamma+1\right] = 2m^2(\gamma+1)$$

$$\Rightarrow M_{st} = \sqrt{2(\gamma+1)}\,m$$

Discussion:

$m,\ \gamma\ \bullet\!\rightarrow \mathbf{v}\quad \mathbf{v}\leftarrow\!\bullet\ m,\ \gamma$ (before)
$M_{cm}\ \bullet$ (after)

In problem 1 (CM frame), the new particle's mass is $M_{cm} = 2\gamma m$. (33)

$m,\ \gamma\ \bullet\!\longrightarrow \mathbf{v}\quad \bullet\ m$ (before)
$M_{st},\ \gamma_{st}\ \bullet\!\longrightarrow \mathbf{V}_{st}$ (after)

In problem 2 (ST frame), the new particle's mass is $M_{st} = \sqrt{2(1+\gamma)}\,m$. (34)

Note that γ is the Lorentz factor of the particle(s) before collision.

In particle physics experiments, $M_{cm}c^2$ or $M_{st}c^2$ is the energy available for the creation of new particles (why not $\gamma_{st}M_{st}c^2$?).

The rest energy of the electron or positron is $mc^2 = 0.511$ MeV. If 2 TeV of energy is needed for particle creation (i.e. $M_{cm}c^2 = 2$ TeV or $M_{st}c^2 = 2$ TeV), then the required γ of the colliding particle(s) is

$$\begin{cases} \text{by (33)}, M_{cm}c^2 = 2\gamma mc^2 = 2\text{ TeV} \Rightarrow \gamma \approx 1.957\times 10^6 & \text{[CM frame]} \\ \text{by (34)}, M_{st}c^2 = \sqrt{2(1+\gamma)}\,mc^2 = 2\text{ TeV} \Rightarrow \gamma \approx 7.66\times 10^{12} & \text{[ST frame]} \end{cases}.$$

The energy associated with γ is to be obtained in an accelerator.

Thus,

$$\frac{\text{kinetic energy needed in CM frame}}{\text{kinetic energy needed in ST frame}} = \frac{2\times(1.957\times10^6-1)}{7.66\times10^{12}-1} \approx 5\times10^{-7}$$

This shows that far less kinetic energy is needed in the CM frame than in the ST frame. In fact, all the kinetic energy of the two colliding particles $[2\times(1.957\times10^6-1)\times 0.511\text{ MeV} = 2\text{ TeV}]$ is put in use in the CM frame, while in the ST frame, 99.99995% of the kinetic energy of the incident particle is wasted! This is why the International Linear Collider (ILC) project plans to accelerate both electrons and positrons to energies up to 1 TeV so that the the collision occurs in the CM frame.

Question: Why use a long linear accelerator instead of a more compact circular accelerator?

Section 3: Covariance of Electrodynamics

In the special theory of relativity, Newton's law has been radically modified. The electromagnetic laws do not need any modification because they are already covariant. However, the *covariance* of these laws (such as Maxwell equations) is not immediately clear from the equations by which they are usually represented.

Our purpose in this section is to prove that the EM laws are indeed covariant by casting them into relations between 4-tensors of the same rank [see (19) and (20)]. We will do this by first defining 4-tensors in terms of known EM quantities and forming equations with 4-tensors of the same rank, then show that one or more existing EM laws are implicit in each equation. This will prove that the laws are convariant and justify the defined quantities to be legitimate 4-tensors.

Furthermore, Lorentz transformations of these tensors will yield the tranformation equations for various EM quantities.

Note: Jackson switches to the Gausian unit system starting from Ch. 11. From here on, we also adopt the Gaussian unit system.

1. Define a 4-current as $\quad (c\rho,\ J_x,\ J_y,\ J_z,\ ) \leftarrow$ Griffiths

$$\mathbf{J} \equiv (J_x,\ J_y,\ J_z,\ ic\rho) = (\mathbf{J},\ ic\rho) \tag{35}$$

and use it to form a relation

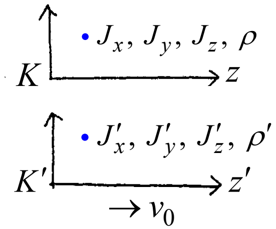
$$\Box \cdot \mathbf{J} = 0 \tag{36}$$

Then, (36) gives the law of conservation of charge

$$\tfrac{\partial}{\partial x} J_x + \tfrac{\partial}{\partial y} J_y + \tfrac{\partial}{\partial z} J_z + \frac{\partial(ic\rho)}{\partial(ict)} = 0 \ \Rightarrow\ \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \tag{5.2}$$

Thus, the definition of $\mathbf{J}$ in (35) as a 4-vector leads to the covariant representation [(36)] of the EM law in (5.2). This in turn justifies the definition of $\mathbf{J}$ as a 4-vector. The Lorentz transformation of $\mathbf{J}$ then gives

$$\begin{cases} J'_x = J_x \\ J'_y = J_y \\ J'_z = \gamma_0 (J_z - v_0 \rho) \\ \rho' = \gamma_0 \left(\rho - \frac{v_0}{c^2} J_z\right) \end{cases} \tag{37}$$



$(\frac{V}{c},\ A_x,\ A_y,\ A_z,\ ) \leftarrow$ Griffiths

2. Define a 4-potential as $\mathbf{A} \equiv (A_x,\ A_y,\ A_z,\ i\Phi)$ (38)

and write the covariant relations:
$$\begin{cases} \Box^2 \mathbf{A} = -\frac{4\pi}{c} \mathbf{J} & (39) \\ \Box \cdot \mathbf{A} = 0 & (40) \end{cases}$$

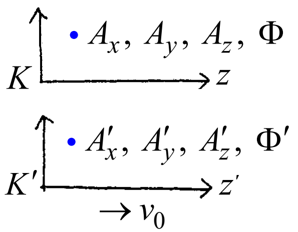
$$(39) \Rightarrow \begin{cases} \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A} = -\frac{4\pi}{c} \mathbf{J} & (6.15) \\ \nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Phi = -4\pi\rho & (6.16) \end{cases}$$

$\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = 0 \ \leftarrow$ Griffiths

$$(40) \Rightarrow\ \nabla \cdot \mathbf{A} + \tfrac{1}{c} \tfrac{\partial}{\partial t} \Phi = 0 \ \ \text{[Lorenz condition]} \tag{6.14}$$

This again shows the consistency of $\mathbf{A}$ being a 4-vector and (6.14)-(6.16) being covariant laws. The Lorentz transformation of $\mathbf{A}$ then gives

$$\begin{cases} A'_x = A_x \\ A'_y = A_y \\ A'_z = \gamma_0 \left(A_z - \frac{v_0}{c} \Phi\right) \\ \Phi' = \gamma_0 \left(\Phi - \frac{v_0}{c} A_z\right) \end{cases} \tag{41}$$



Note: The source-free wave equation can be directly put into the covariant form: $\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi = 0 \ \Rightarrow\ \Box^2 \psi = 0.$ (42)

3. Define a 4-wavenumber as

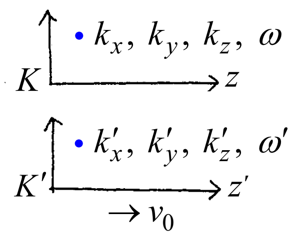
$$\mathbf{k} \equiv \left(k_x,\ k_y,\ k_z,\ \tfrac{i\omega}{c}\right) = \left(\mathbf{k},\ \tfrac{i\omega}{c}\right) \tag{43}$$

Then, $\mathbf{k}' \cdot \mathbf{x}' = \mathbf{k} \cdot \mathbf{x} \Rightarrow\ \mathbf{k}' \cdot \mathbf{x}' - \omega' t' = \mathbf{k} \cdot \mathbf{x} - \omega t$

$\Rightarrow$ Invariance of the phase

By the same argument, we find that $\mathbf{k}$ defined in (43) is a legitimate 4-vector. Thus, its Lorentz transformation gives

$$\begin{cases} k'_x = k_x \\ k'_y = k_y \\ \underbrace{k'_z = \gamma_0 \left(k_z - \frac{v_0}{c^2} \omega\right) \\ \omega' = \gamma_0 (\omega - v_0 k_z)}_{\text{relativistic Doppler shift}} \end{cases} \tag{44}$$



4. Define a field strength tensor of the second rank $\overleftrightarrow{\mathbf{F}}$ [Marion, (14.62)]:

$$\overleftrightarrow{\mathbf{F}} \equiv \begin{bmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{bmatrix} \qquad (45)$$

Then, $\Box\cdot\overleftrightarrow{\mathbf{F}} = \frac{4\pi}{c}\mathbf{J} \Rightarrow \begin{cases} \nabla\cdot\mathbf{E} = 4\pi\rho \\ \nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J} \end{cases}$

$$\text{SI}\begin{cases} \nabla\cdot\mathbf{D} = \rho \\ \nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t} = \mathbf{J} \\ \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \end{cases}$$

In the covariant set of equations [see (16)]

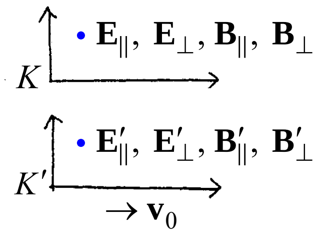
$$\frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0 \quad (\lambda,\mu,\nu = 1\text{-}4) \quad \left[\begin{matrix}F_{\mu\nu}\text{'s are elements} \\ \text{of } \overleftrightarrow{\mathbf{F}} \text{ in (45).}\end{matrix}\right],$$

set $(\lambda,\mu,\nu) = (1,2,3) \Rightarrow \nabla\cdot\mathbf{B} = 0$

set $(\lambda,\mu,\nu) = (1,2,4)$, $(1,3,4)$, and $(2,3,4) \Rightarrow \nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = 0.$

The covariant equations, $\Box\cdot\overleftrightarrow{\mathbf{F}} = \frac{4\pi}{c}\mathbf{J}$ and $\frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0$, give the set of Maxwell equations in free space. This shows that Maxwell equatins are covariant as well as justifies the definition of $\overleftrightarrow{\mathbf{F}}$ as a tensor of the second rank. Thus, $F'_{\mu\nu} = \sum_{\lambda,\sigma} a_{\mu\lambda} a_{\nu\sigma} F_{\lambda\sigma}$ gives the transformation equations for **E** and **B** (see Marion, Sec. 14.6.)

$$\begin{cases} \mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \\ \mathbf{E}'_{\perp} = \gamma_0\left(\mathbf{E}_\perp + \frac{\mathbf{v}_0}{c}\times\mathbf{B}_\perp\right) \\ \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \\ \mathbf{B}'_{\perp} = \gamma_0\left(\mathbf{B}_\perp - \frac{\mathbf{v}_0}{c}\times\mathbf{E}_\perp\right) \end{cases} \qquad (46)$$



In (46), $\mathbf{v}_0$ is the velocity of frame K' relative to frame K, and "$\parallel$" and "$\perp$" refer to the direction of $\mathbf{v}_0$.

See Appendix C for a summary of transformation equations.

5. The covariant equation*, $\frac{d}{d\tau}\mathbf{P} = \frac{e}{mc}\overleftrightarrow{\mathbf{F}}\cdot\mathbf{P}$ ($d\tau$ is a Lorentz scalar), gives (Marion, p.439)

$\Rightarrow \frac{d\mathbf{P}}{dt} = \frac{d\tau}{dt}\frac{e}{mc}\overleftrightarrow{\mathbf{F}}\cdot\mathbf{P}$, where $\mathbf{p} \equiv (p_x, p_y, p_z, \frac{iE}{c})$ and $\mathbf{p} \equiv \gamma m\mathbf{v}$

$$\overleftrightarrow{\mathbf{F}} \equiv \begin{bmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{bmatrix}$$

$$\frac{d}{dt}p_x = \frac{e}{\gamma mc}\left(E_x\frac{E}{c} + \gamma m(v_y B_z - v_z B_y)\right) = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)_x$$

$$\begin{cases} \frac{d}{dt}\mathbf{p} = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right) & \left[\begin{matrix}\text{relativistic equation} \\ \text{of motion}\end{matrix}\right] \\ mc^2\frac{d}{dt}\gamma = e\mathbf{v}\cdot\mathbf{E} & \left[\begin{matrix}\text{This equation is} \\ \text{implicit in (47).}\end{matrix}\right] \end{cases} \qquad (47)$$

*In order for this equation to be covariant, the charge e must be a Lorentz invariant. This has been experimentally established (see Jackson, p.554).

6. In a similar manner, we can demonstrate the covariance of the conservation laws for field/mechanical energy and field/mechanical momentum, as given by Jackson (6.111) and (6.122):

$$\begin{cases} \frac{d}{dt}\left(E_{\text{mech}} + E_{\text{field}}\right) = -\oint_S \mathbf{n}\cdot\mathbf{S}\,da & (6.111) \\ \frac{d}{dt}\left(\mathbf{p}_{\text{mech}} + \mathbf{p}_{\text{field}}\right) = \oint_S \sum_\beta T_{\alpha\beta} n_\beta\, da & (6.122) \end{cases}$$

Consider the general form of the relativistic equation of motion in (47), $\frac{d}{dt}\mathbf{p} = \mathbf{F}$, where $\mathbf{F}$ is any force, such as the gravitational force.

Special case 1: $\mathbf{F} \parallel \mathbf{v}$ (one-dimensional problem)

$$F = \frac{d}{dt}(\gamma m v) = m v \underbrace{\frac{d\gamma}{dt}} + \gamma m \frac{dv}{dt} = \gamma m \frac{dv}{dt}\underbrace{\left(\gamma^2 \frac{v^2}{c^2} + 1\right)} = \gamma^3 m \frac{dv}{dt} \qquad (48)$$

$$\boxed{\begin{aligned} \frac{d}{dt}\gamma &= \frac{d}{dt}\left(1-\frac{v^2}{c^2}\right)^{-1/2} \\ &= \frac{-1}{2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left(\frac{-2v}{c^2}\right)\frac{dv}{dt} = \gamma^3 \frac{v}{c^2}\frac{dv}{dt} \end{aligned}}$$

$$\boxed{\begin{aligned} \gamma^2 \frac{v^2}{c^2} + 1 &= \frac{v^2/c^2}{1-v^2/c^2} + 1 \\ &= \frac{v^2/c^2 + 1 - v^2/c^2}{1-v^2/c^2} = \frac{1}{1-v^2/c^2} = \gamma^2 \end{aligned}}$$

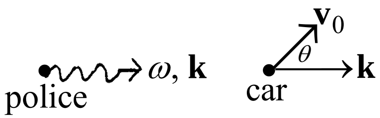
$\Rightarrow F = \gamma^3 m a \Rightarrow$ Constant force does not cause constant acceleration.

Special case 2: $\mathbf{F} \perp \mathbf{v}$ ($\Rightarrow \gamma = const.$, as in uniform circular motion)

$$\Rightarrow \mathbf{F} = \frac{d}{dt}\mathbf{p} = \frac{d}{dt}(\gamma m \mathbf{v}) = \gamma m \frac{d}{dt}\mathbf{v} \quad \text{(Undulator \& Wiggler)} \qquad (49)$$

Questions: (i) It is sometimes said that a particle has two masses, $\gamma^3 m$ and γm. Why? (ii) The acceleration is not necessarily parallel to the force. Give an example. (iii) Relate (48) to (A.23).

Problem 1: A police radar operates on a frequency of ω. What is the frequency received by the police after the signal is reflected from a car moving at the velocity $\mathbf{v}_0$?



shown in police frame

Solution: We do it in 2 steps.

Step 1. In the police frame, the radar sends a wave (ω, $\mathbf{k}$) toward the car, which is moving at velocity $\mathbf{v}_0$ (direction shown in the figure). Transformming ω to the car frame by (44), we obtain

$$\omega' = \gamma_0(\omega - v_0 k_z),$$

where k_z is the component of $\mathbf{k}$ along $\mathbf{v}_0$, i.e. $k_z = k\cos\theta$ (see figure.)

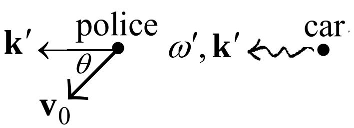
Thus, $\boxed{k = \omega/c}$ $\boxed{\gamma_0 = (1 - v_0^2/c^2)^{-1/2}}$

$$\omega' = \gamma_0(\omega - k_z v_0) = \gamma_0(\omega - k v_0 \cos\theta) = \gamma_0 \omega\left(1 - \frac{v_0\cos\theta}{c}\right)$$

This is the Dopper-shifted frequency detected by the car. It is also the frequency of the wave reflected by the car as seen in the car frame.

Step 2. In the car frame (see figure), the car sends the reflected wave (ω', $\mathbf{k}'$) back to the car at the frequency

$$\omega' = \gamma_0 \omega\left(1 - \frac{v_0\cos\theta}{c}\right)$$



shown in car frame

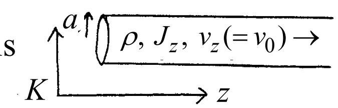
In the car frame, the police is moving at velocity $\mathbf{v}_0$ (direction shown in the fugure) relative to the car. Tranformming ω' to the police frame by (44), we obtain the frequency observed by the police (Doppler shifted again)

$$\omega'' = \gamma_0(\omega' - k_z' v_0) = \gamma_0(\omega' - k' v_0 \cos\theta) = \gamma_0 \omega'\left(1 - \frac{v_0\cos\theta}{c}\right)$$
$$= \gamma_0^2 \omega\left(1 - \frac{v_0\cos\theta}{c}\right)^2 \approx \omega\left(1 - 2\frac{v_0\cos\theta}{c}\right) \quad \text{since } v_0 \ll c.$$

If the radar frequency is $f(=\omega/2\pi) = 10^9$ Hz and the car moves away from the police ($\theta = 0$) at $v_0 = 150$ km/hr, the police would detect a frequency $f''(=\omega''/2\pi)$ shifted by $\Delta f \approx -f\frac{2v_0}{c} \approx -278$ Hz.

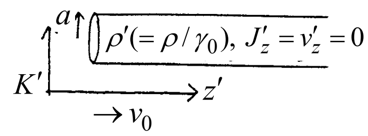
Problem 2: An observer in the laboratory sees an infinite electron beam of radius a and uniform charge density ρ, moving axially at velocity v_0. What force does he see on an electron at a distance r ($\leq a$) from the axis? Assume the electron moves axially at the velocity v_0.

Solution: The problem can be readily solved in the lab frame. Here, we will take a long route for an exercise on some of the transformation equations just derived.



The current density J_z in the lab frame is

$J_z = \rho v_0$. [ρ has a negative value.]



By (37), we have, in the beam frame

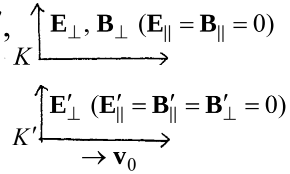
$$J_z' = \gamma_0(J_z - v_0 \rho) = 0,$$

$$\rho' = \gamma_0\left(\rho - \frac{v_0}{c^2}J_z\right) = \gamma_0 \rho\left(1 - \frac{v_0^2}{c^2}\right) = \frac{\rho}{\gamma_0}.$$

We see that the lab frame ρ is greater than the beam frame ρ' by the factor γ_0. This is because every unit length of the beam in its rest frame is contracted by this factor when viewed in the lab frame.

In the beam frame, $J_z' = 0$, $\rho' = \rho/\gamma_0$; hence, there is only a radial electric field. Gauss law, $\oint_{S'} \mathbf{E}' \cdot d\mathbf{a}' = 4\pi \int_{V'} \rho' d^3x'$, then gives $2\pi r' E_r' = 4\pi(\pi\rho' r'^2)$, for $r' \le a$

$$\Rightarrow E_r' = 2\pi\rho' r' = \frac{2\pi\rho r}{\gamma_0}. \quad [r' = r,\ \rho' = \rho/\gamma_0]$$



We now transform $\mathbf{E}_\perp' (= E_r' \mathbf{e}_r)$ into lab-frame $\mathbf{E}_\perp$ and $\mathbf{B}_\perp$ by using the reverse transformation equations in (46), in which we set $\mathbf{v}_0 = v_0 \mathbf{e}_z$.

$$\begin{cases} \mathbf{E}_\perp = \gamma_0(\mathbf{E}_\perp' - \frac{\mathbf{v}_0}{c} \times \cancel{\mathbf{B}_\perp'}) = \gamma_0 \mathbf{E}_\perp' = \gamma_0 \frac{2\pi\rho r}{\gamma_0} \mathbf{e}_r = 2\pi\rho r \mathbf{e}_r \\ \mathbf{B}_\perp = \gamma_0(\cancel{\mathbf{B}_\perp'} + \frac{\mathbf{v}_0}{c} \times \mathbf{E}_\perp') = \gamma_0 (\frac{v_0 \mathbf{e}_z}{c}) \times \frac{2\pi\rho r}{\gamma_0} \mathbf{e}_r = \frac{v_0}{c} 2\pi\rho r \mathbf{e}_\theta \end{cases}$$

Thus, the force $\mathbf{f}$ on an electron (in the lab frame) is

$$\mathbf{f} = -e\left(\mathbf{E} + \tfrac{1}{c}\mathbf{v} \times \mathbf{B}\right) = -e\left[2\pi\rho r \mathbf{e}_r + \tfrac{1}{c}(v_0 \mathbf{e}_z) \times (\tfrac{v_0}{c} 2\pi\rho r \mathbf{e}_\theta)\right]$$

$$= -2\pi e \rho r (1 - \frac{v_0^2}{c^2}) \mathbf{e}_r = -\frac{2\pi e \rho r}{\gamma_0^2} \mathbf{e}_r$$

[$e \equiv |e|$ is positive. For an electron beam, ρ is negative.]

Homework of Chap. 11

Problems: 3, 4, 5, 6, 9
16, 19, 23, 30

Appendix A: Relativity in College Physics

(Ref. Halliday, Resnick, and Walker, "Fundamentals of Physics")

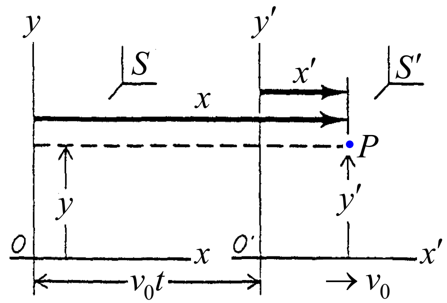
Section 1: The Lorentz Transformation

The Galilean Transformation: Consider 2 *inertial frames* S and S'. Frame S' moves along the common x-axis* with constant speed v_0 relative to frame S. At $t = 0$, the coordinates coincide and, at time t, the position of point P is (x, y, z) in S and (x', y', z') in S'. Then the Galilean transformation gives

$$x' = x - v_0 t,\ y' = y,\ z' = z,\ t' = t \quad \text{[t is unchanged in the transformation.]} \qquad \text{(A.1)}$$

* In the main text, the z-axis is the direction of relative motion. To be consistent with the references cited in this appendix, here we assume that the relative motion is along the x-axis.

Question: How do you determine a reference frame is inertial?



Einstein's Postulates: The laws of classical mehanics do not vary in form under the Galilean transformation. For example, (A.1) shows $\mathbf{F} = m\mathbf{a}$ in frame S transforms to $\mathbf{F} = m\mathbf{a}'$ in frame S'. However, when the same transformation is applied to the wave equation in vacuum, $\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = 0,$ its form changes completely (see Jackson, p. 516.)

So, when Einstein began his work on relativity, there were two approaches to make *all* the laws of physics invariant in form in all inertial frames: (1) Modify the theory of electromagnetism so that it is invariant in form under the Galilean transformation; or (2) Modify the Galilean transformation and the laws of mechanics so that the laws of both mechanics and electromagnetism are invariant in form under the new transformation. Einstein took the second approach. His special theory of relativity is based on 2 postulates:

1. Laws of physics are invariant in form in all inertial frames.
2. The speed of light in vacuum has the same value c in all inertial frames, independent of the motion of the source.

Event and Simultaneity: An event is something (such as the emission of a light pulse by a source) which happens at position (x, y, z) and time t. An event [described collectively as (x, y, z, t) in a given frame] will have different coordinates in different frames. The frames mentioned here and later are all inertial frames.

The time of an event can be measured by methods we normally think of. But, in relativity, time measurement often requires high precision (which can at least be done in a thought experiment) and we must bear in mind the frame in which the time is measured. The simplest way to measure time is to read the clock at the position of the event. If the clock is away from the event, the time of the event is the time shown on the clock (at the instant the light signal of the event reaches the clock) minus the time delay due to the travel of the signal (at speed c) from the event's position to the clock's position. The position of the event and the measured time of the event all refer to the frame in which the observer and the clock are *both* at rest (but the source which generates the event is not necessarily at rest.)

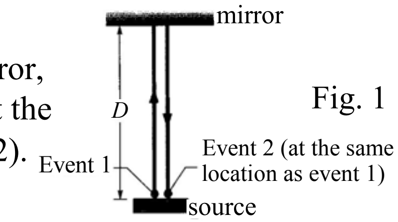
Two events are simultaneous in a reference frame if they have the same time coordinate in that frame, whether or not they have the same spatial coordinates. Simultaneity can be experimentally tested as follows. If two events are detected at the same instant by an observer located midway, they are simultaneous in the observer's frame.

Within a given frame, the concept of space and time in the special theory of relativity is not different from our usual concept of space and time. However, radical differences arise when space and time coordinates of an event measured in one frame are compared with those measured in another frame. In making the comparison, we find that space and time are entangled with each other in relativity. For example, two simultaneous events occurring at different positions in frame S will no longer be simultaneous in frame S', and their time difference in S' depends upon their spatial separation in S. In relativity, space and time coordinates transform according to the Lorentz transformation, which is derived below from postulate 2.

Time Dilation: Consider a pulse of light emitted by a source on a train (event 1). It travels vertically upward for a distance D, then is reflected back by a mirror, and later detected at the source (event 2).

In the train frame (Fig. 1), the time interval between the 2 events is

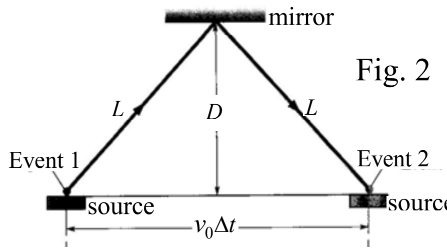
$$\Delta t_0 = \frac{2D}{c}. \qquad \text{(A.2)}$$



In the lab frame (Fig. 2), the train, mirror, and source are all moving at speed v_0, but the light still travels at speed c (by postulate 2). So, the time interval of the 2 events is

$$\Delta t = \frac{2L}{c}, \qquad \text{(A.3)}$$

where $L = [(\frac{1}{2}v_0\Delta t)^2 + D^2]^{1/2}$ (A.4)



Eliminating D and L from (A.2)-(A.4), we obtain

$$\Delta t = \gamma_0 \Delta t_0, \qquad \text{(A.5)}$$

where $\gamma_0 \equiv [1 - v_0^2/c^2]^{-1/2}$. (A.6)

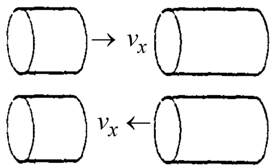
Question: Why is D the same in both frames?

Lengths perpendicular to the direction of motion are the same in both frames, i.e. the y and z coordinates transform as:

$$y' = y, \quad z' = z \qquad \text{(A.7)}$$

The proof of this is by contradiction.

Suppose that we have two identically manufactured pieces of pipe (see figure). They cannot fit inside each other because they have identical radius. Imagine that one pipe is at rest on the ground and the other is at rest on the train. If the motion of the train relative to the ground were to bring about a transverse contraction of the train pipe, then by symmetry, the motion of the ground pipe relative to the train would have to bring about a contraction of the ground pipe. But these two effects are contradictory, since in one case the train pipe would fit inside the ground pipe, and in the other case it would fit outside.



from H. C. Ohanian, "Physics"

Going back to (A.5): $\Delta t = \gamma_0 \Delta t_0$. In this equation, Δt_0 is the time interval of 2 events measured in a special frame in which the 2 events occur at the same position. It is called the <u>proper time</u>. Viewed in any other frame, these 2 events will occur at different positions and, by (A.5), their time interval (Δt) will be greater than the proper time by a factor of γ_0. This is known as the effect of <u>time dilation</u>.

The muon has an average lifetime of 2.2 μsec (between birth and decay) in its rest frame. In a 1977 experiment at CERN, muons were accelerated to a speed of 0.9994c, corresponding to a γ_0 value of 28.87. Within experimental error, the measured average lifetime of these muons was indeed 28.87x2.2 = 63.5 μsec. In another experiment, two synchronized clocks with near perfect precision showed slightly different readings after one had been flown around the world. The difference was again in agreement with (A.5).

Time dilation runs counter to our intuition, because it is rooted in a postulate which also runs counter to our intuition.

The Twin Paradox: Suppose someone travels on a spaceship with a Lorentz factor of $\gamma_0 = 20$ (in the earth frame) and his twin brother stays on earth. Then, by time dilation, every day measured by the traveling twin in the spaceship frame (this is his proper time) will be 20 days when measured by the earth twin in the earth frame. So the earth twin ages faster and his traveling brother will be 19 years younger when he returns to earth after an 1-year journey (neglect the spaceship's acceleration/deceleration periods). The paradox is: if the traveling twin measures the age of his earth twin, will he conclude that he himself ages faster by the same argument of time dilation?

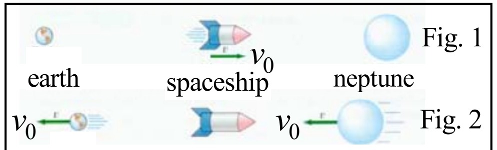
There is no paradox at all. Only the earth twin's measurement is correct because he is always in an inertial frame. The traveling twin will have to be accelerated and decelerated in the spaceship. During these periods, he cannot use the special theory of relativity (Einstein's 2 postulates refer to inertial frames.) In fact, he will confirm the measurement of his earth twin if he uses Einstein's general theory of relativity, which deals with accelerating reference frames.

Length Contraction: Assume that planet neptune is stationary in the earth frame and at a distance L_0 from earth (Fig. 1). A spaceship is traveling at speed v_0 to neptune. The duration of the trip, measured on earth, is

$$\Delta t = L_0 / v_0. \qquad \text{(A.8)}$$

In the spaceship's frame (Fig. 2), both earth and neptune move at speed v_0. The duration of the trip, Δt_0, is the interval between the departure of the earth and the arrival of the neptune. This is the "proper time" of the spaceship because both events occur at the same position. Thus, by (A.5)

$$\Delta t_0 = \Delta t / \gamma_0, \qquad \text{(A.9)}$$



from Giancoli, "Physics for Scientists and Engineers"

Δt_0 can be used to calculate the earth-naptune distance as viewed on the spaceship

$$L = v_0 \Delta t_0. \qquad \text{(A.10)}$$

Eliminating Δt and Δt_0 from (A.8)-(A.10), we obtain

$$L = \frac{L_0}{\gamma_0} \qquad \text{(A.11)}$$

In (A.11), $L = L_0/\gamma_0$, L_0 is the length of an object (or, in the above example, the earth-neptune distance) measured in the rest frame of the object (i.e. the frame in which the object is at rest). Length measured in this special frame is called the proper length. Viewed in any other frame, the object will be moving and, by (A.11), its length will be less than the proper length by a factor of γ_0. This is known as the effect of length contraction. Note that the contraction effect applies only to lengths along the direction of motion.

Length contraction is a direct consequence of time dilation [see (A.9)]. It is therefore not surprising that time dilation can be inferred from length contraction. If, for example, the spaceship has a γ_0 value of 2. The earth-neptune distance, as measured in the spaceship, would be half of that measured on earth. But the speed of earth/neptune relative to the spaceship is still v_0. So, to the spaceship, the journey's duration is only half of that measured on earth. Hence, one minute elapsed in the spaceship will be 2 minutes elapsed on earth.

The Lorentz Transformation: Assume frames S and S' coincide at $t = 0$ and S' moves along the common x-axis with speed v_0 relative to S (see figure). A point P has coordinates (x, y, z, t) in S and (x', y', z', t') in S'. The length x', when measured in S, is $\frac{x'}{\gamma_0}$ (length contraction). So,

$$x = v_0 t + \frac{x'}{\gamma_0} \quad \text{or} \quad x' = \gamma_0 (x - v_0 t). \tag{A.12}$$

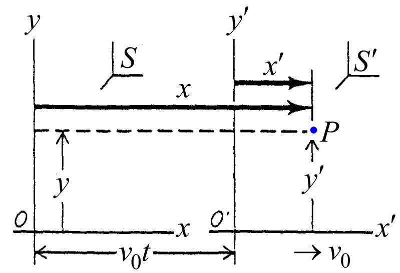
By symmetry or by similar argument, $x = \gamma_0 (x' + v_0 t')$ (A.13)

Eliminating x from (A.12) and (A.13) [using $\gamma_0^2 - 1 = \gamma_0^2 v_0^2 / c^2$],

$$\Rightarrow \quad t' = \gamma_0 \left(t - \frac{v_0}{c^2} x\right) \tag{A.14}$$

(A.7), (A.12), and (A.14) give the Lorentz transformation:

$$\begin{cases} x' = \gamma_0 (x - v_0 t) \\ y' = y \\ z' = z \\ t' = \gamma_0 \left(t - \frac{v_0}{c^2} x\right) \end{cases} \tag{A.15}$$



See Appendix B for a more formal derivation.

Transformation of Coordinate Difference between 2 Events :

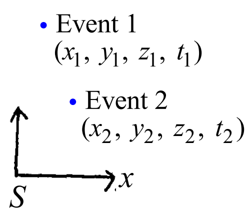
Since the Lorentz transformation is linear, the coordinate differences between 2 events:

in S: $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$, $\Delta z = z_2 - z_1$, $\Delta t = t_2 - t_1$ (A.16)

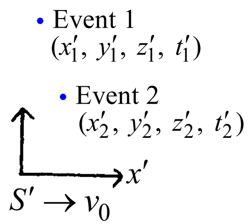
in S': $\Delta x' = x_2' - x_1'$, $\Delta y' = y_2' - y_1'$, $\Delta z' = z_2' - z_1'$, $\Delta t' = t_2' - t_1'$ (A.17)

transform in the same manner. Thus,

$$\begin{cases} \Delta x' = \gamma_0 (\Delta x - v_0 \Delta t) \\ \Delta y' = y \\ \Delta z' = z \\ \Delta t' = \gamma_0 \left(\Delta t - \frac{v_0}{c^2} \Delta x\right) \end{cases} \tag{A.18}$$



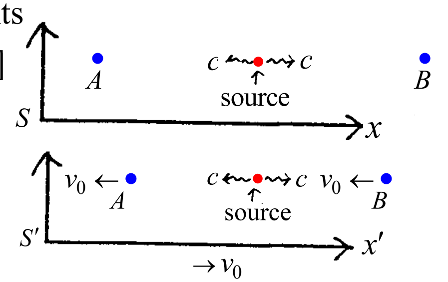
$\Delta x = x_2 - x_1$
$\Delta y = y_2 - y_1$
$\Delta z = z_2 - z_1$
$\Delta t = t_2 - t_1$



$\Delta x' = x_2' - x_1'$
$\Delta y' = y_2' - y_1'$
$\Delta z' = z_2' - z_1'$
$\Delta t' = t_2' - t_1'$

Discussion on simultaneity: Consider the transformation equation for the time interval between two events

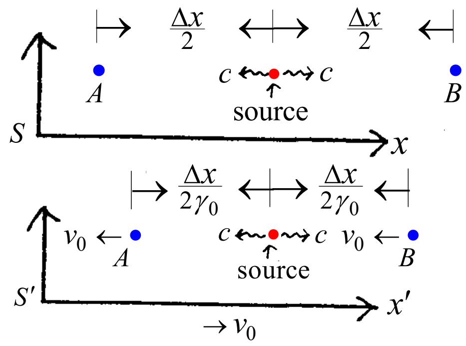
$$\Delta t' = \gamma_0 \left(\Delta t - \frac{v_0}{c^2} \Delta x\right) \quad \text{[from (A.18)]}$$



It indicates that 2 simultaneous events in frame S ($\Delta t = 0$) which occur at different positions ($\Delta x \neq 0$) will not be simultaneous in frame S' ($\Delta t' \neq 0$). This can be explained on the basis of postulate 2 through the following example.

In frame S, a pulse of light emitted midway between points A and B (see figure) will reach A and B at the same time, i.e. the two events (arrivals of the signals at A and B) are simultaneous in frame S. In frame S', the signal still travels at speed c in both directions, but B is moving toward the light and A away from it. So, the signal will reach B first and the two events are no longer simultaneous.

The example discussed above can be examined quantitatively as follows.



Assume that, in frame S, the two events are spatially separated by a distance Δx. Observed in S', the distance is shorter by a factor of γ_0 due to length contraction, i.e.

$$\Delta x' = \frac{\Delta x}{\gamma_0}.$$

Thus, in frame S', the signals reach A and B with a time difference of

$$\Delta t' = t'_B - t'_A = \frac{\frac{\Delta x}{2\gamma_0}}{c+v_0} - \frac{\frac{\Delta x}{2\gamma_0}}{c-v_0} = -\frac{\Delta x}{\gamma_0}\frac{v_0}{c^2-v_0^2} = -\frac{\gamma_0 v_0}{c^2}\Delta x.$$

This is precisely the prediction of (A.18),

$$\Delta t' = \gamma_0\left(\Delta t - \tfrac{v_0}{c^2}\Delta x\right) = -\tfrac{\gamma_0 v_0}{c^2}\Delta x. \quad [\Delta t = 0 \text{ in frame } S]$$

Problem 1: In frame S, events A and B occur at different positions, and event B occurs after event A. Is it posible for event B to precede event A in another frame S' moving at speed v_0 relative to frame S? If so, does this mean that an effect can precede its cause?

Solution: In frame S, let the 2 events have a spatial interval $\Delta x = x_B - x_A$ and time interval $\Delta t = t_B - t_A$. Then the time interval in frame S', $\Delta t' = t'_B - t'_A$, is given in (A.18): $\Delta t' = \gamma_0\left(\Delta t - \frac{v_0}{c^2}\Delta x\right)$.

We see that if $\Delta t < v_0\Delta x / c^2$, then $\Delta t' < 0$, which means that the order of *independent* events in frame S may be reversed in frame S'.

Suppose, however, that the events are connected, i.e. event B is caused by event A. This would require a body, or a signal, to travel from A to B. Rewrite (A.18) as $\Delta t' = \gamma_0 \Delta t\left(1 - \frac{v_0}{c^2}\frac{\Delta x}{\Delta t}\right)$. Since the fastest speed for a signal to travel from A to B is $\frac{\Delta x}{\Delta t} = c$, we must have $v_0 > c$ in order for $\Delta t' < 0$. This is not possible [see (A.6)] and thus the order of connected events (cause and effect) cannot be reversed.

Problem 2: Show that the effects of time dilation and length contraction are implicit in the Lorentz transformation.

Solution: The time interval between 2 events transform as $\Delta t' = \gamma_0\left(\Delta t - \frac{v_0}{c^2}\Delta x\right)$ or $\Delta t = \gamma_0\left(\Delta t' + \frac{v_0}{c^2}\Delta x'\right)$. If $\Delta t'$ is the proper time in S', then the 2 events occur at the same position ($\Delta x' = 0$). So we use the latter equation and obtain

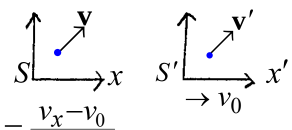
$$\Delta t = \gamma_0 \Delta t' \text{ (time dilation).}$$

The difference in the x coordinates of the 2 events transform as $\Delta x' = \gamma_0\left(\Delta x - v_0\Delta t\right)$ or $\Delta x = \gamma_0(\Delta x' + v_0\Delta t')$. Again, the question is which equation to use. If $\Delta x'$ is the "proper length" in S', then the two end points are at rest and their coordinates do not have to be measured simultaneously (i.e. we do not know $\Delta t'$.) But since the rod is moving in S, its end points must be measured simultaneously in S ($\Delta t = 0$). So we use the former equation and obtain

$$\Delta x = \frac{\Delta x'}{\gamma_0} \text{ (length contraction).}$$

Transformation of Velocity: The velocity of a particle is given by

$$\mathbf{v} = \lim_{\Delta t \to 0}\frac{\Delta \mathbf{x}}{\Delta t} \text{ (in frame } S\text{)}; \quad \mathbf{v}' = \lim_{\Delta t' \to 0}\frac{\Delta \mathbf{x}'}{\Delta t'} \text{ (in frame } S'\text{)}. \qquad (A.19)$$



Let $\Delta t \to 0$, $\boxed{\Delta x = v_x \Delta t}$

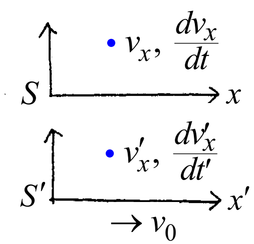
$$\begin{cases} \Delta x' = \gamma_0(\Delta x - v_0\Delta t) \\ \Delta y' = y \\ \Delta z' = z \\ \Delta t' = \gamma_0\left(\Delta t - \frac{v_0}{c^2}\Delta x\right) \end{cases} \Rightarrow \begin{cases} v'_x = \frac{\Delta x'}{\Delta t'} = \frac{\gamma_0(\Delta x - v_0\Delta t)}{\gamma_0\left(\Delta t - \frac{v_0}{c^2}\Delta x\right)} = \frac{v_x - v_0}{1 - \frac{v_0}{c^2}v_x} \\ v'_y = \frac{\Delta y'}{\Delta t'} = \frac{\Delta y}{\gamma_0\left(\Delta t - \frac{v_0}{c^2}\Delta x\right)} = \frac{v_y}{\gamma_0\left(1 - \frac{v_0}{c^2}v_x\right)} \\ v'_z = \frac{\Delta z'}{\Delta t'} = \frac{\Delta z}{\gamma_0\left(\Delta t - \frac{v_0}{c^2}\Delta x\right)} = \frac{v_z}{\gamma_0\left(1 - \frac{v_0}{c^2}v_x\right)} \end{cases} \qquad (A.20)$$

Problem: A spaceship moves away from the earth at speed v_0. A pulse of light is emitted from the earth in the direction toward the sapceship. What is the speed of light measured on the spaceship?

Solution: $\quad v_x = c \;\Rightarrow\; v'_x = \frac{v_x - v_0}{1 - \frac{v_0}{c^2}v_x} = \frac{c - v_0}{1 - \frac{v_0}{c}} = c$

Transformation of Acceleration : For simplicity, we first consider the transformation of acceleration in the direction of relative motion (i.e. $\frac{dv_x}{dt}$)

$$v'_x = \frac{v_x - v_0}{1 - \frac{v_0}{c^2} v_x} \quad \text{[from (A.20)]}$$



$$\Rightarrow dv'_x = \frac{dv_x}{1 - \frac{v_0}{c^2} v_x} - \frac{(v_x - v_0)(-\frac{v_0}{c^2}) dv_x}{(1 - \frac{v_0}{c^2} v_x)^2} = \frac{dv_x}{\gamma_0^2 (1 - \frac{v_0}{c^2} v_x)^2}$$

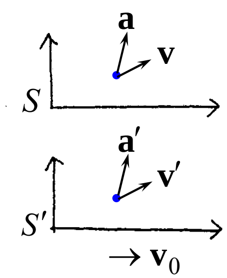
$$t' = \gamma_0 (t - \frac{v_0}{c^2} x) \quad \text{[from (A.15)]}$$

$$\Rightarrow \ dt' = \gamma_0 (dt - \frac{v_0}{c^2} dx) \qquad \boxed{dx = v_x dt}$$

$$\text{Hence, } \frac{dv'_x}{dt'} = \frac{1}{\gamma_0 (dt - \frac{v_0}{c^2} dx)} \frac{dv_x}{\gamma_0^2 (1 - \frac{v_0}{c^2} v_x)^2} = \frac{1}{\gamma_0^3 (1 - \frac{v_0}{c^2} v_x)^3} \frac{dv_x}{dt} \qquad \text{(A.21)}$$

By the same method, we may obtain the transformation equations for acceleration in arbitrary directions (see Jackson Problem 11.5).

$$\begin{cases} \mathbf{a}'_{\parallel} = \dfrac{1}{\gamma_0^3 \left(1 - \frac{\mathbf{v}_0 \cdot \mathbf{v}}{c^2}\right)^3} \mathbf{a}_{\parallel} \\ \mathbf{a}'_{\perp} = \dfrac{1}{\gamma_0^2 \left(1 - \frac{\mathbf{v}_0 \cdot \mathbf{v}}{c^2}\right)^3} \left[\mathbf{a}_{\perp} - \frac{\mathbf{v}_0}{c^2} \times (\mathbf{a} \times \mathbf{v}) \right] \end{cases} \qquad \text{(A.22)}$$

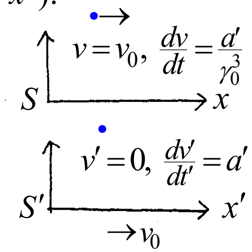


where "$\parallel$" and "$\perp$" refer to the direction of $\mathbf{v}_0$.

Problem : A rocket is launched from the earth into outer space. It moves on a straight line with a constant acceleration (a') with respect to its rest frame (Why is a' specified in the rocket's frame?) Calculate the time required for the rocket to accelerate from zero speed to the final speed v_f, according to earth and rocket clocks.

Solution : Let S be the earth frame, S' be the rocket rest frame, and the one-dimensional motion be along the x-axis. The inverse transformation of (A.21) gives (omitting subscript "x"):

$$a = \frac{dv}{dt} = \frac{1}{\gamma_0^3 (1 + v_0 v'/c^2)^3} \frac{dv'}{dt'}.$$



Lorentz transformations apply to two inertial frames. So, S' is the *instantaneous* rest frame of the rocket, but S' does not accelerate with the rocket. In S', we have $v' = 0$ and $dv'/dt' = a'$. This gives a (acceleration in S) $= a'/\gamma_0^3$. Since S' is the rest frame of the rocket, γ_0 for the transformation equals γ of the rocket in S. Thus,

$$a = a'/\gamma_0^3 = a'/\gamma^3, \text{ where } \gamma = (1 - v^2/c^2)^{-1/2} \quad [\textit{Note}: \ a' \geq a]$$

From the expression of the acceleration in the earth frame (S),

$$a = a'/\gamma^3 = a'(1 - v^2/c^2)^{3/2}, \qquad \text{(A.23)}$$

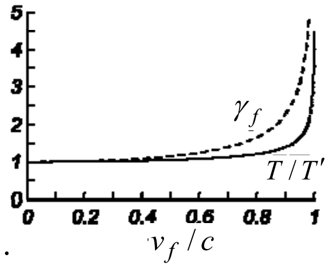
we may evaluate the total acceleration time as measured on earth

$$T = \int_0^T dt = \int_0^{v_f} \frac{1}{a} dv = \int_0^{v_f} \frac{\gamma^3}{a'} dv = \frac{1}{a'} \int_0^{v_f} \frac{dv}{(1 - v^2/c^2)^{3/2}} = \frac{v_f}{a'(1 - v_f^2/c^2)^{1/2}}.$$

The rocket frame is accelerating. So, to find the total acceleration time as measured on the rocket, we must still work in the earth frame by using the relation $dt' = dt/\gamma$.

$$T' = \int_0^{T'} dt' = \int_0^T \frac{1}{\gamma} dt = \int_0^{v_f} \frac{1}{\gamma a} dv = \frac{1}{a'} \int_0^{v_f} \frac{dv}{1 - v^2/c^2} = \frac{c}{2a'} \ln\left(\frac{1 + v_f/c}{1 - v_f/c}\right).$$

We find that, in the limit $v_f/c \rightarrow 0$, both T and T' approach the expected value of v_f/a'. However, T/T' increases rapidly as $v_f/c \rightarrow 1$ due to the effect of time dilation (see figure). In the figure, $\gamma_f = (1 - v_f^2/c^2)^{-\frac{1}{2}}$ is the time dilation factor at the final speed v_f.



Section 2: Relativistic Momentum and Energy

(Ref.: H. C. Ohanian, "Physics," 2nd ed., pp.1013-1014.)

The law of conservation of momentum states that, for an isolated system of particles, $\sum m_i \mathbf{v}_i$(before collision) $= \sum m_i \mathbf{u}_i$(after collision).

Under the Galilean transformation, the statement is true in all (inertial) frames. However, under the Lorentz transformation, $\sum m_i \mathbf{v}_i$, though conserved in one frame, will in general not be conserved in another frame. Thus, postulate 1 is violated if we continue to define the momentum as $m\mathbf{v}$. The theory of relativity takes a major step by redefining (or postulating) the momentum and energy as

$$\begin{cases} \mathbf{p} = \gamma m \mathbf{v} & \text{(A.24)} \\ E = \gamma m c^2 & \text{(A.25)} \end{cases}$$

[*Note*: $\gamma \equiv (1-v^2/c^2)^{-1/2}$ is the Lorentz factor of a particle. It is to be distinguished from the Lorentz factor γ_0 for the transformation.]

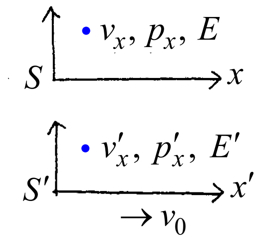
For simplicity, we will consider only one-dimentional motion along the x axis. The momentum and energy of a particle are then

$$p_x = \frac{mv_x}{\sqrt{1-v_x^2/c^2}} \quad \text{and} \quad E = \frac{mc^2}{\sqrt{1-v_x^2/c^2}}$$

From (A.20), the velocity in frame S' is $v'_x = \frac{v_x - v_0}{1-v_x v_0/c^2}$. Hence, the momentum of the particle is (assuming m has the same value in S')

$$p'_x = \frac{mv'_x}{\sqrt{1-v'^2_x/c^2}} = \frac{m(v_x-v_0)}{1-v_xv_0/c^2}\,\frac{1}{\sqrt{1-(1/c^2)[(v_x-v_0)/(1-v_xv_0/c^2)]^2}}$$

$$= \frac{m(v_x-v_0)}{\sqrt{(1-v_xv_0/c^2)^2-(v_x-v_0)^2/c^2}}$$



Since $(1-v_xv_0/c^2)^2 - (v_x-v_0)^2/c^2$ $= (1-v_0^2/c^2)(1-v_x^2/c^2)$, p'_x becomes

$$p'_x = \frac{1}{\sqrt{1-v_0^2/c^2}}\frac{mv_x}{\sqrt{1-v_x^2/c^2}} - \frac{v_0}{\sqrt{1-v_0^2/c^2}}\frac{m}{\sqrt{1-v_x^2/c^2}}$$

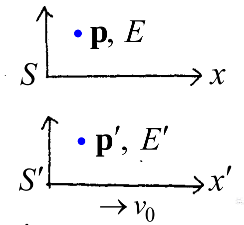
$$= \gamma_0 (p_x - \frac{v_0}{c^2}E) \qquad \text{(A.26)}$$

Similarly, we derive the Lorentz transformation equation for energy:

$$E' = \gamma_0 (E - v_0 p_x) \qquad \text{(A.27)}$$

By the same method, we can extend the motion to 3 dimensions and derive the Lorentz transformation equations for $\mathbf{p}$ and E. The result is

$$\begin{cases} p'_x = \gamma_0 (p_x - \frac{v_0}{c^2}E) \\ p'_y = p_y \\ p'_z = p_z \\ E' = \gamma_0 (E - v_0 p_x) \end{cases} \qquad \text{(A.28)}$$



(A.28) shows that $\mathbf{p}'$ and E' in S' is a *linear* combination of $\mathbf{p}$ and E in S, with constant coefficients (i.e. the coefficients are independent of $\mathbf{p}$ and E of the particle). The same equations will therefore hold true for the *total* momentum and energy ($\sum \mathbf{p}_j$, $\sum E_j$) of a system of particles,

$$\begin{cases} \sum p'_{jx} = \gamma_0 (\sum p_{jx} - \frac{v_0}{c^2}\sum E_j) \\ \sum p'_{jy} = \sum p_{jy} \\ \sum p'_{jz} = \sum p_{jz} \\ \sum E'_j = \gamma_0 (\sum E_j - v_0 \sum p_{jx}) \end{cases} \qquad \text{(A.29)}$$

Rewrite (A.29)
$$\begin{cases} \sum p'_{jx} = \gamma_0 (\sum p_{jx} - \frac{v_0}{c^2}\sum E_j) \\ \sum p'_{jy} = \sum p_{jy} \\ \sum p'_{jz} = \sum p_{jz} \\ \sum E'_j = \gamma_0 (\sum E_j - v_0 \sum p_{jx}) \end{cases}.$$

Form this set of equations, we see that if (and only if) the total momentum ($\sum \mathbf{p}_j$) and total energy ($\sum E_j$) of a system of particles are *both* conserved in S, the total momentum and total energy will be both conserved in S'.

Discussion: (i) This shows that the postulation of $\mathbf{p} = \gamma m \mathbf{v}$ and $E = \gamma mc^2$ will preserve the conservation law under the Lorentz transformation. However, the conservation law must now be extended to include both the momentum and energy.

(ii) Writing $E = \gamma mc^2 = (\gamma - 1)mc^2 + mc^2$, we may divide the total energy into the kinetic enery $(\gamma - 1)mc^2$ (due to motion) and a new form energy mc^2 (an intrinsic energy) called the rest-mass energy.

Problem: A particle of rest mass m moves on the x-axis is attracted to the origin by a force $m\omega^2 x$ ($\omega = const$). It performs oscillations of amplitude a. Express the *relativistic* oscillation period as a definite integral, and obtain the 2 leading terms of this integral for small a.

Solution: The period τ is given by $\tau = 4\int_0^a \frac{dx}{v}$, (A.30)

where the velocity v can be calculated from the energy equation

$$mc^2(1-v^2/c^2)^{-1/2} + \tfrac{1}{2}m\omega^2x^2 = mc^2 + \tfrac{1}{2}m\omega^2a^2 \qquad \text{(A.31)}$$

Substituting v from (A.31) into (A.30), we obtain

$$\tau = \frac{4}{\omega}\int_0^a dx \frac{1+\omega^2(a^2-x^2)/2c^2}{(a^2-x^2)^{1/2}\left[1+\omega^2(a^2-x^2)/(4c^2)\right]^{1/2}}$$

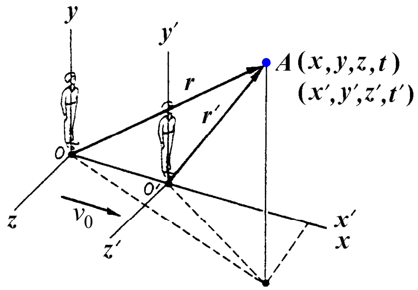
Expanding the integrand in powers of $\omega^2(a^2-x^2)/c^2$ and using $\int_0^b dy \frac{1}{(by-y^2)^{1/2}} = \pi$ and $\int_0^b dy \frac{y}{(by-y^2)^{1/2}} = \frac{b\pi}{2}$ (for $b>0$), we obtain

$$\tau = \frac{2\pi}{\omega}\left[1+\frac{3\omega^2a^2}{16c^2}+\cdots\right]$$

Appendix B: A Formal Derivation of the Lorentz Transformation

In Appendix A, we begin with the derivation of "time dilation" and "length contraction" from postulate 2, followed by a derivation of the Lorentz transformation. Here, we present a more formal (but physically less transparent) approach, whereby the Lorentz transformation is derived directly from postulate 2. The following paragraphs are taken from Alonso & Finn "Physics," p.92.

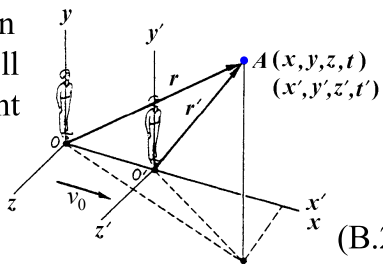
Referring to the figure to the right, suppose that at $t = 0$ a flash of light is emitted at the common position of the two observers. After a time t, observer O will note that the light has reached point A and will write $r = ct$, where c is the speed of light. Since $x^2+y^2+z^2 = r^2$, we may also write



$$x^2+y^2+z^2 = c^2t^2 \qquad \text{(B.1)}$$

Similarly, observer O', whose position is no longer coincident with that of O, will note that the light arrives at the same point A in a time t', but also with velocity c.

Therefore he writes $r' = ct'$, or



$$x'^2+y'^2+z'^2 = c^2t'^2 \qquad \text{(B.2)}$$

Our next task is to obtain a transformation relating (B.1) and (B.2). The symmetry of the problem suggests that $y' = y$ and $z' = z$. Also since $OO' = v_0t$ for observer O, it must be that $x = v_0t$ for $x' = 0$ (point O'). This suggests making $x' = k(x - v_0t)$, where k is a constant to be determined. Since t' is different, we may also assume that $t' = a(t - bx)$, where a and b are constants to be determined (for the Galilean transformation, $k = a = 1$ and $b = 0$). Making all these substitutions in (B.2), we have

$$k^2(x^2 - 2v_0xt + v_0^2t^2) + y^2 + z^2 = c^2a^2(t^2 - 2bxt + b^2x^2) \quad \text{or}$$

$$(k^2 - b^2a^2c^2)x^2 - 2(k^2v_0 - ba^2c^2)xt + y^2 + z^2 = (a^2 - k^2v_0^2/c^2)c^2t^2.$$

This result must be identical to (B.1). Therefore

$$k^2 - b^2a^2c^2 = 0$$

$$k^2v_0 - ba^2c^2 = 0$$

$$a^2 - k^2v_0^2/c^2 = 1$$

Solving this set of equations, for k, a, and b, we have

$$k = a = \frac{1}{\sqrt{1-v_0^2/c^2}} \quad \text{and} \quad b = v_0/c^2$$

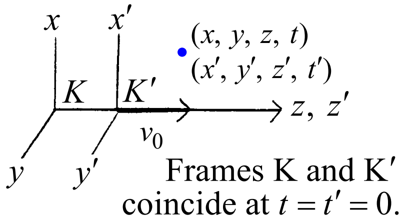
Inserting these values of k, a, and b in $x' = k(x - v_0t)$ and $t' = (a - bx)$, we obtain the Lorentz transformation

$$\begin{cases} x' = \dfrac{x - v_0t}{\sqrt{1-v_0^2/c^2}} \\ y' = y \\ z' = z \\ t' = \dfrac{t - v_0x/c^2}{\sqrt{1-v_0^2/c^2}} \end{cases} \qquad \text{(B.3)}$$

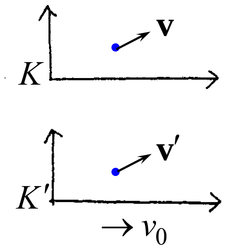
Appendix C: Summary of Lorentz Transformation Equations

For all equations, $\gamma_0 \equiv \left(1-\frac{v_0^2}{c^2}\right)^{-\frac{1}{2}}$. By symmetry, equations for the inverse transformation differ only by the sign of v_0 (or $\mathbf{v}_0$).

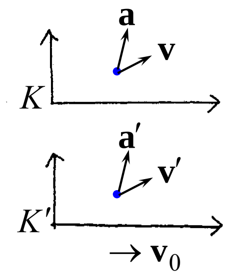
1. $$\begin{cases} x' = x \\ y' = y \\ z' = \gamma_0\left(z - v_0 t\right) \\ t' = \gamma_0\left(t - \frac{v_0}{c^2} z\right) \end{cases}$$



2. $$\begin{cases} v'_x = \dfrac{v_x}{\gamma_0\left(1-\frac{v_0}{c^2}v_z\right)} \\ v'_y = \dfrac{v_y}{\gamma_0\left(1-\frac{v_0}{c^2}v_z\right)} \\ v'_z = \dfrac{v_z - v_0}{1-\frac{v_0}{c^2}v_z} \end{cases}$$



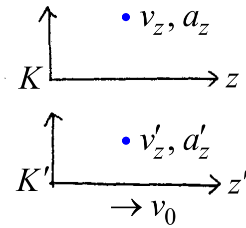
3. $$\begin{cases} \mathbf{a}'_{\parallel} = \dfrac{1}{\gamma_0^3\left(1-\frac{\mathbf{v}_0\cdot\mathbf{v}}{c^2}\right)^3}\mathbf{a}_{\parallel} \\ \mathbf{a}'_{\perp} = \dfrac{1}{\gamma_0^2\left(1-\frac{\mathbf{v}_0\cdot\mathbf{v}}{c^2}\right)^3}\left[\mathbf{a}_{\perp} - \frac{\mathbf{v}_0}{c^2}\times(\mathbf{a}\times\mathbf{v})\right] \end{cases}$$



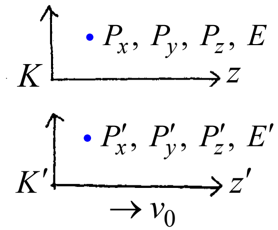
where "$\parallel$" and "$\perp$" refer to the direction of $\mathbf{v}_0$.

Special case: one dimensional motion

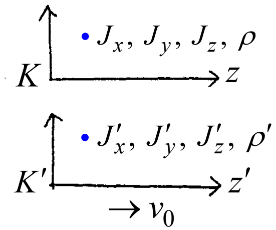
$$a'_z = \frac{1}{\gamma_0^3\left(1-\frac{v_0}{c^2}v_z\right)^3} a_z$$



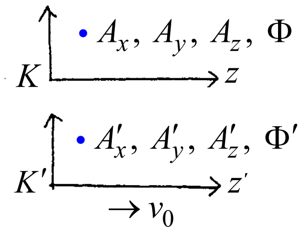
4. $$\begin{cases} p'_x = p_x \\ p'_y = p_y \\ p'_z = \gamma_0\left(p_z - \frac{v_0}{c^2}E\right) \\ E' = \gamma_0\left(E - v_0 p_z\right) \end{cases}$$



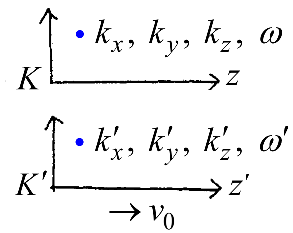
5. $$\begin{cases} J'_x = J_x \\ J'_y = J_y \\ J'_z = \gamma_0\left(J_z - v_0\rho\right) \\ \rho' = \gamma_0\left(\rho - \frac{v_0}{c^2}J_z\right) \end{cases}$$



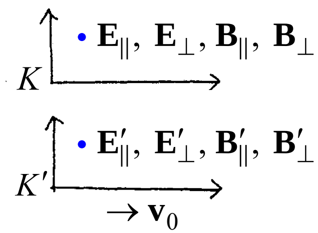
6. $$\begin{cases} A'_x = A_x \\ A'_y = A_y \\ A'_z = \gamma_0\left(A_z - \frac{v_0}{c}\Phi\right) \\ \Phi' = \gamma_0\left(\Phi - \frac{v_0}{c}A_z\right) \end{cases}$$



7. $$\begin{cases} k'_x = k_x \\ k'_y = k_y \\ k'_z = \gamma_0\left(k_z - \frac{v_0}{c^2}\omega\right) \\ \omega' = \gamma_0\left(\omega - v_0 k_z\right) \end{cases}$$



8. $$\begin{cases} \mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \\ \mathbf{E}'_{\perp} = \gamma_0\left(\mathbf{E}_{\perp} + \frac{\mathbf{v}_0}{c}\times\mathbf{B}_{\perp}\right) \\ \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \\ \mathbf{B}'_{\perp} = \gamma_0\left(\mathbf{B}_{\perp} - \frac{\mathbf{v}_0}{c}\times\mathbf{E}_{\perp}\right) \end{cases}$$



where "$\parallel$" and "$\perp$" refer to the direction of $\mathbf{v}_0$.

Chapter 14: Radiation by Moving Charges

Review of Basic Equations : [converted to Gaussian unit system, see p.782 for conversion formulae.]

$$\begin{cases} \nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi = -4\pi\rho\left(\frac{\rho}{\varepsilon_0}\right) & (6.15) \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}\left(\mu_0\mathbf{J}\right) & (6.16) \end{cases}$$

[free-space inhomogeneous wave equations (SI)]

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = -4\pi f(\mathbf{x},t) \qquad (6.32)$$

[general form of (6.15) and (6.16)]

Solution of (6.32) with outgoing-wave b.c.:

$$\psi(\mathbf{x},t) = \psi_{in}(\mathbf{x},t) + \int d^3x' \int dt' G^+(\mathbf{x},t,\mathbf{x}',t') f(\mathbf{x}',t'), \qquad (6.45)$$

where the retarded Green's function

$$G^+(\mathbf{x},t,\mathbf{x}',t') = \delta\left[t' - \left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right] / |\mathbf{x}-\mathbf{x}'| \qquad (6.44)$$

is the solution of (with outgoing-wave b.c.)

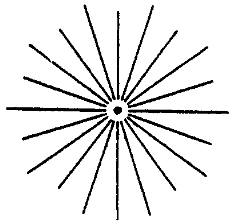
$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G^+(\mathbf{x},t,\mathbf{x}',t') = -4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \qquad (6.41)$$

Apply (6.45) (assuming $\psi_{in} = 0$) to (6.15) & (6.16)

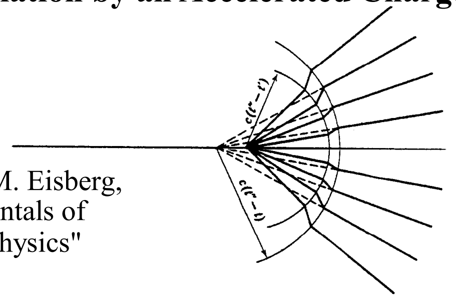
$$\begin{Bmatrix} \Phi(\mathbf{x},t) \\ \mathbf{A}(\mathbf{x},t) \end{Bmatrix} = \int d^3x' \int dt' \frac{\delta\left[t' - \left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right]}{|\mathbf{x}-\mathbf{x}'|} \begin{Bmatrix} \rho(\mathbf{x}',t') \\ \frac{1}{c}\mathbf{J}(\mathbf{x}',t') \end{Bmatrix} \qquad (9.2)$$

Note: We need both Φ and $\mathbf{A}$ to specify $\mathbf{E}$ and $\mathbf{B}$, unless the source has harmonic time dependence (as in Chs. 9 and 10).

A Qualitative Picture of Radiation by an Accelerated Charge:



From R. M. Eisberg, "Fundamentals of Modern Physics"



E-field lines surrounding a stationary charge.

A fraction of **E**-field lines showing the effect of charge acceleration.

14.1 Liénard-Wiechert Potentials and Fields for a Point Charge

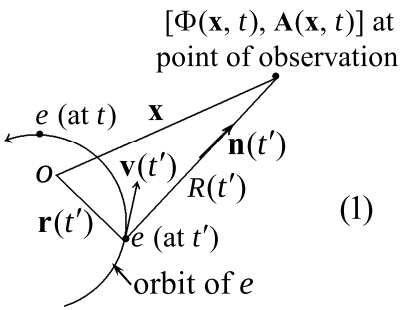
Lienard-Wiechert Potentials for a Point Charge :

Rewrite (9.2): $\begin{Bmatrix} \mathbf{A}(\mathbf{x},t) \\ \Phi(\mathbf{x},t) \end{Bmatrix} = \int d^3x' \int dt' \frac{\delta\left[t' - \left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right]}{|\mathbf{x}-\mathbf{x}'|} \begin{Bmatrix} \frac{1}{c}\mathbf{J}(\mathbf{x}',t') \\ \rho(\mathbf{x}',t') \end{Bmatrix}$

$\rho(\mathbf{x}',t')$, $\mathbf{J}(\mathbf{x}',t')$ due to a point charge e (e carries a sign) moving along the orbit $\mathbf{r}(t')$ at the velocity $\mathbf{v}(t') = d\mathbf{r}(t')/dt'$ can be written

$$\begin{cases} \rho(\mathbf{x}',t') = e\delta\left[\mathbf{x}' - \mathbf{r}(t')\right] \\ \mathbf{J}(\mathbf{x}',t') = e\mathbf{v}(t')\delta\left[\mathbf{x}' - \mathbf{r}(t')\right] \end{cases}$$

$$\Rightarrow \begin{cases} \Phi(\mathbf{x},t) = e\int dt' \frac{\delta\left[t' + \frac{R(t')}{c} - t\right]}{R(t')} \\ \mathbf{A}(\mathbf{x},t) = e\int dt' \frac{\boldsymbol{\beta}(t')\delta\left[t' + \frac{R(t')}{c} - t\right]}{R(t')} \end{cases} \qquad (1)$$



where $R(t') = |\mathbf{x} - \mathbf{r}(t')|$ and $\boldsymbol{\beta}(t') = \mathbf{v}(t')/c$.

Rewrite (1): $\begin{cases} \Phi(\mathbf{x},t) = e\int dt' \frac{\delta\left[t' + \frac{R(t')}{c} - t\right]}{R(t')} = e\int dt' \frac{\delta[f(t') - t]}{R(t')} \\ \mathbf{A}(\mathbf{x},t) = e\int dt' \frac{\boldsymbol{\beta}(t')\delta\left[t' + \frac{R(t')}{c} - t\right]}{R(t')} = e\int dt' \frac{\boldsymbol{\beta}(t')\delta[f(t') - t]}{R(t')} \end{cases}$,

where $f(t') \equiv t' + R(t')/c$. (2)

Using $\int g(x)\delta[f(x) - a]dx = \sum_i \left[\frac{g(x)}{\left|\frac{d}{dx}f(x)\right|}\right]_{x_i}$, we obtain

[x_i is the solution of $f(x_i) = $ a.]

$$\begin{cases} \Phi(\mathbf{x},t) = \left[\frac{e}{R(t')\left|\frac{d}{dt'}f(t')\right|}\right]_{ret} \\ \mathbf{A}(\mathbf{x},t) = \left[\frac{e\boldsymbol{\beta}(t')}{R(t')\left|\frac{d}{dt'}f(t')\right|}\right]_{ret} \end{cases} \qquad (3)$$

where $[\;\;]_{ret}$ implies that quantities in the bracket are to be evaluated at the retarded time t' $[= t - R(t')/c]$.

Question: What information is needed in order to find t'?

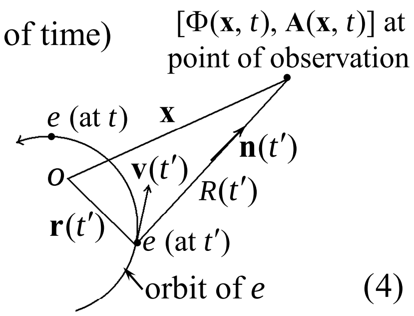
$$\frac{dR(t')}{dt'} = \frac{d|\mathbf{x}-\mathbf{r}(t')|}{dt'} = \frac{d}{dt'}[x^2 - 2\mathbf{x}\cdot\mathbf{r}(t') + \mathbf{r}^2(t')]^{\frac{1}{2}}$$

($\mathbf{x}$ is a fixed position, indep. of time)

$$= \frac{-2\mathbf{x}\cdot\frac{d}{dt'}\mathbf{r}(t') + 2\mathbf{r}(t')\cdot\frac{d}{dt'}\mathbf{r}(t')}{2[x^2-2\mathbf{x}\cdot\mathbf{r}(t')+\mathbf{r}^2(t')]^{\frac{1}{2}}}$$

$$= -\frac{\mathbf{v}(t')\cdot[\mathbf{x}-\mathbf{r}(t')]}{R(t')}$$

$$= -\mathbf{v}(t')\cdot\mathbf{n}(t') \qquad (4)$$



$$\Rightarrow \frac{d}{dt'} f(t') = \frac{d}{dt'}[t' + \frac{R(t')}{c}] = 1 - \boldsymbol{\beta}(t')\cdot\mathbf{n}(t') \equiv \kappa (> 0) \qquad (5)$$

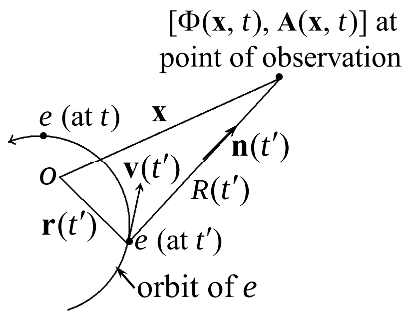
Sub. (5) into (3) gives the Lienard-Wiechert potentials

$$\begin{cases} \Phi(\mathbf{x},t) = \left[\dfrac{e}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{ret} \\ \mathbf{A}(\mathbf{x},t) = \left[\dfrac{e\boldsymbol{\beta}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{ret} \end{cases} \qquad (14.8)$$

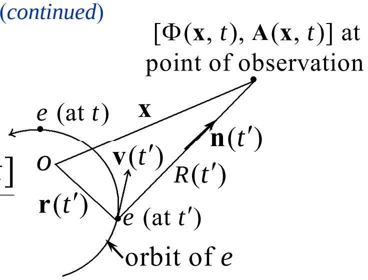
Fields for a Point Charge: Rewrite (1) and (14.8):

$$\begin{cases} \Phi(\mathbf{x},t) = e\int dt' \dfrac{\delta[t'+\frac{R(t')}{c}-t]}{R(t')} \\ \mathbf{A}(\mathbf{x},t) = e\int dt' \dfrac{\boldsymbol{\beta}(t')\delta[t'+\frac{R(t')}{c}-t]}{R(t')} \end{cases} (1), \quad \begin{cases} \Phi(\mathbf{x},t) = \left[\dfrac{e}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{ret} \\ \mathbf{A}(\mathbf{x},t) = \left[\dfrac{e\boldsymbol{\beta}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{ret} \end{cases} (14.8)$$

To obtain $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$, we need to differentiate $\Phi(\mathbf{x},t)$ and $\mathbf{A}(\mathbf{x},t)$ with respect to $\mathbf{x}$. The RHS of (14.8) depends on $\mathbf{x}$ through $\mathbf{n}$ and R, but the RHS of (1) depends on $\mathbf{x}$ through R only. Hence, it is more convenient to use (1).



Rewrite (1):
$$\begin{cases} \Phi(\mathbf{x},t) = e\int dt' \dfrac{\delta[t'+\frac{R(t')}{c}-t]}{R(t')} \\ \mathbf{A}(\mathbf{x},t) = e\int dt' \dfrac{\boldsymbol{\beta}(t')\delta[t'+\frac{R(t')}{c}-t]}{R(t')} \end{cases}$$



Let $F(R)$ be any function of R, then

$$\nabla_{\mathbf{x}} F(R) = \frac{dF}{dR}\nabla_{\mathbf{x}} R = \frac{dF}{dR}\underbrace{\nabla_{\mathbf{x}}|\mathbf{x}-\mathbf{r}(t')|}_{=\mathbf{n}(t')} = \mathbf{n}(t')\frac{dF}{dR} \qquad (6)$$

Use $\nabla_{\mathbf{x}}|\mathbf{x}-\mathbf{x}'|^n = n|\mathbf{x}-\mathbf{x}'|^{n-2}(\mathbf{x}-\mathbf{x}')$.
See Sec. 1.3 of lecture notes.

$$(1) \,\&\, (6) \Rightarrow \begin{cases} \nabla\Phi(\mathbf{x},t) = e\int \mathbf{n}(t')\left[\dfrac{-\delta[t'+\frac{R(t')}{c}-t]}{R^2(t')} + \dfrac{\delta'[t'+\frac{R(t')}{c}-t]}{cR(t')}\right]dt' \\ \dfrac{1}{c}\dfrac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t) = -\dfrac{e}{c}\int \dfrac{\boldsymbol{\beta}(t')\delta'[t'+\frac{R(t')}{c}-t]}{R(t')}dt' \end{cases}$$

Thus,

$f(t') \equiv t' + \frac{R(t')}{c} \Rightarrow dt' = \frac{dt'}{df(t')}df(t') = \frac{1}{\kappa}df(t')$, where $\kappa \equiv 1 - \boldsymbol{\beta}(t')\cdot\mathbf{n}(t')$. Use (5)

$$\mathbf{E}(\mathbf{x},t) = -\nabla\Phi - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}$$

$$= e\int\left[\frac{\mathbf{n}}{R^2}\delta[t'+\frac{R(t')}{c}-t] + \frac{\boldsymbol{\beta}-\mathbf{n}}{Rc}\delta'[t'+\frac{R(t')}{c}-t]\right]dt'$$

$$= e\int\left[\frac{\mathbf{n}}{\kappa R^2}\delta[f(t')-t] + \frac{\boldsymbol{\beta}-\mathbf{n}}{\kappa Rc}\delta'[f(t')-t]\right]df(t') \text{ [see note below]}$$

$\int g(x)\delta'(x-a)dx = -g'(a)$

$$= e\left[\frac{\mathbf{n}}{\kappa R^2} + \frac{1}{c}\frac{d}{df(t')}\left(\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}\right)\right]_{ret}$$

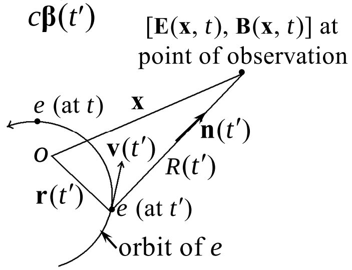
$\frac{d}{df(t')} = \frac{dt'}{df(t')}\frac{d}{dt'} = \frac{1}{\kappa}\frac{d}{dt'}$

$$= e\left[\frac{\mathbf{n}}{\kappa R^2} + \frac{1}{\kappa c}\frac{d}{dt'}\left(\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}\right)\right]_{ret} \qquad (7)$$

Note: Because of the $\delta[f(t')-t]$ factor in the integrand, integration over $f(t')$ demands $f(t') = t$ or $t' = t - \frac{R(t')}{c}$. But $\mathbf{n}$, $\boldsymbol{\beta}$, R, and κ in the integrand are all functions of t' [not $f(t')$]. Hnece, $\mathbf{n}$, $\boldsymbol{\beta}$, R, and κ are to be evaluated at the retarded time t' [not t].

To put **E** in a simpler form, we need to evaluate $\frac{d\mathbf{n}(t')}{dt'}$ and $\frac{d}{dt'}(\kappa R)$.

$$\frac{d\mathbf{n}(t')}{dt'} = \frac{d}{dt'}\frac{\mathbf{x}-\mathbf{r}(t')}{R(t')} = -\underbrace{\frac{\mathbf{x}-\mathbf{r}(t')}{R^2(t')}}_{\frac{\mathbf{n}(t')}{R(t')}}\underbrace{\frac{dR(t')}{dt'}}_{\substack{-c\boldsymbol{\beta}\cdot\mathbf{n}\\ \text{by (4)}}} - \frac{1}{R(t')}\underbrace{\frac{d\mathbf{r}(t')}{dt'}}_{c\boldsymbol{\beta}(t')} = \frac{c}{R}[\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\beta})-\boldsymbol{\beta}] \quad (8)$$



$$\frac{d}{dt'}(\kappa R) = \frac{d}{dt'}\{[1-\boldsymbol{\beta}(t')\cdot\mathbf{n}(t')]R\}$$

$$= (1-\boldsymbol{\beta}\cdot\mathbf{n})\underbrace{\frac{d}{dt'}R}_{-c\boldsymbol{\beta}\cdot\mathbf{n}} - R\frac{d}{dt'}(\boldsymbol{\beta}\cdot\mathbf{n})$$

$$= -c(1-\boldsymbol{\beta}\cdot\mathbf{n})(\boldsymbol{\beta}\cdot\mathbf{n}) - R\dot{\boldsymbol{\beta}}\cdot\mathbf{n} - R\boldsymbol{\beta}\cdot\frac{d\mathbf{n}}{dt'} \quad \boxed{\text{Sub. (8) for } \frac{d\mathbf{n}}{dt'}}$$

$$= -c(1-\boldsymbol{\beta}\cdot\mathbf{n})(\boldsymbol{\beta}\cdot\mathbf{n}) - R\dot{\boldsymbol{\beta}}\cdot\mathbf{n} - c\left[(\mathbf{n}\cdot\boldsymbol{\beta})^2 - \beta^2\right]$$

$$= -c(\boldsymbol{\beta}\cdot\mathbf{n})(1-\boldsymbol{\beta}\cdot\mathbf{n}+\boldsymbol{\beta}\cdot\mathbf{n}) + c\beta^2 - R\dot{\boldsymbol{\beta}}\cdot\mathbf{n}$$

$$= c\beta^2 - c(\boldsymbol{\beta}\cdot\mathbf{n}) - R(\dot{\boldsymbol{\beta}}\cdot\mathbf{n}) \quad (9)$$

$$\mathbf{E}(\mathbf{x},t) = e\left[\frac{\mathbf{n}}{\kappa R^2} + \frac{1}{c\kappa^2 R}\frac{d}{dt'}(\mathbf{n}-\boldsymbol{\beta}) + \frac{\mathbf{n}-\boldsymbol{\beta}}{c\kappa}\frac{d}{dt'}\left(\frac{1}{\kappa R}\right)\right]_{ret} \quad \boxed{\text{from (7)}} \quad \boxed{\text{Use (8), (9)}}$$

$$= e\left\{\frac{\mathbf{n}}{\kappa R^2} + \frac{1}{c\kappa^2 R}\left[\frac{c}{R}[\mathbf{n}(\boldsymbol{\beta}\cdot\mathbf{n})-\boldsymbol{\beta}] - \dot{\boldsymbol{\beta}}\right] - \frac{\mathbf{n}-\boldsymbol{\beta}}{c\kappa^3 R^2}\left[c\beta^2 - c(\boldsymbol{\beta}\cdot\mathbf{n}) - R(\dot{\boldsymbol{\beta}}\cdot\mathbf{n})\right]\right\}_{ret}$$

$$= e\left\{\frac{1}{R^2}\left[\underbrace{\frac{\mathbf{n}}{\kappa} + \frac{\mathbf{n}(\boldsymbol{\beta}\cdot\mathbf{n})-\boldsymbol{\beta}}{\kappa^2}}_{=\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa^2} = \frac{(\mathbf{n}-\boldsymbol{\beta})[1-(\boldsymbol{\beta}\cdot\mathbf{n})]}{\kappa^3}} - \frac{(\mathbf{n}-\boldsymbol{\beta})(\beta^2-\boldsymbol{\beta}\cdot\mathbf{n})}{\kappa^3}\right] + \frac{1}{R}\left[\frac{-\dot{\boldsymbol{\beta}}}{c\kappa^2} + \frac{(\mathbf{n}-\boldsymbol{\beta})(\dot{\boldsymbol{\beta}}\cdot\mathbf{n})}{c\kappa^3}\right]\right\}_{ret} \quad \boxed{\kappa \equiv 1-\boldsymbol{\beta}\cdot\mathbf{n}}$$

$$= e\left\{\begin{array}{l}\frac{1}{\kappa^3 R^2}\left[(\mathbf{n}-\boldsymbol{\beta})(1-\boldsymbol{\beta}\cdot\mathbf{n}) - (\mathbf{n}-\boldsymbol{\beta})(\beta^2-\boldsymbol{\beta}\cdot\mathbf{n})\right] \\ + \frac{1}{c\kappa^3 R}\left[-\dot{\boldsymbol{\beta}}(1-\boldsymbol{\beta}\cdot\mathbf{n}) + (\mathbf{n}-\boldsymbol{\beta})(\dot{\boldsymbol{\beta}}\cdot\mathbf{n})\right]\end{array}\right\}_{ret}$$

$$= e\left\{\frac{1}{\kappa^3 R^2}[(\mathbf{n}-\boldsymbol{\beta})(\underbrace{1-\beta^2}_{1/\gamma^2})] + \frac{1}{c\kappa^3 R}\left[\underbrace{\mathbf{n}(\dot{\boldsymbol{\beta}}\cdot\mathbf{n}) - \dot{\boldsymbol{\beta}}}_{\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})} - \underbrace{[\boldsymbol{\beta}(\dot{\boldsymbol{\beta}}\cdot\mathbf{n}) - \dot{\boldsymbol{\beta}}\cdot(\boldsymbol{\beta}\cdot\mathbf{n})]}_{\mathbf{n}\times(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})}\right]\right\}_{ret}$$

$$= e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R^2}\right]_{ret} + \frac{e}{c}\left[\frac{\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]_{ret} \quad (14.14)$$

To derive $\mathbf{B}(\mathbf{x},t)$, we write (7)

$$\mathbf{E}(\mathbf{x},t) = e\left[\frac{\mathbf{n}}{\kappa R^2} + \frac{1}{\kappa c}\frac{d}{dt'}\left(\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}\right)\right]_{ret}$$

$$\Rightarrow \mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t) = e\left[\frac{1}{\kappa c}\mathbf{n}\times\frac{d}{dt'}\left(\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}\right)\right]_{ret}$$

$$= -e\left[\frac{1}{\kappa c}\frac{d}{dt'}\left(\frac{\mathbf{n}\times\boldsymbol{\beta}}{\kappa R}\right) + \frac{\mathbf{n}\times\boldsymbol{\beta}}{\kappa R^2}\right]_{ret}$$

$$\boxed{\begin{aligned}&\mathbf{n}\times\frac{d}{dt'}\left(\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}\right) \quad \text{Use (8)}\\ &= \frac{d}{dt'}\left[\frac{\mathbf{n}\times(\mathbf{n}-\boldsymbol{\beta})}{\kappa R}\right] - \frac{d\mathbf{n}}{dt'}\times\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}\\ &= -\frac{d}{dt'}\left(\frac{\mathbf{n}\times\boldsymbol{\beta}}{\kappa R}\right) - \frac{c[\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\beta})-\boldsymbol{\beta}]}{R}\times\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}\\ &= -\frac{d}{dt'}\left(\frac{\mathbf{n}\times\boldsymbol{\beta}}{\kappa R}\right) - \frac{c\mathbf{n}\times\boldsymbol{\beta}}{R^2}\end{aligned}}$$

$\boxed{\begin{array}{l}\nabla \text{ operates on } R(t') \text{ only}\\ [\text{only } R(t') \text{ depends on } \mathbf{x}]\end{array}}$

$$\mathbf{B}(\mathbf{x},t) = \nabla\times\mathbf{A} \underset{\boxed{(1)}}{=} e\int dt'\nabla\times\left[\frac{\boldsymbol{\beta}(t')\delta[t'+R(t')/c-t]}{R(t')}\right]$$

$$= e\int dt'\left[\nabla\frac{\delta[t'+R(t')/c-t]}{R(t')}\right]\times\boldsymbol{\beta}(t') \quad \boxed{\nabla\times\psi\mathbf{a} = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}}$$

$$= e\int dt'\left[-\frac{\delta[t'+R(t')/c-t]}{R^2} + \frac{\delta'[t'+R(t')/c-t]}{cR}\right]\overbrace{\nabla R(t')}^{\mathbf{n}(t')}\times\boldsymbol{\beta}(t')$$

$$= -e\left[\frac{1}{\kappa c}\frac{d}{dt'}\left(\frac{\mathbf{n}\times\boldsymbol{\beta}}{\kappa R}\right) + \frac{\mathbf{n}\times\boldsymbol{\beta}}{\kappa R^2}\right]_{ret} \quad \boxed{\begin{array}{l}\text{following the}\\ \text{same steps as}\\ \text{in deriving (7)}\end{array}}$$

$$\Rightarrow \mathbf{B}(\mathbf{x},t) = \mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t) \quad (14.13)$$

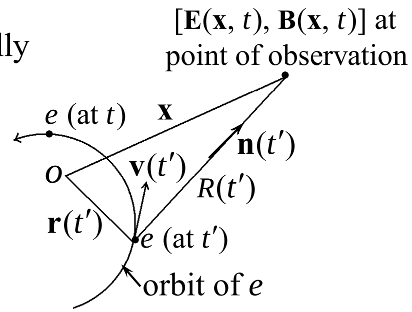
Rewrite

$$\begin{cases}\mathbf{E}(\mathbf{x},t) = \underbrace{e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R^2}\right]_{ret}}_{\text{velocity field } (\propto\frac{1}{R^2})} + \underbrace{\frac{e}{c}\left[\frac{\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]_{ret}}_{\substack{\text{acceleration field}\\ (\propto\frac{\dot{\beta}}{R} \text{ and } \perp\mathbf{n})}} & (14.14)\\ \mathbf{B}(\mathbf{x},t) = \mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t) & (14.13)\end{cases}$$

Discussion:

(i) The velocity fields are essentially static fields falling off as $1/R^2$.

(ii) For the acceleration fields, (14.13) and (14.14) show that $\mathbf{E}(\mathbf{x},t)$, $\mathbf{B}(\mathbf{x},t)$, and $\mathbf{n}(t')$ are mutually orthogonal, as is typical of radiation fields.

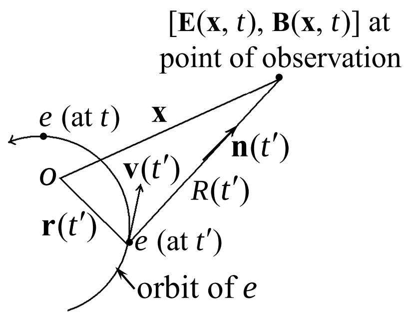


Note: (i) Unit vector $\mathbf{n}(t')$ points from the *retarded* position to **x**.

(ii) t and t' are quantities in the same reference frame.

Rewrite
$$\mathbf{E}(\mathbf{x},t)=e\underbrace{\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret}}_{\text{velocity field }(\propto\frac{1}{R^2})}+\frac{e}{c}\underbrace{\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\right]_{ret}}_{\substack{\text{acceleration field}\\(\propto\frac{\dot\beta}{R}\text{ and }\perp\mathbf{n})}}\qquad(14.14)$$
$$\mathbf{B}(\mathbf{x},t)=\mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t)\qquad(14.13)$$

(iii) **E** and **B** in general have a broad frequency spectrum. Since we have derived (14.13) and (14.14) from (9.2), which applies to a non-dispersive medium (in this case, the vacuum), singals at all frequencies travel at speed c. Hence, **E** and **B** at t depend only on the *instantaneous* motion of the point charge at a *single* retarded position $\mathbf{r}(t')$.

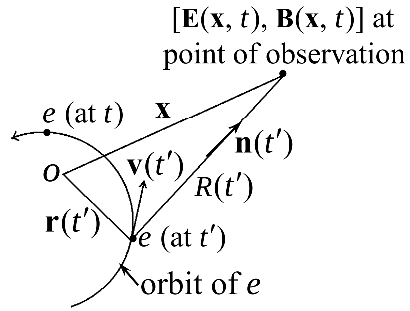


Rewrite
$$\mathbf{E}(\mathbf{x},t)=e\underbrace{\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret}}_{\text{velocity field }(\propto\frac{1}{R^2})}+\frac{e}{c}\underbrace{\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\right]_{ret}}_{\substack{\text{acceleration field}\\(\propto\frac{\dot\beta}{R}\text{ and }\perp\mathbf{n})}}\qquad(14.14)$$
$$\mathbf{B}(\mathbf{x},t)=\mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t)\qquad(14.13)$$

(iv) Quantities in the brackets are to be evaluated at the retarded time t', which is the solution of

$$t'+|\mathbf{x}-\mathbf{r}(t')|/c=t,$$

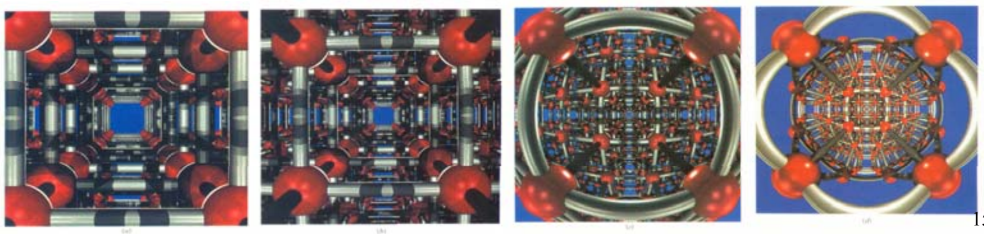
where the orbit $\mathbf{r}(t')$ is a specified function of t'. Thus, t' depends on $\mathbf{x}$ and t. This makes the final expression for **E** a function of $\mathbf{x}$ and t, as shown on the LHS of (14.14). For the same reason, the unit vector $\mathbf{n}(t')$ in (14.13), hence the final expression for **B**, also depends on $\mathbf{x}$ and t [see (14.17a) below].



(v) The relation between observer's time and the retarded time, $t'=t-|\mathbf{x}-\mathbf{r}(t')|/c$, indicates that a signal from the charge travels at speed c toward the observer, independent of the motion of the charge (Einstein's postulate 2).

An Illustration of Time Retardation and Length Contraction: Computer generated graphics show the visual appearance of a three-dimensional lattice of rods and balls moving toward you at various speeds. (from Benson, "University Physics")

The normal view at rest	At 0.5c, the rods appear straight.	At 0.95c, the rods appear bent.	At 0.99c, the lattice appears severely distorted.



Charge in Uniform Motion : $\mathbf{v}=const.$

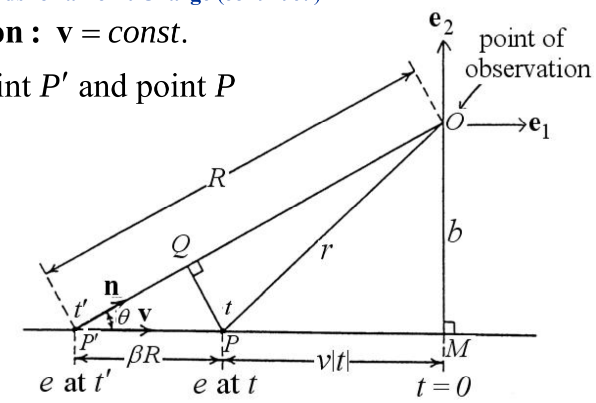
$P'P=$ distance between point P' and point P

$$=v\frac{R}{c}=\beta R$$

$$P'Q=\beta R\cos\theta=\boldsymbol{\beta}\cdot\mathbf{n}R$$

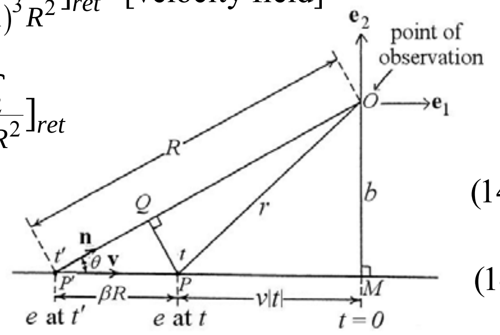
$$OQ=R-P'Q=R(1-\boldsymbol{\beta}\cdot\mathbf{n})$$

$$(OQ)^2=[R(1-\boldsymbol{\beta}\cdot\mathbf{n})]^2$$
$$=r^2-(PQ)^2$$
$$=\overbrace{r^2}^{b^2+v^2t^2}-\beta^2\overbrace{R^2\sin^2\theta}^{b^2}$$
$$=b^2+v^2t^2-\beta^2b^2=\tfrac{1}{\gamma^2}(b^2+\gamma^2v^2t^2)$$



In the above expressions, R and $\mathbf{n}$ are retarded quantities ($\boldsymbol{\beta}$ is a constant). Hence, $[R(1-\boldsymbol{\beta}\cdot\mathbf{n})]_{ret}=\frac{1}{\gamma}(b^2+\gamma^2v^2t^2)^{1/2}$

$$\mathbf{v} = const. \Rightarrow \mathbf{E} = e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R^2}\right]_{ret} \quad \text{[velocity field]}$$



$$\Rightarrow E_2 = \mathbf{E}\cdot\mathbf{e}_2 = e\left[\frac{\overbrace{\mathbf{n}\cdot\mathbf{e}_2}^{b/R} - \overbrace{\boldsymbol{\beta}\cdot\mathbf{e}_2}^{0}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R^2}\right]_{ret}$$

$$= e\left[\frac{b}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R^3}\right]_{ret} \qquad (14.17b)$$

$$= \frac{e\gamma b}{\left(b^2+\gamma^2 v^2 t^2\right)^{3/2}} \qquad (14.17a)$$

[same as (11.152)]

$$\boxed{\left[R(1-\boldsymbol{\beta}\cdot\mathbf{n})\right]_{ret} = \tfrac{1}{\gamma}(b^2+\gamma^2 v^2 t^2)^{1/2}, \text{ last page}}$$

$$E_1 = \mathbf{E}\cdot\mathbf{e}_1 = e\left[\frac{\overbrace{\mathbf{n}\cdot\mathbf{e}_1}^{\cos\theta} - \overbrace{\boldsymbol{\beta}\cdot\mathbf{e}_1}^{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R^2}\right]_{ret} = e\left[\frac{\cos\theta-\beta}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R^2}\right]_{ret} = \frac{e\gamma v|t|}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

$E_3 = 0$ by symmetry.

$\cos\theta - \beta = \frac{\beta R + v|t|}{R} - \beta = \frac{v|t|}{R}$.
$t < 0$ on the left side of the origin ($t = 0$).

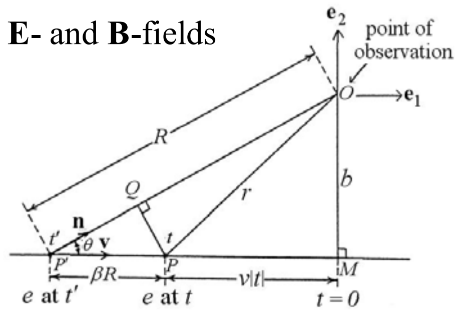
$$\mathbf{B} = \mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t) = (\cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2)\times(E_1\mathbf{e}_1 + E_2\mathbf{e}_2)$$
$$= (E_2\cos\theta - E_1\sin\theta)\,\mathbf{e}_3$$

So, the only nonvanishing component of $\mathbf{B}$ is B_3

$$B_3 = E_2 \underbrace{\cos\theta}_{\frac{\beta R+v|t|}{R}} - E_1 \underbrace{\sin\theta}_{\frac{b}{R}} = \frac{e\gamma}{\left(b^2+\gamma^2 v^2 t^2\right)^{3/2}}\left[\tfrac{b}{R}(\beta R + v|t|) - v|t|\tfrac{b}{R}\right] = \beta E_2$$

Discussion: (i) Rewrite the **E**- and **B**-fields

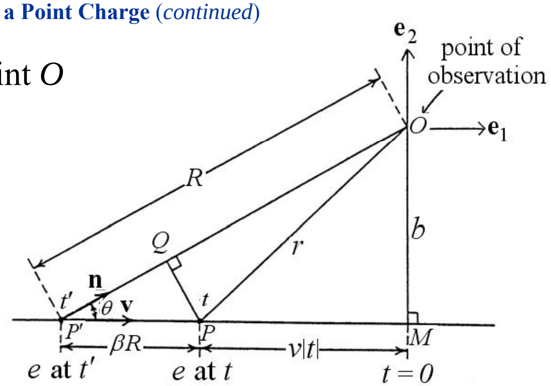
$$\begin{cases} E_1 = \dfrac{e\gamma v|t|}{\left(b^2+\gamma^2 v^2 t^2\right)^{3/2}} \\ E_2 = \dfrac{e\gamma b}{\left(b^2+\gamma^2 v^2 t^2\right)^{3/2}} \\ B_3 = \beta E_2 \end{cases}$$



As expected, the final expressions for **E** and **B** are functions of the observer's position ($\mathbf{x} = b\mathbf{e}_2$) and time ($t$), although the fields are generated by the charge at the retarded position (P') and time (t').

(ii) Rewrite the **E**-field at point O

$$\begin{cases} E_1 = \dfrac{e\gamma v|t|}{\left(b^2+\gamma^2 v^2 t^2\right)^{3/2}} \\ E_2 = \dfrac{e\gamma b}{\left(b^2+\gamma^2 v^2 t^2\right)^{3/2}} \end{cases}$$



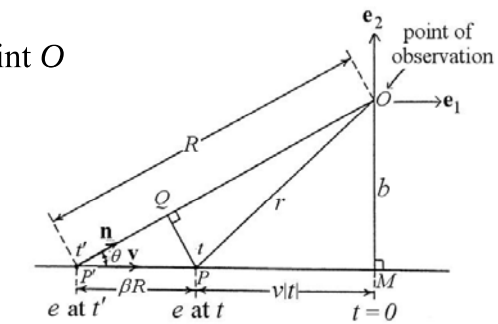
$$\Rightarrow \frac{E_1}{E_2} = \frac{v|t|}{b}$$

$\Rightarrow$ If $e > 0$, **E** is directed from the charge's *present* position P (i.e. position at the time of observation) to the observation point O, although **E** is generated by the charge at the retarded position P'.

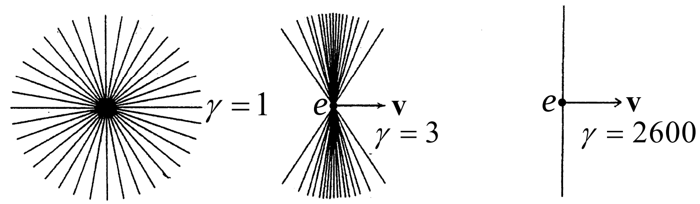
$\Rightarrow$ Since b and t can be given arbitrary (positive or negative) values, this direction relation applies to all observation points around the charge. Thus, **E**-field lines around the charge are straight lines emanating from (or, if $e < 0$, converging to) the present position P.

(iii) Rewrite the **E**-field at point O

$$\begin{cases} E_1 = \dfrac{e\gamma v|t|}{\left(b^2+\gamma^2 v^2 t^2\right)^{3/2}} \\ E_2 = \dfrac{e\gamma b}{\left(b^2+\gamma^2 v^2 t^2\right)^{3/2}} \end{cases}$$

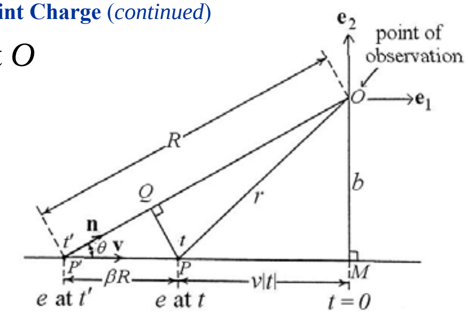


E-field lines $\rightarrow$



(iv) Rewrite the **E**-field at point O

$$\begin{cases} E_1 = \dfrac{e\gamma v|t|}{\left(b^2+\gamma^2 v^2 t^2\right)^{3/2}} \\ E_2 = \dfrac{e\gamma b}{\left(b^2+\gamma^2 v^2 t^2\right)^{3/2}} \end{cases}$$



E_2 has a maximum value at $t = 0$, when e passes through point M.

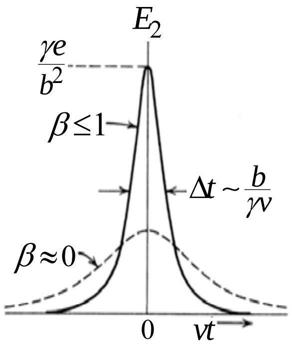
$$E_2^{\max} = E_2(t=0) = \frac{\gamma e}{b^2}$$

E_2 is down to $\frac{1}{2\sqrt{2}} E_2^{\max}$ at $t = \frac{b}{\gamma v}$.

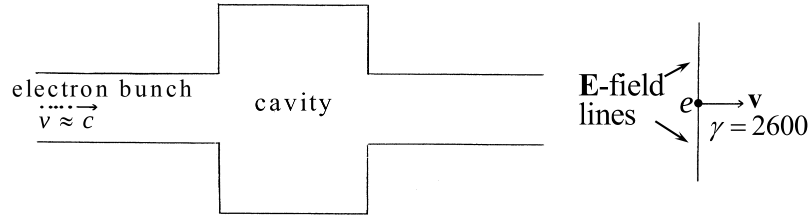
$$\frac{E_2\left(t=\frac{b}{\gamma v}\right)}{E_2^{\max}} = \frac{1}{2\sqrt{2}}$$

same as (11.153)

$\Rightarrow$ Duration of appeciable E_2: $\overbrace{\Delta t \approx \frac{b}{\gamma v}}$

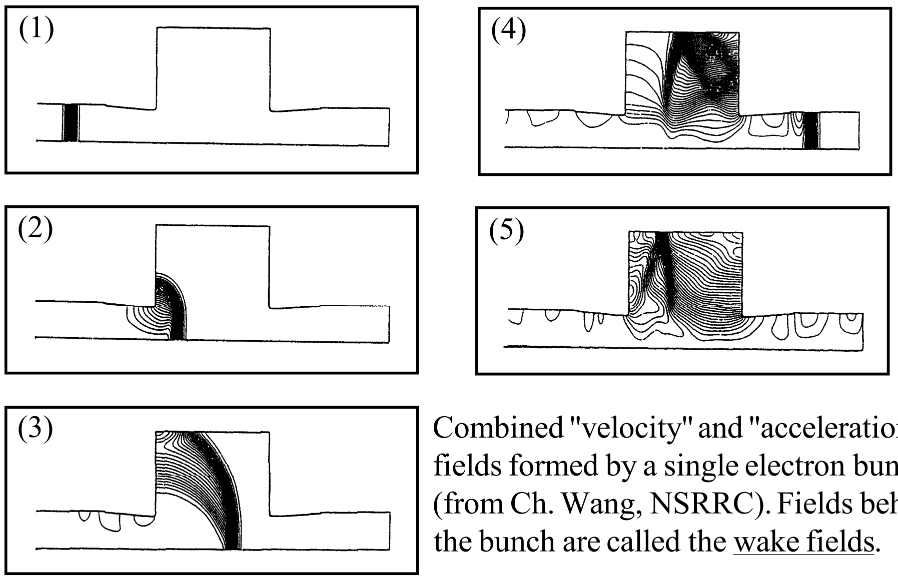


Electrodynamics in a Cavity : As shown in the figure, an electron bunch moving uniformly on the axis with $\gamma = 2600$ is about to enter a cavity. Since $E_\perp = (2600)^3 E_\parallel$, the **E**-field lines of every electron are concentrated in a flat disk with the electron at the center (velocity field). As a result, the electrons hardly "see" each other, because the (axial) electric forces between these electrons are negligible*. Then, as the bunch enters the cavity, the acceleration fields emerge (next page).



**Question*: The negligible electric force betwen any 2 electrons implies that the axial acceleration of either electron is negligible. However, the acceleration will be non-negligible when it is viewed in the lab frame. Why? [See lecture notes, Ch. 11, Eq. (A.23).]

Fields in the cavity produced by a $\gamma = 2600$ electron bunch



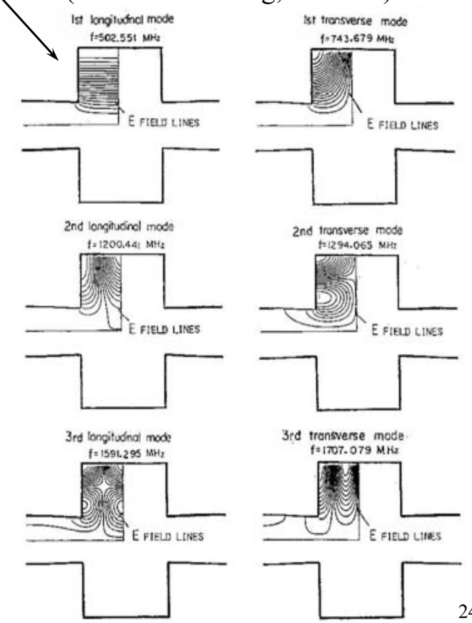
Combined "velocity" and "acceleration" fields formed by a single electron bunch (from Ch. Wang, NSRRC). Fields behind the bunch are called the wake fields.

Question: How do the electrons get decelerated in the cavity?

The lowest order (TM_{010}) mode is excited by the injection of high power microwaves from a klystron. The axial electric field of this mode is used to accelerate the electrons.

Wake fields left in the cavity by the electron bunch can be viewed as the superposition of the complete set of cavity eigenmodes. One or more of the higher-order modes may thus be resonantly reinforced by a succession of electron bunches to grow to significant amplitude and interfere with the acceleration process.

E-field lines of several cavity modes (from L. H. Chang, NSRRC)



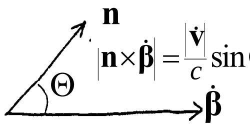
14.2 Total Power Radiated by an Accelerated Charge

Rewrite (14.14): $\mathbf{E}(\mathbf{x},t) = \underbrace{e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret}}_{\text{velocity field}} + \underbrace{\frac{e}{c}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\right]_{ret}}_{\text{acceleration field}}$

$$\mathbf{S}(\mathbf{x},t) = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{c}{4\pi}\mathbf{E}(\mathbf{x},t)\times[\mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t)] = \frac{c}{4\pi}|\mathbf{E}(\mathbf{x},t)|^2\,\mathbf{n}(t')$$

Larmor's Formula: Neglect the velocity field and take the limit $\beta\to 0$ ($\Rightarrow$ retarded γ, R, $\boldsymbol{\beta}$, $\mathbf{n}\approx$ present γ, R, $\boldsymbol{\beta}$, $\mathbf{n}$). Then,

$$\lim_{\beta\to 0}\mathbf{E}(\mathbf{x},t)\approx\frac{e}{cR}\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})$$



$$|\mathbf{n}\times\dot{\boldsymbol{\beta}}| = \frac{|\dot{\mathbf{v}}|}{c}\sin\Theta$$

$$\Rightarrow \lim_{\beta\to 0}\mathbf{S}\cdot\mathbf{n} = \frac{e^2}{4\pi cR^2}|\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})|^2$$

$$\Rightarrow \lim_{\beta\to 0}\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}|\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})|^2 = \frac{e^2}{4\pi c}|\mathbf{n}\times\dot{\boldsymbol{\beta}}|^2 \qquad (14.20)$$

$$= \frac{e^2}{4\pi c^3}|\dot{\mathbf{v}}|^2\sin^2\Theta\ \left[\frac{\text{power radiated}}{\text{unit solid angle}},\ \text{peak at }\Theta=\frac{\pi}{2}\right] \qquad (14.21)$$

$$\Rightarrow \lim_{\beta\to 0}P = \int\frac{dP}{d\Omega}d\Omega = \frac{2e^2}{3c^3}|\dot{\mathbf{v}}|^2 = \frac{2e^2}{3m^2c^3}\left|\frac{d\mathbf{p}}{dt}\right|^2\ \left[\text{Larmor's formula}\right] \qquad (14.23)$$

Note that all quantities in Secs. 14.1-14.4 are real. Hence,

$$\left|\frac{d\mathbf{p}}{dt}\right|^2 = \frac{d\mathbf{p}}{dt}\cdot\frac{d\mathbf{p}}{dt}.\quad\left[\text{In Jackson, this is denoted by }\left(\frac{d\mathbf{p}}{dt}\right)^2\right]$$

Relativistic Generalization: The expression in (14.23) can be generalized to a relativistic form in which P is a Lorentz invariant and applicable to all electron energies. The procedure is as follows.

$$\begin{cases}\mathbf{p}\to\mathbf{P}=(\mathbf{p},\ \frac{iE}{c})\ \text{(4-vector)}\\ t\to\tau\ \text{(Lorentz scalar)}\end{cases}\Rightarrow\frac{d\mathbf{p}}{dt}\to\frac{d\mathbf{P}}{d\tau}\Rightarrow P=\frac{2e^2}{3m^2c^3}\left|\frac{d\mathbf{P}}{d\tau}\right|^2 \qquad (14.24)$$

In terms of $\mathbf{p}$ and E: $P = \frac{2e^2}{3m^2c^3}\left[\left|\frac{d\mathbf{p}}{d\tau}\right|^2 - \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2\right]$ (14.25)

Convert to lab time by $d\tau=\frac{dt}{\gamma}$: $P = \frac{2e^2}{3m^2c^3}\gamma^2\left[\left|\frac{d\mathbf{p}}{dt}\right|^2 - \frac{1}{c^2}\left(\frac{dE}{dt}\right)^2\right]$ (10)

(10) agrees with results derived directly from (14.14) (See Sec. 14.3).

$P = \frac{2e^2}{3m^2c^3}\gamma^2\left[\left|\frac{d\mathbf{p}}{dt}\right|^2 - \frac{1}{c^2}\left(\frac{dE}{dt}\right)^2\right]$ in (10) can be put in different forms:

$$\gamma = \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\Rightarrow\gamma^2 = \left(1-\frac{v^2}{c^2}\right)^{-1} = \left(1-\frac{p^2}{\gamma^2m^2c^2}\right)^{-1}$$

$$\Rightarrow\ \gamma^2 = 1+\frac{p^2}{m^2c^2}\Rightarrow\gamma = \left(1+\frac{p^2}{m^2c^2}\right)^{\frac{1}{2}}$$

$$\frac{dE}{dt} = mc^2\frac{d}{dt}\gamma = mc^2\frac{d}{dt}\left(1+\frac{p^2}{m^2c^2}\right)^{\frac{1}{2}}$$

$$= mc^2\frac{2\frac{p}{m^2c^2}\frac{d}{dt}p}{2\left(1+\frac{p^2}{m^2c^2}\right)^{\frac{1}{2}}} = \frac{p}{\gamma m}\frac{dp}{dt} = v\frac{dp}{dt}$$

Note: $\frac{d\mathbf{p}}{dt}$ expresses both direction and amplitude variations of $\mathbf{p}$, but $\frac{dp}{dt}$ only expresses the amplitude variation of $\mathbf{p}$.

Sub. $v\frac{dp}{dt}$ for $\frac{dE}{dt}$ in (10)

$$\Rightarrow P = \frac{2e^2}{3m^2c^3}\gamma^2\left[\left|\frac{d\mathbf{p}}{dt}\right|^2 - \beta^2\left(\frac{dp}{dt}\right)^2\right] \qquad (11)$$

$$\left|\frac{d\mathbf{p}}{dt}\right|^2 - \frac{1}{c^2}\left(\frac{dE}{dt}\right)^2$$

$$= m^2c^2\left|\boldsymbol{\beta}\frac{d\gamma}{dt}+\gamma\dot{\boldsymbol{\beta}}\right|^2 - m^2c^2\left(\frac{d\gamma}{dt}\right)^2$$

$$\frac{d\gamma}{dt} = \frac{d}{dt}(1-\boldsymbol{\beta}\cdot\boldsymbol{\beta})^{-\frac{1}{2}} = -\frac{1}{2}\frac{-\dot{\boldsymbol{\beta}}\cdot\boldsymbol{\beta}-\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}}{(1-\boldsymbol{\beta}\cdot\boldsymbol{\beta})^{\frac{3}{2}}} = \gamma^3\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}$$

$$= m^2c^2\left[\beta^2\left(\frac{d\gamma}{dt}\right)^2 + 2\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\gamma\frac{d\gamma}{dt} + \gamma^2|\dot{\boldsymbol{\beta}}|^2 - \left(\frac{d\gamma}{dt}\right)^2\right]$$

$$= m^2c^2\left[-\frac{1}{\gamma^2}\gamma^6(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2 + 2\gamma^4(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2 + \gamma^2|\dot{\boldsymbol{\beta}}|^2\right]$$

$$= \gamma^4m^2c^2\left[(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2 + (1-\beta^2)|\dot{\boldsymbol{\beta}}|^2\right] = \gamma^4m^2c^2\left[|\dot{\boldsymbol{\beta}}|^2 + (\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2 - \beta^2\dot{\beta}^2\right]$$

$$= \gamma^4m^2c^2\left[|\dot{\boldsymbol{\beta}}|^2 - |\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}|^2\right] \qquad (12)$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \Rightarrow |\mathbf{A}\times\mathbf{B}|^2 = A^2B^2 - |\mathbf{A}\cdot\mathbf{B}|^2$$

Sub. (12) into (10)

$$\Rightarrow P = \frac{2e^2}{3c}\gamma^6\left[|\dot{\boldsymbol{\beta}}|^2 - |\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}|^2\right] \qquad (14.26)$$

Example 1: Linear accelerator (**p** || accelerating force **F**)

Rewrite (11): $P=\frac{2e^2}{3m^2c^3}\gamma^2\left[\left|\frac{d\mathbf{p}}{dt}\right|^2-\beta^2\left(\frac{dp}{dt}\right)^2\right]$

For linear acceleration, $\left|\frac{d\mathbf{p}}{dt}\right|^2=\left(\frac{dp}{dt}\right)^2 \Rightarrow \quad P=\frac{2e^2}{3m^2c^3}\left(\frac{dp}{dt}\right)^2$ (14.27)

$$\Rightarrow P=\frac{2e^2}{3m^2c^3}\left(\frac{dE}{dx}\right)^2 \longleftarrow \boxed{\left.\begin{matrix}dp=Fdt\\ dE=Fdx\end{matrix}\right\}\Rightarrow \frac{dp}{dt}=\frac{dE}{dx}=F} \qquad (14.28)$$

$$\frac{P}{\left(\frac{dE}{dt}\right)}=\frac{\frac{2e^2}{3m^2c^3}\frac{dE}{dx}\frac{1}{v}\cancel{\frac{dE}{dt}}}{\cancel{\frac{dE}{dt}}}=\frac{2e^2}{3m^2c^3}\frac{1}{v}\frac{dE}{dx}\underset{v\approx c}{\approx}\underbrace{\frac{2\left(\frac{e^2}{mc^2}\right)}{3mc^2}}_{3.7\times10^{-15}\text{ m/MeV}}\frac{dE}{dx}$$

$\boxed{P\text{: radiated power. } \frac{dE}{dt}\text{: externally supplied power}}$

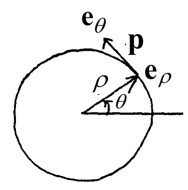
Typically, $\frac{dE}{dx}<50$ MeV/m $\Rightarrow$ [Radiation losses are completely negligible in linear accelerators.]

Example 2: Circular accelerator (e.g. synchrotron)

Rewrite (11): $P=\frac{2e^2}{3m^2c^3}\gamma^2\left[\left|\frac{d\mathbf{p}}{dt}\right|^2-\beta^2\left(\frac{dp}{dt}\right)^2\right]$

For circular accelerators, $\left|\frac{d\mathbf{p}}{dt}\right|\gg\frac{dp}{dt}$. Thus,

$$P\approx\frac{2e^2}{3m^2c^3}\gamma^2\left|\frac{d\mathbf{p}}{dt}\right|^2$$



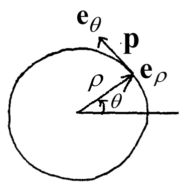
$$\frac{d\mathbf{p}}{dt}=\frac{d(p\mathbf{e}_\theta)}{dt}=p\frac{d}{dt}\mathbf{e}_\theta+\mathbf{e}_\theta\underbrace{\frac{dp}{dt}}_{\text{negligible}}\approx p\underbrace{\frac{d\mathbf{e}_\theta}{d\theta}}_{-\mathbf{e}_\rho}\underbrace{\frac{d\theta}{dt}}_{\omega}=-\omega p\mathbf{e}_\rho$$

$\Rightarrow\left|\frac{d\mathbf{p}}{dt}\right|\approx\omega p.$ $\quad\boxed{\omega=\frac{v}{\rho},\ p=\gamma mv}$

$$\Rightarrow P\approx\frac{2e^2}{3m^2c^3}\gamma^2\omega^2p^2\overset{\downarrow}{=}\frac{2e^2c}{3\rho^2}\beta^4\gamma^4 \qquad (14.31)$$

Note that (14.31) is an exact expression for P if the particle is in uniform circular motion, i.e. if $\frac{dp}{dt}=0$.

Rewrite (14.31): $P\approx\frac{2e^2c}{3\rho^2}\beta^4\gamma^4$



$\Rightarrow\delta E=$ radiation loss per revolution

$$=\frac{2\pi\rho}{v}P$$

$$\approx\frac{4\pi}{3}\frac{e^2}{\rho}\beta^3\gamma^4\overset{\beta\approx1}{\approx}8.85\times10^{-2}\frac{[E(\text{in GeV})]^4}{\rho(\text{in meters})}\text{ MeV}$$

$$\approx\begin{cases}1\text{ keV}, & \text{for early synchrotrons (accelerators)}\\ 72\text{ keV}, & \text{for the 1.3 GeV NSRRC synchrontron}\\ 8.85\text{ MeV}, & \text{for the 10 GeV Cornell synchrontron}\end{cases}\ \left.\right\}\text{ storage rings}$$

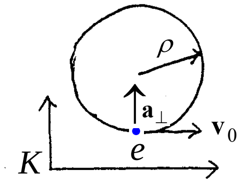
Total power radiated in circular electron accelerators:

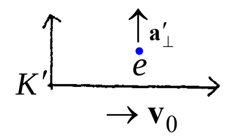
$$P(\text{in watts})=10^6\times\delta E(\text{in MeV})\times J(\text{in amp})$$

Problem: If a charge is in uniform circular motion, (14.31) is an exact expression for the total power it radiates. Show that the total power has the same value as viewed in the rest frame of the particle.

Solution: Consider an instantaneous position of the charge located at the bottom of its orbit, where the charge moves horizontally to the right at velocity $\mathbf{v}_0$ (upper figure) and the acceleration $\mathbf{a}_\perp$ points vertically upward with $a_\perp=v_0^2/\rho$ (ρ is the radius of the circle). Viewed in the rest frame of the charge (lower figure), we have [see Eq. (A.22) in Ch. 11 of lecture notes]

$$\begin{cases}\mathbf{a}'_{\|}=\dfrac{1}{\gamma_0^3(1-\frac{\mathbf{v}_0\cdot\mathbf{v}}{c^2})^3}\underbrace{\mathbf{a}_{\|}}_{0}\\[2ex] \mathbf{a}'_{\perp}=\dfrac{1}{\gamma_0^2(1-\frac{\mathbf{v}_0\cdot\overset{\mathbf{v}_0}{\overbrace{\mathbf{v}}}}{c^2})^3}\underbrace{[\mathbf{a}_\perp-\frac{\mathbf{v}_0}{c^2}\times(\overset{\mathbf{a}_\perp}{\overbrace{\mathbf{a}}}\times\overset{\mathbf{v}_0}{\overbrace{\mathbf{v}}})]}_{\mathbf{a}_\perp(1-v_0^2/c^2)}\end{cases}$$



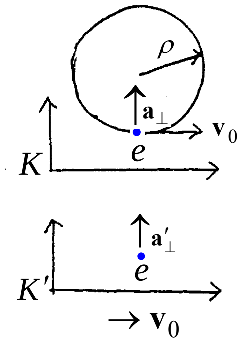


Thus, $\mathbf{a}'_{\|}=0$ and $\mathbf{a}'_\perp=\gamma_0^2\mathbf{a}_\perp$.

Thus, the acceleration of the charge is vertically upward in both frames and they are related by

$$\mathbf{a}'_\perp = \gamma^2 \mathbf{a}_\perp$$

Since the charge is at rest in frame K', Larmor's formula in (14.23) becomes exact, which gives



$$P' = \frac{2e^2}{3c^3}|\dot{\mathbf{v}}'|^2 = \frac{2e^2}{3c^3}|\mathbf{a}'_\perp|^2 = \frac{2e^2}{3c^3}\gamma^4 a_\perp^2$$

$$= \frac{2e^2}{3c^3}\gamma^4\frac{v_0^4}{\rho^2} = \frac{2e^2 c}{3\rho^2}\beta^4\gamma^4 \qquad \boxed{a_\perp = \frac{v_0^2}{\rho}}$$

This is the same power as viewed in the lab frame [see (14.31)]. The result here, $P = P'$, is consistent with the fact the total radiated power is a Lorentz invariant [see (14.24)]. However, the angular distribution of radiation will be different in the two frames. We will show later in (14.44) that for the same acceleration, the angular distribution depends sensitively on particle's velocity.

14.3 Angular Distribution of Radiation Emitted by an Accelerated Charge

Rewrite (14.14): $\mathbf{E}(\mathbf{x},t) = e\underbrace{\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret}}_{\text{velocity field}} + \frac{e}{c}\underbrace{\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\right]_{ret}}_{\text{acceleration field}}$

power per unit area at observation point

$$\mathbf{S}(\mathbf{x},t) = \frac{c}{4\pi}\mathbf{E}(\mathbf{x},t)\times\mathbf{B}(\mathbf{x},t)$$

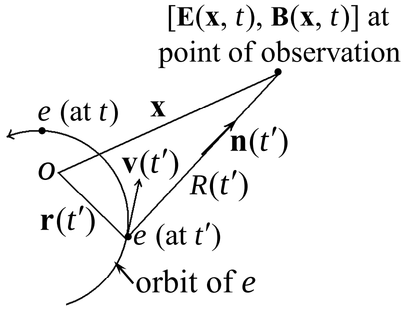
$$= \frac{c}{4\pi}\mathbf{E}(\mathbf{x},t)\times[\mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t)]$$

$$= \frac{c}{4\pi}|\mathbf{E}(\mathbf{x},t)|^2\,\mathbf{n}(t')$$

$$\Rightarrow \mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}(t') = \frac{c}{4\pi}|\mathbf{E}(\mathbf{x},t)|^2$$

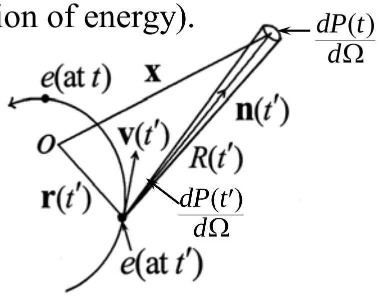
(Neglect the velocity field)

$$= \frac{e^2}{4\pi c}\left\{\frac{1}{R^2}\left|\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right|^2\right\}_{ret} \qquad (14.35)$$



In this section (as in Sec. 14.2), we are interested in the angular distribution of power *radiated* by the charge. But $\mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}(t') = \frac{c}{4\pi}|\mathbf{E}(\mathbf{x},t)|^2$ in (14.35) gives the power per unit area *received* at the observation point. Power radiated by the charge into a unit solid angle [$dP(t')/d\Omega$] is in general different from the power received over the area subtending the solid angle [$dP(t)/d\Omega$]. The reason is that motion of the charge toward (away from) the observation point will shorten (lengthen) the radiated pulse, which results in increased (decreased) power at the observation point because the total energy received must equal the total energy radiated (conservation of energy).

Thus, to express the power *radiated* in terms of the power *received*, we need to determine the ratio of dt (received pulse length) to dt' (radiated pulse length).



Observation time t and radiation time t' are related by

$$t = t' + \frac{R(t')}{c}. \qquad \boxed{\text{Use (4): } \frac{dR(t')}{dt'} = -\mathbf{v}(t')\cdot\mathbf{n}(t')}$$

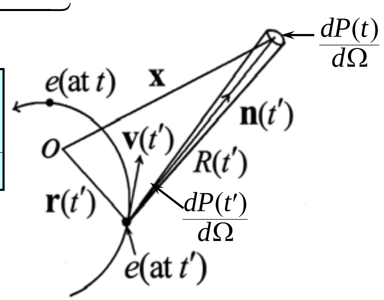
Thus, $\frac{dt}{dt'} = 1 + \frac{1}{c}\frac{dR(t')}{dt'} = 1 - \boldsymbol{\beta}(t')\cdot\mathbf{n}(t')$

$\Rightarrow$ A pulse of duration dt received at $\mathbf{x}$ and t is radiated by the charge at $\mathbf{r}(t')$ and t' for a duration of $dt' = dt/[1-\boldsymbol{\beta}(t')\cdot\mathbf{n}(t')]$. Note that dt and dt' are quantities in the same reference frame (lab frame).

$$\underbrace{R^2(t')\mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}(t')}_{dP(t)/d\Omega}\,dt = \underbrace{R^2(t')\mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}(t')\frac{dt}{dt'}}_{dP(t')/d\Omega}\,dt'$$

$dP(t)/d\Omega$: $\boxed{\dfrac{\text{power received at } \mathbf{x} \text{ and } t}{\text{unit solid angle}}}$

$dP(t')/d\Omega$: $\boxed{\dfrac{\text{power radiated by charge at } \mathbf{r}(t') \text{ and } t'}{\text{unit solid angle}}}$

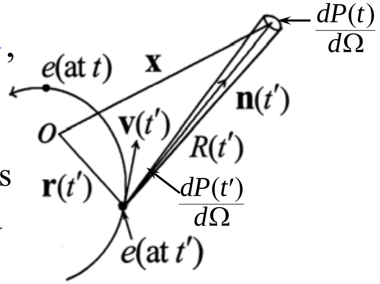


In both $\frac{dP(t')}{d\Omega}$ and $\frac{dP(t)}{d\Omega}$, $d\Omega$ is with respect to the charge.

Rewrite $\underbrace{R^2(t')\mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}(t')}_{dP(t)/d\Omega}dt = \underbrace{R^2(t')\mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}(t')\frac{dt}{dt'}}_{dP(t')/d\Omega}dt'$

$$\Rightarrow \frac{dP(t')}{d\Omega} = \frac{dP(t)}{d\Omega}\underbrace{\frac{dt}{dt'}}_{\boxed{=1-\boldsymbol{\beta}(t')\cdot\mathbf{n}(t')}} = R^2(t')\underbrace{\mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}(t')}_{\boxed{=\frac{e^2}{4\pi c}\left\{\frac{1}{R^2}\left|\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right|^2\right\}_{ret}\text{ by (14.35)}}}[1-\boldsymbol{\beta}(t')\cdot\mathbf{n}(t')]$$

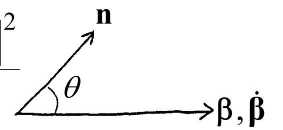
$$\Rightarrow \frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c}\frac{\left|\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right|^2}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^5}, \qquad (14.38)$$



where $\mathbf{n}$, $\boldsymbol{\beta}$, $\dot{\boldsymbol{\beta}}$ are to be evaluated at the retarded time t'. (14.38) gives the power radiated into a unit solid angle in the direction of $\mathbf{n}$ in terms of the charge e and instantaneous $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ of the particle.

Case 1: $\boldsymbol{\beta}\parallel\dot{\boldsymbol{\beta}}$

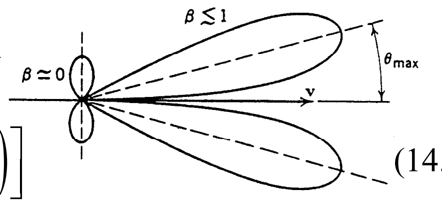
Rewrite (14.38): $\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c}\frac{\left|\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right|^2}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^5}$



$$\left.\begin{cases}\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}=0\\ \left|\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\right|^2=\left|\dot{\boldsymbol{\beta}}\right|^2\sin^2\theta\end{cases}\right\}\Rightarrow \frac{dP(t')}{d\Omega} = \frac{e^2\dot{v}^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \qquad (14.39)$$

$$\Rightarrow P(t') = \int\frac{dP(t')}{d\Omega}d\Omega = \frac{2}{3}\frac{e^2}{c^3}\dot{v}^2\gamma^6 \quad \left[\begin{array}{l}\text{agree with (14.26)}\\ \text{and (14.27)}\end{array}\right] \qquad (14.43)$$

For $\beta \ll 1$, (14.39) reduces to Larmor's result (14.21), with the radiation peaking at $\theta = 90^\circ$. But as $\beta\to 1$, the angular distribution is tipped forward more and more and increases in magnitude, with the maximum intensity at



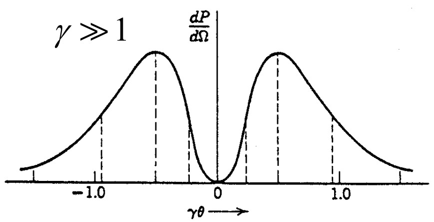
$$\theta_{\max} = \cos^{-1}\left[\frac{1}{3\beta}\left(\sqrt{1+15\beta^2}-1\right)\right] \qquad (14.40)$$

As $\beta\to 1$, we have $\theta \ll 1$. Hence,

$$1-\beta\cos\theta \approx 1-\beta\left(1-\tfrac{1}{2}\theta^2\right)$$

$$=1-\beta+\frac{\overbrace{\beta}^{\approx 1}}{2}\theta^2 \approx \frac{(1-\beta)(1+\overbrace{\beta}^{\approx 1})}{2}+\frac{\theta^2}{2}$$

$$=\frac{1-\beta^2}{2}+\frac{\theta^2}{2}=\frac{1}{2\gamma^2}\left(1+\gamma^2\theta^2\right)$$

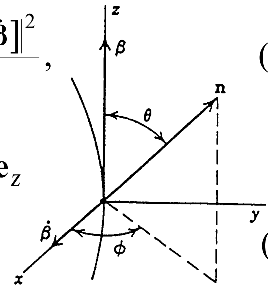


$$\Rightarrow \lim_{\beta\to 1}\frac{dP(t')}{d\Omega} = \frac{e^2\dot{v}^2}{4\pi c^3}\frac{\overbrace{\sin^2\theta}^{\approx\theta^2}}{(1-\beta\cos\theta)^5} \approx \frac{8}{\pi}\frac{e^2\dot{v}^2}{c^3}\gamma^8\frac{(\gamma\theta)^2}{(1+\gamma^2\theta^2)^5} \qquad (14.41)$$

$$\Rightarrow \begin{cases}\theta_{\max} = \frac{1}{2\gamma}\ \text{[angle of maximum intensity]} & (14.40)\\ <\theta^2>^{\frac{1}{2}} = \left[\dfrac{\int\theta^2\frac{dP(t')}{d\Omega}d\Omega}{\int\frac{dP(t')}{d\Omega}d\Omega}\right]^{\frac{1}{2}} = \frac{1}{\gamma} = \frac{mc^2}{E}\ \left[\begin{array}{l}\text{root mean}\\ \text{square angle}\end{array}\right] & (14.42)\end{cases}$$

Case 2: $\boldsymbol{\beta}\perp\dot{\boldsymbol{\beta}}$. In $\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c}\frac{\left|\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]\right|^2}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^5}$, $\qquad (14.38)$

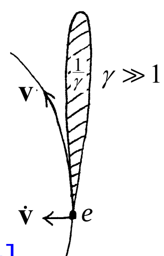
$$\text{let}\begin{cases}\boldsymbol{\beta}\parallel\mathbf{e}_z,\ \dot{\boldsymbol{\beta}}\parallel\mathbf{e}_x\\ \mathbf{n}=\sin\theta\cos\phi\,\mathbf{e}_x+\sin\theta\sin\phi\,\mathbf{e}_y+\cos\theta\,\mathbf{e}_z\end{cases}$$



$$\Rightarrow \frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c^3}\frac{|\dot{\mathbf{v}}|^2}{(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right] \qquad (14.44)$$

$$\Rightarrow P(t') = \int\frac{dP(t')}{d\Omega}d\Omega = \frac{2}{3}\frac{e^2}{c^3}\underbrace{|\dot{\mathbf{v}}|^2}_{\frac{v^2}{\rho}}\gamma^4 = \begin{cases}\frac{2}{3}\frac{e^2c}{\rho^2}\beta^4\gamma^4\ \text{[agree with (14.31)]}\\ \frac{2}{3}\frac{e^2}{m^2c^3}\gamma^2\left|\frac{d\mathbf{p}}{dt}\right|^2\end{cases} \qquad (14.47)$$

$$\lim_{\beta\to 1}\frac{dP(t')}{d\Omega} = \frac{2e^2}{\pi c^3}\gamma^6\frac{|\dot{\mathbf{v}}|^2}{(1+\gamma^2\theta^2)^3}\left[1-\frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta^2)^2}\right] \qquad (14.45)$$



$$\Rightarrow\begin{cases}\theta_{\max}=0\ \text{[angle of maximum intensity]}\\ <\theta^2>^{\frac{1}{2}}=\frac{1}{\gamma}\ \text{[}\Rightarrow\text{ narrow cone like a searchlight]}\end{cases}$$

14.4 Radiation Emitted by a Charge in Arbitrary, Extremely Relativistic Motion

In Secs. 14.2 and 14.3, we examined the radiation problem from the viewpoint of the charged particle and expresssed the radiated power in terms of the instantaneous $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ of the particle.

From here on, we will switch our viewpoint to the observer. The emphasis will also be switched from the power of radiation to the *frequency spectrum* of the signal received at the observervation point.

To find the spectrum, we need to first know the time history of the observed ratiation. Hence, we can no longer stick to instantaneous quantities as in Secs. 14.2 and 14.3. We must now follow the particle's orbit. As the particle travels along its orbit, it continuously radiates toward the observer. A Fourier transform of the time-dependent signal received then reveals its sprectral contents.

We will be interested only in perpendicular acceleration ($\dot{\boldsymbol{\beta}} \perp \boldsymbol{\beta}$). The reason is as follows.

Rewrite

$$\begin{cases} P(t') = \dfrac{2e^2}{3m^2c^3}\left(\dfrac{dp}{dt}\right)^2, & \text{for } \dot{\boldsymbol{\beta}} \parallel \boldsymbol{\beta} \quad (14.27) \\ P(t') = \dfrac{2}{3}\dfrac{e^2}{m^2c^3}\gamma^2\left|\dfrac{d\mathbf{p}}{dt}\right|^2, & \text{for } \dot{\boldsymbol{\beta}} \perp \boldsymbol{\beta} \quad (14.47) \end{cases}$$

which implies $P(\dot{\boldsymbol{\beta}} \perp \boldsymbol{\beta}) = \gamma^2 P(\dot{\boldsymbol{\beta}} \parallel \boldsymbol{\beta})$ for the same accelerating force.

Hence, for a charge with $\gamma \gg 1$ in arbitrary motion, we may neglect $P(t')$ due to $\dot{\boldsymbol{\beta}} \parallel \boldsymbol{\beta}$ and consider only $P(t')$ due to $\dot{\boldsymbol{\beta}} \perp \boldsymbol{\beta}$. The instantaneous radius of curvature ρ can be expressed in terms of the perpendicular component of the acceleration ($\dot{v}_\perp$) as follows.

$$F_\perp = \frac{\gamma m v^2}{\rho} = \gamma m \dot{v}_\perp$$

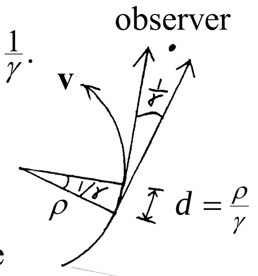
$$\Rightarrow \rho = \frac{v^2}{\dot{v}_\perp} \underset{\gamma \gg 1}{\approx} \frac{c^2}{\dot{v}_\perp} \qquad (14.48)$$

[For acceleration $\perp$ to $\mathbf{v}$, the effective mass is γm. See lecture notes, Ch. 11, Eq. (49).]

The Spectral Width for $\dot{\boldsymbol{\beta}} \perp \boldsymbol{\beta}$:

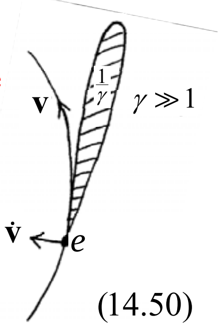
Angular distribution of radiation: $<\theta^2>^{\frac{1}{2}} \approx \frac{1}{\gamma}$.

$\Rightarrow$ The observer is illuminated by light emitted in an arc of length $d \approx \frac{\rho}{\gamma}$, corresponding to a (retarded time) interval of emission $\Delta t' \approx \frac{\rho}{\gamma v}$.



In the interval $\Delta t'$, the front edge of the pulse travels a distance $D = c\Delta t' = \frac{\rho}{\gamma\beta}$, while the rear edge of pulse is behind the front edge by a distance

$$L = D - d = \left(\tfrac{1}{\beta} - 1\right)\frac{\rho}{\gamma} = \frac{1-\beta}{\beta}\frac{\rho}{\gamma} \approx \frac{(1-\beta)(1+\beta)}{2\beta}\frac{\rho}{\gamma} \approx \frac{\rho}{2\gamma^3}$$



$\Rightarrow$ Pulse duration (to the observer): $T = L/c$

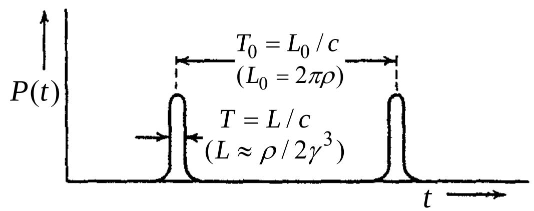
$\Rightarrow$ A broad spectrum ranging from near 0 up to a critical frequency of $\omega_c \sim \frac{1}{T} \sim \frac{c}{L} \sim \frac{c}{\rho}\gamma^3$, (14.50)

where ω_c is the maximum frequency of appreciable radiation.

Synchrotron Radiation-A Qualitative Discussion: If the charge is in circular motion with rotation frequency ω_0, then $\omega_0\rho \approx c$ and

$$\omega_c \sim \frac{c}{\rho}\gamma^3 \approx \omega_0\gamma^3$$

The pulses occur at the observation point at regular intervals of



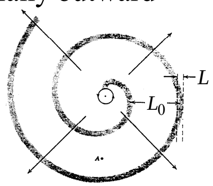
$$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi\rho}{v} \approx \frac{2\pi\rho}{c}$$

Eexample: Cornell 10 GeV synchrotron

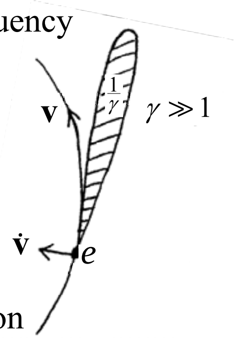
$$\begin{cases} \gamma \approx 2\times 10^4 \\ \omega_0 \approx 3\times 10^6 \text{ / sec} \end{cases}$$

$\Rightarrow \omega_c \approx 2.4\times 10^{19}$ / sec (16 keV x-rays)

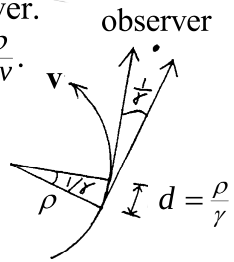
Pulses of synchrotron radiation propagating radially outward



Discussion: In (14.50), $\omega_c \sim \frac{c}{\rho}\gamma^3$, the critical frequency ω_c (maximum frequency of appreciable radiation) scales as γ^3, which explains the extremely high frequency radiation from a synchrotron. The factor γ^3 is due to the short duration of the pluse seen by the observer. The pulse is shortened by two effects:



1. Because the angular width ($1/\gamma$) of the radiation is very narrow, only the radiation emitted by an electron over an arc of length d ($=\rho/\gamma$) can reach the observer. Thus, to the electron, the emission interval is $\Delta t' \approx \frac{\rho}{\gamma v}$.



2. The electron is "chasing" its radiation. Hence, to the observer, the received pulse length is not $\Delta t'$. Instead, it is $\Delta t'$ compressed by a factor of

$$\frac{dt}{dt'} = 1 - \boldsymbol{\beta}(t')\cdot\mathbf{n}(t') = 1-\beta \approx \frac{(1-\beta)(1+\beta)}{2} = \frac{1}{2\gamma^2}$$

Effect 2 is exploited in a device called the free electron laser (FEL).

Example: As a practical example of the pulse duration to the observer, consider again the Cornell 10 GeV synchrotron, for which we have

$$\omega_c \approx 2.4\times10^{19}\,/\,\text{sec}.$$

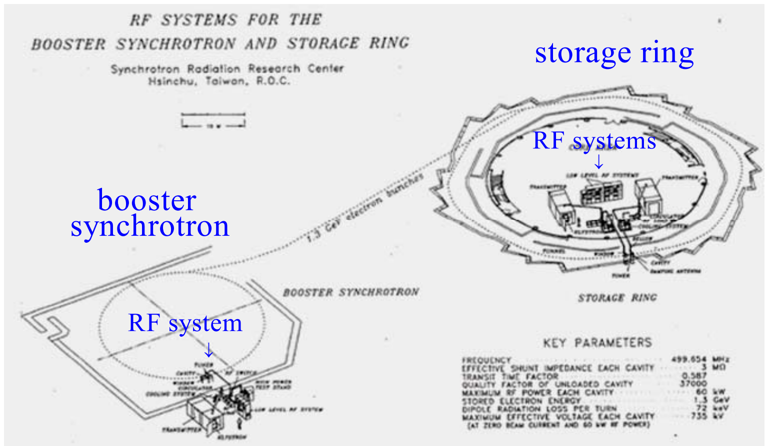
Since $\omega_c \sim \frac{1}{T}$, the pulse duration T of a single electron is incredibly short,

$$T \sim \frac{1}{\omega_c} \approx 4.2\times10^{-20}\ \text{sec}.$$

This explains the broad spectrum. However, the actual pluse in a synchrotron does not come from a single electron, but from an electron bunch of finite length (typically a few mm). Electrons in the bunch radiate incoherently. So the spectrum of the bunch is the same as that of a single electron, but the pulse duration (τ) equals the passage time of the electron bunch ($\tau \approx$ bunch length/c). For example, for a bunch length of 6 mm, we have $\tau \approx 2\times10^{-11}$ sec.

The Synchrotron as a Light Source : The synchrotron emits *intense radiation* with a *very broad frequency spectrum* in a beam of *extremely small angular spread* ($1/\gamma$). It is a unique research tool and can also be used for micro-fabrication and other applications. The photo below shows the light source facility at the National Synchrotron Radiation Research Center (NSRRC) in Taiwan.

Electron bunches are first accelerated to an energy of 1.3 GeV in the booster synchrotron, and then sent to the storage ring (also a synchrotron), where the energy is maintaind at 1.3 GeV while the electrons provide synchrotron radiation to users around the ring. The electrons are powered by microwaves from the RF systems.



The RF system

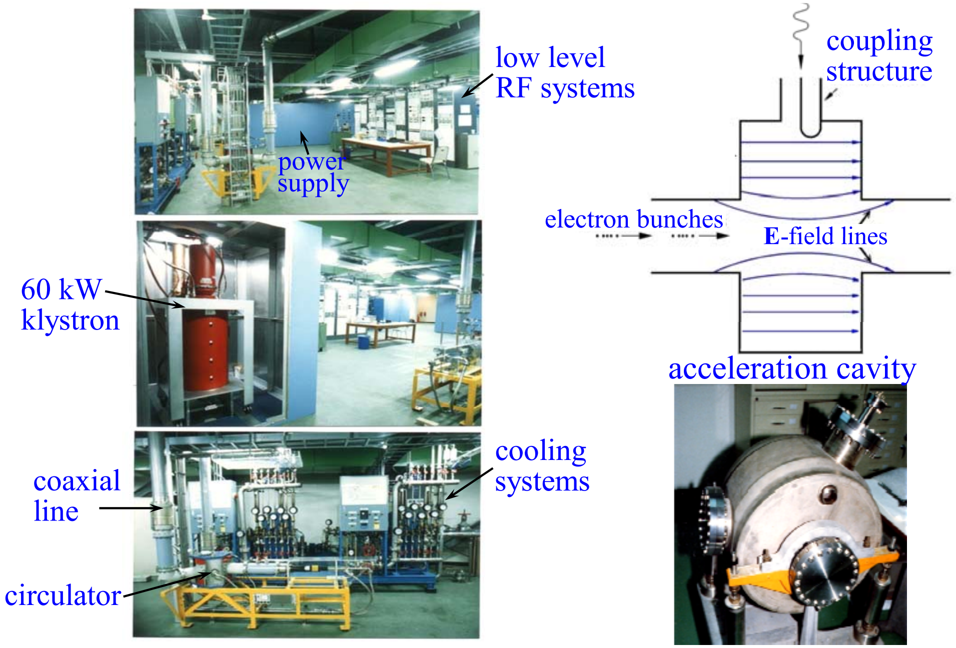
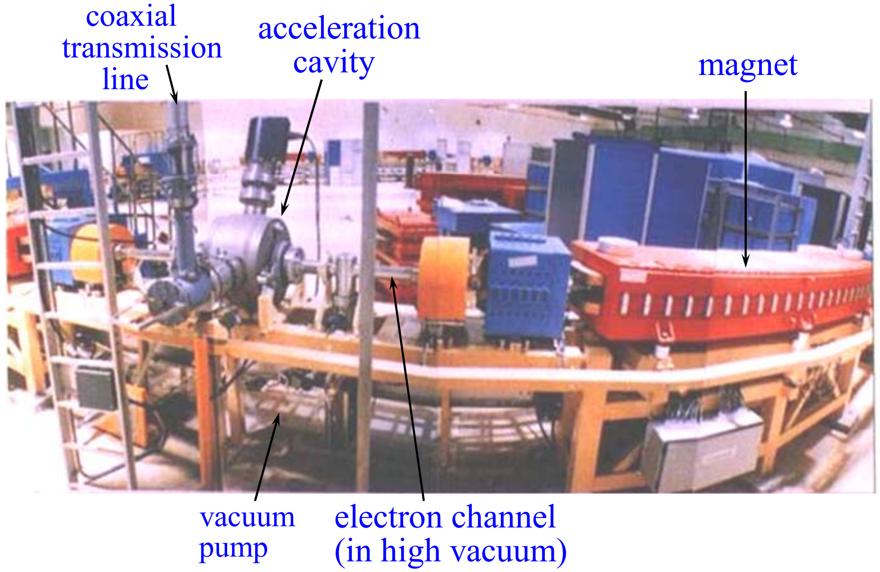


Photo of the NSRRC booster synchrotron showing some key components of the accelerator



Research stations around the NSRRC storage ring

14.7 Undulators and Wigglers for Synchrotron Light Sources

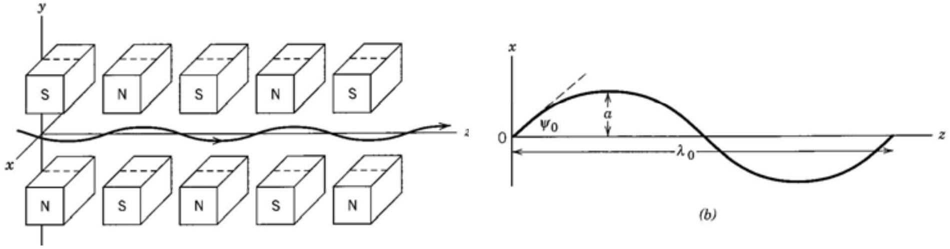
The broad spectrum of radiation emitted by relativistic electrons bent by the magnetic fields of synchrotron storage rings provides a useful source of energetic photons.

As application grew, the need for brighter sources with the radiation more concentrated in frequency led to the magnetic "insertion devices" called wigglers and undulators to be placed in the synchrotron ring.

The magnetic properties of these devices cause the electrons to undergo special motion that results in the concentration of the radiation into a much more monochromatic spectrum or series of seperated peaks.

Essential Idea of Undulators and Wigglers

The essential idea of undulators and wigglers is that a charge particle, usually an electron and usually moving relativistically ($\gamma \gg 1$), is caused to move transversely to its general forward motion by magnetic fields that alternate periodically.

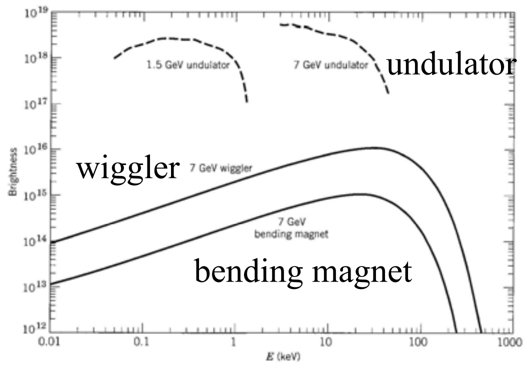


The external magnetic fields induce small transverse oscillations in the motion; the associated accelerations cause radiation to be emitted.

Classification of Undulators and Wigglers

(a) Wiggler ($\psi_0 \gg \Delta\theta$): An observer detectors a series of flicks of the searchlight beam. $\Delta\theta$: angular width of the radiation about the path.

(b) Undulator ($\psi_0 \ll \Delta\theta$): The searchlight beam of radiation moves *negligibly* compared to its own angular width. The radiation detected by the observer is an almost coherent superposition of the contributions from all the oscillations of the trajectory.



Homework of Chap. 14

Problems: 1, 4, 5, 9