Chapter 11: Special Theory of Relativity

(Ref.: Marion & Heald, "Classical Electromagnetic Radiation," 3rd ed., Ch. 14)

Einstein's special theory of relativity is based on two postulates:

1. Laws of physics are invariant in form in all Lorentz frames (In relativity, we often call the inertial frame a Lorentz frame.)
2. The speed of light in vacuum has the same value c in all Lorentz frames, independent of the motion of the source.

The basics of the theory are covered in Appendix A on an elementary level with an emphasis on the Lorentz transformation and relativistic momentum/energy. Here, we examine relativity in the four-dimensional space of $\mathbf{x}$ and t, which provides the framework for us to examine the laws of mechanics and electromagnetism. The contents of the lecture notes depart considerably from Ch.11 of Jackson. Instead, we follow Ch. 14 of Marion.

In the lecture notes, section numbers do not follow Jackson.

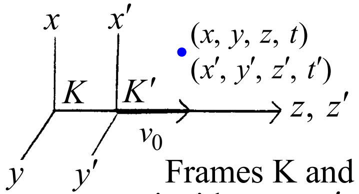
Section 1: Definitions and Operation Rules of Tensors of Different Ranks in the 4-Dimensional Space

The Lorentz Transformation:

Consider two Lorentz frames, K and K'. Frame K' moves along the common z-axis with constant speed v_0 relative to frame K.

Assume that at $t = t' = 0$, coordinate axes of frames K and K' overlap. Postulate 2 leads to the following Lorentz transformation for space and time coordinates. [derived in Appendix A, Eq. (A.15), where the relative motion is assumed to be along the x-axis.]

$$\begin{cases} x' = x \\ y' = y \\ z' = \gamma_0 (z - v_0 t) \\ t' = \gamma_0 (t - \frac{v_0}{c^2} z) \end{cases} \quad (1)$$



Frames K and K′ coincide at $t = t' = 0$.

where $\gamma_0 \equiv (1 - \frac{v_0^2}{c^2})^{-\frac{1}{2}}$ is the Lorentz factor for the transformation.

A note about notation: In many books, the relative speed between two frames is denoted by v and the particle velocity in a given frame is denoted by $\mathbf{u}$. This eventually leads to two definitions for the same notation γ:

$$\gamma \equiv (1 - \frac{v^2}{c^2})^{-\frac{1}{2}} \quad \begin{bmatrix} \text{Lorentz factor for the transformation,} \\ \text{Jackson (11.17)} \end{bmatrix}$$

$$\gamma \equiv (1 - \frac{u^2}{c^2})^{-\frac{1}{2}} \quad \begin{bmatrix} \text{Lorentz factor of a particle in a given frame,} \\ \text{Jackson (11.46) and (11.51)} \end{bmatrix}.$$

To avoid confusion with the notation γ (e.g. when we perform a Lorentz transformation of the Lorentz factor of a particle), we will denote the relative speed between two frames by v_0 and the particle velocity by v throughout this chapter, and thus define

$$\gamma_0 \equiv (1 - \frac{v_0^2}{c^2})^{-\frac{1}{2}} \quad \text{[Lorentz factor for the transformation]}$$

$$\gamma \equiv (1 - \frac{v^2}{c^2})^{-\frac{1}{2}} \quad \text{[Lorentz factor of a particle in a given frame].}$$

Four-Dimension Space Quantities and Operation Rules:

Define a position vector in the 4-dimensional space of $\mathbf{x}$ and t as

$$\mathrm{X} \equiv (x, y, z, ict) = (\mathbf{x}, ict)$$

4-vector (X); spatial vector ($\mathbf{x}$)

and a 4-D matrix as

$$a_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_0 & i\gamma_0\beta_0 \\ 0 & 0 & -i\gamma_0\beta_0 & \gamma_0 \end{bmatrix}, \quad \beta_0 = v_0 / c$$

$\mu = 1-4$, row number; $\nu = 1-4$, column number

then, the Lorentz transformation in (1) can be written

$$\begin{bmatrix} x' \\ y' \\ z' \\ ict' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_0 & i\gamma_0\beta_0 \\ 0 & 0 & -i\gamma_0\beta_0 & \gamma_0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ ict \end{bmatrix} \quad \text{or} \quad x'_\mu = \sum_{\nu=1}^{4} a_{\mu\nu} x_\nu \quad (2)$$

and the inverse Lorentz transformation is: $x_\nu = \sum_{\mu=1}^{4} a_{\mu\nu} x'_\mu$. (3)

The $a_{\mu\nu}$ matrix in (2) shows that the Lorentz transformation is an orthogonal transformation because it satisfies

$$\sum_{\mu} a_{\mu\nu} a_{\mu\lambda} = \delta_{\nu\lambda} \quad \begin{bmatrix}\text{definition of orthogonal} \\ \text{transformation*}\end{bmatrix} \qquad (4)$$

*See H. Goldstein, "Classical Mechanics," 2nd edition, p.134.

$$\text{Thus, } \sum_{\mu} x_{\mu}'^2 = \sum_{\mu}\overbrace{\sum_{\nu} a_{\mu\nu} x_{\nu}}^{x'_\mu \text{ by (2)}} \overbrace{\sum_{\lambda} a_{\mu\lambda} x_{\lambda}}^{x'_\mu \text{ by (2)}} = \sum_{\nu,\lambda} \overbrace{\sum_{\mu} a_{\mu\nu} a_{\mu\lambda}}^{\delta_{\nu\lambda} \text{ by (4)}} x_{\nu} x_{\lambda} = \sum_{\lambda} x_{\lambda}^2$$

$$\Rightarrow x'^2 + y'^2 + z'^2 - c^2 t'^2 = x^2 + y^2 + z^2 - c^2 t^2,$$

which is a statement of postulate 2 [see Eqs. (B.1) and (B.2) in Appendix B.]

Just as the 3-dimensional vectors (and tensors in general) are defined by their transformation properties in the **x**-space, we may define 4-vectors (and 4-tensors in general) by their transformation properties in the $(\mathbf{x}, t)$ space and find rules for their operation.

1. Any set of 4 quantities A_μ $(\mu = 1-4)$ or $\mathbf{A} = (A_1, A_2, A_3, A_4)$, which transform in the same way as x_μ, i.e.

$$A'_\mu = \sum_{\nu} a_{\mu\nu} A_\nu, \qquad (5)$$

is called a 4-vector (or 4-tensor of the first rank).

The position vector $\mathbf{x}\, [\equiv (x, y, z, ict)]$ of a point in the 4-D space is obviously a 4-vector. As another example, the momentum vector of a particle in the 4-D space, defined as

$$\mathbf{p} \equiv (p_x, p_y, p_z, \tfrac{iE}{c}) = (\mathbf{p}, \tfrac{iE}{c}),$$

is a 4-vector because it transforms as [see Eq. (A.28), Appendix A.]

$$\begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ \frac{iE'}{c} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_0 & i\gamma_0\beta_0 \\ 0 & 0 & -i\gamma_0\beta_0 & \gamma_0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ \frac{iE}{c} \end{bmatrix} \quad \text{or} \quad p'_\mu = \sum_{\nu=1}^{4} a_{\mu\nu} p_\nu$$

2. If a quantity Φ is unchanged under the Lorentz transformation, it is called a Lorentz scalar (or 4-vector of the zeroth rank). The Lorentz scalar is also called a Lorentz invariant.

   The Lorentz scalar is in general a function of the components of a 4-vector. For example, we have just shown that

$$\sum_{\mu} x_{\mu}'^2 = \sum_{\lambda} x_{\lambda}^2$$

   Hence, $\sum_{\lambda} x_{\lambda}^2$ is a Lorentz scalar.

3. Define the 4-D operator, $\Box \equiv [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial (ict)}]$, as the counterpart of the operator ∇ in the **x**-space. Then, the 4-gradient of a Lorentz scalar, $\Box\Phi \equiv [\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z}, \frac{\partial \Phi}{\partial (ict)}]$, is a 4-vector.

$$\textit{Proof}: \ (\Box'\Phi)_\mu = \frac{\partial \Phi}{\partial x'_\mu} = \sum_{\nu} \frac{\partial \Phi}{\partial x_\nu} \underbrace{\frac{\partial x_\nu}{\partial x'_\mu}}_{= a_{\mu\nu} \text{ by (3)}} = \sum_{\nu} a_{\mu\nu} \frac{\partial \Phi}{\partial x_\nu} = \underbrace{\sum_{\nu} a_{\mu\nu} (\Box\Phi)_\nu}_{\text{Transforms as a 4-vector.}} \qquad (6)$$

4. The 4-divergence of a 4-vector, $\Box\cdot\mathbf{A} \equiv \sum_{\mu} \frac{\partial A_\mu}{\partial x_\mu}$, is a Lorentz scalar.

Proof :

A: 4-vector
A_μ: component of **A**

$$\Box'\cdot\mathbf{A}' = \sum_{\nu} \frac{\partial A'_\nu}{\partial x'_\nu} = \sum_{\nu}\sum_{\mu} \underbrace{\frac{\partial x_\mu}{\partial x'_\nu}}_{= a_{\nu\mu} \text{ by (3)}} \frac{\partial \overbrace{A'_\nu}^{= \frac{\partial}{\partial x_\mu}\sum_{\lambda} a_{\nu\lambda} A_\lambda \text{ by (5)}}}{\partial x_\mu} = \sum_{\mu\lambda} \underbrace{\sum_{\nu} a_{\nu\mu} a_{\nu\lambda}}_{= \delta_{\mu\lambda} \text{ by (4)}} \frac{\partial A_\lambda}{\partial x_\mu} = \sum_{\mu} \frac{\partial A_\mu}{\partial x_\mu} = \Box\cdot\mathbf{A} \qquad (7)$$

$\Rightarrow$ $\Box\cdot\mathbf{A}$ is unchanged under the Lorentz transformation

5. The 4-Laplacian operator, $\Box^2 \equiv \Box \cdot \Box = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$, (8)

is a Lorentz scalar operator, i.e. $\Box'^2 \Phi = \Box^2 \Phi$ [Φ : a Lorentz scalar].

Proof : We have shown in item 4 that the divergence of a 4-vector is a Lorentz scalar, i.e. $\Box' \cdot \mathbf{A}' = \Box \cdot \mathbf{A}$. Let Φ be a Lorentz scalar, then $\mathbf{A}' = \Box' \Phi$ and $\mathbf{A} = \Box \Phi$ are both 4-vectors (see item 3). Hence,

$$\Box' \cdot \mathbf{A}' = \Box \cdot \mathbf{A} \Rightarrow \Box' \cdot \Box' \Phi = \Box \cdot \Box \Phi \;\Rightarrow\; \Box'^2 \Phi = \Box^2 \Phi.$$

6. The dot product of two 4-vectors, $\mathbf{A} \cdot \mathbf{B} \equiv \sum_\mu A_\mu B_\mu$, is a Lorentz scalar.

Proof :

$$\mathbf{A}' \cdot \mathbf{B}' = \sum_\sigma A'_\sigma B'_\sigma = \sum_\sigma \overbrace{\sum_\nu a_{\sigma\nu} A_\nu}^{A'_\sigma} \overbrace{\sum_\lambda a_{\sigma\lambda} B_\lambda}^{B'_\sigma} = \sum_{\nu\lambda} \overbrace{\sum_\sigma a_{\sigma\nu} a_{\sigma\lambda}}^{=\delta_{\nu\lambda} \text{ by (4)}} A_\nu B_\lambda$$

$$= \sum_\lambda A_\lambda B_\lambda = \mathbf{A} \cdot \mathbf{B} \qquad (9)$$

$$\Rightarrow \mathbf{A}' \cdot \mathbf{A}' = \mathbf{A} \cdot \mathbf{A} \Rightarrow \sum_\mu A'^2_\mu = \sum_\mu A^2_\mu \quad \left[\begin{array}{l}\text{a useful property of the} \\ \text{orthogonal transformation}\end{array}\right]$$

Example : In frame K, a particle's position changes by $d\mathbf{x}$ in a time interval dt. Then, $d\mathbf{x} = (dx,\ dy,\ dz,\ icdt)$ is a 4-vector. Hence, $d\mathbf{x} \cdot d\mathbf{x}$ $(= \sum_\mu dx_\mu dx_\mu)$ is a Lorentz invariant, i.e. in frame K', $d\mathbf{x}' \cdot d\mathbf{x}'$ $(= \sum_\mu dx'_\mu dx'_\mu)$ is given by $d\mathbf{x} \cdot d\mathbf{x}$.

Special case : The particle is at rest in frame K' (the rest frame of the particle). Hence, $d\mathbf{x}' = 0$ and $d\mathbf{x}' = (0,\ 0,\ 0,\ icd\tau)$, where we have denoted the differential time in frame K' by $d\tau$ instead of dt', because frame K' is a unique frame. $d\tau$ is called the proper time of the particle.

$$\sum_\mu dx'_\mu dx'_\mu = \sum_\mu dx_\mu dx_\mu \Rightarrow -c^2 d\tau^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$$

$$\Rightarrow\; d\tau = dt\sqrt{1 - \frac{v^2}{c^2}} = \frac{dt}{\gamma} \quad \text{[a Lorentz invariant]} \qquad (10)$$

where $\mathbf{v} = \frac{dx}{dt}\mathbf{e}_x + \frac{dy}{dt}\mathbf{e}_y + \frac{dz}{dt}\mathbf{e}_z$ is the velocity of the particle in frame K.

Discussion : (i) For the special case that K' is the rest frame of the particle, v is also the relative velocity of the 2 frames. Hence, $\gamma = \gamma_0$.

(ii) The Lorentz transformation applies only to inertial frames. If the particle has an acceleration, $d\tau (= \frac{dt}{\gamma})$ in (10) is the differential time in the *instantaneous* rest frame of the particle, in which the particle has zero velocity but a finite acceleration. In general, the speed of the rest frame (hence γ) is a function of time, i. e. $d\tau = \frac{dt}{\gamma(t)}$ [Jackson, (11.26)].

(iii) Consider a special case of *constant* particle velocity. The muon has a lifetime of 2.2 μsec in its rest frame between birth and decay. If the lifetime is measured in a Lorentz frame in which the muon has a constant γ, then by (10), the rest-frame lifetime (τ_d) and the measured lifetime t_d are related by

$$\int_{\tau_{birth}}^{\tau_{decay}} d\tau = \int_{t_{birth}}^{t_{decay}} \frac{dt}{\gamma} = \frac{1}{\gamma}\int_{t_{birth}}^{t_{decay}} dt \Rightarrow \quad \tau_d = \frac{t_d}{\gamma}.$$

This expresses the phenomenon of time dilation; namely, when the time interval of a clock's rest time (e.g. τ_d above) is observed in a moving frame, it is greater by a factor of γ. The invariance of τ_d $(= \frac{t_d}{\gamma})$ means that $\frac{t_d}{\gamma}$ will have the same value in all Lorentz frames.

7. A 4-tensor of the second rank ($\overleftrightarrow{\mathbf{T}}$) is a set of 16 quantities, $T_{\mu\nu} (\mu, \nu = 1-4)$, which transform according to

$$T'_{\mu\nu} = \sum_{\lambda,\sigma} a_{\mu\lambda} a_{\nu\sigma} T_{\lambda\sigma} \qquad (11)$$

8. The dot product of a 4-tensor of the second rank and a 4-vector, $(\overleftrightarrow{\mathbf{T}} \cdot \mathbf{A})_\mu \equiv \sum_\nu T_{\mu\nu} A_\nu$, is a 4-vector.

Proof : ($T'_{\mu\nu} = \sum_{\lambda\sigma} a_{\mu\lambda} a_{\nu\sigma} T_{\lambda\sigma}$; $A'_\nu = \sum_\alpha a_{\nu\alpha} A_\alpha$)

$$(\overleftrightarrow{\mathbf{T}}' \cdot \mathbf{A}')_\mu = \sum_\nu T'_{\mu\nu} A'_\nu = \sum_{\lambda,\sigma,\alpha} a_{\mu\lambda} \overbrace{\sum_\nu a_{\nu\sigma} a_{\nu\alpha}}^{\delta_{\sigma\alpha}} T_{\lambda\sigma} A_\alpha$$

$$= \sum_\lambda a_{\mu\lambda} \sum_\sigma T_{\lambda\sigma} A_\sigma = \underbrace{\sum_\lambda a_{\mu\lambda} (\overleftrightarrow{\mathbf{T}} \cdot \mathbf{A})_\lambda}_{\text{Transform as a 4-vector.}} \qquad (12)$$

9. The 4-divergence of a 4-tensor of the second rank, $(\Box\cdot\overleftrightarrow{\mathbf{T}})_\mu \equiv \sum_\nu \frac{\partial T_{\mu\nu}}{\partial x_\nu}$, is a 4-vector.

Proof :

$$(\Box'\cdot\overleftrightarrow{\mathbf{T}}')_\mu = \sum_\nu \frac{\partial T'_{\mu\nu}}{\partial x'_\nu} = \sum_\nu \frac{\partial}{\partial x'_\nu}\sum_{\lambda,\sigma} a_{\mu\lambda}a_{\nu\sigma}T_{\lambda\sigma} = \sum_\nu\sum_\alpha \underbrace{\frac{\partial x_\alpha}{\partial x'_\nu}}_{a_{\nu\alpha}} \frac{\partial}{\partial x_\alpha}\sum_{\lambda\sigma} a_{\mu\lambda}a_{\nu\sigma}T_{\lambda\sigma}$$

$$= \sum_{\lambda,\sigma,\alpha} a_{\mu\lambda}\overbrace{\sum_\nu a_{\nu\alpha}a_{\nu\sigma}}^{\delta_{\alpha\sigma}}\frac{\partial T_{\lambda\sigma}}{\partial x_\alpha} = \sum_\lambda a_{\mu\lambda}\sum_\sigma \frac{\partial T_{\lambda\sigma}}{\partial x_\sigma} = \underbrace{\sum_\lambda a_{\mu\lambda}(\Box\cdot\overleftrightarrow{\mathbf{T}})_\lambda}_{\text{Transform as a 4-vector.}} \qquad (13)$$

10. A 4-tensor of the third rank is a set of 64 quantities, $G_{\lambda\mu\nu}(\lambda,\mu,\nu=1\text{-}4)$, which transform according to

$$G'_{\lambda\mu\nu} = \sum_{ijk} a_{\lambda i}a_{\mu j}a_{\nu k}G_{ijk} \qquad (14)$$

Problem 1: If $F_{\mu\nu}$ is a 4-tensor of the second rank, show that $\frac{\partial F_{\mu\nu}}{\partial x_\lambda}(\lambda,\mu,\nu=1\text{-}4)$ is a 4-tensor of the third rank.

Solution: $F'_{\mu\nu} = \sum_{jk} a_{\mu j}a_{\nu k}F_{jk}$

$$\Rightarrow \frac{\partial F'_{\mu\nu}}{\partial x'_\lambda} = \sum_{jk} a_{\mu j}a_{\nu k}\sum_i \frac{\partial F_{jk}}{\partial x_i}\overbrace{\frac{\partial x_i}{\partial x'_\lambda}}^{a_{\lambda i}} = \underbrace{\sum_{ijk} a_{\lambda i}a_{\mu j}a_{\nu k}\frac{\partial F_{jk}}{\partial x_i}}_{\text{Transform as a 4-tensor of the third rank.}} \qquad (15)$$

Problem 2: Show that the set of equations,

$$\frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0 \ (\lambda,\mu,\nu=1\text{-}4) \qquad (16)$$

is invariant in form under the Lorentz transformation.

Solution: Rewrite (15): $\frac{\partial F'_{\mu\nu}}{\partial x'_\lambda} = \sum_{ijk} a_{\lambda i}a_{\mu j}a_{\nu k}\frac{\partial F_{jk}}{\partial x_i}$

Change indecies in (15) as follows: $\begin{cases}\lambda\to\nu,\ \mu\to\lambda,\ \nu\to\mu\\ i\to k,\ \ k\to j,\ j\to i\end{cases}$

$$\Rightarrow \frac{\partial F'_{\lambda\mu}}{\partial x'_\nu} = \sum_{ijk} a_{\lambda i}a_{\mu j}a_{\nu k}\frac{\partial F_{ij}}{\partial x_k} \qquad (17)$$

Change indecies in (17) as follows: $\begin{cases}\nu\to\mu,\ \lambda\to\nu,\ \mu\to\lambda\\ k\to j,\ i\to k,\ \ j\to i\end{cases}$

$$\Rightarrow \frac{\partial F'_{\nu\lambda}}{\partial x'_\mu} = \sum_{ijk} a_{\lambda i}a_{\mu j}a_{\nu k}\frac{\partial F_{ki}}{\partial x_j} \qquad (18)$$

Combine (15), (17), and (18),

$$\Rightarrow \frac{\partial F'_{\mu\nu}}{\partial x'_\lambda} + \frac{\partial F'_{\lambda\mu}}{\partial x'_\nu} + \frac{\partial F'_{\nu\lambda}}{\partial x'_\mu} = \sum_{ijk} a_{\lambda i}a_{\mu j}a_{\nu k}\overbrace{\left(\frac{\partial F_{jk}}{\partial x_i} + \frac{\partial F_{ij}}{\partial x_k} + \frac{\partial F_{ki}}{\partial x_j}\right)}^{=0 \text{ by } (16)} = 0$$

11. If a physical law can be expressed as a relation between 4-tensors of the same rank, then it's form is invariant in all Lorentz frames.

Example 1: If the physical law in frame K is of the form $\mathbf{A}=\mathbf{B}$, then, $A'_\nu = \sum_\mu a_{\mu\nu}\underbrace{A_\mu}_{B_\mu} = \sum_\mu a_{\mu\nu}B_\mu = B'_\nu$, i.e. $\mathbf{A}=\mathbf{B}\Rightarrow\mathbf{A}'=\mathbf{B}'$. (19)

Example 2: If the physical law in frame K is of the form $\overleftrightarrow{\mathbf{T}}=\overleftrightarrow{\mathbf{F}}$, then, $T'_{\mu\nu} = \sum_{\lambda\sigma} a_{\mu\lambda}a_{\nu\sigma}\underbrace{T_{\lambda\sigma}}_{F_{\mu\nu}} = \sum_{\lambda\sigma} a_{\mu\lambda}a_{\nu\sigma}F_{\mu\nu} = F'_{\mu\nu}$, i.e.

$$\overleftrightarrow{\mathbf{T}}=\overleftrightarrow{\mathbf{F}}\Rightarrow\overleftrightarrow{\mathbf{T}}'=\overleftrightarrow{\mathbf{F}}' \ \text{[i.e. invariant in form]} \qquad (20)$$

In the following section, we examine relativistic mechanics in 4-vector formalism. In Sec. 3, we will demonstrate that laws of electromagnetism are in variant under the Lorentz transformation by expressing them as relations between tensors of the same rank. From the Lorentz transformation of these tensors, we also obtain the transformation equations for various electromagnetic quantities.

Section 2: Relativistic Mechanics

We begin with a note on the terms "conservation", "invariance", and "covariance".

The conservation of a quantity means that it remains unchanged in time in a given Lorentz frame. For example, the relativistic momentum and energy of an isolated system of particles are both conserved after a collision. This is a fundamental law to be discussed in this Section.

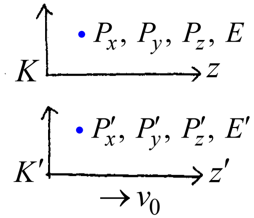
The invariance of a quantity means that it is invariant in value under a Lorentz transformation. Such a quantity is called a Lorentz invariant or Lorentz scalar. For example, the dot product of two 4-vectors is a Lorentz invariant. However, it may or may not be a conserved quantity. An example will be provided in this section.

The term covariance refers to physical laws. A physical law is "covariant" if it is "invariant in form under the Lorentz transformation." As will be shown, the new laws of relativistic mechanics and existing laws of electromagnetism are all covariant.

The 4 - Momentum (p) of a Single Particle :

As shown in (A.28), if we define the momentum of a particle as $\mathbf{p} \equiv \gamma m \mathbf{v}$ and energy as $E \equiv \gamma mc^2$ (m is called the rest mass*), then the 4-momentum, $\mathsf{p} \equiv (p_x, p_y, p_z, \frac{iE}{c})$, is a 4-vector, which transforms as

$$\begin{cases} p_x' = p_x & (21.1) \\ p_y' = p_y & (21.2) \\ p_z' = \gamma_0 (p_z - \frac{v_0}{c^2} E) & (21.3) \\ E' = \gamma_0 (E - v_0 p_z) & (21.4) \end{cases}$$



*Throughout this chapter, m and M denote the rest mass.

Discussion: In Appendix A, we first define $\mathbf{p} = \gamma m \mathbf{v}$ and $E = \gamma mc^2$, then show that the law of conservation of momentum and energy is covariant. Conversely, from the requirement of the covariance of this conservation law, we can deduce the definitions of $\mathbf{p} = \gamma m \mathbf{v}$ and $E = \gamma mc^2$ (see Jackson Sec. 11.5).

The dot product of two 4-vectors is a Lorentz scalar, hence

$$\mathsf{p} \cdot \mathsf{p} = \mathsf{p}' \cdot \mathsf{p}' \Rightarrow p^2 - \tfrac{E^2}{c^2} = p'^2 - \tfrac{E'^2}{c^2} \qquad (22)$$

i.e. $E^2 - p^2c^2$ is a Lorentz scalar (invariant).

If frame K' is the rest frame of the particle (i.e. $p' = 0$, $E' = mc^2$) then $\mathsf{p}' = (0,\ 0,\ 0,\ imc)$ and $\mathsf{p} \cdot \mathsf{p} = \mathsf{p}' \cdot \mathsf{p}'$ gives $p^2 - \frac{E^2}{c^2} = -m^2c^2$, or

$$E^2 - p^2c^2 = m^2c^4 \qquad (23)$$

Since $E^2 - p^2c^2$ is a Lorentz invariant, (23) shows that the rest mass m is a Lorentz invariant. This has in fact been assumed in Sec. 2 of Appendix A, where we derive the Lorentz transformation equations for $\mathbf{p}$ $(= \gamma m \mathbf{v})$ and E $(= \gamma mc^2)$. (23) is a useful formula for it relates the particle's total energy (E) to its momentum (p). (Momentum in particle physics is often expressed in unit of GeV/c.)

For a relativistic particle, we can still speak of its (macroscopic) kinetic energy K, defined as: $K = E - mc^2 = (\gamma - 1)mc^2$. (24)

The 4 - Momentum (P) of a System of Particles

Consider a system of particles, each with the 4-momentum

$$\mathsf{p}_j = (p_{xj}, p_{yj}, p_{zj}, iE_j/c) = (\mathbf{p}_j, iE_j/c),\ j = 1,\ 2,\ 3, \cdots$$

Since the Lorentz transformation is a linear transformation, the sum of any number of 4-vectors also obeys the Lorentz tansformation. Thus, $\mathsf{P} = \sum_j \mathsf{p}_j$ is a 4-vector and its components transform as

$$\begin{cases} \sum_j p_{xj}' = \sum_j p_{xj} & (25.1) \\ \sum_j p_{yj}' = \sum_j p_{yj} & (25.2) \\ \sum_j p_{zj}' = \gamma_0 (\sum_j p_{zj} - \frac{v_0}{c^2} \sum_j E_j) & (25.3) \\ \sum_j E_j' = \gamma_0 (\sum_j E_j - v_0 \sum_j p_{zj}) & (25.4) \end{cases}$$

and $\mathsf{P} \cdot \mathsf{P} = (\sum_j \mathsf{p}_j) \cdot (\sum_j \mathsf{p}_j) = (\sum_j \mathbf{p}_j) \cdot (\sum_j \mathbf{p}_j) - (\sum_j E_j/c)^2$ (26)

is a Lorentz invariant.

Rewrite (25):
$$\begin{cases}\sum_j p'_{xj} = \sum_j p_{xj} \\ \sum_j p'_{yj} = \sum_j p_{yj} \\ \sum_j p'_{zj} = \gamma_0(\sum_j p_{zj} - \frac{v_0}{c^2}\sum_j E_j) \\ \sum_j E'_j = \gamma_0(\sum_j E_j - v_0 \sum_j p_{zj})\end{cases}$$

We see from (25) that only when all the components of $\mathbf{P}$ (i.e. the three components of total momentum plus the total energy) are each conserved in frame K will all the components of $\mathbf{P}'$ be conserved. If one component of $\mathbf{P}$ is not conserved, a rotation of the spatial coordinate system can make any component of $\mathbf{P}'$ (momentum or energy) unconserved in the new spatial coordinate system. Thus, the relativistic law of conservation must take the form as described below in order for it to be a covariant law.

Law of Conservation of Momentum and Energy :
For reasons just discussed, in relativity, the conservation of momentum and energy comes in one law rather than separate laws for the momentum and energy as in nonrelativistic mechanics. The law states that, for an *isolated* system of particles,

$$\mathbf{P}(\text{before collision}) = \mathbf{P}(\text{after collision}), \qquad (27)$$

which implies that $\sum_j p_{xj}$, $\sum_j p_{yj}$, $\sum_j p_{zj}$, and $\sum_j E_j$ are each conserved, i.e.

$$\sum_j \mathbf{p}_j(\text{before collision}) = \sum_j \mathbf{p}_j(\text{after collision}) \qquad (28)$$

$$\sum_j E_j(\text{before collision}) = \sum_j E_j(\text{after collision}) \qquad (29)$$

Since the law in (27) is expressed as a 4-vector relation, it has the same form in all Lorentz frames [see (19)]. Thus, in frame K', we have $\mathbf{P}'(\text{before collision}) = \mathbf{P}'(\text{after collision})$.

If $\mathbf{P}$ is conserved, the dot product $\mathbf{P}\cdot\mathbf{P}$ must also be conserved. Thus,

$$\underbrace{(\sum_j \mathbf{p}_j)\cdot(\sum_j \mathbf{p}_j) - (\sum_j \frac{E_j}{c})^2}_{\text{before collision}} = \underbrace{(\sum_j \mathbf{p}_j)\cdot(\sum_j \mathbf{p}_j) - (\sum_j \frac{E_j}{c})^2}_{\text{after collision}} \qquad (30)$$

Discussion :

(i) $\mathbf{P}\cdot\mathbf{P}$ for an *isolated* system is both a Lorentz invariant [see (26)] and a conserved quantity [see (30)]. If the system is not isolated, it is still a Lorentz invariant, but no longer a conserved quantity.

(ii) $\mathbf{P}(\text{before collision}) = \mathbf{P}(\text{after collision})$ in (27) is a fundamental law (rather than a derived relation), in which the nonrelativistic law of conservation of momentum has been extended to include the energy, $E = \gamma mc^2$. A very important aspect of this law is that it applies to all processes in an isolated system, such as elastic and inelastic collisions, nuclear reactions, and particle decays. As a result, the total rest mass of the system may not be conserved, as is illustrated in the following two problems.

Problem 1: Two identical particles of rest mass m and equal and opposite velocities $\pm\mathbf{v}$ collide head-on inelastically to form a single particle. Find the mass and velocity of the new particle.

Solution :

$m,\ \gamma\ \bullet \rightarrow \mathbf{v}\quad \mathbf{v} \leftarrow \bullet\ m,\ \gamma$	(before)
$M_{cm}\ \bullet$	(after)

The total momentum before the collision is $\gamma m\mathbf{v} - \gamma m\mathbf{v} = 0$.
So the collision occurs in the center-of-momentum (CM) frame, i.e. the frame in which the sum of the momentum of all particles vanishes. For later comparison with the result in problem 2, we denote the mass of the new particle by M_{cm} to indicate that it is created in the CM frame.

$$\begin{cases}\text{Conservation of momentum} \Rightarrow \text{The new particle is stationary.} \\ \text{Conservation of energy} \Rightarrow \gamma m + \gamma m = M_{cm} \Rightarrow M_{cm} = 2\gamma m\end{cases}$$

$m,\ \gamma\ \bullet\!\rightarrow \mathbf{v}\quad \mathbf{v}\leftarrow\!\bullet\ m,\ \gamma$ (before)
$M_{cm}\ \bullet$ (after)

Discussion: In this problem, we find $M_{cm} = 2\gamma m > 2m$, i.e. rest mass has been created from the kinetic energy $[(\gamma\text{-}1)mc^2]$ of the colliding particles. There is no need to know what's inside the new particle. We only need to know its rest mass and hence the energy associated with it. A hot object has a rest mass greater than when it's cold. The difference in rest mass due to an increase in temperature can in principle be measured by its acceleration under a known force, and we know that at least some of the added mass is in the form of thermal energy. In many other cases, it's not possible to know what's inside.

Nuclear fusion and fission reactions are examples of non-conservation of rest mass. The total rest mass is reduced after the reaction and the mass deficit appears as kinetic energies and radiation. In fact, all reactions (chemical or nuclear) in which energy is absorbed (e.g. photosynthesis) or released (e.g. digestion of food) involve a corresponding change of the reactants' total rest mass.

Problem 2: A particle of rest mass m and velocity $\mathbf{v}$ collides with a stationary particle of the same rest mass and is absorbed by it. Find the rest mass and velocity of the new particle.

Solution: The collision occurs in the stationary-target (ST) frame. So, we denote the new particle mass by M_{st}, velocity by $\mathbf{V}_{st}$, and Lorentz factor by γ_{st} $[=(1-V_{st}^2/c^2)^{-1/2}]$. ($m, \gamma, \mathbf{v}$ are also ST frame quantities.)

$$\begin{cases}\text{Conservation of momentum} \Rightarrow \gamma m\mathbf{v} = \gamma_{st} M_{st}\mathbf{V}_{st} & (31)\\ \text{Conservation of energy} \quad \Rightarrow (\gamma+1)m = \gamma_{st} M_{st} & (32)\end{cases}$$

$m,\ \gamma\ \bullet\!\longrightarrow \mathbf{v}\quad \bullet\ m$ (before)
$M_{st},\ \gamma_{st}\ \bullet\!\longrightarrow \mathbf{V}_{st}$ (after)

$$\frac{(31)}{(32)} \Rightarrow \mathbf{V}_{st} = \frac{\gamma}{\gamma+1}\mathbf{v}$$

$$(32) \Rightarrow M_{st} = \frac{\gamma+1}{\gamma_{st}} m$$

$$\Rightarrow M_{st}^2 = m^2\frac{(\gamma+1)^2}{\gamma_{st}^2} = m^2(\gamma+1)^2(1-\frac{V_{st}^2}{c^2}) = m^2(\gamma+1)^2[1-\frac{\gamma^2 v^2}{c^2(\gamma+1)^2}]$$

$$= m^2(\gamma^2+2\gamma+1-\gamma^2\frac{v^2}{c^2}) = m^2[\gamma^2(1-\frac{v^2}{c^2})+2\gamma+1] = 2m^2(\gamma+1)$$

$$\Rightarrow M_{st} = \sqrt{2(\gamma+1)}\,m$$

Discussion:

$m,\ \gamma\ \bullet\!\rightarrow \mathbf{v}\quad \mathbf{v}\leftarrow\!\bullet\ m,\ \gamma$ (before)
$M_{cm}\ \bullet$ (after)

In problem 1 (CM frame), the new particle's mass is $M_{cm} = 2\gamma m$. (33)

$m,\ \gamma\ \bullet\!\longrightarrow \mathbf{v}\quad \bullet\ m$ (before)
$M_{st},\ \gamma_{st}\ \bullet\!\longrightarrow \mathbf{V}_{st}$ (after)

In problem 2 (ST frame), the new particle's mass is $M_{st} = \sqrt{2(1+\gamma)}\,m$. (34)

Note that γ is the Lorentz factor of the particle(s) before collision.

In particle physics experiments, $M_{cm}c^2$ or $M_{st}c^2$ is the energy available for the creation of new particles (why not $\gamma_{st}M_{st}c^2$?).

The rest energy of the electron or positron is $mc^2 = 0.511$ MeV. If 2 TeV of energy is needed for particle creation (i.e. $M_{cm}c^2 = 2$ TeV or $M_{st}c^2 = 2$ TeV), then the required γ of the colliding particle(s) is

$$\begin{cases}\text{by (33), } M_{cm}c^2 = 2\gamma mc^2 = 2\text{ TeV} \Rightarrow \gamma \approx 1.957\times 10^6 & \text{[CM frame]}\\ \text{by (34), } M_{st}c^2 = \sqrt{2(1+\gamma)}\,mc^2 = 2\text{ TeV} \Rightarrow \gamma \approx 7.66\times 10^{12} & \text{[ST frame]}\end{cases}.$$

The energy associated with γ is to be obtained in an accelerator.

Thus,

$$\frac{\text{kinetic energy needed in CM frame}}{\text{kinetic energy needed in ST frame}} = \frac{2\times(1.957\times 10^6-1)}{7.66\times 10^{12}-1} \approx 5\times 10^{-7}$$

This shows that far less kinetic energy is needed in the CM frame than in the ST frame. In fact, all the kinetic energy of the two colliding particles $[2\times(1.957\times 10^6-1)\times 0.511\text{ MeV} = 2\text{ TeV}]$ is put in use in the CM frame, while in the ST frame, 99.99995% of the kinetic energy of the incident particle is wasted! This is why the International Linear Collider (ILC) project plans to accelerate both electrons and positrons to energies up to 1 TeV so that the the collision occurs in the CM frame.

Question: Why use a long linear accelerator instead of a more compact circular accelerator?

Section 3: Covariance of Electrodynamics

In the special theory of relativity, Newton's law has been radically modified. The electromagnetic laws do not need any modification because they are already covariant. However, the *covariance* of these laws (such as Maxwell equations) is not immediately clear from the equations by which they are usually represented.

Our purpose in this section is to prove that the EM laws are indeed covariant by casting them into relations between 4-tensors of the same rank [see (19) and (20)]. We will do this by first defining 4-tensors in terms of known EM quantities and forming equations with 4-tensors of the same rank, then show that one or more existing EM laws are implicit in each equation. This will prove that the laws are convariant and justify the defined quantities to be legitimate 4-tensors.

Furthermore, Lorentz transformations of these tensors will yield the tranformation equations for various EM quantities.

Note: Jackson switches to the Gausian unit system starting from Ch. 11. From here on, we also adopt the Gaussian unit system.

1. Define a 4-current as $(c\rho,\ J_x,\ J_y,\ J_z,\ ) \leftarrow$ Griffiths

$$\mathbf{J} \equiv (J_x,\ J_y,\ J_z,\ ic\rho) = (\mathbf{J},\ ic\rho) \qquad (35)$$

and use it to form a relation

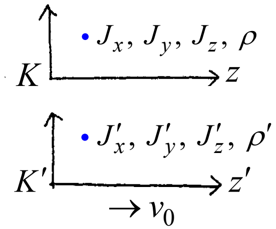
$$\Box \cdot \mathbf{J} = 0 \qquad (36)$$

Then, (36) gives the law of conservation of charge

$$\frac{\partial}{\partial x} J_x + \frac{\partial}{\partial y} J_y + \frac{\partial}{\partial z} J_z + \frac{\partial (ic\rho)}{\partial (ict)} = 0 \ \Rightarrow\ \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \qquad (5.2)$$

Thus, the definition of **J** in (35) as a 4-vector leads to the covariant representation [(36)] of the EM law in (5.2). This in turn justifies the definition of **J** as a 4-vector. The Lorentz transformation of **J** then gives

$$\begin{cases} J_x' = J_x \\ J_y' = J_y \\ J_z' = \gamma_0 (J_z - v_0 \rho) \\ \rho' = \gamma_0 \left(\rho - \frac{v_0}{c^2} J_z\right) \end{cases} \qquad (37)$$



 $(\frac{V}{c},\ A_x,\ A_y,\ A_z,\ ) \leftarrow$ Griffiths

2. Define a 4-potential as $\mathbf{A} \equiv (A_x,\ A_y,\ A_z,\ i\Phi)$ (38)

and write the covariant relations:

$$\begin{cases} \Box^2 \mathbf{A} = -\frac{4\pi}{c} \mathbf{J} & (39) \\ \Box \cdot \mathbf{A} = 0 & (40) \end{cases}$$

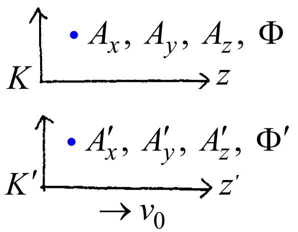
$$(39) \Rightarrow \begin{cases} \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A} = -\frac{4\pi}{c} \mathbf{J} & (6.15) \\ \nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Phi = -4\pi\rho & (6.16) \end{cases}$$

$\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = 0 \leftarrow$ Griffiths

$$(40) \Rightarrow\ \nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial}{\partial t} \Phi = 0 \quad \text{[Lorenz condition]} \qquad (6.14)$$

This again shows the consistency of **A** being a 4-vector and (6.14)-(6.16) being covariant laws. The Lorentz transformation of **A** then gives

$$\begin{cases} A_x' = A_x \\ A_y' = A_y \\ A_z' = \gamma_0 \left(A_z - \frac{v_0}{c} \Phi\right) \\ \Phi' = \gamma_0 \left(\Phi - \frac{v_0}{c} A_z\right) \end{cases} \qquad (41)$$



Note: The source-free wave equation can be directly put into the covariant form: $\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi = 0 \ \Rightarrow\ \Box^2 \psi = 0.$ (42)

3. Define a 4-wavenumber as

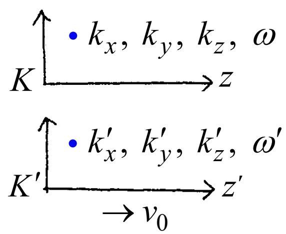
$$\mathbf{k} \equiv \left(k_x,\ k_y,\ k_z,\ \frac{i\omega}{c}\right) = \left(\mathbf{k},\ \frac{i\omega}{c}\right) \qquad (43)$$

Then, $\mathbf{k}' \cdot \mathbf{x}' = \mathbf{k} \cdot \mathbf{x} \Rightarrow\ \mathbf{k}' \cdot \mathbf{x}' - \omega' t' = \mathbf{k} \cdot \mathbf{x} - \omega t$

$\Rightarrow$ Invariance of the phase

By the same argument, we find that **k** defined in (43) is a legitimate 4-vector. Thus, its Lorentz transformation gives

$$\underbrace{\begin{cases} k_x' = k_x \\ k_y' = k_y \\ k_z' = \gamma_0 \left(k_z - \frac{v_0}{c^2} \omega\right) \\ \omega' = \gamma_0 (\omega - v_0 k_z) \end{cases}}_{\text{relativistic Doppler shift}} \qquad (44)$$



4. Define a field strength tensor of the second rank $\overleftrightarrow{\mathbf{F}}$ [Marion, (14.62)]:

$$\overleftrightarrow{\mathbf{F}} \equiv \begin{bmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{bmatrix} \quad (45)$$

Then, $\Box\cdot\overleftrightarrow{\mathbf{F}} = \frac{4\pi}{c}\mathbf{J} \Rightarrow \begin{cases} \nabla\cdot\mathbf{E} = 4\pi\rho \\ \nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J} \end{cases}$

SI $\begin{cases} \nabla\cdot\mathbf{D} = \rho \\ \nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t} = \mathbf{J} \\ \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \end{cases}$

In the covariant set of equations [see (16)]

$$\frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0 \quad (\lambda,\mu,\nu = 1\text{-}4)$$

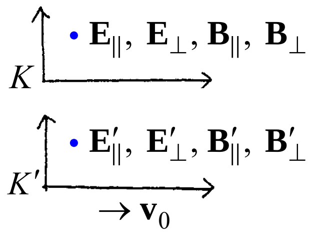
[$F_{\mu\nu}$'s are elements of $\overleftrightarrow{\mathbf{F}}$ in (45).],

set $(\lambda,\mu,\nu) = (1,2,3) \Rightarrow \nabla\cdot\mathbf{B} = 0$

set $(\lambda,\mu,\nu) = (1,2,4)$, $(1,3,4)$, and $(2,3,4) \Rightarrow \nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = 0$.

The covariant equations, $\Box\cdot\overleftrightarrow{\mathbf{F}} = \frac{4\pi}{c}\mathbf{J}$ and $\frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0$, give the set of Maxwell equations in free space. This shows that Maxwell equatins are covariant as well as justifies the definition of $\overleftrightarrow{\mathbf{F}}$ as a tensor of the second rank. Thus, $F'_{\mu\nu} = \sum_{\lambda,\sigma} a_{\mu\lambda}a_{\nu\sigma}F_{\lambda\sigma}$ gives the transformation equations for **E** and **B** (see Marion, Sec. 14.6.)

$$\begin{cases} \mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \\ \mathbf{E}'_{\perp} = \gamma_0\left(\mathbf{E}_{\perp} + \frac{\mathbf{v}_0}{c}\times\mathbf{B}_{\perp}\right) \\ \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \\ \mathbf{B}'_{\perp} = \gamma_0\left(\mathbf{B}_{\perp} - \frac{\mathbf{v}_0}{c}\times\mathbf{E}_{\perp}\right) \end{cases} \quad (46)$$



In (46), $\mathbf{v}_0$ is the velocity of frame K' relative to frame K, and "$\parallel$" and "$\perp$" refer to the direction of $\mathbf{v}_0$.

See Appendix C for a summary of transformation equations.

5. The covariant equation*, $\frac{d}{d\tau}\mathbf{P} = \frac{e}{mc}\overleftrightarrow{\mathbf{F}}\cdot\mathbf{P}$ ($d\tau$ is a Lorentz scalar), gives (Marion, p.439)

$\Rightarrow \frac{d\mathbf{P}}{dt} = \frac{d\tau}{dt}\frac{e}{mc}\overleftrightarrow{\mathbf{F}}\cdot\mathbf{P}$, where $\mathbf{p} \equiv (p_x, p_y, p_z, \frac{iE}{c})$ and $\mathbf{p} \equiv \gamma m\mathbf{v}$

$$\overleftrightarrow{\mathbf{F}} \equiv \begin{bmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{bmatrix}$$

$$\frac{d}{dt}p_x = \frac{e}{\gamma mc}\left(E_x\frac{E}{c} + \gamma m(v_yB_z - v_zB_y)\right) = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)_x$$

$$\begin{cases} \frac{d}{dt}\mathbf{p} = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right) & \text{[relativistic equation of motion]} \\ mc^2\frac{d}{dt}\gamma = e\mathbf{v}\cdot\mathbf{E} & \text{[This equation is implicit in (47).]} \end{cases} \quad (47)$$

*In order for this equation to be covariant, the charge e must be a Lorentz invariant. This has been experimentally established (see Jackson, p.554).

6. In a similar manner, we can demonstrate the covariance of the conservation laws for field/mechanical energy and field/mechanical momentum, as given by Jackson (6.111) and (6.122):

$$\begin{cases} \frac{d}{dt}\left(E_{\text{mech}} + E_{\text{field}}\right) = -\oint_S \mathbf{n}\cdot\mathbf{S}\,da & (6.111) \\ \frac{d}{dt}\left(\mathbf{p}_{\text{mech}} + \mathbf{p}_{\text{field}}\right) = \oint_S \sum_\beta T_{\alpha\beta}n_\beta\,da & (6.122) \end{cases}$$

Consider the general form of the relativistic equation of motion in (47), $\frac{d}{dt}\mathbf{p} = \mathbf{F}$, where $\mathbf{F}$ is any force, such as the gravitational force.

Special case 1: $\mathbf{F} \parallel \mathbf{v}$ (one-dimensional problem)

$$F = \frac{d}{dt}(\gamma m v) = mv\underbrace{\frac{d\gamma}{dt}} + \gamma m\frac{dv}{dt} = \gamma m\frac{dv}{dt}\underbrace{\left(\gamma^2\frac{v^2}{c^2}+1\right)} = \gamma^3 m\frac{dv}{dt} \qquad (48)$$

$$\frac{d}{dt}\gamma = \frac{d}{dt}\left(1-\frac{v^2}{c^2}\right)^{-1/2} = \frac{-1}{2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left(\frac{-2v}{c^2}\right)\frac{dv}{dt} = \gamma^3\frac{v}{c^2}\frac{dv}{dt}$$

$$\gamma^2\frac{v^2}{c^2}+1 = \frac{v^2/c^2}{1-v^2/c^2}+1 = \frac{v^2/c^2+1-v^2/c^2}{1-v^2/c^2} = \frac{1}{1-v^2/c^2} = \gamma^2$$

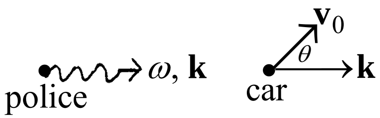
$\Rightarrow F = \gamma^3 ma \Rightarrow$ Constant force does not cause constant acceleration.

Special case 2: $\mathbf{F} \perp \mathbf{v}$ ($\Rightarrow \gamma = const.$, as in uniform circular motion)

$$\Rightarrow \mathbf{F} = \frac{d}{dt}\mathbf{p} = \frac{d}{dt}(\gamma m\mathbf{v}) = \gamma m\frac{d}{dt}\mathbf{v} \quad \text{(Undulator \& Wiggler)} \qquad (49)$$

Questions: (i) It is sometimes said that a particle has two masses, $\gamma^3 m$ and γm. Why? (ii) The acceleration is not necessarily parallel to the force. Give an example. (iii) Relate (48) to (A.23).

Problem 1: A police radar operates on a frequency of ω. What is the frequency received by the police after the signal is reflected from a car moving at the velocity $\mathbf{v}_0$?



shown in police frame

Solution: We do it in 2 steps.

Step 1. In the police frame, the radar sends a wave (ω, $\mathbf{k}$) toward the car, which is moving at velocity $\mathbf{v}_0$ (direction shown in the figure). Transformming ω to the car frame by (44), we obtain

$$\omega' = \gamma_0(\omega - v_0 k_z),$$

where k_z is the component of $\mathbf{k}$ along $\mathbf{v}_0$, i.e. $k_z = k\cos\theta$ (see figure.)

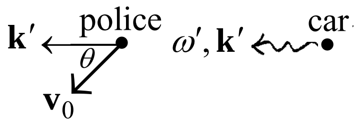
Thus, $k = \omega/c$ $\quad \gamma_0 = (1 - v_0^2/c^2)^{-1/2}$

$$\omega' = \gamma_0(\omega - k_z v_0) = \gamma_0(\omega - kv_0\cos\theta) = \gamma_0\omega\left(1 - \frac{v_0\cos\theta}{c}\right)$$

This is the Dopper-shifted frequency detected by the car. It is also the frequency of the wave reflected by the car as seen in the car frame.

Step 2. In the car frame (see figure), the car sends the reflected wave (ω', $\mathbf{k}'$) back to the car at the frequency

$$\omega' = \gamma_0\omega\left(1 - \frac{v_0\cos\theta}{c}\right)$$



shown in car frame

In the car frame, the police is moving at velocity $\mathbf{v}_0$ (direction shown in the fugure) relative to the car. Tranformming ω' to the police frame by (44), we obtain the frequency observed by the police (Doppler shifted again)

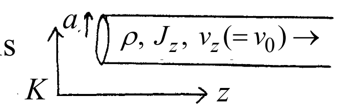
$$\omega'' = \gamma_0(\omega' - k_z' v_0) = \gamma_0(\omega' - k' v_0\cos\theta) = \gamma_0\omega'\left(1 - \frac{v_0\cos\theta}{c}\right)$$

$$= \gamma_0^2\omega\left(1 - \frac{v_0\cos\theta}{c}\right)^2 \approx \omega\left(1 - 2\frac{v_0\cos\theta}{c}\right) \quad \text{since } v_0 \ll c.$$

If the radar frequency is $f\,(=\omega/2\pi) = 10^9$ Hz and the car moves away from the police ($\theta = 0$) at $v_0 = 150$ km/hr, the police would detect a frequency $f''(=\omega''/2\pi)$ shifted by $\Delta f \approx -f\frac{2v_0}{c} \approx -278$ Hz.

Problem 2: An observer in the laboratory sees an infinite electron beam of radius a and uniform charge density ρ, moving axially at velocity v_0. What force does he see on an electron at a distance $r\,(\le a)$ from the axis? Assume the electron moves axially at the velocity v_0.

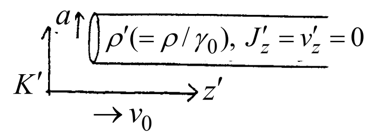
Solution: The problem can be readily solved in the lab frame. Here, we will take a long route for an exercise on some of the transformation equations just derived.



The current density J_z in the lab frame is

$J_z = \rho v_0$. [ρ has a negative value.]

By (37), we have, in the beam frame



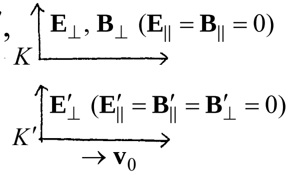
$$J_z' = \gamma_0(J_z - v_0\rho) = 0,$$

$$\rho' = \gamma_0\left(\rho - \frac{v_0}{c^2}J_z\right) = \gamma_0\rho\left(1 - \frac{v_0^2}{c^2}\right) = \frac{\rho}{\gamma_0}.$$

We see that the lab frame ρ is greater than the beam frame ρ' by the factor γ_0. This is because every unit length of the beam in its rest frame is contracted by this factor when viewed in the lab frame.

In the beam frame, $J'_z = 0$, $\rho' = \rho/\gamma_0$; hence, there is only a radial electric field. Gauss law, $\oint_{S'} \mathbf{E}' \cdot d\mathbf{a}' = 4\pi \int_{V'} \rho' d^3x'$, then gives $2\pi r' E'_r = 4\pi(\pi\rho' r'^2)$, for $r' \le a$

$$\Rightarrow E'_r = 2\pi\rho' r' = \frac{2\pi\rho r}{\gamma_0}. \quad [r' = r,\ \rho' = \rho/\gamma_0]$$



We now transform $\mathbf{E}'_\perp (= E'_r \mathbf{e}_r)$ into lab-frame $\mathbf{E}_\perp$ and $\mathbf{B}_\perp$ by using the reverse transformation equations in (46), in which we set $\mathbf{v}_0 = v_0 \mathbf{e}_z$.

$$\begin{cases} \mathbf{E}_\perp = \gamma_0(\mathbf{E}'_\perp - \frac{\mathbf{v}_0}{c} \times \cancel{\mathbf{B}'_\perp}) = \gamma_0 \mathbf{E}'_\perp = \gamma_0 \frac{2\pi\rho r}{\gamma_0} \mathbf{e}_r = 2\pi\rho r \mathbf{e}_r \\ \mathbf{B}_\perp = \gamma_0(\cancel{\mathbf{B}'_\perp} + \frac{\mathbf{v}_0}{c} \times \mathbf{E}'_\perp) = \gamma_0 (\frac{v_0 \mathbf{e}_z}{c}) \times \frac{2\pi\rho r}{\gamma_0} \mathbf{e}_r = \frac{v_0}{c} 2\pi\rho r \mathbf{e}_\theta \end{cases}$$

Thus, the force $\mathbf{f}$ on an electron (in the lab frame) is

$$\mathbf{f} = -e\left(\mathbf{E} + \tfrac{1}{c}\mathbf{v} \times \mathbf{B}\right) = -e\left[2\pi\rho r \mathbf{e}_r + \tfrac{1}{c}(v_0 \mathbf{e}_z) \times (\tfrac{v_0}{c} 2\pi\rho r \mathbf{e}_\theta)\right]$$

$$= -2\pi e\rho r(1 - \tfrac{v_0^2}{c^2})\mathbf{e}_r = -\frac{2\pi e\rho r}{\gamma_0^2} \mathbf{e}_r$$

[$e \equiv |e|$ is positive. For an electron beam, ρ is negative.]

Homework of Chap. 11

Problems: 3, 4, 5, 6, 9
16, 19, 23, 30

Appendix A: Relativity in College Physics

(Ref. Halliday, Resnick, and Walker, "Fundamentals of Physics")

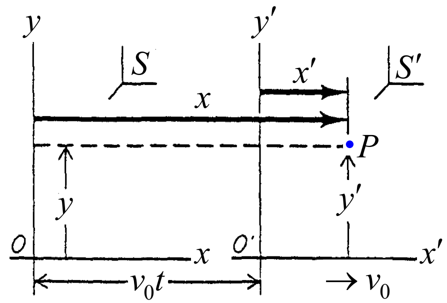
Section 1: The Lorentz Transformation

The Galilean Transformation: Consider 2 *inertial frames* S and S'. Frame S' moves along the common x-axis* with constant speed v_0 relative to frame S. At $t = 0$, the coordinates coincide and, at time t, the position of point P is (x, y, z) in S and (x', y', z') in S'. Then the Galilean transformation gives

$$x' = x - v_0 t,\ y' = y,\ z' = z,\ t' = t \quad \text{[t is unchanged in the transformation.]} \quad \text{(A.1)}$$

* In the main text, the z-axis is the direction of relative motion. To be consistent with the references cited in this appendix, here we assume that the relative motion is along the x-axis.

Question: How do you determine a reference frame is inertial?



Einstein's Postulates: The laws of classical mehanics do not vary in form under the Galilean transformation. For example, (A.1) shows $\mathbf{F} = m\mathbf{a}$ in frame S transforms to $\mathbf{F} = m\mathbf{a}'$ in frame S'. However, when the same transformation is applied to the wave equation in vacuum, $\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = 0$, its form changes completely (see Jackson, p. 516.)

So, when Einstein began his work on relativity, there were two approaches to make *all* the laws of physics invariant in form in all inertial frames: (1) Modify the theory of electromagnetism so that it is invariant in form under the Galilean transformation; or (2) Modify the Galilean transformation and the laws of mechanics so that the laws of both mechanics and electromagnetism are invariant in form under the new transformation. Einstein took the second approach. His special theory of relativity is based on 2 postulates:

1. Laws of physics are invariant in form in all inertial frames.
2. The speed of light in vacuum has the same value c in all inertial frames, independent of the motion of the source.

Event and Simultaneity: An event is something (such as the emission of a light pulse by a source) which happens at position (x, y, z) and time t. An event [described collectively as (x, y, z, t) in a given frame] will have different coordinates in different frames. The frames mentioned here and later are all inertial frames.

The time of an event can be measured by methods we normally think of. But, in relativity, time measurement often requires high precision (which can at least be done in a thought experiment) and we must bear in mind the frame in which the time is measured. The simplest way to measure time is to read the clock at the position of the event. If the clock is away from the event, the time of the event is the time shown on the clock (at the instant the light signal of the event reaches the clock) minus the time delay due to the travel of the signal (at speed c) from the event's position to the clock's position. The position of the event and the measured time of the event all refer to the frame in which the observer and the clock are *both* at rest (but the source which generates the event is not necessarily at rest.)

Two events are simultaneous in a reference frame if they have the same time coordinate in that frame, whether or not they have the same spatial coordinates. Simultaneity can be experimentally tested as follows. If two events are detected at the same instant by an observer located midway, they are simultaneous in the observer's frame.

Within a given frame, the concept of space and time in the special theory of relativity is not different from our usual concept of space and time. However, radical differences arise when space and time coordinates of an event measured in one frame are compared with those measured in another frame. In making the comparison, we find that space and time are entangled with each other in relativity. For example, two simultaneous events occurring at different positions in frame S will no longer be simultaneous in frame S', and their time difference in S' depends upon their spatial separation in S. In relativity, space and time coordinates transform according to the Lorentz transformation, which is derived below from postulate 2.

Time Dilation: Consider a pulse of light emitted by a source on a train (event 1). It travels vertically upward for a distance D, then is reflected back by a mirror, and later detected at the source (event 2).

In the train frame (Fig. 1), the time interval between the 2 events is

$$\Delta t_0 = \frac{2D}{c}. \qquad \text{(A.2)}$$

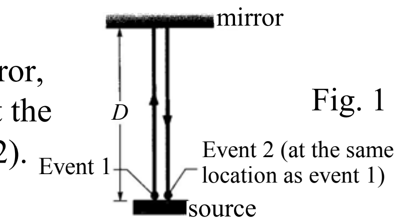


Fig. 1

In the lab frame (Fig. 2), the train, mirror, and source are all moving at speed v_0, but the light still travels at speed c (by postulate 2). So, the time interval of the 2 events is

$$\Delta t = \frac{2L}{c}, \qquad \text{(A.3)}$$

where $L = [(\frac{1}{2}v_0\Delta t)^2 + D^2]^{1/2}$ (A.4)

Eliminating D and L from (A.2)-(A.4), we obtain

$$\Delta t = \gamma_0 \Delta t_0, \qquad \text{(A.5)}$$

where $\gamma_0 \equiv [1 - v_0^2/c^2]^{-1/2}$. (A.6)

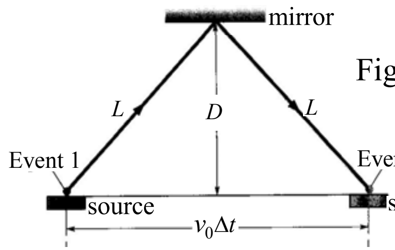


Fig. 2

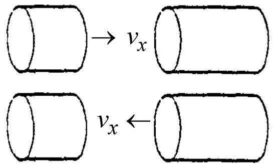
Question: Why is D the same in both frames?

Lengths perpendicular to the direction of motion are the same in both frames, i.e. the y and z coordinates transform as:

$$y' = y, \quad z' = z \qquad \text{(A.7)}$$

The proof of this is by contradiction.

Suppose that we have two identically manufactured pieces of pipe (see figure). They cannot fit inside each other because they have identical radius. Imagine that one pipe is at rest on the ground and the other is at rest on the train. If the motion of the train relative to the ground were to bring about a transverse contraction of the train pipe, then by symmetry, the motion of the ground pipe relative to the train would have to bring about a contraction of the ground pipe. But these two effects are contradictory, since in one case the train pipe would fit inside the ground pipe, and in the other case it would fit outside.



from H. C. Ohanian, "Physics"

Going back to (A.5): $\Delta t = \gamma_0 \Delta t_0$. In this equation, Δt_0 is the time interval of 2 events measured in a special frame in which the 2 events occur at the same position. It is called the proper time. Viewed in any other frame, these 2 events will occur at different positions and, by (A.5), their time interval (Δt) will be greater than the proper time by a factor of γ_0. This is known as the effect of time dilation.

The muon has an average lifetime of 2.2 μsec (between birth and decay) in its rest frame. In a 1977 experiment at CERN, muons were accelerated to a speed of 0.9994c, corresponding to a γ_0 value of 28.87. Within experimental error, the measured average lifetime of these muons was indeed 28.87x2.2 = 63.5 μsec. In another experiment, two synchronized clocks with near perfect precision showed slightly different readings after one had been flown around the world. The difference was again in agreement with (A.5).

Time dilation runs counter to our intuition, because it is rooted in a postulate which also runs counter to our intuition.

The Twin Paradox: Suppose someone travels on a spaceship with a Lorentz factor of $\gamma_0 = 20$ (in the earth frame) and his twin brother stays on earth. Then, by time dilation, every day measured by the traveling twin in the spaceship frame (this is his proper time) will be 20 days when measured by the earth twin in the earth frame. So the earth twin ages faster and his traveling brother will be 19 years younger when he returns to earth after an 1-year journey (neglect the spaceship's acceleration/deceleration periods). The paradox is: if the traveling twin measures the age of his earth twin, will he conclude that he himself ages faster by the same argument of time dilation?

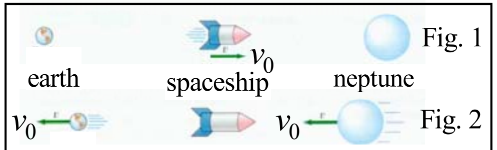
There is no paradox at all. Only the earth twin's measurement is correct because he is always in an inertial frame. The traveling twin will have to be accelerated and decelerated in the spaceship. During these periods, he cannot use the special theory of relativity (Einstein's 2 postulates refer to inertial frames.) In fact, he will confirm the measurement of his earth twin if he uses Einstein's general theory of relativity, which deals with accelerating reference frames.

Length Contraction: Assume that planet neptune is stationary in the earth frame and at a distance L_0 from earth (Fig. 1). A spaceship is traveling at speed v_0 to neptune. The duration of the trip, measured on earth, is

$$\Delta t = L_0 / v_0. \qquad \text{(A.8)}$$

In the spaceship's frame (Fig. 2), both earth and neptune move at speed v_0. The duration of the trip, Δt_0, is the interval between the departure of the earth and the arrival of the neptune. This is the "proper time" of the spaceship because both events occur at the same position. Thus, by (A.5)

$$\Delta t_0 = \Delta t / \gamma_0, \qquad \text{(A.9)}$$



from Giancoli, "Physics for Scientists and Engineers"

Δt_0 can be used to calculate the earth-naptune distance as viewed on the spaceship

$$L = v_0 \Delta t_0. \qquad \text{(A.10)}$$

Eliminating Δt and Δt_0 from (A.8)-(A.10), we obtain

$$L = \frac{L_0}{\gamma_0} \qquad \text{(A.11)}$$

In (A.11), $L = L_0/\gamma_0$, L_0 is the length of an object (or, in the above example, the earth-neptune distance) measured in the rest frame of the object (i.e. the frame in which the object is at rest). Length measured in this special frame is called the proper length. Viewed in any other frame, the object will be moving and, by (A.11), its length will be less than the proper length by a factor of γ_0. This is known as the effect of length contraction. Note that the contraction effect applies only to lengths along the direction of motion.

Length contraction is a direct consequence of time dilation [see (A.9)]. It is therefore not surprising that time dilation can be inferred from length contraction. If, for example, the spaceship has a γ_0 value of 2. The earth-neptune distance, as measured in the spaceship, would be half of that measured on earth. But the speed of earth/neptune relative to the spaceship is still v_0. So, to the spaceship, the journey's duration is only half of that measured on earth. Hence, one minute elapsed in the spaceship will be 2 minutes elapsed on earth.

The Lorentz Transformation: Assume frames S and S' coincide at $t = 0$ and S' moves along the common x-axis with speed v_0 relative to S (see figure). A point P has coordinates (x, y, z, t) in S and (x', y', z', t') in S'. The length x', when measured in S, is $\frac{x'}{\gamma_0}$ (length contraction). So,

$$x = v_0 t + \frac{x'}{\gamma_0} \quad \text{or} \quad x' = \gamma_0 (x - v_0 t). \qquad \text{(A.12)}$$

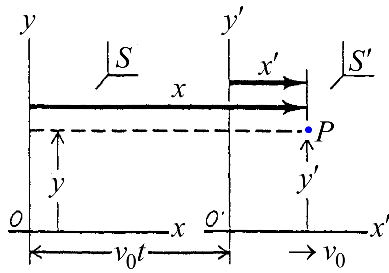
By symmetry or by similar argument, $x = \gamma_0 (x' + v_0 t')$ (A.13)

Eliminating x from (A.12) and (A.13) [using $\gamma_0^2 - 1 = \gamma_0^2 v_0^2 / c^2$],

$$\Rightarrow \quad t' = \gamma_0 (t - \frac{v_0}{c^2} x) \qquad \text{(A.14)}$$

(A.7), (A.12), and (A.14) give the Lorentz transformation:

$$\begin{cases} x' = \gamma_0 (x - v_0 t) \\ y' = y \\ z' = z \\ t' = \gamma_0 (t - \frac{v_0}{c^2} x) \end{cases} \qquad \text{(A.15)}$$



See Appendix B for a more formal derivation.

Transformation of Coordinate Difference between 2 Events :

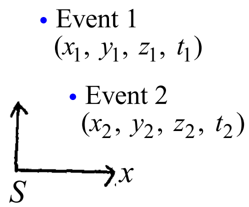
Since the Lorentz transformation is linear, the coordinate differences between 2 events:

in S: $\Delta x = x_2 - x_1,\ \Delta y = y_2 - y_1,\ \Delta z = z_2 - z_1,\ \Delta t = t_2 - t_1$ (A.16)

in S': $\Delta x' = x_2' - x_1',\ \Delta y' = y_2' - y_1',\ \Delta z' = z_2' - z_1',\ \Delta t' = t_2' - t_1'$ (A.17)

transform in the same manner. Thus,

$$\begin{cases} \Delta x' = \gamma_0 (\Delta x - v_0 \Delta t) \\ \Delta y' = y \\ \Delta z' = z \\ \Delta t' = \gamma_0 (\Delta t - \frac{v_0}{c^2} \Delta x) \end{cases} \qquad \text{(A.18)}$$

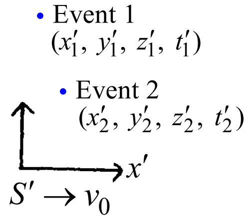


$\Delta x = x_2 - x_1$

$\Delta y = y_2 - y_1$

$\Delta z = z_2 - z_1$

$\Delta t = t_2 - t_1$



$\Delta x' = x_2' - x_1'$

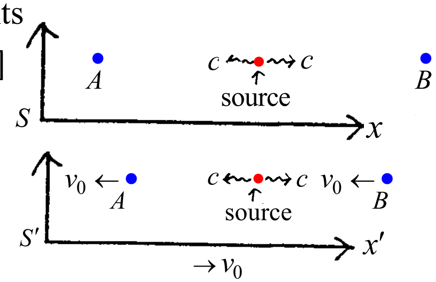
$\Delta y' = y_2' - y_1'$

$\Delta z' = z_2' - z_1'$

$\Delta t' = t_2' - t_1'$

Discussion on simultaneity: Consider the transformation equation for the time interval between two events

$$\Delta t' = \gamma_0 (\Delta t - \frac{v_0}{c^2} \Delta x) \quad \text{[from (A.18)]}$$



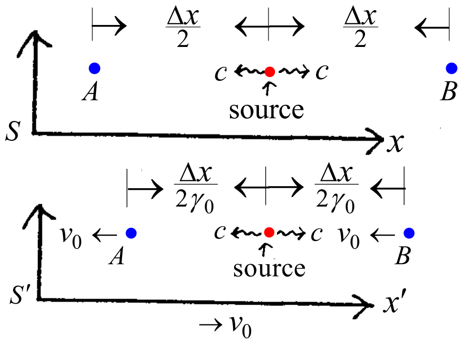
It indicates that 2 simultaneous events in frame S ($\Delta t = 0$) which occur at different positions ($\Delta x \neq 0$) will not be simultaneous in frame S' ($\Delta t' \neq 0$). This can be explained on the basis of postulate 2 through the following example.

In frame S, a pulse of light emitted midway between points A and B (see figure) will reach A and B at the same time, i.e. the two events (arrivals of the signals at A and B) are simultaneous in frame S. In frame S', the signal still travels at speed c in both directions, but B is moving toward the light and A away from it. So, the signal will reach B first and the two events are no longer simultaneous.

The example discussed above can be examined quantitatively as follows.

Assume that, in frame S, the two events are spatially separated by a distance Δx. Observed in S', the distance is shorter by a factor of γ_0 due to length contraction, i.e.

$$\Delta x' = \frac{\Delta x}{\gamma_0}.$$



Thus, in frame S', the signals reach A and B with a time difference of

$$\Delta t' = t'_B - t'_A = \frac{\frac{\Delta x}{2\gamma_0}}{c+v_0} - \frac{\frac{\Delta x}{2\gamma_0}}{c-v_0} = -\frac{\Delta x}{\gamma_0}\frac{v_0}{c^2-v_0^2} = -\frac{\gamma_0 v_0}{c^2}\Delta x.$$

This is precisely the prediction of (A.18),

$$\Delta t' = \gamma_0(\Delta t - \tfrac{v_0}{c^2}\Delta x) = -\frac{\gamma_0 v_0}{c^2}\Delta x. \quad [\Delta t = 0 \text{ in frame } S]$$

Problem 1: In frame S, events A and B occur at different positions, and event B occurs after event A. Is it posible for event B to precede event A in another frame S' moving at speed v_0 relative to frame S? If so, does this mean that an effect can precede its cause?

Solution: In frame S, let the 2 events have a spatial interval $\Delta x = x_B - x_A$ and time interval $\Delta t = t_B - t_A$. Then the time interval in frame S', $\Delta t' = t'_B - t'_A$, is given in (A.18): $\Delta t' = \gamma_0(\Delta t - \frac{v_0}{c^2}\Delta x)$.

We see that if $\Delta t < v_0 \Delta x / c^2$, then $\Delta t' < 0$, which means that the order of *independent* events in frame S may be reversed in frame S'.

Suppose, however, that the events are connected, i.e. event B is caused by event A. This would require a body, or a signal, to travel from A to B. Rewrite (A.18) as $\Delta t' = \gamma_0 \Delta t(1 - \frac{v_0}{c^2}\frac{\Delta x}{\Delta t})$. Since the fastest speed for a signal to travel from A to B is $\frac{\Delta x}{\Delta t} = c$, we must have $v_0 > c$ in order for $\Delta t' < 0$. This is not possible [see (A.6)] and thus the order of connected events (cause and effect) cannot be reversed.

Problem 2: Show that the effects of time dilation and length contraction are implicit in the Lorentz transformation.

Solution: The time interval between 2 events transform as $\Delta t' = \gamma_0(\Delta t - \frac{v_0}{c^2}\Delta x)$ or $\Delta t = \gamma_0(\Delta t' + \frac{v_0}{c^2}\Delta x')$. If $\Delta t'$ is the proper time in S', then the 2 events occur at the same position ($\Delta x' = 0$). So we use the latter equation and obtain

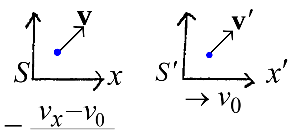
$$\Delta t = \gamma_0 \Delta t' \text{ (time dilation).}$$

The difference in the x coordinates of the 2 events transform as $\Delta x' = \gamma_0(\Delta x - v_0 \Delta t)$ or $\Delta x = \gamma_0(\Delta x' + v_0 \Delta t')$. Again, the question is which equation to use. If $\Delta x'$ is the "proper length" in S', then the two end points are at rest and their coordinates do not have to be measured simultaneously (i.e. we do not know $\Delta t'$.) But since the rod is moving in S, its end points must be measured simultaneously in S ($\Delta t = 0$). So we use the former equation and obtain

$$\Delta x = \frac{\Delta x'}{\gamma_0} \text{ (length contraction).}$$

Transformation of Velocity: The velocity of a particle is given by

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{x}}{\Delta t} \text{ (in frame } S\text{)}; \quad \mathbf{v}' = \lim_{\Delta t' \to 0} \frac{\Delta \mathbf{x}'}{\Delta t'} \text{ (in frame } S'\text{)}. \quad \text{(A.19)}$$



Let $\Delta t \to 0$,

$\Delta x = v_x \Delta t$

$$\begin{cases} \Delta x' = \gamma_0(\Delta x - v_0 \Delta t) \\ \Delta y' = y \\ \Delta z' = z \\ \Delta t' = \gamma_0(\Delta t - \frac{v_0}{c^2}\Delta x) \end{cases} \Rightarrow \begin{cases} v'_x = \frac{\Delta x'}{\Delta t'} = \frac{\gamma_0(\Delta x - v_0 \Delta t)}{\gamma_0(\Delta t - \frac{v_0}{c^2}\Delta x)} = \frac{v_x - v_0}{1 - \frac{v_0}{c^2}v_x} \\ v'_y = \frac{\Delta y'}{\Delta t'} = \frac{\Delta y}{\gamma_0(\Delta t - \frac{v_0}{c^2}\Delta x)} = \frac{v_y}{\gamma_0(1 - \frac{v_0}{c^2}v_x)} \\ v'_z = \frac{\Delta z'}{\Delta t'} = \frac{\Delta z}{\gamma_0(\Delta t - \frac{v_0}{c^2}\Delta x)} = \frac{v_z}{\gamma_0(1 - \frac{v_0}{c^2}v_x)} \end{cases} \quad \text{(A.20)}$$

Problem: A spaceship moves away from the earth at speed v_0. A pulse of light is emitted from the earth in the direction toward the sapceship. What is the speed of light measured on the spaceship?

Solution: $\quad v_x = c \Rightarrow v'_x = \frac{v_x - v_0}{1 - \frac{v_0}{c^2}v_x} = \frac{c - v_0}{1 - \frac{v_0}{c}} = c$

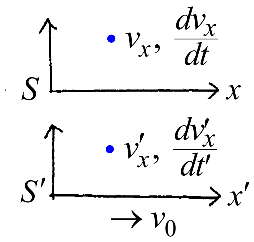
Transformation of Acceleration: For simplicity, we first consider the transformation of acceleration in the direction of relative motion (i.e. $\frac{dv_x}{dt}$)

$$v'_x = \frac{v_x - v_0}{1-\frac{v_0}{c^2}v_x} \quad \text{[from (A.20)]}$$

$$\Rightarrow dv'_x = \frac{dv_x}{1-\frac{v_0}{c^2}v_x} - \frac{(v_x-v_0)(-\frac{v_0}{c^2})dv_x}{(1-\frac{v_0}{c^2}v_x)^2} = \frac{dv_x}{\gamma_0^2(1-\frac{v_0}{c^2}v_x)^2}$$

$$t' = \gamma_0(t - \frac{v_0}{c^2}x) \quad \text{[from (A.15)]}$$

$$\Rightarrow \; dt' = \gamma_0(dt - \frac{v_0}{c^2}dx)$$

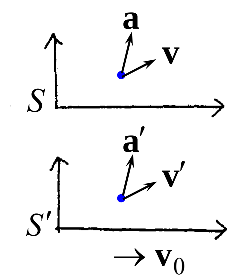


$dx = v_x dt$

$$\text{Hence, } \frac{dv'_x}{dt'} = \frac{1}{\gamma_0(dt-\frac{v_0}{c^2}dx)}\frac{dv_x}{\gamma_0^2(1-\frac{v_0}{c^2}v_x)^2} = \frac{1}{\gamma_0^3(1-\frac{v_0}{c^2}v_x)^3}\frac{dv_x}{dt} \quad \text{(A.21)}$$

By the same method, we may obtain the transformation equations for acceleration in arbitrary directions (see Jackson Problem 11.5).

$$\begin{cases} \mathbf{a}'_{\parallel} = \dfrac{1}{\gamma_0^3\left(1-\frac{\mathbf{v}_0\cdot\mathbf{v}}{c^2}\right)^3}\mathbf{a}_{\parallel} \\ \mathbf{a}'_{\perp} = \dfrac{1}{\gamma_0^2\left(1-\frac{\mathbf{v}_0\cdot\mathbf{v}}{c^2}\right)^3}\left[\mathbf{a}_{\perp} - \frac{\mathbf{v}_0}{c^2}\times(\mathbf{a}\times\mathbf{v})\right] \end{cases} \quad \text{(A.22)}$$

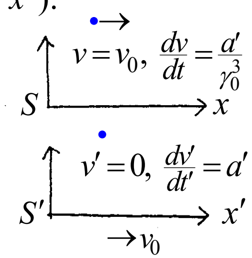


where "$\parallel$" and "$\perp$" refer to the direction of $\mathbf{v}_0$.

Problem: A rocket is launched from the earth into outer space. It moves on a straight line with a constant acceleration (a') with respect to its rest frame (Why is a' specified in the rocket's frame?) Calculate the time required for the rocket to accelerate from zero speed to the final speed v_f, according to earth and rocket clocks.

Solution: Let S be the earth frame, S' be the rocket rest frame, and the one-dimensional motion be along the x-axis. The inverse transformation of (A.21) gives (omitting subscript "x"):

$$a = \frac{dv}{dt} = \frac{1}{\gamma_0^3(1+v_0 v'/c^2)^3}\frac{dv'}{dt'}.$$



Lorentz transformations apply to two inertial frames. So, S' is the *instantaneous* rest frame of the rocket, but S' does not accelerate with the rocket. In S', we have $v' = 0$ and $dv'/dt' = a'$. This gives a (acceleration in S) $= a'/\gamma_0^3$. Since S' is the rest frame of the rocket, γ_0 for the transformation equals γ of the rocket in S. Thus,

$$a = a'/\gamma_0^3 = a'/\gamma^3, \text{ where } \gamma = (1-v^2/c^2)^{-1/2} \quad [\textit{Note}: \; a' \geq a]$$

From the expression of the acceleration in the earth frame (S),

$$a = a'/\gamma^3 = a'(1-v^2/c^2)^{3/2}, \quad \text{(A.23)}$$

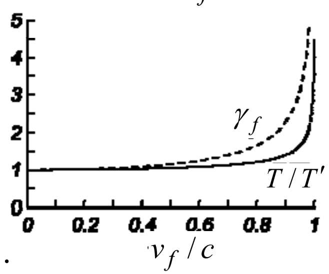
we may evaluate the total acceleration time as measured on earth

$$T = \int_0^T dt = \int_0^{v_f}\frac{1}{a}dv = \int_0^{v_f}\frac{\gamma^3}{a'}dv = \frac{1}{a'}\int_0^{v_f}\frac{dv}{(1-v^2/c^2)^{3/2}} = \frac{v_f}{a'(1-v_f^2/c^2)^{1/2}}.$$

The rocket frame is accelerating. So, to find the total acceleration time as measured on the rocket, we must still work in the earth frame by using the relation $dt' = dt/\gamma$.

$$T' = \int_0^{T'} dt' = \int_0^T \frac{1}{\gamma}dt = \int_0^{v_f}\frac{1}{\gamma a}dv = \frac{1}{a'}\int_0^{v_f}\frac{dv}{1-v^2/c^2} = \frac{c}{2a'}\,ln(\frac{1+v_f/c}{1-v_f/c}).$$

We find that, in the limit $v_f/c \rightarrow 0$, both T and T' approach the expected value of v_f/a'. However, T/T' increases rapidly as $v_f/c \rightarrow 1$ due to the effect of time dilation (see figure). In the figure, $\gamma_f = (1-v_f^2/c^2)^{-\frac{1}{2}}$ is the time dilation factor at the final speed v_f.



Section 2: Relativistic Momentum and Energy

(Ref.: H. C. Ohanian, "Physics," 2nd ed., pp.1013-1014.)

The law of conservation of momentum states that, for an isolated system of particles, $\sum m_i \mathbf{v}_i$(before collision) $= \sum m_i \mathbf{u}_i$(after collision).

Under the Galilean transformation, the statement is true in all (inertial) frames. However, under the Lorentz transformation, $\sum m_i \mathbf{v}_i$, though conserved in one frame, will in general not be conserved in another frame. Thus, postulate 1 is violated if we continue to define the momentum as $m\mathbf{v}$. The theory of relativity takes a major step by redefining (or postulating) the momentum and energy as

$$\begin{cases} \mathbf{p} = \gamma m \mathbf{v} & \text{(A.24)} \\ E = \gamma m c^2 & \text{(A.25)} \end{cases}$$

[*Note*: $\gamma \equiv (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor of a particle. It is to be distinguished from the Lorentz factor γ_0 for the transformation.]

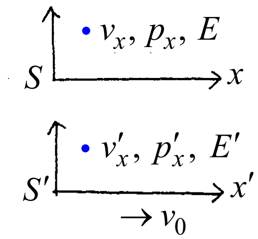
For simplicity, we will consider only one-dimentional motion along the x axis. The momentum and energy of a particle are then

$$p_x = \frac{m v_x}{\sqrt{1 - v_x^2/c^2}} \quad \text{and} \quad E = \frac{mc^2}{\sqrt{1 - v_x^2/c^2}}$$

From (A.20), the velocity in frame S' is $v'_x = \dfrac{v_x - v_0}{1 - v_x v_0/c^2}$. Hence, the momentum of the particle is (assuming m has the same value in S')

$$p'_x = \frac{m v'_x}{\sqrt{1 - v'^2_x/c^2}} = \frac{m(v_x - v_0)}{1 - v_x v_0/c^2} \frac{1}{\sqrt{1 - (1/c^2)[(v_x - v_0)/(1 - v_x v_0/c^2)]^2}}$$

$$= \frac{m(v_x - v_0)}{\sqrt{(1 - v_x v_0/c^2)^2 - (v_x - v_0)^2/c^2}}$$



Since $(1 - v_x v_0/c^2)^2 - (v_x - v_0)^2/c^2$
$= (1 - v_0^2/c^2)(1 - v_x^2/c^2)$, p'_x becomes

$$p'_x = \frac{1}{\sqrt{1 - v_0^2/c^2}} \frac{m v_x}{\sqrt{1 - v_x^2/c^2}} - \frac{v_0}{\sqrt{1 - v_0^2/c^2}} \frac{m}{\sqrt{1 - v_x^2/c^2}}$$

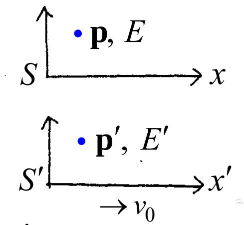
$$= \gamma_0 (p_x - \frac{v_0}{c^2} E) \qquad \text{(A.26)}$$

Similarly, we derive the Lorentz transformation equation for energy:

$$E' = \gamma_0 (E - v_0 p_x) \qquad \text{(A.27)}$$

By the same method, we can extend the motion to 3 dimensions and derive the Lorentz transformation equations for $\mathbf{p}$ and E. The result is

$$\begin{cases} p'_x = \gamma_0 (p_x - \frac{v_0}{c^2} E) \\ p'_y = p_y \\ p'_z = p_z \\ E' = \gamma_0 (E - v_0 p_x) \end{cases} \qquad \text{(A.28)}$$



(A.28) shows that $\mathbf{p}'$ and E' in S' is a *linear* combination of $\mathbf{p}$ and E in S, with constant coefficients (i.e. the coefficients are independent of $\mathbf{p}$ and E of the particle). The same equations will therefore hold true for the *total* momentum and energy ($\sum \mathbf{p}_j$, $\sum E_j$) of a system of particles,

$$\begin{cases} \sum p'_{jx} = \gamma_0 (\sum p_{jx} - \frac{v_0}{c^2} \sum E_j) \\ \sum p'_{jy} = \sum p_{jy} \\ \sum p'_{jz} = \sum p_{jz} \\ \sum E'_j = \gamma_0 (\sum E_j - v_0 \sum p_{jx}) \end{cases} \qquad \text{(A.29)}$$

Rewrite (A.29)
$$\begin{cases} \sum p'_{jx} = \gamma_0 (\sum p_{jx} - \frac{v_0}{c^2} \sum E_j) \\ \sum p'_{jy} = \sum p_{jy} \\ \sum p'_{jz} = \sum p_{jz} \\ \sum E'_j = \gamma_0 (\sum E_j - v_0 \sum p_{jx}) \end{cases}.$$

Form this set of equations, we see that if (and only if) the total momentum ($\sum \mathbf{p}_j$) and total energy ($\sum E_j$) of a system of particles are *both* conserved in S, the total momentum and total energy will be both conserved in S'.

Discussion: (i) This shows that the postulation of $\mathbf{p} = \gamma m \mathbf{v}$ and $E = \gamma mc^2$ will preserve the conservation law under the Lorentz transformation. However, the conservation law must now be extended to include both the momentum and energy.

(ii) Writing $E = \gamma mc^2 = (\gamma - 1)mc^2 + mc^2$, we may divide the total energy into the kinetic enery $(\gamma - 1)mc^2$ (due to motion) and a new form energy mc^2 (an intrinsic energy) called the rest-mass energy.

Problem: A particle of rest mass m moves on the x-axis is attracted to the origin by a force $m\omega^2 x$ ($\omega = const$). It performs oscillations of amplitude a. Express the *relativistic* oscillation period as a definite integral, and obtain the 2 leading terms of this integral for small a.

Solution: The period τ is given by $\tau = 4\int_0^a \frac{dx}{v}$, (A.30)

where the velocity v can be calculated from the energy equation

$$mc^2(1 - v^2/c^2)^{-1/2} + \tfrac{1}{2}m\omega^2 x^2 = mc^2 + \tfrac{1}{2}m\omega^2 a^2 \qquad \text{(A.31)}$$

Substituting v from (A.31) into (A.30), we obtain

$$\tau = \frac{4}{\omega}\int_0^a dx \frac{1 + \omega^2(a^2 - x^2)/2c^2}{(a^2 - x^2)^{1/2}\left[1 + \omega^2(a^2 - x^2)/(4c^2)\right]^{1/2}}$$

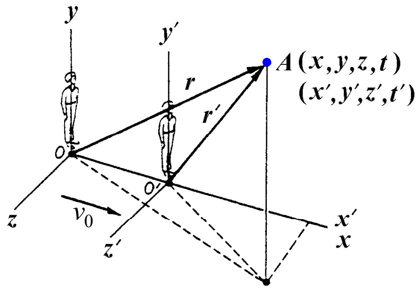
Expanding the integrand in powers of $\omega^2(a^2 - x^2)/c^2$ and using $\int_0^b dy \frac{1}{(by - y^2)^{1/2}} = \pi$ and $\int_0^b dy \frac{y}{(by - y^2)^{1/2}} = \frac{b\pi}{2}$ (for $b > 0$), we obtain

$$\tau = \frac{2\pi}{\omega}\left[1 + \frac{3\omega^2 a^2}{16c^2} + \cdots\right]$$

Appendix B: A Formal Derivation of the Lorentz Transformation

In Appendix A, we begin with the derivation of "time dilation" and "length contraction" from postulate 2, followed by a derivation of the Lorentz transformation. Here, we present a more formal (but physically less transparent) approach, whereby the Lorentz transformation is derived directly from postulate 2. The following paragraphs are taken from Alonso & Finn "Physics," p.92.

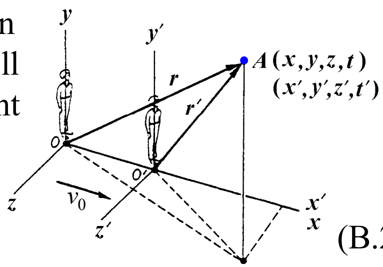
Referring to the figure to the right, suppose that at $t = 0$ a flash of light is emitted at the common position of the two observers. After a time t, observer O will note that the light has reached point A and will write $r = ct$, where c is the speed of light. Since $x^2 + y^2 + z^2 = r^2$, we may also write



$$x^2 + y^2 + z^2 = c^2 t^2 \qquad \text{(B.1)}$$

Similarly, observer O', whose position is no longer coincident with that of O, will note that the light arrives at the same point A in a time t', but also with velocity c.

Therefore he writes $r' = ct'$, or



$$x'^2 + y'^2 + z'^2 = c^2 t'^2 \qquad \text{(B.2)}$$

Our next task is to obtain a transformation relating (B.1) and (B.2). The symmetry of the problem suggests that $y' = y$ and $z' = z$. Also since $OO' = v_0 t$ for observer O, it must be that $x = v_0 t$ for $x' = 0$ (point O'). This suggests making $x' = k(x - v_0 t)$, where k is a constant to be determined. Since t' is different, we may also assume that $t' = a(t - bx)$, where a and b are constants to be determined (for the Galilean transformation, $k = a = 1$ and $b = 0$). Making all these substitutions in (B.2), we have

$$k^2(x^2 - 2v_0 xt + v_0^2 t^2) + y^2 + z^2 = c^2 a^2(t^2 - 2bxt + b^2 x^2) \quad \text{or}$$

$$(k^2 - b^2 a^2 c^2)x^2 - 2(k^2 v_0 - ba^2 c^2)xt + y^2 + z^2 = (a^2 - k^2 v_0^2/c^2)c^2 t^2.$$

This result must be identical to (B.1). Therefore

$$k^2 - b^2 a^2 c^2 = 0$$

$$k^2 v_0 - ba^2 c^2 = 0$$

$$a^2 - k^2 v_0^2 / c^2 = 1$$

Solving this set of equations, for k, a, and b, we have

$$k = a = \frac{1}{\sqrt{1 - v_0^2/c^2}} \quad \text{and} \quad b = v_0/c^2$$

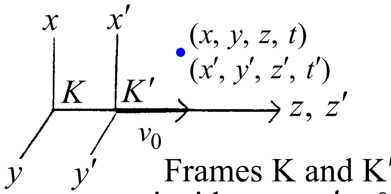
Inserting these values of k, a, and b in $x' = k(x - v_0 t)$ and $t' = (a - bx)$, we obtain the Lorentz transformation

$$\begin{cases} x' = \dfrac{x - v_0 t}{\sqrt{1 - v_0^2/c^2}} \\ y' = y \\ z' = z \\ t' = \dfrac{t - v_0 x/c^2}{\sqrt{1 - v_0^2/c^2}} \end{cases} \qquad \text{(B.3)}$$

Appendix C: Summary of Lorentz Transformation Equations

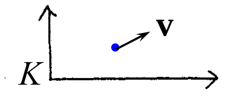
For all equations, $\gamma_0 \equiv \left(1-\frac{v_0^2}{c^2}\right)^{-\frac{1}{2}}$. By symmetry, equations for the inverse transformation differ only by the sign of v_0 (or $\mathbf{v}_0$).

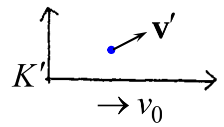
$$1.\quad \begin{cases} x' = x \\ y' = y \\ z' = \gamma_0\,(z - v_0 t) \\ t' = \gamma_0\left(t - \frac{v_0}{c^2} z\right) \end{cases}$$



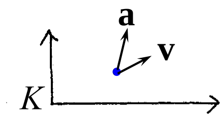
Frames K and K′ coincide at $t = t' = 0$.

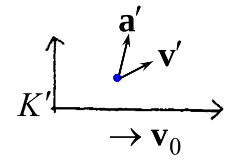
$$2.\quad \begin{cases} v'_x = \dfrac{v_x}{\gamma_0\left(1-\frac{v_0}{c^2}v_z\right)} \\ v'_y = \dfrac{v_y}{\gamma_0\left(1-\frac{v_0}{c^2}v_z\right)} \\ v'_z = \dfrac{v_z - v_0}{1-\frac{v_0}{c^2}v_z} \end{cases}$$





$$3.\quad \begin{cases} \mathbf{a}'_{\parallel} = \dfrac{1}{\gamma_0^3\left(1-\frac{\mathbf{v}_0\cdot\mathbf{v}}{c^2}\right)^3}\,\mathbf{a}_{\parallel} \\ \mathbf{a}'_{\perp} = \dfrac{1}{\gamma_0^2\left(1-\frac{\mathbf{v}_0\cdot\mathbf{v}}{c^2}\right)^3}\left[\mathbf{a}_{\perp} - \frac{\mathbf{v}_0}{c^2}\times(\mathbf{a}\times\mathbf{v})\right] \end{cases}$$

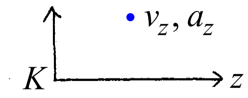


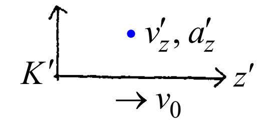


where "$\parallel$" and "$\perp$" refer to the direction of $\mathbf{v}_0$.

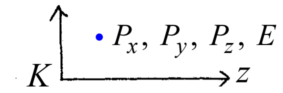
Special case: one dimensional motion

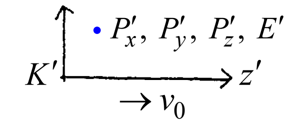
$$a'_z = \frac{1}{\gamma_0^3\left(1-\frac{v_0}{c^2}v_z\right)^3}\,a_z$$



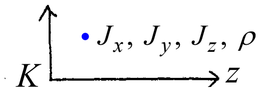


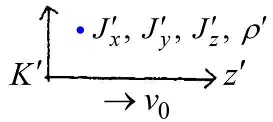
$$4.\quad \begin{cases} p'_x = p_x \\ p'_y = p_y \\ p'_z = \gamma_0\left(p_z - \frac{v_0}{c^2}E\right) \\ E' = \gamma_0(E - v_0 p_z) \end{cases}$$



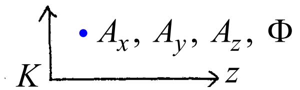


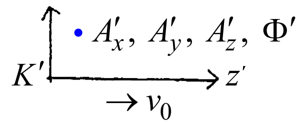
$$5.\quad \begin{cases} J'_x = J_x \\ J'_y = J_y \\ J'_z = \gamma_0(J_z - v_0\rho) \\ \rho' = \gamma_0\left(\rho - \frac{v_0}{c^2}J_z\right) \end{cases}$$



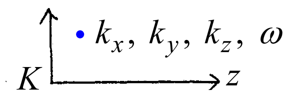


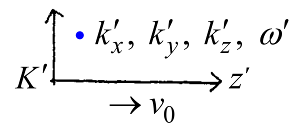
$$6.\quad \begin{cases} A'_x = A_x \\ A'_y = A_y \\ A'_z = \gamma_0\left(A_z - \frac{v_0}{c}\Phi\right) \\ \Phi' = \gamma_0\left(\Phi - \frac{v_0}{c}A_z\right) \end{cases}$$



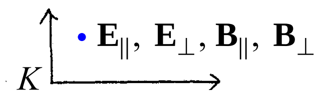


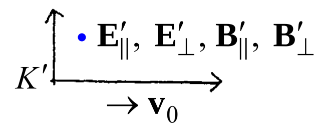
$$7.\quad \begin{cases} k'_x = k_x \\ k'_y = k_y \\ k'_z = \gamma_0\left(k_z - \frac{v_0}{c^2}\omega\right) \\ \omega' = \gamma_0(\omega - v_0 k_z) \end{cases}$$





$$8.\quad \begin{cases} \mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \\ \mathbf{E}'_{\perp} = \gamma_0\left(\mathbf{E}_{\perp} + \frac{\mathbf{v}_0}{c}\times\mathbf{B}_{\perp}\right) \\ \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \\ \mathbf{B}'_{\perp} = \gamma_0\left(\mathbf{B}_{\perp} - \frac{\mathbf{v}_0}{c}\times\mathbf{E}_{\perp}\right) \end{cases}$$





where "$\parallel$" and "$\perp$" refer to the direction of $\mathbf{v}_0$.