Chapter 14: Radiation by Moving Charges

Review of Basic Equations : [converted to Gaussian unit system, see p.782 for conversion formulae.]

$$\begin{cases} \nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi = -4\pi\rho\left(\frac{\rho}{\varepsilon_0}\right) & (6.15) \\ \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}\left(\mu_0\mathbf{J}\right) & (6.16) \end{cases}$$

[free-space inhomogeneous wave equations (SI)]

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = -4\pi f(\mathbf{x},t) \quad \text{[general form of (6.15) and (6.16)]} \qquad (6.32)$$

Solution of (6.32) with outgoing-wave b.c.:

$$\psi(\mathbf{x},t) = \psi_{in}(\mathbf{x},t) + \int d^3x' \int dt' G^+(\mathbf{x},t,\mathbf{x}',t') f(\mathbf{x}',t'), \qquad (6.45)$$

where the retarded Green's function

$$G^+(\mathbf{x},t,\mathbf{x}',t') = \delta\left[t' - \left(t - \tfrac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right] / |\mathbf{x}-\mathbf{x}'| \qquad (6.44)$$

is the solution of (with outgoing-wave b.c.)

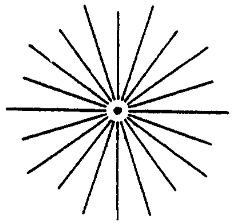
$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G^+(\mathbf{x},t,\mathbf{x}',t') = -4\pi\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \qquad (6.41)$$

Apply (6.45) (assuming $\psi_{in} = 0$) to (6.15) & (6.16)

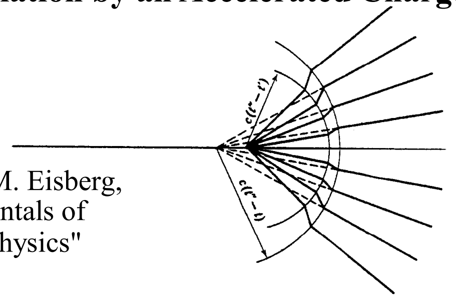
$$\begin{Bmatrix} \Phi(\mathbf{x},t) \\ \mathbf{A}(\mathbf{x},t) \end{Bmatrix} = \int d^3x' \int dt' \frac{\delta\left[t' - \left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right]}{|\mathbf{x}-\mathbf{x}'|} \begin{Bmatrix} \rho(\mathbf{x}',t') \\ \frac{1}{c}\mathbf{J}(\mathbf{x}',t') \end{Bmatrix} \qquad (9.2)$$

Note: We need both Φ and **A** to specify **E** and **B**, unless the source has harmonic time dependence (as in Chs. 9 and 10).

A Qualitative Picture of Radiation by an Accelerated Charge:



From R. M. Eisberg, "Fundamentals of Modern Physics"



E-field lines surrounding a stationary charge.

A fraction of **E**-field lines showing the effect of charge acceleration.

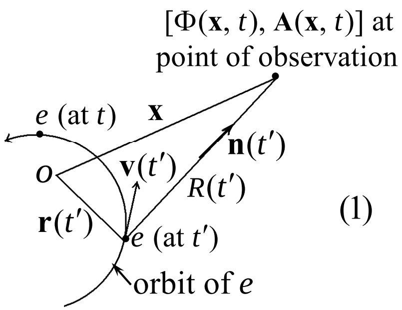
14.1 Liénard-Wiechert Potentials and Fields for a Point Charge

Lienard-Wiechert Potentials for a Point Charge :

Rewrite (9.2):
$$\begin{Bmatrix} \mathbf{A}(\mathbf{x},t) \\ \Phi(\mathbf{x},t) \end{Bmatrix} = \int d^3x' \int dt' \frac{\delta\left[t' - \left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right]}{|\mathbf{x}-\mathbf{x}'|} \begin{Bmatrix} \frac{1}{c}\mathbf{J}(\mathbf{x}',t') \\ \rho(\mathbf{x}',t') \end{Bmatrix}$$

$\rho(\mathbf{x}',t')$, $\mathbf{J}(\mathbf{x}',t')$ due to a point charge e (e carries a sign) moving along the orbit $\mathbf{r}(t')$ at the velocity $\mathbf{v}(t') = d\mathbf{r}(t')/dt'$ can be written

$$\begin{cases} \rho(\mathbf{x}',t') = e\delta\left[\mathbf{x}' - \mathbf{r}(t')\right] \\ \mathbf{J}(\mathbf{x}',t') = e\mathbf{v}(t')\delta\left[\mathbf{x}' - \mathbf{r}(t')\right] \end{cases}$$



$$\Rightarrow \begin{cases} \Phi(\mathbf{x},t) = e\int dt' \frac{\delta\left[t' + \frac{R(t')}{c} - t\right]}{R(t')} \\ \mathbf{A}(\mathbf{x},t) = e\int dt' \frac{\boldsymbol{\beta}(t')\delta\left[t' + \frac{R(t')}{c} - t\right]}{R(t')} \end{cases} \qquad (1)$$

where $R(t') = |\mathbf{x} - \mathbf{r}(t')|$ and $\boldsymbol{\beta}(t') = \mathbf{v}(t')/c$.

Rewrite (1):
$$\begin{cases} \Phi(\mathbf{x},t) = e\int dt' \frac{\delta\left[t' + \frac{R(t')}{c} - t\right]}{R(t')} = e\int dt' \frac{\delta\left[f(t') - t\right]}{R(t')} \\ \mathbf{A}(\mathbf{x},t) = e\int dt' \frac{\boldsymbol{\beta}(t')\delta\left[t' + \frac{R(t')}{c} - t\right]}{R(t')} = e\int dt' \frac{\boldsymbol{\beta}(t')\delta\left[f(t') - t\right]}{R(t')} \end{cases},$$

where $f(t') \equiv t' + R(t')/c$. (2)

Using $\int g(x)\delta\left[f(x) - a\right]dx = \sum_i \left[\frac{g(x)}{\left|\frac{d}{dx}f(x)\right|}\right]_{x_i}$, we obtain

[x_i is the solution of $f(x_i) = a$.]

$$\begin{cases} \Phi(\mathbf{x},t) = \left[\frac{e}{R(t')\left|\frac{d}{dt'}f(t')\right|}\right]_{ret} \\ \mathbf{A}(\mathbf{x},t) = \left[\frac{e\boldsymbol{\beta}(t')}{R(t')\left|\frac{d}{dt'}f(t')\right|}\right]_{ret} \end{cases} \qquad (3)$$

where $[\ \ ]_{ret}$ implies that quantities in the bracket are to be evaluated at the retarded time t' $[= t - R(t')/c]$.

Question: What information is needed in order to find t'?

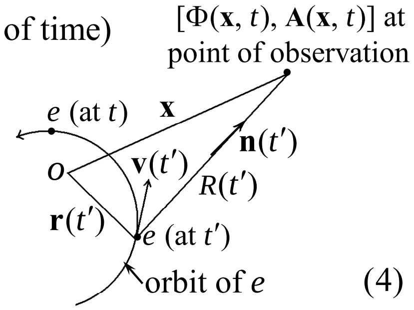
$$\frac{dR(t')}{dt'} = \frac{d\left|\mathbf{x}-\mathbf{r}(t')\right|}{dt'} = \frac{d}{dt'}[x^2 - 2\mathbf{x}\cdot\mathbf{r}(t') + \mathbf{r}^2(t')]^{\frac{1}{2}}$$

($\mathbf{x}$ is a fixed position, indep. of time)

$$= \frac{-2\mathbf{x}\cdot\frac{d}{dt'}\mathbf{r}(t') + 2\mathbf{r}(t')\cdot\frac{d}{dt'}\mathbf{r}(t')}{2[x^2 - 2\mathbf{x}\cdot\mathbf{r}(t') + \mathbf{r}^2(t')]^{\frac{1}{2}}}$$

$$= -\frac{\mathbf{v}(t')\cdot[\mathbf{x}-\mathbf{r}(t')]}{R(t')}$$

$$= -\mathbf{v}(t')\cdot\mathbf{n}(t') \qquad (4)$$



$$\Rightarrow \frac{d}{dt'} f(t') = \frac{d}{dt'}[t' + \frac{R(t')}{c}] = 1 - \boldsymbol{\beta}(t')\cdot\mathbf{n}(t') \equiv \kappa (> 0) \qquad (5)$$

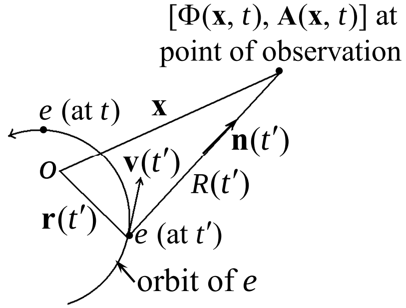
Sub. (5) into (3) gives the Lienard-Wiechert potentials

$$\begin{cases} \Phi(\mathbf{x},t) = \left[\dfrac{e}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{ret} \\ \mathbf{A}(\mathbf{x},t) = \left[\dfrac{e\boldsymbol{\beta}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{ret} \end{cases} \qquad (14.8)$$

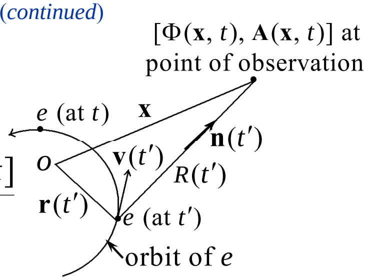
Fields for a Point Charge: Rewrite (1) and (14.8):

$$\begin{cases} \Phi(\mathbf{x},t) = e\int dt' \dfrac{\delta[t'+\frac{R(t')}{c}-t]}{R(t')} \\ \mathbf{A}(\mathbf{x},t) = e\int dt' \dfrac{\boldsymbol{\beta}(t')\delta[t'+\frac{R(t')}{c}-t]}{R(t')} \end{cases} (1), \quad \begin{cases} \Phi(\mathbf{x},t) = \left[\dfrac{e}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{ret} \\ \mathbf{A}(\mathbf{x},t) = \left[\dfrac{e\boldsymbol{\beta}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})R}\right]_{ret} \end{cases} (14.8)$$

To obtain $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$, we need to differentiate $\Phi(\mathbf{x},t)$ and $\mathbf{A}(\mathbf{x},t)$ with respect to $\mathbf{x}$. The RHS of (14.8) depends on $\mathbf{x}$ through $\mathbf{n}$ and R, but the RHS of (1) depends on $\mathbf{x}$ through R only. Hence, it is more convenient to use (1).



Rewrite (1):
$$\begin{cases} \Phi(\mathbf{x},t) = e\int dt' \dfrac{\delta[t'+\frac{R(t')}{c}-t]}{R(t')} \\ \mathbf{A}(\mathbf{x},t) = e\int dt' \dfrac{\boldsymbol{\beta}(t')\delta[t'+\frac{R(t')}{c}-t]}{R(t')} \end{cases}$$



Let $F(R)$ be any function of R, then

$$\nabla_{\mathbf{x}} F(R) = \frac{dF}{dR}\nabla_{\mathbf{x}} R = \frac{dF}{dR}\underbrace{\nabla_{\mathbf{x}}\left|\mathbf{x}-\mathbf{r}(t')\right|}_{=\mathbf{n}(t')} = \mathbf{n}(t')\frac{dF}{dR} \qquad (6)$$

Use $\nabla_{\mathbf{x}} |\mathbf{x}-\mathbf{x}'|^n = n|\mathbf{x}-\mathbf{x}'|^{n-2}(\mathbf{x}-\mathbf{x}')$.
See Sec. 1.3 of lecture notes.

$$(1)\ \&\ (6) \Rightarrow \begin{cases} \nabla\Phi(\mathbf{x},t) = e\int \mathbf{n}(t')\left[\dfrac{-\delta[t'+\frac{R(t')}{c}-t]}{R^2(t')} + \dfrac{\delta'[t'+\frac{R(t')}{c}-t]}{cR(t')}\right]dt' \\ \dfrac{1}{c}\dfrac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t) = -\dfrac{e}{c}\int \dfrac{\boldsymbol{\beta}(t')\delta'[t'+\frac{R(t')}{c}-t]}{R(t')}dt' \end{cases}$$

Thus,

$f(t') \equiv t' + \frac{R(t')}{c} \Rightarrow dt' = \frac{dt'}{df(t')}df(t') = \frac{1}{\kappa}df(t')$,
where $\kappa \equiv 1-\boldsymbol{\beta}(t')\cdot\mathbf{n}(t')$. Use (5)

$$\mathbf{E}(\mathbf{x},t) = -\nabla\Phi - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}$$

$$= e\int\left[\frac{\mathbf{n}}{R^2}\delta[t'+\frac{R(t')}{c}-t] + \frac{\boldsymbol{\beta}-\mathbf{n}}{Rc}\delta'[t'+\frac{R(t')}{c}-t]\right]dt'$$

$$= e\int\left[\frac{\mathbf{n}}{\kappa R^2}\delta[f(t')-t] + \frac{\boldsymbol{\beta}-\mathbf{n}}{\kappa Rc}\delta'[f(t')-t]\right]df(t') \text{ [see note below]}$$

$$\int g(x)\delta'(x-a)dx = -g'(a)$$

$$= e\left[\frac{\mathbf{n}}{\kappa R^2} + \frac{1}{c}\frac{d}{df(t')}\left(\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}\right)\right]_{ret}$$

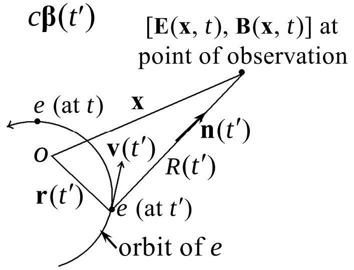
$$\frac{d}{df(t')} = \frac{dt'}{df(t')}\frac{d}{dt'} = \frac{1}{\kappa}\frac{d}{dt'}$$

$$= e\left[\frac{\mathbf{n}}{\kappa R^2} + \frac{1}{\kappa c}\frac{d}{dt'}\left(\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}\right)\right]_{ret} \qquad (7)$$

Note: Because of the $\delta[f(t')-t]$ factor in the integrand, integration over $f(t')$ demands $f(t') = t$ or $t' = t - \frac{R(t')}{c}$. But $\mathbf{n}$, $\boldsymbol{\beta}$, R, and κ in the integrand are all functions of t' [not $f(t')$]. Hnece, $\mathbf{n}$, $\boldsymbol{\beta}$, R, and κ are to be evaluated at the retarded time t' [not t].

To put $\mathbf{E}$ in a simpler form, we need to evaluate $\frac{d\mathbf{n}(t')}{dt'}$ and $\frac{d}{dt'}(\kappa R)$.

$$\frac{d\mathbf{n}(t')}{dt'} = \frac{d}{dt'}\frac{\mathbf{x}-\mathbf{r}(t')}{R(t')} = -\underbrace{\frac{\mathbf{x}-\mathbf{r}(t')}{R^2(t')}}_{\frac{\mathbf{n}(t')}{R(t')}}\underbrace{\frac{dR(t')}{dt'}}_{\substack{-c\boldsymbol{\beta}\cdot\mathbf{n}\\ \text{by (4)}}} - \frac{1}{R(t')}\underbrace{\frac{d\mathbf{r}(t')}{dt'}}_{c\boldsymbol{\beta}(t')} = \frac{c}{R}[\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\beta})-\boldsymbol{\beta}] \quad (8)$$



$$\frac{d}{dt'}(\kappa R) = \frac{d}{dt'}\{[1-\boldsymbol{\beta}(t')\cdot\mathbf{n}(t')]R\}$$

$$= (1-\boldsymbol{\beta}\cdot\mathbf{n})\underbrace{\frac{d}{dt'}R}_{-c\boldsymbol{\beta}\cdot\mathbf{n}} - R\frac{d}{dt'}(\boldsymbol{\beta}\cdot\mathbf{n})$$

$$= -c(1-\boldsymbol{\beta}\cdot\mathbf{n})(\boldsymbol{\beta}\cdot\mathbf{n}) - R\dot{\boldsymbol{\beta}}\cdot\mathbf{n} - R\boldsymbol{\beta}\cdot\frac{d\mathbf{n}}{dt'} \quad \leftarrow \boxed{\text{Sub. (8) for } \frac{d\mathbf{n}}{dt'}}$$

$$= -c(1-\boldsymbol{\beta}\cdot\mathbf{n})(\boldsymbol{\beta}\cdot\mathbf{n}) - R\dot{\boldsymbol{\beta}}\cdot\mathbf{n} - c\left[(\mathbf{n}\cdot\boldsymbol{\beta})^2 - \beta^2\right]$$

$$= -c(\boldsymbol{\beta}\cdot\mathbf{n})(1-\boldsymbol{\beta}\cdot\mathbf{n}+\boldsymbol{\beta}\cdot\mathbf{n}) + c\beta^2 - R\dot{\boldsymbol{\beta}}\cdot\mathbf{n}$$

$$= c\beta^2 - c(\boldsymbol{\beta}\cdot\mathbf{n}) - R(\dot{\boldsymbol{\beta}}\cdot\mathbf{n}) \quad (9)$$

$$\mathbf{E}(\mathbf{x},t) = e\left[\frac{\mathbf{n}}{\kappa R^2} + \frac{1}{c\kappa^2 R}\frac{d}{dt'}(\mathbf{n}-\boldsymbol{\beta}) + \frac{\mathbf{n}-\boldsymbol{\beta}}{c\kappa}\frac{d}{dt'}\left(\frac{1}{\kappa R}\right)\right]_{ret} \quad \leftarrow \boxed{\text{from (7)}} \quad \leftarrow \boxed{\text{Use (8), (9)}}$$

$$= e\left\{\frac{\mathbf{n}}{\kappa R^2} + \frac{1}{c\kappa^2 R}\left[\frac{c}{R}[\mathbf{n}(\boldsymbol{\beta}\cdot\mathbf{n})-\boldsymbol{\beta}] - \dot{\boldsymbol{\beta}}\right] - \frac{\mathbf{n}-\boldsymbol{\beta}}{c\kappa^3 R^2}\left[c\beta^2 - c(\boldsymbol{\beta}\cdot\mathbf{n}) - R(\dot{\boldsymbol{\beta}}\cdot\mathbf{n})\right]\right\}_{ret}$$

$$= e\left\{\frac{1}{R^2}\left[\underbrace{\frac{\mathbf{n}}{\kappa} + \frac{\mathbf{n}(\boldsymbol{\beta}\cdot\mathbf{n})-\boldsymbol{\beta}}{\kappa^2}}_{=\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa^2} = \frac{(\mathbf{n}-\boldsymbol{\beta})[1-(\boldsymbol{\beta}\cdot\mathbf{n})]}{\kappa^3}} - \frac{(\mathbf{n}-\boldsymbol{\beta})(\beta^2-\boldsymbol{\beta}\cdot\mathbf{n})}{\kappa^3}\right] + \frac{1}{R}\left[\frac{-\dot{\boldsymbol{\beta}}}{c\kappa^2} + \frac{(\mathbf{n}-\boldsymbol{\beta})(\dot{\boldsymbol{\beta}}\cdot\mathbf{n})}{c\kappa^3}\right]\right\}_{ret}$$

$$\boxed{\kappa \equiv 1-\boldsymbol{\beta}\cdot\mathbf{n}}$$

$$= e\left\{\begin{array}{l}\frac{1}{\kappa^3 R^2}\left[(\mathbf{n}-\boldsymbol{\beta})(1-\boldsymbol{\beta}\cdot\mathbf{n}) - (\mathbf{n}-\boldsymbol{\beta})(\beta^2-\boldsymbol{\beta}\cdot\mathbf{n})\right] \\ +\frac{1}{c\kappa^3 R}\left[-\dot{\boldsymbol{\beta}}(1-\boldsymbol{\beta}\cdot\mathbf{n}) + (\mathbf{n}-\boldsymbol{\beta})(\dot{\boldsymbol{\beta}}\cdot\mathbf{n})\right]\end{array}\right\}_{ret}$$

$$= e\left\{\frac{1}{\kappa^3 R^2}[(\mathbf{n}-\boldsymbol{\beta})(\underbrace{1-\beta^2}_{1/\gamma^2})] + \frac{1}{c\kappa^3 R}\left[\underbrace{\mathbf{n}(\dot{\boldsymbol{\beta}}\cdot\mathbf{n})-\dot{\boldsymbol{\beta}}}_{\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})} - [\underbrace{\boldsymbol{\beta}(\dot{\boldsymbol{\beta}}\cdot\mathbf{n})-\dot{\boldsymbol{\beta}}\cdot(\boldsymbol{\beta}\cdot\mathbf{n})}_{\mathbf{n}\times(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})}]\right]\right\}_{ret}$$

$$= e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R^2}\right]_{ret} + \frac{e}{c}\left[\frac{\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]_{ret} \quad (14.14)$$

To derive $\mathbf{B}(\mathbf{x},t)$, we write (7)

$$\mathbf{E}(\mathbf{x},t) = e\left[\frac{\mathbf{n}}{\kappa R^2} + \frac{1}{\kappa c}\frac{d}{dt'}\left(\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}\right)\right]_{ret}$$

$$\Rightarrow \mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t) = e\left[\frac{1}{\kappa c}\mathbf{n}\times\frac{d}{dt'}\left(\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}\right)\right]_{ret}$$

$$= -e\left[\frac{1}{\kappa c}\frac{d}{dt'}\left(\frac{\mathbf{n}\times\boldsymbol{\beta}}{\kappa R}\right) + \frac{\mathbf{n}\times\boldsymbol{\beta}}{\kappa R^2}\right]_{ret}$$

$$\mathbf{n}\times\frac{d}{dt'}\left(\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}\right)$$

Use (8)

$$= \frac{d}{dt'}\left[\frac{\mathbf{n}\times(\mathbf{n}-\boldsymbol{\beta})}{\kappa R}\right] - \frac{d\mathbf{n}}{dt'}\times\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}$$

$$= -\frac{d}{dt'}\left(\frac{\mathbf{n}\times\boldsymbol{\beta}}{\kappa R}\right) - \frac{c[\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\beta})-\boldsymbol{\beta}]}{R}\times\frac{\mathbf{n}-\boldsymbol{\beta}}{\kappa R}$$

$$= -\frac{d}{dt'}\left(\frac{\mathbf{n}\times\boldsymbol{\beta}}{\kappa R}\right) - \frac{c\mathbf{n}\times\boldsymbol{\beta}}{R^2}$$

∇ operates on $R(t')$ only
[only $R(t')$ depends on $\mathbf{x}$]

$$\mathbf{B}(\mathbf{x},t) = \nabla\times\mathbf{A} = e\int dt'\nabla\times\left[\frac{\boldsymbol{\beta}(t')\delta[t'+R(t')/c-t]}{R(t')}\right]$$

(1)

$$= e\int dt'\left[\nabla\frac{\delta[t'+R(t')/c-t]}{R(t')}\right]\times\boldsymbol{\beta}(t') \quad \leftarrow \boxed{\nabla\times\psi\mathbf{a} = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}}$$

$$= e\int dt'\left[-\frac{\delta[t'+R(t')/c-t]}{R^2} + \frac{\delta'[t'+R(t')/c-t]}{cR}\right]\overbrace{\nabla R(t')}^{\mathbf{n}(t')}\times\boldsymbol{\beta}(t')$$

$$= -e\left[\frac{1}{\kappa c}\frac{d}{dt'}\left(\frac{\mathbf{n}\times\boldsymbol{\beta}}{\kappa R}\right) + \frac{\mathbf{n}\times\boldsymbol{\beta}}{\kappa R^2}\right]_{ret}$$

following the same steps as in deriving (7)

$$\Rightarrow \mathbf{B}(\mathbf{x},t) = \mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t) \quad (14.13)$$

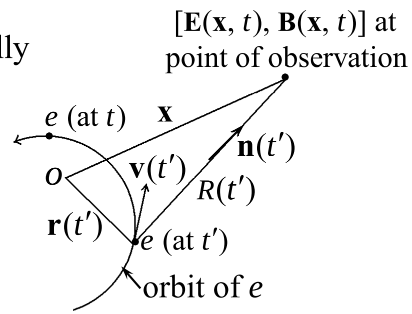
Rewrite

$$\left\{\begin{array}{ll}\mathbf{E}(\mathbf{x},t) = e\underbrace{\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R^2}\right]_{ret}}_{\text{velocity field } (\propto\frac{1}{R^2})} + \frac{e}{c}\underbrace{\left[\frac{\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]_{ret}}_{\substack{\text{acceleration field}\\ (\propto\frac{\dot{\beta}}{R} \text{ and } \perp\mathbf{n})}} & (14.14) \\ \mathbf{B}(\mathbf{x},t) = \mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t) & (14.13)\end{array}\right.$$

Discussion:

(i) The velocity fields are essentially static fields falling off as $1/R^2$.

(ii) For the acceleration fields, (14.13) and (14.14) show that $\mathbf{E}(\mathbf{x},t)$, $\mathbf{B}(\mathbf{x},t)$, and $\mathbf{n}(t')$ are mutually orthogonal, as is typical of radiation fields.

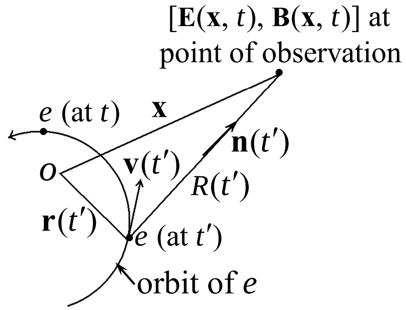


Note: (i) Unit vector $\mathbf{n}(t')$ points from the *retarded* position to $\mathbf{x}$.

(ii) t and t' are quantities in the same reference frame.

Rewrite
$$\left\{\begin{aligned}&\mathbf{E}(\mathbf{x},t)=e\underbrace{\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret}}_{\text{velocity field }(\propto\frac{1}{R^2})}+\frac{e}{c}\underbrace{\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\right]_{ret}}_{\substack{\text{acceleration field}\\(\propto\frac{\dot\beta}{R}\text{ and }\perp\mathbf{n})}} &&(14.14)\\&\mathbf{B}(\mathbf{x},t)=\mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t)&&(14.13)\end{aligned}\right.$$

(iii) **E** and **B** in general have a broad frequency spectrum. Since we have derived (14.13) and (14.14) from (9.2), which applies to a non-dispersive medium (in this case, the vacuum), singals at all frequencies travel at speed c. Hence, **E** and **B** at t depend only on the *instantaneous* motion of the point charge at a *single* retarded position $\mathbf{r}(t')$.

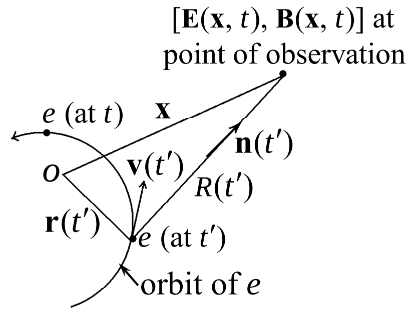


Rewrite
$$\left\{\begin{aligned}&\mathbf{E}(\mathbf{x},t)=e\underbrace{\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret}}_{\text{velocity field }(\propto\frac{1}{R^2})}+\frac{e}{c}\underbrace{\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\right]_{ret}}_{\substack{\text{acceleration field}\\(\propto\frac{\dot\beta}{R}\text{ and }\perp\mathbf{n})}} &&(14.14)\\&\mathbf{B}(\mathbf{x},t)=\mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t)&&(14.13)\end{aligned}\right.$$

(iv) Quantities in the brackets are to be evaluated at the retarded time t', which is the solution of

$$t'+|\mathbf{x}-\mathbf{r}(t')|/c=t,$$

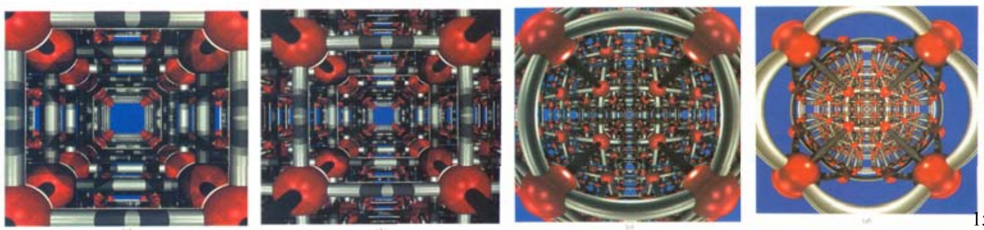
where the orbit $\mathbf{r}(t')$ is a specified function of t'. Thus, t' depends on $\mathbf{x}$ and t. This makes the final expression for $\mathbf{E}$ a function of $\mathbf{x}$ and t, as shown on the LHS of (14.14). For the same reason, the unit vector $\mathbf{n}(t')$ in (14.13), hence the final expression for $\mathbf{B}$, also depends on $\mathbf{x}$ and t [see (14.17a) below].



(v) The relation between observer's time and the retarded time, $t'=t-|\mathbf{x}-\mathbf{r}(t')|/c$, indicates that a signal from the charge travels at speed c toward the observer, independent of the motion of the charge (Einstein's postulate 2).

An Illustration of Time Retardation and Length Contraction: Computer generated graphics show the visual appearance of a three-dimensional lattice of rods and balls moving toward you at various speeds. (from Benson, "University Physics")

The normal view at rest	At 0.5c, the rods appear straight.	At 0.95c, the rods appear bent.	At 0.99c, the lattice appears severely distorted.



Charge in Uniform Motion : $\mathbf{v}=const.$

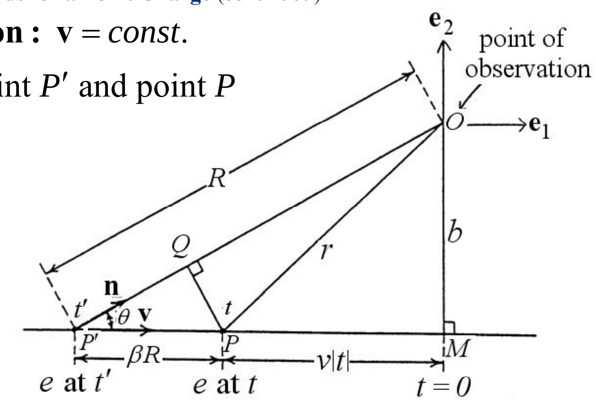
$P'P=$ distance between point P' and point P

$$=v\frac{R}{c}=\beta R$$

$$P'Q=\beta R\cos\theta=\boldsymbol{\beta}\cdot\mathbf{n}R$$

$$OQ=R-P'Q=R(1-\boldsymbol{\beta}\cdot\mathbf{n})$$

$$(OQ)^2=[R(1-\boldsymbol{\beta}\cdot\mathbf{n})]^2$$
$$=r^2-(PQ)^2$$
$$=\overbrace{r^2}^{b^2+v^2t^2}-\beta^2\overbrace{R^2\sin^2\theta}^{b^2}$$
$$=b^2+v^2t^2-\beta^2b^2=\frac{1}{\gamma^2}(b^2+\gamma^2v^2t^2)$$



In the above expressions, R and $\mathbf{n}$ are retarded quantities ($\boldsymbol{\beta}$ is a constant). Hence, $[R(1-\boldsymbol{\beta}\cdot\mathbf{n})]_{ret}=\frac{1}{\gamma}(b^2+\gamma^2v^2t^2)^{1/2}$

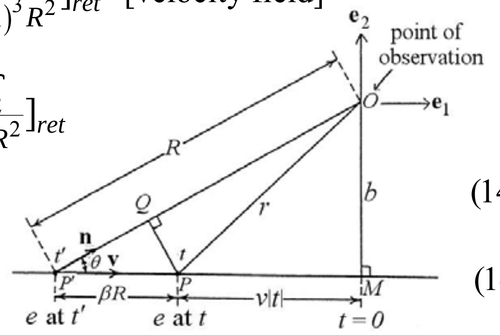
$\mathbf{v}=const. \Rightarrow \mathbf{E}=e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret}$ [velocity field]

$$\Rightarrow E_2=\mathbf{E}\cdot\mathbf{e}_2=e\left[\frac{\overbrace{\mathbf{n}\cdot\mathbf{e}_2}^{b/R}-\overbrace{\boldsymbol{\beta}\cdot\mathbf{e}_2}^{0}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret}$$

$$=e\left[\frac{b}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^3}\right]_{ret} \quad (14.17b)$$

$$=\frac{e\gamma b}{\left(b^2+\gamma^2v^2t^2\right)^{3/2}} \quad (14.17a)$$

[same as (11.152)]



$\boxed{[R(1-\boldsymbol{\beta}\cdot\mathbf{n})]_{ret}=\frac{1}{\gamma}(b^2+\gamma^2v^2t^2)^{1/2}\text{, last page}}$

$$E_1=\mathbf{E}\cdot\mathbf{e}_1=e\left[\frac{\overbrace{\mathbf{n}\cdot\mathbf{e}_1}^{\cos\theta}-\overbrace{\boldsymbol{\beta}\cdot\mathbf{e}_1}^{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret}=e\left[\frac{\cos\theta-\beta}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret}=\frac{e\gamma v|t|}{(b^2+\gamma^2v^2t^2)^{3/2}}$$

$E_3=0$ by symmetry.

$\cos\theta-\beta=\frac{\beta R+v|t|}{R}-\beta=\frac{v|t|}{R}$.
$t<0$ on the left side of the origin ($t=0$).

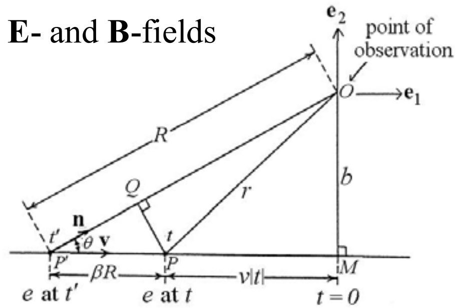
$$\mathbf{B}=\mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t)=\left(\cos\theta\mathbf{e}_1+\sin\theta\mathbf{e}_2\right)\times\left(E_1\mathbf{e}_1+E_2\mathbf{e}_2\right)$$
$$=\left(E_2\cos\theta-E_1\sin\theta\right)\mathbf{e}_3$$

So, the only nonvanishing component of $\mathbf{B}$ is B_3

$$B_3=E_2\underbrace{\cos\theta}_{\frac{\beta R+v|t|}{R}}-E_1\underbrace{\sin\theta}_{\frac{b}{R}}=\frac{e\gamma}{\left(b^2+\gamma^2v^2t^2\right)^{3/2}}\left[\tfrac{b}{R}(\beta R+v|t|)-v|t|\tfrac{b}{R}\right]=\beta E_2$$

Discussion: (i) Rewrite the **E**- and **B**-fields

$$\begin{cases} E_1=\dfrac{e\gamma v|t|}{\left(b^2+\gamma^2v^2t^2\right)^{3/2}} \\ E_2=\dfrac{e\gamma b}{\left(b^2+\gamma^2v^2t^2\right)^{3/2}} \\ B_3=\beta E_2 \end{cases}$$

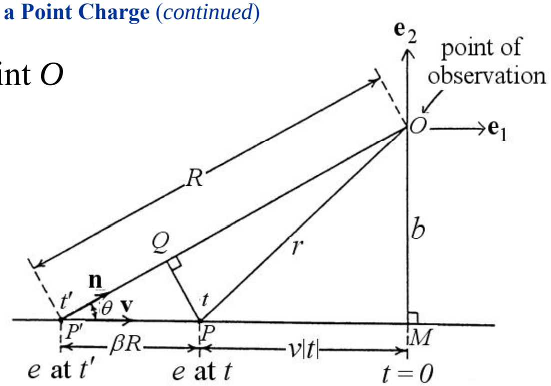


As expected, the final expressions for **E** and **B** are functions of the observer's position ($\mathbf{x}=b\mathbf{e}_2$) and time (t), although the fields are generated by the charge at the retarded position (P') and time (t').

(ii) Rewrite the **E**-field at point O

$$\begin{cases} E_1=\dfrac{e\gamma v|t|}{\left(b^2+\gamma^2v^2t^2\right)^{3/2}} \\ E_2=\dfrac{e\gamma b}{\left(b^2+\gamma^2v^2t^2\right)^{3/2}} \end{cases}$$

$\Rightarrow \frac{E_1}{E_2}=\frac{v|t|}{b}$

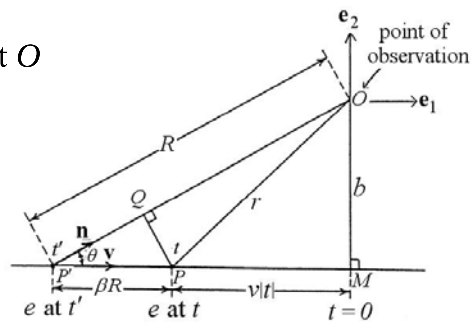


$\Rightarrow$ If $e>0$, **E** is directed from the charge's *present* position P (i.e. position at the time of observation) to the observation point O, although **E** is generated by the charge at the retarded position P'.

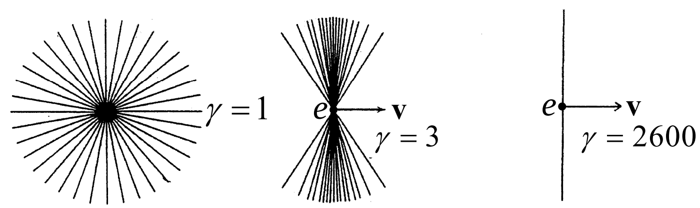
$\Rightarrow$ Since b and t can be given arbitrary (positive or negative) values, this direction relation applies to all observation points around the charge. Thus, **E**-field lines around the charge are straight lines emanating from (or, if $e<0$, converging to) the present position P.

(iii) Rewrite the **E**-field at point O

$$\begin{cases} E_1=\dfrac{e\gamma v|t|}{\left(b^2+\gamma^2v^2t^2\right)^{3/2}} \\ E_2=\dfrac{e\gamma b}{\left(b^2+\gamma^2v^2t^2\right)^{3/2}} \end{cases}$$

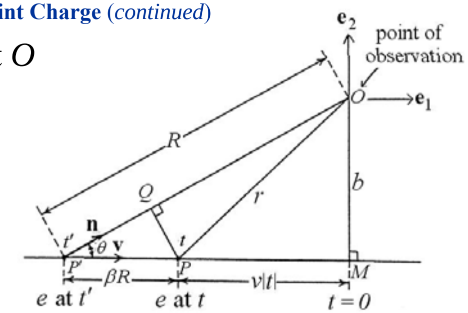


E-field lines $\rightarrow$



(iv) Rewrite the **E**-field at point O

$$\begin{cases} E_1 = \dfrac{e\gamma v|t|}{\left(b^2+\gamma^2 v^2 t^2\right)^{3/2}} \\ E_2 = \dfrac{e\gamma b}{\left(b^2+\gamma^2 v^2 t^2\right)^{3/2}} \end{cases}$$



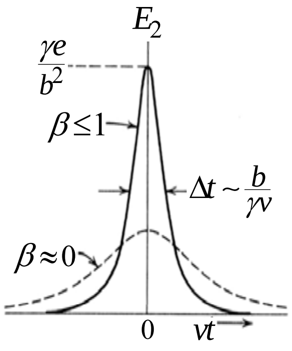
E_2 has a maximum value at $t=0$, when e passes through point M.

$$E_2^{\max} = E_2(t=0) = \frac{\gamma e}{b^2}$$

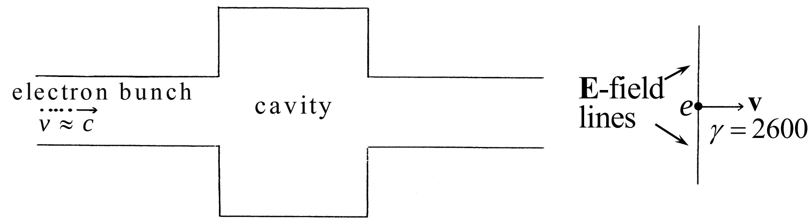
E_2 is down to $\frac{1}{2\sqrt{2}} E_2^{\max}$ at $t = \frac{b}{\gamma v}$.

$$\frac{E_2(t=\frac{b}{\gamma v})}{E_2^{\max}} = \frac{1}{2\sqrt{2}}$$ same as (11.153)

$\Rightarrow$ Duration of appeciable E_2: $\overbrace{\Delta t \approx \frac{b}{\gamma v}}$

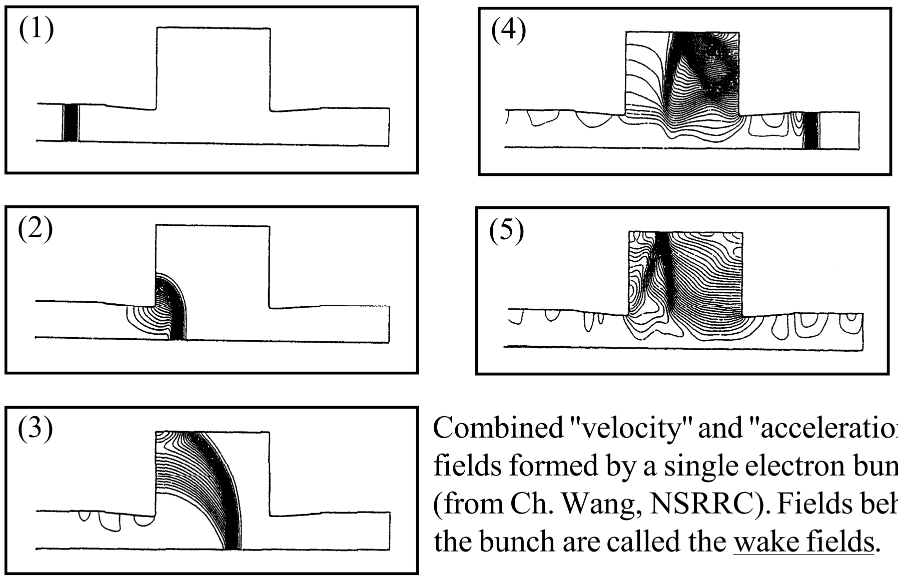


Electrodynamics in a Cavity : As shown in the figure, an electron bunch moving uniformly on the axis with $\gamma = 2600$ is about to enter a cavity. Since $E_\perp = (2600)^3 E_\parallel$, the **E**-field lines of every electron are concentrated in a flat disk with the electron at the center (velocity field). As a result, the electrons hardly "see" each other, because the (axial) electric forces between these electrons are negligible*. Then, as the bunch enters the cavity, the acceleration fields emerge (next page).



**Question*: The negligible electric force betwen any 2 electrons implies that the axial acceleration of either electron is negligible. However, the acceleration will be non-negligible when it is viewed in the lab frame. Why? [See lecture notes, Ch. 11, Eq. (A.23).]

Fields in the cavity produced by a $\gamma = 2600$ electron bunch



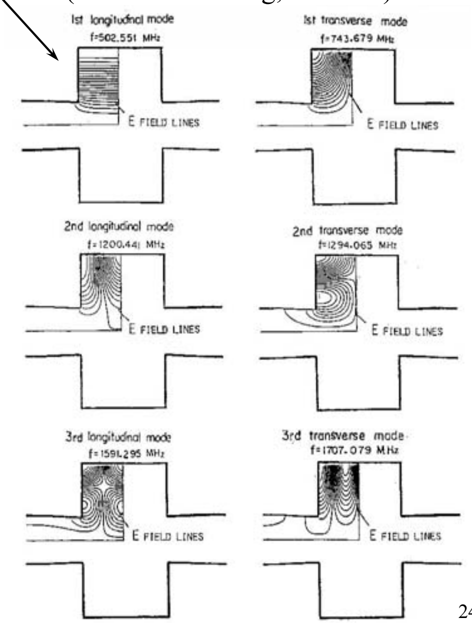
Combined "velocity" and "acceleration" fields formed by a single electron bunch (from Ch. Wang, NSRRC). Fields behind the bunch are called the wake fields.

Question: How do the electrons get decelerated in the cavity?

The lowest order (TM_{010}) mode is excited by the injection of high power microwaves from a klystron. The axial electric field of this mode is used to accelerate the electrons.

Wake fields left in the cavity by the electron bunch can be viewed as the superposition of the complete set of cavity eigenmodes. One or more of the higher-order modes may thus be resonantly reinforced by a succession of electron bunches to grow to significant amplitude and interfere with the acceleration process.

E-field lines of several cavity modes (from L. H. Chang, NSRRC)



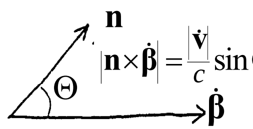
14.2 Total Power Radiated by an Accelerated Charge

Rewrite (14.14): $\mathbf{E}(\mathbf{x},t)=\underbrace{e\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R^2}\right]_{ret}}_{\text{velocity field}}+\underbrace{\frac{e}{c}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\right]_{ret}}_{\text{acceleration field}}$

$$\mathbf{S}(\mathbf{x},t)=\frac{c}{4\pi}\mathbf{E}\times\mathbf{B}=\frac{c}{4\pi}\mathbf{E}(\mathbf{x},t)\times[\mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t)]=\frac{c}{4\pi}|\mathbf{E}(\mathbf{x},t)|^2\,\mathbf{n}(t')$$

Larmor's Formula: Neglect the velocity field and take the limit $\beta\to 0$ ($\Rightarrow$ retarded γ, R, $\boldsymbol{\beta}$, $\mathbf{n}\approx$ present γ, R, $\boldsymbol{\beta}$, $\mathbf{n}$). Then,

$$\lim_{\beta\to 0}\mathbf{E}(\mathbf{x},t)\approx\frac{e}{cR}\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})$$



$$|\mathbf{n}\times\dot{\boldsymbol{\beta}}|=\frac{|\dot{\mathbf{v}}|}{c}\sin\Theta$$

$$\Rightarrow\lim_{\beta\to 0}\mathbf{S}\cdot\mathbf{n}=\frac{e^2}{4\pi cR^2}|\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})|^2$$

$$\Rightarrow\lim_{\beta\to 0}\frac{dP}{d\Omega}=\frac{e^2}{4\pi c}|\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})|^2=\frac{e^2}{4\pi c}|\mathbf{n}\times\dot{\boldsymbol{\beta}}|^2\qquad(14.20)$$

$$=\frac{e^2}{4\pi c^3}|\dot{\mathbf{v}}|^2\sin^2\Theta\quad\left[\frac{\text{power radiated}}{\text{unit solid angle}},\ \text{peak at }\Theta=\frac{\pi}{2}\right]\qquad(14.21)$$

$$\Rightarrow\lim_{\beta\to 0}P=\int\frac{dP}{d\Omega}d\Omega=\frac{2e^2}{3c^3}|\dot{\mathbf{v}}|^2=\frac{2e^2}{3m^2c^3}\left|\frac{d\mathbf{p}}{dt}\right|^2\quad\left[\begin{array}{l}\text{Larmor's}\\\text{formula}\end{array}\right]\qquad(14.23)$$

Note that all quantities in Secs. 14.1-14.4 are real. Hence,

$$\left|\frac{d\mathbf{p}}{dt}\right|^2=\frac{d\mathbf{p}}{dt}\cdot\frac{d\mathbf{p}}{dt}.\quad\left[\text{In Jackson, this is denoted by }\left(\frac{d\mathbf{p}}{dt}\right)^2\right]$$

Relativistic Generalization: The expression in (14.23) can be generalized to a relativistic form in which P is a Lorentz invariant and applicable to all electron energies. The procedure is as follows.

$$\left\{\begin{array}{l}\mathbf{p}\to\mathbf{P}=(\mathbf{p},\ \frac{iE}{c})\ \text{(4-vector)}\\ t\to\tau\ \text{(Lorentz scalar)}\end{array}\right.\Rightarrow\frac{d\mathbf{p}}{dt}\to\frac{d\mathbf{P}}{d\tau}\Rightarrow P=\frac{2e^2}{3m^2c^3}\left|\frac{d\mathbf{P}}{d\tau}\right|^2\qquad(14.24)$$

In terms of $\mathbf{p}$ and E: $P=\frac{2e^2}{3m^2c^3}\left[\left|\frac{d\mathbf{p}}{d\tau}\right|^2-\frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2\right]$ (14.25)

Convert to lab time by $d\tau=\frac{dt}{\gamma}$: $P=\frac{2e^2}{3m^2c^3}\gamma^2\left[\left|\frac{d\mathbf{p}}{dt}\right|^2-\frac{1}{c^2}\left(\frac{dE}{dt}\right)^2\right]$ (10)

(10) agrees with results derived directly from (14.14) (See Sec. 14.3).

$P=\frac{2e^2}{3m^2c^3}\gamma^2\left[\left|\frac{d\mathbf{p}}{dt}\right|^2-\frac{1}{c^2}\left(\frac{dE}{dt}\right)^2\right]$ in (10) can be put in different forms:

$$\gamma=\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\Rightarrow\gamma^2=\left(1-\frac{v^2}{c^2}\right)^{-1}=\left(1-\frac{p^2}{\gamma^2m^2c^2}\right)^{-1}$$

$$\Rightarrow\gamma^2=1+\frac{p^2}{m^2c^2}\Rightarrow\gamma=\left(1+\frac{p^2}{m^2c^2}\right)^{\frac{1}{2}}$$

$$\frac{dE}{dt}=mc^2\frac{d}{dt}\gamma=mc^2\frac{d}{dt}\left(1+\frac{p^2}{m^2c^2}\right)^{\frac{1}{2}}$$

$$=mc^2\frac{2\frac{p}{m^2c^2}\frac{d}{dt}p}{2\left(1+\frac{p^2}{m^2c^2}\right)^{\frac{1}{2}}}=\frac{p}{\gamma m}\frac{dp}{dt}=v\frac{dp}{dt}$$

Sub. $v\frac{dp}{dt}$ for $\frac{dE}{dt}$ in (10)

$$\Rightarrow P=\frac{2e^2}{3m^2c^3}\gamma^2\left[\left|\frac{d\mathbf{p}}{dt}\right|^2-\beta^2\left(\frac{dp}{dt}\right)^2\right]\qquad(11)$$

Note: $\frac{d\mathbf{p}}{dt}$ expresses both direction and amplitude variations of $\mathbf{p}$, but $\frac{dp}{dt}$ only expresses the amplitude variation of $\mathbf{p}$.

$$\left|\frac{d\mathbf{p}}{dt}\right|^2-\frac{1}{c^2}\left(\frac{dE}{dt}\right)^2$$

$$=m^2c^2\left|\boldsymbol{\beta}\frac{d\gamma}{dt}+\gamma\dot{\boldsymbol{\beta}}\right|^2-m^2c^2\left(\frac{d\gamma}{dt}\right)^2$$

$$\frac{d\gamma}{dt}=\frac{d}{dt}(1-\boldsymbol{\beta}\cdot\boldsymbol{\beta})^{-\frac{1}{2}}=-\frac{1}{2}\frac{-\dot{\boldsymbol{\beta}}\cdot\boldsymbol{\beta}-\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}}{(1-\boldsymbol{\beta}\cdot\boldsymbol{\beta})^{\frac{3}{2}}}=\gamma^3\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}$$

$$=m^2c^2\left[\beta^2\left(\frac{d\gamma}{dt}\right)^2+2\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}\gamma\frac{d\gamma}{dt}+\gamma^2|\dot{\boldsymbol{\beta}}|^2-\left(\frac{d\gamma}{dt}\right)^2\right]$$

$$=m^2c^2\left[-\frac{1}{\gamma^2}\gamma^6(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2+2\gamma^4(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2+\gamma^2|\dot{\boldsymbol{\beta}}|^2\right]$$

$$=\gamma^4m^2c^2\left[(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2+(1-\beta^2)|\dot{\boldsymbol{\beta}}|^2\right]=\gamma^4m^2c^2\left[|\dot{\boldsymbol{\beta}}|^2+(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2-\beta^2\dot{\beta}^2\right]$$

$$=\gamma^4m^2c^2\left[|\dot{\boldsymbol{\beta}}|^2-|\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}|^2\right]\qquad(12)$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})\Rightarrow|\mathbf{A}\times\mathbf{B}|^2=A^2B^2-|\mathbf{A}\cdot\mathbf{B}|^2$$

Sub. (12) into (10)

$$\Rightarrow P=\frac{2e^2}{3c}\gamma^6\left[|\dot{\boldsymbol{\beta}}|^2-|\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}|^2\right]\qquad(14.26)$$

Example 1: Linear accelerator (**p** || accelerating force **F**)

Rewrite (11): $P = \frac{2e^2}{3m^2c^3}\gamma^2\left[\left|\frac{d\mathbf{p}}{dt}\right|^2 - \beta^2\left(\frac{dp}{dt}\right)^2\right]$

For linear acceleration, $\left|\frac{d\mathbf{p}}{dt}\right|^2 = \left(\frac{dp}{dt}\right)^2 \Rightarrow \quad P = \frac{2e^2}{3m^2c^3}\left(\frac{dp}{dt}\right)^2$ (14.27)

$$\Rightarrow P = \frac{2e^2}{3m^2c^3}\left(\frac{dE}{dx}\right)^2 \longleftarrow \boxed{\left.\begin{matrix} dp = Fdt \\ dE = Fdx \end{matrix}\right\} \Rightarrow \frac{dp}{dt} = \frac{dE}{dx} = F} \qquad (14.28)$$

$$\frac{P}{\left(\frac{dE}{dt}\right)} = \frac{\frac{2e^2}{3m^2c^3}\frac{dE}{dx}\frac{1}{v}\frac{\cancel{dE}}{\cancel{dt}}}{\frac{\cancel{dE}}{\cancel{dt}}} = \frac{2e^2}{3m^2c^3}\frac{1}{v}\frac{dE}{dx} \underset{v\approx c}{\approx} \underbrace{\frac{2\left(\frac{e^2}{mc^2}\right)}{3mc^2}}_{3.7\times10^{-15}\text{ m/MeV}}\frac{dE}{dx}$$

$\boxed{P\text{: radiated power. } \frac{dE}{dt}\text{: externally supplied power}}$

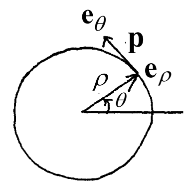
Typically, $\frac{dE}{dx} < 50$ MeV/m $\Rightarrow$ [Radiation losses are completely negligible in linear accelerators.]

Example 2: Circular accelerator (e.g. synchrotron)

Rewrite (11): $P = \frac{2e^2}{3m^2c^3}\gamma^2\left[\left|\frac{d\mathbf{p}}{dt}\right|^2 - \beta^2\left(\frac{dp}{dt}\right)^2\right]$

For circular accelerators, $\left|\frac{d\mathbf{p}}{dt}\right| \gg \frac{dp}{dt}$. Thus,

$$P \approx \frac{2e^2}{3m^2c^3}\gamma^2\left|\frac{d\mathbf{p}}{dt}\right|^2$$



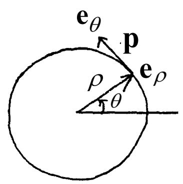
$$\frac{d\mathbf{p}}{dt} = \frac{d(p\mathbf{e}_\theta)}{dt} = p\frac{d}{dt}\mathbf{e}_\theta + \mathbf{e}_\theta\underbrace{\frac{dp}{dt}}_{\text{negligible}} \approx p\underbrace{\frac{d\mathbf{e}_\theta}{d\theta}}_{-\mathbf{e}_\rho}\underbrace{\frac{d\theta}{dt}}_{\omega} = -\omega p\mathbf{e}_\rho$$

$\Rightarrow \left|\frac{d\mathbf{p}}{dt}\right| \approx \omega p.$ $\boxed{\omega = \frac{v}{\rho},\ p = \gamma mv}$

$$\Rightarrow P \approx \frac{2e^2}{3m^2c^3}\gamma^2\omega^2p^2 \downarrow= \frac{2e^2c}{3\rho^2}\beta^4\gamma^4 \qquad (14.31)$$

Note that (14.31) is an exact expression for P if the particle is in uniform circular motion, i.e. if $\frac{dp}{dt} = 0$.

Rewrite (14.31): $P \approx \frac{2e^2c}{3\rho^2}\beta^4\gamma^4$



$\Rightarrow \delta E$ = radiation loss per revolution

$$= \frac{2\pi\rho}{v}P$$

$$\approx \frac{4\pi}{3}\frac{e^2}{\rho}\beta^3\gamma^4 \overset{\beta\approx1}{\approx} 8.85\times10^{-2}\frac{[E(\text{in GeV})]^4}{\rho(\text{in meters})}\text{ MeV}$$

$$\approx \begin{cases} 1\text{ keV}, & \text{for early synchrotrons (accelerators)} \\ 72\text{ keV}, & \text{for the 1.3 GeV NSRRC synchrontron} \\ 8.85\text{ MeV}, & \text{for the 10 GeV Cornell synchrontron} \end{cases} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{storage rings}$$

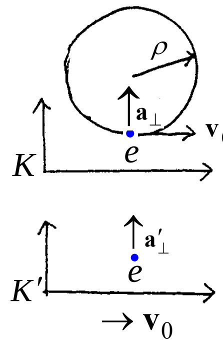
Total power radiated in circular electron accelerators:

$$P(\text{in watts}) = 10^6\times\delta E(\text{in MeV})\times J(\text{in amp})$$

Problem: If a charge is in uniform circular motion, (14.31) is an exact expression for the total power it radiates. Show that the total power has the same value as viewed in the rest frame of the particle.

Solution: Consider an instantaneous position of the charge located at the bottom of its orbit, where the charge moves horizontally to the right at velocity $\mathbf{v}_0$ (upper figure) and the acceleration $\mathbf{a}_\perp$ points vertically upward with $a_\perp = v_0^2/\rho$ (ρ is the radius of the circle). Viewed in the rest frame of the charge (lower figure), we have [see Eq. (A.22) in Ch. 11 of lecture notes]

$$\begin{cases} \mathbf{a}'_{\|} = \frac{1}{\gamma_0^3(1-\frac{\mathbf{v}_0\cdot\mathbf{v}}{c^2})^3}\underbrace{\mathbf{a}_{\|}}_{0} \\ \mathbf{a}'_\perp = \frac{1}{\gamma_0^2(1-\frac{\mathbf{v}_0\cdot\overset{\mathbf{v}_0}{\overbrace{\mathbf{v}}}}{c^2})^3}\underbrace{[\mathbf{a}_\perp - \frac{\mathbf{v}_0}{c^2}\times(\overset{\mathbf{a}_\perp}{\overbrace{\mathbf{a}}}\times\overset{\mathbf{v}_0}{\overbrace{\mathbf{v}}})]}_{\mathbf{a}_\perp(1-v_0^2/c^2)} \end{cases}$$

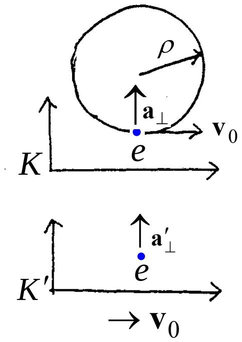


Thus, $\mathbf{a}'_{\|} = 0$ and $\mathbf{a}'_\perp = \gamma_0^2\mathbf{a}_\perp$.

Thus, the acceleration of the charge is vertically upward in both frames and they are related by

$$\mathbf{a}'_\perp = \gamma^2 \mathbf{a}_\perp$$

Since the charge is at rest in frame K', Larmor's formula in (14.23) becomes exact, which gives



$$P' = \frac{2e^2}{3c^3}|\dot{\mathbf{v}}'|^2 = \frac{2e^2}{3c^3}|\mathbf{a}'_\perp|^2 = \frac{2e^2}{3c^3}\gamma^4 a_\perp^2$$

$$= \frac{2e^2}{3c^3}\gamma^4 \frac{v_0^4}{\rho^2} = \frac{2e^2 c}{3\rho^2}\beta^4\gamma^4 \qquad \boxed{a_\perp = \frac{v_0^2}{\rho}}$$

This is the same power as viewed in the lab frame [see (14.31)]. The result here, $P = P'$, is consistent with the fact the total radiated power is a Lorentz invariant [see (14.24)]. However, the angular distribution of radiation will be different in the two frames. We will show later in (14.44) that for the same acceleration, the angular distribution depends sensitively on particle's velocity.

14.3 Angular Distribution of Radiation Emitted by an Accelerated Charge

Rewrite (14.14): $\mathbf{E}(\mathbf{x},t) = e\underbrace{\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R^2}\right]_{ret}}_{\text{velocity field}} + \frac{e}{c}\underbrace{\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3 R}\right]_{ret}}_{\text{acceleration field}}$

power per unit area at observation point

$$\mathbf{S}(\mathbf{x},t) = \frac{c}{4\pi}\mathbf{E}(\mathbf{x},t)\times\mathbf{B}(\mathbf{x},t)$$

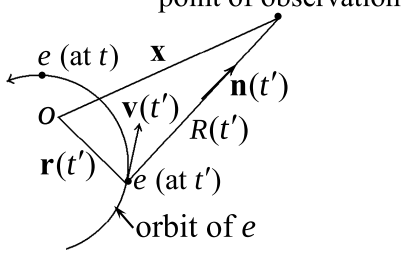
$$= \frac{c}{4\pi}\mathbf{E}(\mathbf{x},t)\times[\mathbf{n}(t')\times\mathbf{E}(\mathbf{x},t)]$$

$$= \frac{c}{4\pi}|\mathbf{E}(\mathbf{x},t)|^2\,\mathbf{n}(t')$$

$$\Rightarrow \mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}(t') = \frac{c}{4\pi}|\mathbf{E}(\mathbf{x},t)|^2$$

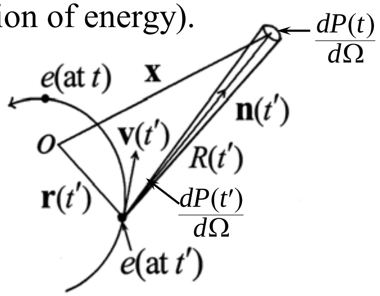
(Neglect the velocity field)

$$= \frac{e^2}{4\pi c}\left\{\frac{1}{R^2}\left|\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right|^2\right\}_{ret} \qquad (14.35)$$



In this section (as in Sec. 14.2), we are interested in the angular distribution of power *radiated* by the charge. But $\mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}(t') = \frac{c}{4\pi}|\mathbf{E}(\mathbf{x},t)|^2$ in (14.35) gives the power per unit area *received* at the observation point. Power radiated by the charge into a unit solid angle [$dP(t')/d\Omega$] is in general different from the power received over the area subtending the solid angle [$dP(t)/d\Omega$]. The reason is that motion of the charge toward (away from) the observation point will shorten (lengthen) the radiated pulse, which results in increased (decreased) power at the observation point because the total energy received must equal the total energy radiated (conservation of energy).

Thus, to express the power *radiated* in terms of the power *received*, we need to determine the ratio of dt (received pulse length) to dt' (radiated pulse length).



Observation time t and radiation time t' are related by

$$t = t' + \frac{R(t')}{c}. \qquad \boxed{\text{Use (4): } \frac{dR(t')}{dt'} = -\mathbf{v}(t')\cdot\mathbf{n}(t')}$$

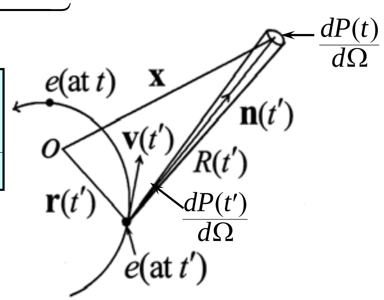
Thus, $\frac{dt}{dt'} = 1 + \frac{1}{c}\frac{dR(t')}{dt'} = 1 - \boldsymbol{\beta}(t')\cdot\mathbf{n}(t')$

$\Rightarrow$ A pulse of duration dt received at $\mathbf{x}$ and t is radiated by the charge at $\mathbf{r}(t')$ and t' for a duration of $dt' = dt/[1-\boldsymbol{\beta}(t')\cdot\mathbf{n}(t')]$. Note that dt and dt' are quantities in the same reference frame (lab frame).

$$\underbrace{R^2(t')\mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}(t')}_{dP(t)/d\Omega}\,dt = \underbrace{R^2(t')\mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}(t')\frac{dt}{dt'}}_{dP(t')/d\Omega}\,dt'$$

$\boxed{\frac{\text{power received at } \mathbf{x} \text{ and } t}{\text{unit solid angle}}}$ $\boxed{\frac{\text{power radiated by charge at } \mathbf{r}(t') \text{ and } t'}{\text{unit solid angle}}}$

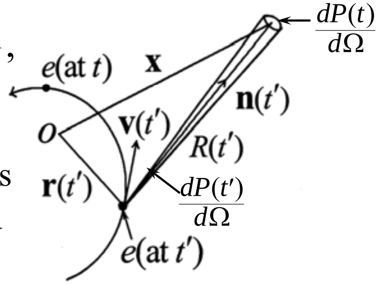
In both $\frac{dP(t')}{d\Omega}$ and $\frac{dP(t)}{d\Omega}$, $d\Omega$ is with respect to the charge.



Rewrite $\underbrace{R^2(t')\mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}(t')}_{dP(t)/d\Omega}dt = \underbrace{R^2(t')\mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}(t')\tfrac{dt}{dt'}}_{dP(t')/d\Omega}dt'$

$$\Rightarrow \frac{dP(t')}{d\Omega} = \frac{dP(t)}{d\Omega}\underbrace{\frac{dt}{dt'}}_{\boxed{=1-\boldsymbol{\beta}(t')\cdot\mathbf{n}(t')}} = R^2(t')\underbrace{\mathbf{S}(\mathbf{x},t)\cdot\mathbf{n}(t')}_{\boxed{=\frac{e^2}{4\pi c}\left\{\frac{1}{R^2}\left|\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right|^2\right\}_{ret}\text{ by (14.35)}}}[1-\boldsymbol{\beta}(t')\cdot\mathbf{n}(t')]$$

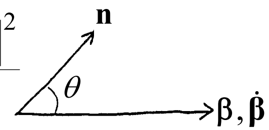
$$\Rightarrow \frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c}\frac{\left|\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]\right|^2}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^5}, \quad (14.38)$$



where $\mathbf{n}$, $\boldsymbol{\beta}$, $\dot{\boldsymbol{\beta}}$ are to be evaluated at the retarded time t'. (14.38) gives the power radiated into a unit solid angle in the direction of $\mathbf{n}$ in terms of the charge e and instantaneous $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ of the particle.

Case 1: $\boldsymbol{\beta} \parallel \dot{\boldsymbol{\beta}}$

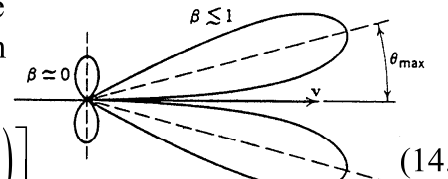
Rewrite (14.38): $\dfrac{dP(t')}{d\Omega} = \dfrac{e^2}{4\pi c}\dfrac{\left|\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]\right|^2}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^5}$



$$\left.\begin{cases}\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}=0\\ \left|\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})\right|^2=|\dot{\boldsymbol{\beta}}|^2\sin^2\theta\end{cases}\right\}\Rightarrow \frac{dP(t')}{d\Omega} = \frac{e^2\dot{v}^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \quad (14.39)$$

$$\Rightarrow P(t') = \int\frac{dP(t')}{d\Omega}d\Omega = \frac{2}{3}\frac{e^2}{c^3}\dot{v}^2\gamma^6 \quad \left[\begin{matrix}\text{agree with (14.26)}\\ \text{and (14.27)}\end{matrix}\right] \quad (14.43)$$

For $\beta \ll 1$, (14.39) reduces to Larmor's result (14.21), with the radiation peaking at $\theta = 90^\circ$. But as $\beta \to 1$, the angular distribution is tipped forward more and more and increases in magnitude, with the maximum intensity at



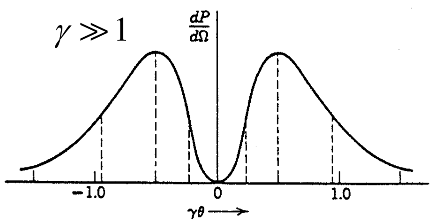
$$\theta_{\max} = \cos^{-1}\left[\frac{1}{3\beta}\left(\sqrt{1+15\beta^2}-1\right)\right] \quad (14.40)$$

As $\beta \to 1$, we have $\theta \ll 1$. Hence,

$$1-\beta\cos\theta \approx 1-\beta\left(1-\tfrac{1}{2}\theta^2\right)$$

$$= 1-\beta+\frac{\overbrace{\beta}^{\approx 1}}{2}\theta^2 \approx \frac{(1-\beta)(1+\overbrace{\beta}^{\approx 1})}{2}+\frac{\theta^2}{2}$$

$$= \frac{1-\beta^2}{2}+\frac{\theta^2}{2} = \frac{1}{2\gamma^2}\left(1+\gamma^2\theta^2\right)$$

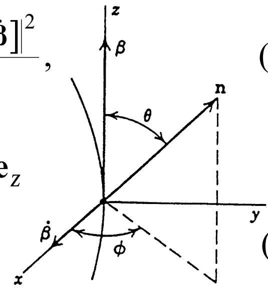


$$\Rightarrow \lim_{\beta\to 1}\frac{dP(t')}{d\Omega} = \frac{e^2\dot{v}^2}{4\pi c^3}\frac{\overbrace{\sin^2\theta}^{\approx\theta^2}}{(1-\beta\cos\theta)^5} \approx \frac{8}{\pi}\frac{e^2\dot{v}^2}{c^3}\gamma^8\frac{(\gamma\theta)^2}{(1+\gamma^2\theta^2)^5} \quad (14.41)$$

$$\Rightarrow \begin{cases}\theta_{\max} = \dfrac{1}{2\gamma} \text{ [angle of maximum intensity]} & (14.40)\\ <\theta^2>^{\frac{1}{2}} = \left[\dfrac{\int\theta^2\frac{dP(t')}{d\Omega}d\Omega}{\int\frac{dP(t')}{d\Omega}d\Omega}\right]^{\frac{1}{2}} = \dfrac{1}{\gamma} = \dfrac{mc^2}{E}\ \left[\begin{matrix}\text{root mean}\\ \text{square angle}\end{matrix}\right] & (14.42)\end{cases}$$

Case 2: $\boldsymbol{\beta} \perp \dot{\boldsymbol{\beta}}$. In $\dfrac{dP(t')}{d\Omega} = \dfrac{e^2}{4\pi c}\dfrac{\left|\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]\right|^2}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^5}$, (14.38)

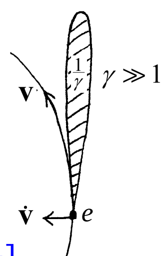
let $\begin{cases}\boldsymbol{\beta}\parallel\mathbf{e}_z,\ \dot{\boldsymbol{\beta}}\parallel\mathbf{e}_x\\ \mathbf{n} = \sin\theta\cos\phi\,\mathbf{e}_x+\sin\theta\sin\phi\,\mathbf{e}_y+\cos\theta\,\mathbf{e}_z\end{cases}$



$$\Rightarrow \frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c^3}\frac{|\dot{\mathbf{v}}|^2}{(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right] \quad (14.44)$$

$$\Rightarrow P(t') = \int\frac{dP(t')}{d\Omega}d\Omega = \frac{2}{3}\frac{e^2}{c^3}\underbrace{|\dot{\mathbf{v}}|^2}_{\frac{v^2}{\rho}}\gamma^4 = \begin{cases}\dfrac{2}{3}\dfrac{e^2c}{\rho^2}\beta^4\gamma^4 \text{ [agree with (14.31)]}\\ \dfrac{2}{3}\dfrac{e^2}{m^2c^3}\gamma^2\left|\dfrac{d\mathbf{p}}{dt}\right|^2\end{cases} \quad (14.47)$$

$$\lim_{\beta\to 1}\frac{dP(t')}{d\Omega} = \frac{2e^2}{\pi c^3}\gamma^6\frac{|\dot{\mathbf{v}}|^2}{(1+\gamma^2\theta^2)^3}\left[1-\frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta^2)^2}\right] \quad (14.45)$$



$$\Rightarrow \begin{cases}\theta_{\max} = 0 \text{ [angle of maximum intensity]}\\ <\theta^2>^{\frac{1}{2}} = \dfrac{1}{\gamma} \text{ [}\Rightarrow\text{ narrow cone like a searchlight]}\end{cases}$$

14.4 Radiation Emitted by a Charge in Arbitrary, Extremely Relativistic Motion

In Secs. 14.2 and 14.3, we examined the radiation problem from the viewpoint of the charged particle and expresssed the radiated power in terms of the instantaneous $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ of the particle.

From here on, we will switch our viewpoint to the observer. The emphasis will also be switched from the power of radiation to the *frequency spectrum* of the signal received at the observervation point.

To find the spectrum, we need to first know the time history of the observed ratiation. Hence, we can no longer stick to instantaneous quantities as in Secs. 14.2 and 14.3. We must now follow the particle's orbit. As the particle travels along its orbit, it continuously radiates toward the observer. A Fourier transform of the time-dependent signal received then reveals its sprectral contents.

We will be interested only in perpendicular acceleration ($\dot{\boldsymbol{\beta}} \perp \boldsymbol{\beta}$). The reason is as follows.

Rewrite
$$\begin{cases} P(t') = \dfrac{2e^2}{3m^2c^3}\left(\dfrac{dp}{dt}\right)^2, \quad \text{for } \dot{\boldsymbol{\beta}} \parallel \boldsymbol{\beta} & (14.27) \\[2ex] P(t') = \dfrac{2}{3}\dfrac{e^2}{m^2c^3}\gamma^2\left|\dfrac{d\mathbf{p}}{dt}\right|^2, \quad \text{for } \dot{\boldsymbol{\beta}} \perp \boldsymbol{\beta} & (14.47) \end{cases}$$

which implies $P(\dot{\boldsymbol{\beta}} \perp \boldsymbol{\beta}) = \gamma^2 P(\dot{\boldsymbol{\beta}} \parallel \boldsymbol{\beta})$ for the same accelerating force.

Hence, for a charge with $\gamma \gg 1$ in arbitrary motion, we may neglect $P(t')$ due to $\dot{\boldsymbol{\beta}} \parallel \boldsymbol{\beta}$ and consider only $P(t')$ due to $\dot{\boldsymbol{\beta}} \perp \boldsymbol{\beta}$. The instantaneous radius of curvature ρ can be expressed in terms of the perpendicular component of the acceleration ($\dot{v}_\perp$) as follows.

$$F_\perp = \frac{\gamma m v^2}{\rho} = \gamma m \dot{v}_\perp$$

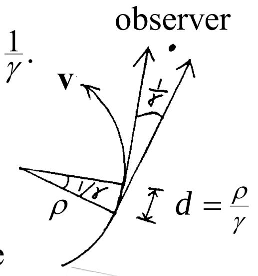
$$\Rightarrow \rho = \frac{v^2}{\dot{v}_\perp} \underset{\gamma \gg 1}{\approx} \frac{c^2}{\dot{v}_\perp} \qquad (14.48)$$

[For acceleration $\perp$ to $\mathbf{v}$, the effective mass is γm. See lecture notes, Ch. 11, Eq. (49).]

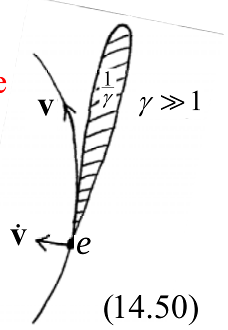
The Spectral Width for $\dot{\boldsymbol{\beta}} \perp \boldsymbol{\beta}$:

Angular distribution of radiation: $<\theta^2>^{\frac{1}{2}} \approx \frac{1}{\gamma}$.

$\Rightarrow$ The observer is illuminated by light emitted in an arc of length $d \approx \frac{\rho}{\gamma}$, corresponding to a (retarded time) interval of emission $\Delta t' \approx \frac{\rho}{\gamma v}$.



In the interval $\Delta t'$, the front edge of the pulse travels a distance $D = c\Delta t' = \frac{\rho}{\gamma\beta}$, while the rear edge of pulse is behind the front edge by a distance



$$L = D - d = \left(\tfrac{1}{\beta} - 1\right)\frac{\rho}{\gamma} = \frac{1-\beta}{\beta}\frac{\rho}{\gamma} \approx \frac{(1-\beta)(1+\beta)}{2\beta}\frac{\rho}{\gamma} \approx \frac{\rho}{2\gamma^3}$$

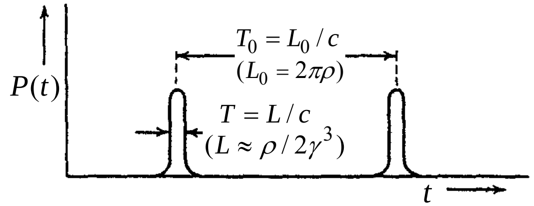
$\Rightarrow$ Pulse duration (to the observer): $T = L/c$

$\Rightarrow$ A broad spectrum ranging from near 0 up to a critical frequency of $\omega_c \sim \frac{1}{T} \sim \frac{c}{L} \sim \frac{c}{\rho}\gamma^3$, (14.50)

where ω_c is the maximum frequency of appreciable radiation.

Synchrotron Radiation-A Qualitative Discussion : If the charge is in circular motion with rotation frequency ω_0, then $\omega_0 \rho \approx c$ and

$$\omega_c \sim \frac{c}{\rho}\gamma^3 \approx \omega_0 \gamma^3$$



The pulses occur at the observation point at regular intervals of

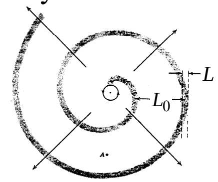
$$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi\rho}{v} \approx \frac{2\pi\rho}{c}$$

Eexample: Cornell 10 GeV synchrotron

$$\begin{cases} \gamma \approx 2 \times 10^4 \\ \omega_0 \approx 3 \times 10^6 / \text{sec} \end{cases}$$

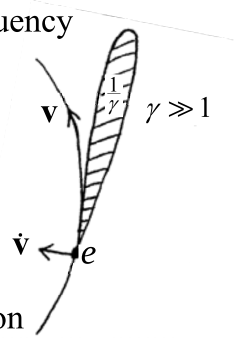
$\Rightarrow \omega_c \approx 2.4 \times 10^{19}$ / sec (16 keV x-rays)

Pulses of synchrotron radiation propagating radially outward

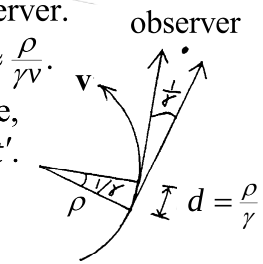


Discussion: In (14.50), $\omega_c \sim \frac{c}{\rho}\gamma^3$, the critical frequency ω_c (maximum frequency of appreciable radiation) scales as γ^3, which explains the extremely high frequency radiation from a synchrotron. The factor γ^3 is due to the short duration of the pluse seen by the observer. The pulse is shortened by two effects:

1. Because the angular width ($1/\gamma$) of the radiation is very narrow, only the radiation emitted by an electron over an arc of length d ($=\rho/\gamma$) can reach the observer. Thus, to the electron, the emission interval is $\Delta t' \approx \frac{\rho}{\gamma v}$.



2. The electron is "chasing" its radiation. Hence, to the observer, the received pulse length is not $\Delta t'$. Instead, it is $\Delta t'$ compressed by a factor of



$$\frac{dt}{dt'} = 1 - \boldsymbol{\beta}(t') \cdot \mathbf{n}(t') = 1 - \beta \approx \frac{(1-\beta)(1+\beta)}{2} = \frac{1}{2\gamma^2}$$

Effect 2 is exploited in a device called the free electron laser (FEL).

Example: As a practical example of the pulse duration to the observer, consider again the Cornell 10 GeV synchrotron, for which we have

$$\omega_c \approx 2.4 \times 10^{19} \text{ / sec}.$$

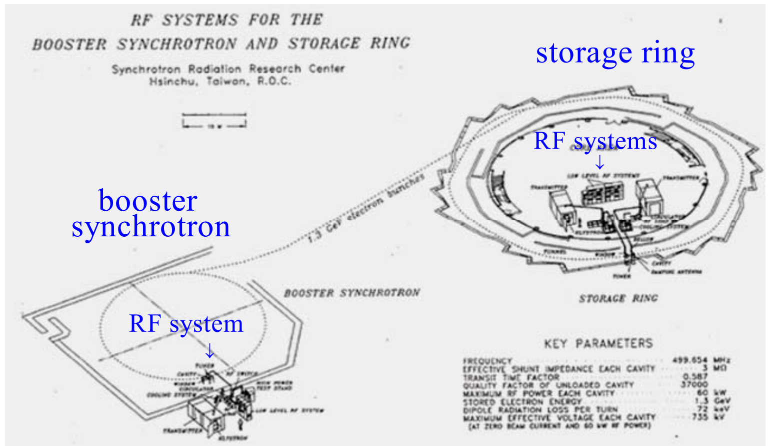
Since $\omega_c \sim \frac{1}{T}$, the pulse duration T of a single electron is incredibly short,

$$T \sim \frac{1}{\omega_c} \approx 4.2 \times 10^{-20} \text{ sec}.$$

This explains the broad spectrum. However, the actual pluse in a synchrotron does not come from a single electron, but from an electron bunch of finite length (typically a few mm). Electrons in the bunch radiate incoherently. So the spectrum of the bunch is the same as that of a single electron, but the pulse duration (τ) equals the passage time of the electron bunch ($\tau \approx$ bunch length/c). For example, for a bunch length of 6 mm, we have $\tau \approx 2 \times 10^{-11}$ sec.

The Synchrotron as a Light Source: The synchrotron emits *intense radiation* with a *very broad frequency spectrum* in a beam of *extremely small angular spread* ($1/\gamma$). It is a unique research tool and can also be used for micro-fabrication and other applications. The photo below shows the light source facility at the National Synchrotron Radiation Research Center (NSRRC) in Taiwan.

Electron bunches are first accelerated to an energy of 1.3 GeV in the booster synchrotron, and then sent to the storage ring (also a synchrotron), where the energy is maintaind at 1.3 GeV while the electrons provide synchrotron radiation to users around the ring. The electrons are powered by microwaves from the RF systems.



The RF system

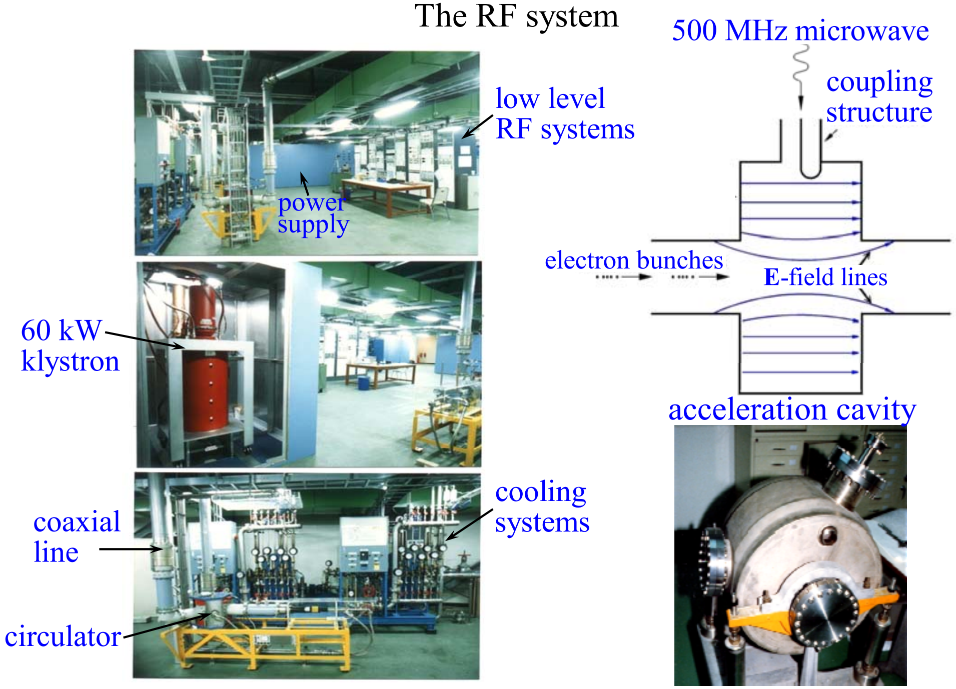
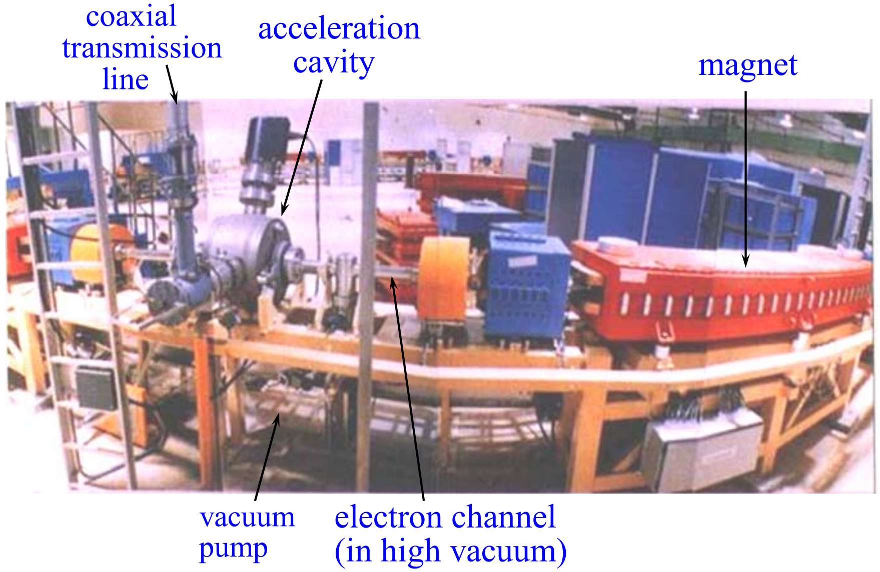


Photo of the NSRRC booster synchrotron showing some key components of the accelerator



Research stations around the NSRRC storage ring

14.7 Undulators and Wigglers for Synchrotron Light Sources

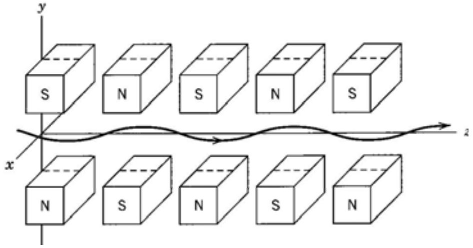
The broad spectrum of radiation emitted by relativistic electrons bent by the magnetic fields of synchrotron storage rings provides a useful source of energetic photons.

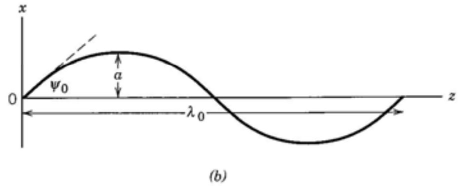
As application grew, the need for brighter sources with the radiation more concentrated in frequency led to the magnetic "insertion devices" called wigglers and undulators to be placed in the synchrotron ring.

The magnetic properties of these devices cause the electrons to undergo special motion that results in the concentration of the radiation into a much more monochromatic spectrum or series of seperated peaks.

Essential Idea of Undulators and Wigglers

The essential idea of undulators and wigglers is that a charge particle, usually an electron and usually moving relativistically ($\gamma \gg 1$), is caused to move transversely to its general forward motion by magnetic fields that alternate periodically.



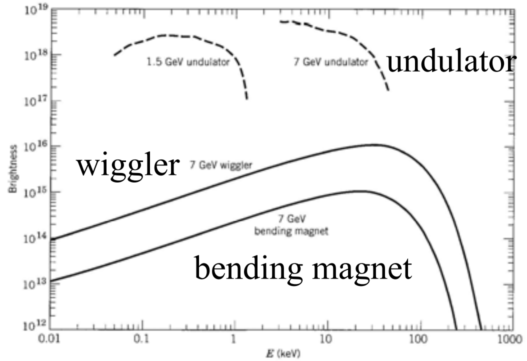


The external magnetic fields induce small transverse oscillations in the motion; the associated accelerations cause radiation to be emitted.

Classification of Undulators and Wigglers

(a) Wiggler ($\psi_0 \gg \Delta\theta$): An observer detectors a series of flicks of the searchlight beam. $\Delta\theta$: angular width of the radiation about the path.

(b) Undulator ($\psi_0 \ll \Delta\theta$): The searchlight beam of radiation moves *negligibly* compared to its own angular width. The radiation detected by the observer is an almost coherent superposition of the contributions from all the oscillations of the trajectory.



Homework of Chap. 14

Problems: 1, 4, 5, 9