1.3

Using Dirac delta functions in the appropriate coordinates, express the following charge distributions as three-dimensional charge densities ρ(**x**).

(a) In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R.

(b) In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius b.

(c) In cylindrical coordinates, a charge Q spread uniformly over a flat circular disc of negligible thickness and radius R.

(d) The same as part (c), but using spherical coordinates.

Solution :

(a)

$$\int \rho(r) d^3r = Q$$

$$\rho = a\delta(r-R)$$

$$d^3x = r^2 \sin\theta dr d\theta d\varphi$$

$$\int \rho(r) d^3r = 4\pi \int r^2 dr \rho(r) = 4\pi R^2 a = Q$$

$$\Rightarrow \rho = \frac{Q}{4\pi R^2}\delta(r-R)$$

(b)

$$\rho(r) = a\delta(r-b)$$

$$\lambda = \int \rho(r) r dr d\theta = 2\pi ba \Rightarrow \rho = \frac{\lambda}{2\pi b}\delta(r-b)$$

(c)

$$\rho(r) = a\delta(z)\theta(R-r)$$

where

$$\theta(x) = \begin{cases} 0, \text{if } x<0 \\ 1, \text{if } x>0 \end{cases}$$

$$\Rightarrow \rho(r) = \frac{Q}{\pi R^2}\delta(z)\theta(R-r)$$

(d) When ,

$$\rho(r) = c\delta(z)\theta(R-r) = c\delta(r\cos\theta)\theta(R-r) = \frac{c}{r}\delta(\cos\theta)\theta(R-r)$$

$$Q = \int \rho(r) r^2 \sin\theta dr d\theta d\varphi = 2\pi c \int_0^\infty \frac{1}{r}\theta(R-r) r^2 dr \int_\pi^0 \delta(\cos\theta) d(\cos\theta) = \pi c R^2$$

$$\Rightarrow c = \frac{Q}{\pi R^2} \Rightarrow \rho(r) = \frac{Q}{\pi R^2 r}\delta(\cos\theta)\theta(R-r)$$

1.4

Each of three charged spheres of radius a, one conducting, one having a uniform charge density within its volume, and one having a spherically symmetric charge density that varies radially as r^n (n>-3) , has a total charge Q. Use Gauss's theorem to obtain the electric fields both inside and outside each sphere. Sketch the behavior of the fields as a function of radius for the first two spheres, and for the third with n = -2 ,+2.

Solution :

1.

$$\int E\cdot da=\frac{Q}{\varepsilon_0}\Rightarrow E=\begin{cases}0,\ \text{if } r<a\\ \dfrac{Q}{4\pi\varepsilon_0 r^2},\ \text{if } r>a\end{cases}$$

2.

$$\rho(r)=\frac{Q}{\frac{4}{3}\pi a^3},\ r<a$$

$$\int E\cdot da=\int\frac{\rho(r)}{\varepsilon_0}r^2\sin\theta drd\theta d\varphi\Rightarrow 4\pi r^2E(r)=\frac{Q}{\frac{4}{3}\pi a^3}\frac{4\pi r^3}{3\varepsilon_0},\ \text{if } r<a$$

$$\Rightarrow E(r)=\begin{cases}\dfrac{Q}{4\pi\varepsilon_0}\dfrac{r}{a^3},\ \text{if } r<a\\ \dfrac{Q}{4\pi\varepsilon_0 r^2},\text{if } r>a\end{cases}$$

3.

$$\rho(r)=Cr^n$$

$$Q=\int_0^a Cr^n r^2 4\pi dr=4\pi C\frac{a^{n+3}}{n+3}$$

$$\Rightarrow C=\frac{(n+3)Q}{4\pi a^{n+3}}$$

$$\text{where } r<a,\ E(r)\frac{C}{\varepsilon_0 r^2}\frac{r^{n+3}}{n+3}=\frac{1}{4\pi\varepsilon_0 r^2}\int_0^r Cr'^n\left(4\pi r'^2\right)dr'=\frac{C}{\varepsilon_0 r^2}\frac{r^{n+3}}{n+3}=\frac{Q}{4\pi\varepsilon_0 r^2}\left(\frac{r}{a}\right)^{n+3}$$

$$\Rightarrow E(r)=\begin{cases}\dfrac{Q}{4\pi\varepsilon_0 ra},\ \text{when } n=-2\\ \dfrac{Qr^3}{4\pi\varepsilon_0 a^5},\ \text{when } n=2\end{cases}$$

1.5

The time-averaged potential of an neutral hydrogen atom is given by

$$\Phi=\frac{q}{4\pi\varepsilon_0}\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right)$$

Where q is the magnitude of the electronic charge, and , a_0 being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically.

Solution :

$$\rho=-\varepsilon_0\nabla^2\Phi=-\frac{q}{4\pi}\nabla^2\left(\frac{1}{r}e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)\right)=-\frac{q}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\left[e^{-\alpha r}\frac{1}{r}\left(1+\frac{\alpha r}{2}\right)\right]$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\left[\left(\frac{e^{-\alpha r}}{r}+\frac{\alpha e^{-\alpha r}}{2}\right)\right]=\frac{1}{r^2}\frac{\partial}{\partial r}\left[\left(r^2\frac{\partial}{\partial r}\frac{e^{-\alpha r}}{r}+r^2\frac{\partial}{\partial r}\frac{\alpha e^{-\alpha r}}{2}\right)\right]=\frac{1}{r^2}\frac{\partial}{\partial r}\left[\left(r^2e^{-\alpha r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)-\alpha re^{-\alpha r}-r^2\frac{\alpha^2e^{-\alpha r}}{2}\right)\right]$$

$$=\frac{1}{r^2}\left[\left(e^{-\alpha r}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right)+\left(\frac{\partial}{\partial r}\left(e^{-\alpha r}\right)\right)r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)-\left(\frac{\partial}{\partial r}(\alpha r)\right)e^{-\alpha r}-\alpha r\frac{\partial}{\partial r}e^{-\alpha r}-\left(\frac{\partial}{\partial r}r^2\right)\frac{\alpha^2e^{-\alpha r}}{2}-r^2\frac{\partial}{\partial r}\frac{\alpha^2e^{-\alpha r}}{2}\right)\right]$$

$$=\frac{1}{r^2}\left[\left(e^{-\alpha r}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right)+\alpha e^{-\alpha r}-\alpha e^{-\alpha r}+\alpha^2re^{-\alpha r}-\alpha^2re^{-\alpha r}+r^2\frac{\alpha^3e^{-\alpha r}}{2}\right)\right]=e^{-\alpha r}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right)+\frac{\alpha^3e^{-\alpha r}}{2}$$

$$=e^{-\alpha r}\nabla^2\frac{1}{r}+\frac{\alpha^3e^{-\alpha r}}{2}=-4\pi e^{-\alpha r}\delta(\vec{r})+\frac{\alpha^3e^{-\alpha r}}{2}=-4\pi\delta(\vec{r})+\frac{\alpha^3e^{-\alpha r}}{2}$$

$$\Rightarrow(-4\pi e^{-\alpha r}\delta(\vec{r})=-4\pi\delta(\vec{r})\Rightarrow\delta(\vec{r})=0,when\ \ r\neq0)$$

$$\rho=-\varepsilon_0\nabla^2\Phi=-\frac{q}{4\pi}\left[-4\pi\delta(\vec{r})+\frac{\alpha^3e^{-\alpha r}}{2}\right]=q\delta(\vec{r})-\frac{q\alpha^3}{8\pi}e^{-\alpha r}$$

1.6

A simple capacitor is a device formed by two insulated conductors adjacent to each other. If equal and opposite charges are placed on the conductors, there will be a certain difference of potential between them. The ratio of the magnitude of the charge on one conductor to the magnitude of the potential difference is called the capacitance (in SI unit it is measured in farads). Using Gauss's law, calculate the capacitance of

(a) Two large, flat, conducting sheets of area A, separated by a small distance d;

(b) Two concentric conducting spheres with radii a, b (b>a);

(c) Two concentric conducting cylinders of length L, large compared to their radii a, b(b>a).

(d) What is the inner diameter of the outer conductor in an air-filled coaxial cable whose center conductor is a cylindrical wire of diameter 1 mm and whose capacitance is 3×10^{-11} F/m? 3×10^{-12} F/m?

Solution :

(a)

$$2AE=\frac{Q}{\varepsilon_0}\Rightarrow E=\frac{Q}{2A\varepsilon_0}=\frac{\sigma}{2\varepsilon_0}\Rightarrow E_{tot.}=2E=\frac{\sigma}{\varepsilon_0}\Rightarrow V=E_{tot.}d\Rightarrow C=\frac{Q}{V}=\frac{A\varepsilon_0}{d}$$

(b)

$$E\cdot4\pi r^2=\frac{Q}{\varepsilon_0}\Rightarrow E=\frac{Q}{4\pi\varepsilon_0r^2}\Rightarrow V=\int_a^b\frac{Q}{4\pi\varepsilon_0r^2}dr=\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a}-\frac{1}{b}\right]\Rightarrow C=\frac{4\pi\varepsilon_0ab}{b-a}$$

(c)

$$E\cdot2\pi rL=\frac{Q}{\varepsilon_0}\Rightarrow E=\frac{Q}{2\pi r\varepsilon_0L}$$

$$V=\int_a^b\frac{Q}{2\pi r\varepsilon_0L}dr=\frac{Q}{2\pi\varepsilon_0L}\ln\left(\frac{b}{a}\right)\Rightarrow C=\frac{Q}{V}=2\pi\varepsilon_0\frac{L}{\ln\left(\frac{b}{a}\right)}$$

1.8

(a) For the three capacitor geometries in Problem 1.6 calculate the total electrostatic energy and express it alternatively in terms of the equal and opposite charges Q and –Q placed on the conductors and the potential difference between them.

(b) Sketch the energy density of the electrostatic filed in each case as a function of the appropriate linear coordinate.

Solution :

(a)

Plane plates:

$$W=\frac{\varepsilon_0}{2}\int |E|^2 d^3x=\frac{\varepsilon_0}{2}\left(\frac{\sigma}{\varepsilon_0}\right)^2 Ad=\frac{Q^2}{A^2}\frac{Ad}{2\varepsilon_0}=\frac{Q^2 d}{2\varepsilon_0 A}=\frac{QV}{2}=\frac{1}{2}CV^2$$

Spherical:

$$W=\frac{\varepsilon_0}{2}\int_a^b\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2 4\pi r^2 dr=\frac{Q^2}{8\pi\varepsilon_0}\int_a^b\frac{dr}{r^2}=\frac{Q^2}{8\pi\varepsilon_0}\frac{b-a}{ab}=\frac{Q^2}{2C}=\frac{1}{2}CV^2$$

Cylindrical:

$$W=\frac{\varepsilon_0}{2}\int_a^b\left(\frac{Q}{2\pi Lr}\right)^2 2\pi Lrdr=\frac{Q^2}{4\pi\varepsilon_0 L}\ln\frac{b}{a}=\frac{Q^2}{2C}=\frac{1}{2}CV^2$$

(b)

Plane plates:

$$W(r)=\frac{\varepsilon_0}{2}\left(\frac{Q}{A\varepsilon_0}\right)^2 \quad when \quad 0<r<d \quad , \quad and \quad W(r)=0 \quad , \quad otherwise$$

Concentric spheres:

$$W(r)=\frac{\varepsilon_0}{2}\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2 \quad when \quad a<r<b \quad , \quad and \quad W(r)=0 \quad , \quad otherwise$$

Concentric cylinders:

$$W(r)=\frac{\varepsilon_0}{2}\left(\frac{Q}{2\pi r\varepsilon_0 L}\right)^2 \quad when \quad a<r<b \quad , \quad and \quad W(r)=0 \quad , \quad otherwise$$

1.9

Calculate the attractive force between conductors in the parallel plate capacitor (Problem 1.6a) and the parallel cylinder capacitor (Problem 1.7) for

(a) Fixed charges on each conductor;

(b) Fixed potential difference between conductors.

Solution :

(a) Plane plate capacitor:

$$W=\frac{\varepsilon_0}{2}\int |E|^2 d^3x=\frac{\varepsilon_0}{2}\left(\frac{\sigma}{\varepsilon_0}\right)^2 Ad=\frac{Q^2}{A^2}\frac{Ad}{2\varepsilon_0}=\frac{Q^2 d}{2\varepsilon_0 A}=\frac{Q^2 z}{2\varepsilon_0 A}$$

$$\Rightarrow We\ choose\ the\ z\ direction \Rightarrow \vec{F}=-\nabla W=-\nabla\frac{Q^2 z}{2\varepsilon_0 A}=-\frac{Q^2}{2\varepsilon_0 A}\hat{e}_z$$

(b) Parallel cylinder capacitor

$$C=\frac{\pi\varepsilon_0}{\ln\left(\frac{d}{a}\right)}$$, the capacitance per unit length

$$C=\frac{\pi\varepsilon_0}{\ln\left(\frac{d}{a}\right)}=\frac{\left(Q/L\right)}{V}\Rightarrow V=\frac{Q}{L\pi\varepsilon_0}\ln\left(\frac{d}{a}\right)=\frac{\lambda}{\pi\varepsilon_0}\ln\left(\frac{d}{a}\right)$$

$$W=\frac{1}{2}\frac{\lambda^2}{C}=\frac{\lambda^2}{2\pi\varepsilon_0}\ln\left(\frac{d}{a}\right)=\frac{\lambda^2}{2\pi\varepsilon_0}\ln\left(\frac{z}{a}\right)\Rightarrow \textit{We choose the z direction}$$

$$\vec{F}=-\nabla W=-\nabla\frac{\lambda^2}{2\pi\varepsilon_0}\ln\left(\frac{z}{a}\right)=-\hat{e}_z\left.\frac{\lambda^2}{2\pi\varepsilon_0 z}\right|_{z=d}=-\frac{\lambda^2}{2\pi\varepsilon_0 d}\hat{e}_z=-\frac{\pi\varepsilon_0 V^2}{2d\left[\ln\left(\frac{d}{a}\right)\right]^2}\hat{e}_z$$

the *force* per unit length

(c) Plane plate capacitor:

$$C=\frac{Q}{V}=\frac{A\varepsilon_0}{d}\Rightarrow W=\frac{1}{2}CV^2=\frac{1}{2}\frac{A\varepsilon_0}{d}V^2=\frac{1}{2}\frac{A\varepsilon_0}{z}V^2$$

$\Rightarrow$ *We choose the z direction*

$$\vec{F}=-\nabla W=-\nabla\frac{1}{2}\frac{A\varepsilon_0}{z}V^2=\hat{e}_z\left.\frac{1}{2}\frac{A\varepsilon_0}{z^2}V^2\right|_{z=d}=\hat{e}_z\frac{1}{2}\frac{A\varepsilon_0}{d^2}V^2$$

$$=\hat{e}_z\frac{1}{2A\varepsilon_0}\left(\frac{A\varepsilon_0}{d}\right)^2V^2=\hat{e}_z\frac{Q^2}{2\varepsilon_0 A}$$

Parallel cylinder capacitor

$$C=\frac{\pi\varepsilon_0}{\ln\left(\frac{d}{a}\right)}$$, the capacitance per unit length

$$W=\frac{1}{2}CV^2=\frac{1}{2}\frac{\pi\varepsilon_0}{\ln\left(\frac{d}{a}\right)}V^2=\frac{1}{2}\frac{\pi\varepsilon_0}{\ln\left(\frac{z}{a}\right)}V^2\Rightarrow \textit{We choose the z direction}$$

$$\vec{F}=-\nabla W=-\nabla\frac{1}{2}\frac{\pi\varepsilon_0}{\ln\left(\frac{z}{a}\right)}V^2=\hat{e}_z\left.\frac{1}{2}\frac{\pi\varepsilon_0}{\left(\ln\left(\frac{z}{a}\right)\right)^2}\frac{V^2}{z}\right|_{z=d}=\hat{e}_z\frac{\pi\varepsilon_0 V^2}{2d\left(\ln\left(\frac{d}{a}\right)\right)^2}$$

the *force* per unit length

1.12

Prove Green's reciprocation theorem: If Φ is the potential due to a volume-charge density ρ within a volume V and a surface-charge density σ on the conducting surface S bounding the volume V, while Φ' is the potential due to another charge distribution ρ' and σ' , then

$$\int_V \rho\Phi' d^3x+\int_s \sigma\Phi' da=\int_V \rho'\Phi d^3x+\int_s \sigma'\Phi da$$

Solution :

$$\int_V \left(\varphi\nabla^2\Psi - \Psi\nabla^2\varphi\right) d^3x = \oint_S \left(\varphi\frac{\partial\Psi}{\partial n} - \Psi\frac{\partial\varphi}{\partial n}\right) da$$

$$\text{Let}\begin{cases}\varphi = \Phi \\ \Psi = \Phi'\end{cases}$$

$$\nabla^2\Phi = -\frac{\rho}{\varepsilon_0} \text{ , } \nabla^2\Phi' = -\frac{\rho'}{\varepsilon_0} \text{ , } \frac{\partial\Phi}{\partial n} = \frac{\sigma}{\varepsilon_0} \text{ , } \frac{\partial\Phi'}{\partial n'} = \frac{\sigma'}{\varepsilon_0}$$

$$\int_V \left[\Phi\left(-\frac{\rho'}{\varepsilon_0}\right) - \Phi'\left(-\frac{\rho}{\varepsilon_0}\right)\right] d^3x = \oint_S \left(\Phi\left(\frac{\sigma'}{\varepsilon_0}\right) - \Phi'\left(\frac{\sigma}{\varepsilon_0}\right)\right) da$$

$$\int_V \rho'\Phi d^3x + \oint_S \sigma'\Phi da = \int_V \rho\Phi' d^3x + \oint_S \sigma\Phi' da$$

1.14

Consider the electrostatic Green functions of Section 1.10 for Dirichlet and Neumann boundary conditions on the surface S bounding the volume V. Apply Green's theorem (1.35) with integration variable y and Φ=G(x,y) , φ= G(x',y), with $\nabla^2_y G(z,y)=-4\pi\delta(y-z)$ Find an expression for the difference [G(x,x')- G(x',x)] in terms of an integral over the boundary surface S.

(a) For Dirichlet boundary conditions one the potential and the associated boundary condition on the Green function, show that $G_D(x,x')$ must be symmetric in x and x' .

(b) For Neumann boundary conditions, use the boundary condition (1.45) for $G_N(x,x')$ to show that $G_N(x',x)$ is not symmetric in general, but that $G_N(x,x')-F(x)$ is symmetric in x and x', where

$$F(x) = \frac{1}{S}\oint_s G_N(x,y) da_y$$

(c) Show that the addition of F(x) to the Green function does not affect the potential Φ(x). See problem 3.36 for an example of the Neumann Green function.

Solution :

$$\int_v \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) d^3x = \oint_s \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) da \text{ , let } \phi = G(x,y) \ \& \ \psi = G(x',y)$$

$$\Rightarrow \int_v \left(G(x,y)\nabla_y^2 G(x',y) - G(x',y)\nabla_y^2 G(x,y)\right) d^3y = \oint_s \left(G(x,y)\frac{\partial G(x',y)}{\partial n} - G(x',y)\frac{\partial G(x,y)}{\partial n}\right) da_y$$

$$\int_v \left(G(x,y)\left(-4\pi\delta(x'-y)\right) - G(x',y)\left(-4\pi\delta(x-y)\right)\right) d^3y = \oint_s \left(G(x,y)\frac{\partial G(x',y)}{\partial n} - G(x',y)\frac{\partial G(x,y)}{\partial n}\right) da_y$$

$$G(x,x') - G(x',x) = -\frac{1}{4\pi}\oint_s \left(G(x,y)\frac{\partial G(x',y)}{\partial n} - G(x',y)\frac{\partial G(x,y)}{\partial n}\right) da_y$$

(a)

For Dirichlet boundary conditions we domand (1.43):

$G_D(x,y) = 0$ for y on S

$$G_D(x,x') - G_D(x',x) = -\frac{1}{4\pi}\oint_s \left(G_D(x,y)\frac{\partial G_D(x',y)}{\partial n} - G_D(x',y)\frac{\partial G_D(x,y)}{\partial n}\right) da_y$$

$$= -\frac{1}{4\pi}\oint_s \left(0\cdot\frac{\partial G_D(x',y)}{\partial n} - 0\cdot\frac{\partial G_D(x,y)}{\partial n}\right) da_y = 0$$

$\Rightarrow G_D(x,x')$ *is* clearly symmetric in x and x′

(b)

$$Eq.(1.45)\ ,\ \frac{\partial G_N(x,x')}{\partial n'}=-\frac{4\pi}{S}\quad \text{for } x' \ on\ S$$

$$G_N(x,x')-G_N(x',x)=-\frac{1}{4\pi}\oint_s\left(G_N(x,y)\frac{\partial G_N(x',y)}{\partial n}-G_N(x',y)\frac{\partial G_N(x,y)}{\partial n}\right)da_y$$

$$=\frac{1}{S}\oint_s G_N(x,y)da_y-\frac{1}{S}\oint_s G_N(x',y)da_y=F(x)-F(x')\neq 0$$

$G_N(x,x')$ *is* not symmetric in x and x′

$$But\ G_N(x,x')-G_N(x',x)=F(x)-F(x')\Rightarrow G_N(x,x')-F(x)=G_N(x',x)-F(x')$$

$$let\ G'_N(x,x')=G_N(x,x')-F(x)\ \ \&\ \ G'_N(x',x)=G_N(x',x)-F(x')\Rightarrow G'_N(x,x')=G'_N(x',x)$$

$\Rightarrow G'_N(x,x')=G_N(x,x')-F(x)$ *is* clearly symmetric in x and x′

(*c*)

$$Eq.(1.46)\ ,\ \Phi(x)=\langle\Phi\rangle_s+\frac{1}{4\pi\varepsilon_0}\int_v\rho(x')G_N(x,x')d^3x'+\frac{1}{4\pi}\oint_s\frac{\partial\Phi(x)}{\partial n'}G_N(x,x')da'$$

$$G_N(x,x')\to G_N(x,x')-F(x)\ ,\ F(x)=\frac{1}{S}\oint_s G_N(x,y)da_y$$

$$\Rightarrow\Phi'(x)=\langle\Phi\rangle_s+\frac{1}{4\pi\varepsilon_0}\int_v\rho(x')\left[G_N(x,x')-F(x)\right]d^3x'+\frac{1}{4\pi}\oint_s\frac{\partial\Phi(x')}{\partial n'}\left[G_N(x,x')-F(x)\right]da'$$

$$=\langle\Phi\rangle_s+\frac{1}{4\pi\varepsilon_0}\int_v\rho(x')G_N(x,x')d^3x'+\frac{1}{4\pi}\oint_s\frac{\partial\Phi(x)}{\partial n'}G_N(x,x')da'-\frac{1}{4\pi\varepsilon_0}\int_v\rho(x')F(x)d^3x'-\frac{1}{4\pi}\oint_s\frac{\partial\Phi(x')}{\partial n'}F(x)da'$$

$$=\Phi(x)-\frac{F(x)}{4\pi\varepsilon_0}\int_v\rho(x')d^3x'-\frac{F(x)}{4\pi}\oint_s\frac{\partial\Phi(x')}{\partial n'}da'=\Phi(x)+\frac{F(x)}{4\pi}\int_v\nabla'\cdot(\nabla'\Phi(x'))d^3x'-\frac{F(x)}{4\pi}\oint_s\frac{\partial\Phi(x')}{\partial n'}da'$$

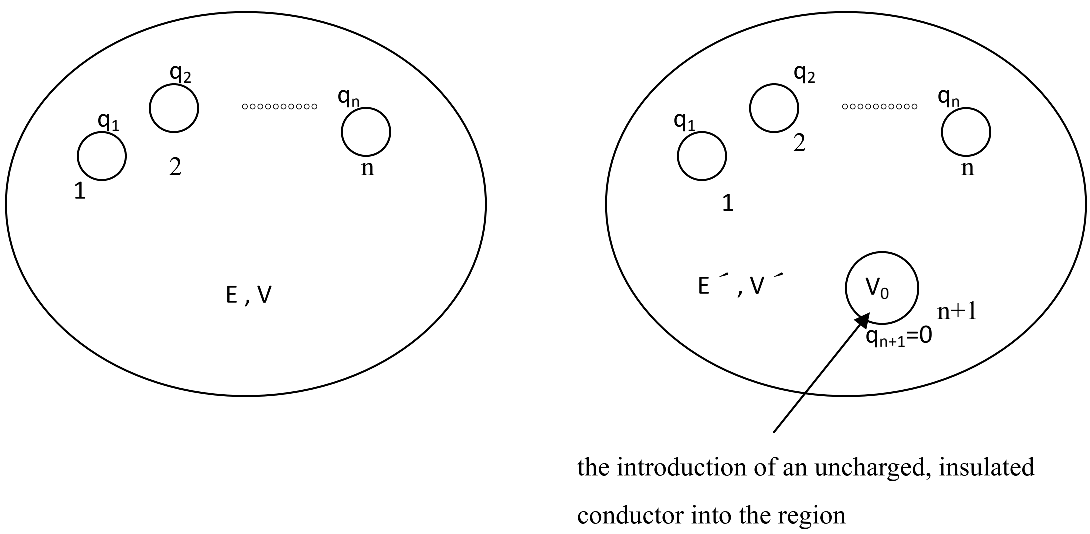
$$=\Phi(x)+\frac{F(x)}{4\pi}\oint_s\frac{\partial\Phi(x')}{\partial n'}da'-\frac{F(x)}{4\pi}\oint_s\frac{\partial\Phi(x')}{\partial n'}da'=\Phi(x)$$

the addition of $F(x)$ to the Green function doesnot affect the potential $\Phi(x)$

1.16

Prove the following theorem: If a number of surfaces are fixed in position with a given total charge on each, the introduction of an uncharged, insulated conductor into the region bounded by the surfaces lowers the electrostatic energy.

Solution :



Initial electrostatic energy : $W=\frac{1}{2}\varepsilon_0\int_V \vec{E}\cdot\vec{E}d^3x \ \Rightarrow V$:不包含導體的體積

Final electrostatic energy: $W'=\frac{1}{2}\varepsilon_0\int_{V'} \vec{E}'\cdot\vec{E}'d^3x \ \Rightarrow V'$:不包含導體的體積

V_0 : 新加入導體的體積

$\Pr ove: W'-W\le 0$

$V=V'+V_0$

$$W'-W=\frac{1}{2}\varepsilon_0\int_{V'}\vec{E}'\cdot\vec{E}'d^3x-\frac{1}{2}\varepsilon_0\int_V\vec{E}\cdot\vec{E}d^3x=\frac{1}{2}\varepsilon_0\int_{V'}\vec{E}'\cdot\vec{E}'d^3x-\frac{1}{2}\varepsilon_0\int_{V'+V_0}\vec{E}\cdot\vec{E}d^3x$$

$$=\frac{1}{2}\varepsilon_0\left(\int_{V'}\left(\vec{E}'\cdot\vec{E}'-\vec{E}\cdot\vec{E}\right)d^3x-\int_{V_0}\vec{E}\cdot\vec{E}d^3x\right)=\frac{1}{2}\varepsilon_0\left(\int_{V'}\left(\vec{E}'\cdot\vec{E}'-\vec{E}\cdot\vec{E}\right)d^3x\right)-\frac{1}{2}\varepsilon_0\int_{V_0}\left|\vec{E}\right|^2d^3x$$

$$\left(\vec{E}'-\vec{E}\right)^2=\vec{E}'\cdot\vec{E}'+\vec{E}\cdot\vec{E}-2\vec{E}'\cdot\vec{E}=-\vec{E}'\cdot\vec{E}'+\vec{E}\cdot\vec{E}-2\vec{E}'\cdot\vec{E}+2\vec{E}'\cdot\vec{E}'=-\left(\vec{E}'\cdot\vec{E}'-\vec{E}\cdot\vec{E}\right)+2\vec{E}'\cdot\left(\vec{E}'-\vec{E}\right)$$

$$\left(\vec{E}'\cdot\vec{E}'-\vec{E}\cdot\vec{E}\right)=-\left(\vec{E}'-\vec{E}\right)^2+2\vec{E}'\cdot\left(\vec{E}'-\vec{E}\right)$$

$$=\frac{1}{2}\varepsilon_0\left(\int_{V'}\left(\vec{E}'\cdot\vec{E}'-\vec{E}\cdot\vec{E}\right)d^3x-\int_{V_0}\vec{E}\cdot\vec{E}d^3x\right)=\frac{1}{2}\varepsilon_0\left(\int_{V'}\left(-\left(\vec{E}'-\vec{E}\right)^2+2\vec{E}'\cdot\left(\vec{E}'-\vec{E}\right)\right)d^3x\right)-\frac{1}{2}\varepsilon_0\int_{V_0}\left|\vec{E}\right|^2d^3x$$

$$=\varepsilon_0\left(\int_{V'}\vec{E}'\cdot\left(\vec{E}'-\vec{E}\right)d^3x\right)-\frac{1}{2}\varepsilon_0\left(\int_{V_0}\left|\vec{E}\right|^2d^3x+\int_{V'}\left|\vec{E}'-\vec{E}\right|^2d^3x\right)$$

其中

$$\int_{V'}\vec{E}'\cdot\left(\vec{E}'-\vec{E}\right)d^3x=-\int_{V'}\nabla\phi'\cdot\left(\vec{E}'-\vec{E}\right)d^3x=-\int_{V'}\nabla\cdot\left(\phi'\left(\vec{E}'-\vec{E}\right)\right)d^3x+\int_{V'}\phi'\nabla\cdot\left(\vec{E}'-\vec{E}\right)d^3x$$

$$=-\oint_{S'}\phi'\left(\vec{E}'-\vec{E}\right)\cdot\hat{n}'da'+\int_{V'}\phi'\nabla\cdot\left(\vec{E}'-\vec{E}\right)d^3x=A+B$$

$\Rightarrow$

$$A=-\oint_{S'}\phi'\left(\vec{E}'-\vec{E}\right)\cdot\hat{n}'da'=-\sum_{i=1}^{n+1}\oint_{S_i'}\phi_i'\left(\vec{E}_i'-\vec{E}_i\right)\cdot\hat{n}_i'da_i'=-\sum_{i=1}^{n+1}\phi_i'\oint_{S_i'}\left(\vec{E}_i'-\vec{E}_i\right)\cdot\hat{n}_i'da_i'$$

equal potential on conductor surface S_i

$$=-\sum_{i=1}^{n+1}\phi_i'\oint_{S_i'}\left(\vec{E}_i'-\vec{E}_i\right)\cdot\hat{n}_i'da_i'=-\sum_{i=1}^{n+1}\phi_i'\oint_{S_i'}\left(\sigma_i'-\sigma_i\right)da_i'=-\sum_{i=1}^{n+1}\phi_i'\left(q_i'-q_i\right)=0$$

$q_i'=q_i$, equal charge on each conductor

$$B=-\int_{V'}\phi'\nabla\cdot\left(\vec{E}'-\vec{E}\right)d^3x=-\int_{V'}\phi'\left(\rho'-\rho\right)d^3x=0 \text{ (No charge inside } V'\text{)}$$

$$\Rightarrow-\oint_{S'}\phi'\left(\vec{E}'-\vec{E}\right)\cdot\hat{n}'da'+\int_{V'}\phi'\nabla\cdot\left(\vec{E}'-\vec{E}\right)d^3x=A+B=0$$

$$W'-W=\varepsilon_0\left(\int_{V'}\vec{E}'\cdot\left(\vec{E}'-\vec{E}\right)d^3x\right)-\frac{1}{2}\varepsilon_0\left(\int_{V_0}\left|\vec{E}\right|^2d^3x+\int_{V'}\left|\vec{E}'-\vec{E}\right|^2d^3x\right)$$

$$=-\frac{1}{2}\varepsilon_0\left(\int_{V_0}\left|\vec{E}\right|^2d^3x+\int_{V'}\left|\vec{E}'-\vec{E}\right|^2d^3x\right)\le 0$$

The insulated conductor into the region lowers the electrostatic energy

1.17

A volume V in vacuum is bounded by a surface S consisting of several separate conducting surfaces Si. One conductor is held at unit potential and all the other conductors at zero potential.

(a) Show that the capacitance of the one conductor is given by

$$C = \varepsilon_0 \int_V |\nabla \Phi|^2 d^3x$$

where Φ(x) is the solution for the potential.

(b) Show that the true capacitance C is always less than or equal to the quantity

$$C[\Psi] = \varepsilon_0 \int_V |\nabla \Psi|^2 d^3x$$

where Ψ is any trial function satisfying the boundary conditions on the conductors. This is a variation principle for the capacitance that yields an upper bound.

(a)

$$W = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij} V_i V_j$$

$$= \frac{\varepsilon_0}{2}\int_V |\nabla \phi|^2 dV = \frac{\varepsilon_0}{2}\int_V |\nabla \phi|^2 dV = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij} V_i V_j = \frac{1}{2} C_{11} V_1^2 = \frac{1}{2} C$$

$$\Rightarrow C = \varepsilon_0 \int_V |\nabla \phi|^2 dV$$

Because there is only one conductor, so $V_i = 0$, $i \neq 1$

(b)

Let $\Psi(x,\lambda) = \phi(x) + \lambda f(x)$, $f(x)$ *is arbitrary function*

$\Psi(x,\lambda)$ & $\phi(x)$ *have same* B.C.

$$\Psi(x,\lambda)\big|_{Si} = \phi(x)\big|_{Si} = \phi(x)\big|_{Si} + \lambda f(x)\big|_{Si} \Rightarrow \lambda f(x)\big|_{Si} = 0$$

$$C[\Psi] = \varepsilon_0 \int_V \left|\nabla(\phi(x) + \lambda f(x))\right|^2 d^3x$$

$$= \varepsilon_0 \int_V |\nabla\phi(x)|^2 d^3x + 2\varepsilon_0\lambda\int_V \nabla\phi(x)\cdot\nabla f(x) d^3x + \varepsilon_0\lambda^2\int_V |\nabla f(x)|^2 d^3x$$

$$= C + 2\varepsilon_0\lambda\int_V \nabla\phi(x)\cdot\nabla f(x) d^3x + \varepsilon_0\lambda^2\int_V |\nabla f(x)|^2 d^3x$$

$$\int_V \nabla\phi(x)\cdot\nabla f(x) d^3x = \int_V \nabla\cdot(f(x)\nabla\phi(x)) d^3x - \int_V (f(x)\nabla^2\phi(x)) d^3x$$

$$= \oint_S \hat{n}\cdot(f(x)\nabla\phi(x)) da - \int_V (f(x)\nabla^2\phi(x)) d^3x = \sum_i \oint_{S_i} \hat{n}_i\cdot(f(x)\nabla\phi(x)) da_i - \int_V (f(x)\nabla^2\phi(x)) d^3x$$

$$= -\sum_i \oint_{S_i} \sigma_i f(x) da_i - \int_V (f(x)\nabla^2\phi(x)) d^3x = -\int_V (f(x)\nabla^2\phi(x)) d^3x$$

$\Rightarrow$ *B.C.* $f(x) = 0$, on conductor surface S_i

$$C[\Psi] = C - \int_V (f(x)\nabla^2\phi(x)) d^3x + \varepsilon_0\lambda^2\int_V |\nabla f(x)|^2 d^3x = C + \varepsilon_0\lambda^2\int_V |\nabla f(x)|^2 d^3x$$

$\left[\nabla^2\phi(x) = 0 \ \ (\textit{No} \text{ charge } \textit{on} \text{ V})\right]$

$$\frac{\partial C[\Psi]}{\partial\lambda} = 2\varepsilon_0\lambda\int_V |\nabla f(x)|^2 d^3x\varepsilon_0 = 0 \Rightarrow \lambda = 0, \ C[\Psi] = C \text{ is minimun}$$

$$\Rightarrow C[\Psi] = C + \varepsilon_0\lambda^2\int_V |\nabla f(x)|^2 d^3x \geq C$$

1.19

For the cylindrical capacitor of Problem 1.6c, evaluate the variation upper bound of Problem 1.17b with the naïve trial function ,Ψ(ρ)=(b-ρ)/(b-a). Compare the variation result with the exact result for b/a=1.5,2,3. Explain the trend of your results in terms of the functional form of Ψ_1. An improved trial function is treated by Collin(pp. 275-277).

$$C=\frac{4\pi\varepsilon_0 ab}{b-a}, \text{ where } b>a$$

$$\Psi(\rho)=\frac{b-\rho}{b-a}\Rightarrow \nabla\Psi(\rho)=\frac{\partial\Psi}{\partial\rho}\hat{\rho}=\frac{-1}{b-a}\hat{\rho}, \text{ By 1.17(b)}$$

$$C[\Psi]=\varepsilon_0\int_a^b\left|\nabla\Psi(\rho)\right|^2 4\pi r^2 dr=\frac{4\pi\varepsilon_0}{(b-a)^2}\int_a^b r^2 dr=\frac{4\pi\varepsilon_0\left(b^3-a^3\right)}{3(b-a)^2}$$

$$\frac{C[\Psi]}{C}=\frac{\left(\frac{b}{a}\right)^3-1}{3\frac{b}{a}\left(\frac{b}{a}-1\right)}$$

$b/a=1.5$

$$\frac{C[\Psi]}{C}=\frac{(1.5)^3-1}{3\times 1.5(1.5-1)}=1.05556$$

$b/a=2$

$$\frac{C[\Psi]}{C}=\frac{2^3-1}{3\times 2(2-1)}=1.16667$$

$b/a=3$

$$\frac{C[\Psi]}{C}=\frac{3^3-1}{3\times 3(3-1)}=1.44444$$