2.1

A point charge q is brought to a position a distance d away from an infinite plane conductor held at zero potential. Use the method of images, find:

(a) the surface-charge density induced on the plane, and plot it;

(b) the force between the plane and the charge by using Coulomb's law for the force between the charge and its image;

(c) the total force acting on the plane by integrating $\sigma^2/2\varepsilon_0$ over the whole plane;

(d) the work necessary to remove the charge q from its position to infinity;

(e) the potential energy between the charge q and its image[compare the answer to part d and discuss].

(f) Find the answer to part d in electron volts for an electron originally one angstrom from the surface.

(a)

$$\Phi_+(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{x}-\vec{x}'|} = \frac{q}{4\pi\varepsilon_0}\left(x^2+y^2+(z-d)^2\right)^{-1/2}$$

$$\Phi_-(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{-q}{|\vec{x}-\vec{x}''|} = \frac{-q}{4\pi\varepsilon_0}\left(x^2+y^2+(z+d)^2\right)^{-1/2}$$

$$\Rightarrow \Phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\left(x^2+y^2+(z-d)^2\right)^{-1/2} - \left(x^2+y^2+(z+d)^2\right)^{-1/2}\right]$$

$$\sigma = -\varepsilon_0\left.\frac{\partial\Phi}{\partial z}\right|_{z=0} = \varepsilon_0\frac{q}{4\pi\varepsilon_0}\left[\frac{(z-d)}{\left(x^2+y^2+(z-d)^2\right)^{3/2}} - \frac{(z+d)}{\left(x^2+y^2+(z+d)^2\right)^{3/2}}\right]_{z=0}$$

$$= -\frac{qd}{2\pi}\left(x^2+y^2+d^2\right)^{-3/2}$$

$$= -\frac{qd}{2\pi}\left(r^2+d^2\right)^{-3/2}$$

(b)

$$\vec{F} = q\vec{E} = q\frac{-q}{4\pi\varepsilon_0(2d)^2}$$

$$= -\frac{q^2}{16\pi\varepsilon_0 d^2}\hat{z}$$

(c)

$$F = \int\frac{\sigma^2}{2\varepsilon_0}dA = \int\frac{\left(\frac{qd}{2\pi}\right)^2\left(r^2+d^2\right)^{-3}}{2\varepsilon_0}dA = \int_0^\infty\int_0^{2\pi} rdrd\varphi\frac{\left(\frac{qd}{2\pi}\right)^2\left(r^2+d^2\right)^{-3}}{2\varepsilon_0}$$

$$= 2\pi\frac{\left(\frac{qd}{2\pi}\right)^2}{2\varepsilon_0}\int_0^\infty rdr\left(r^2+d^2\right)^{-3}$$

$$Let\, u = r^2 + d^2, dr = \frac{du}{2r}$$

$$F = 2\pi \frac{\left(\frac{qd}{2\pi}\right)^2}{2\varepsilon_0} \int r \frac{du}{2r} \frac{1}{u^3} = 2\pi \frac{\left(\frac{qd}{2\pi}\right)^2}{2\varepsilon_0} \frac{1}{2} \int duu^{-3} = 2\pi \frac{\left(\frac{qd}{2\pi}\right)^2}{2\varepsilon_0} \frac{1}{2} \left[\frac{-1}{2} \left(r^2 + d^2\right)^{-2} \right]_0^\infty$$

$$= \frac{q^2}{16\pi\varepsilon_0 d^2}$$

(d)

$$W = \int F dz = \int_d^\infty -\frac{q^2}{16\pi\varepsilon_0 z^2} dz = \frac{q^2}{16\pi\varepsilon_0} \left[z^{-1}\right]_d^\infty$$

$$= -\frac{q^2}{16\pi\varepsilon_0 d}$$

(e)

$$W = \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \sum_{i \neq j} \frac{q_i q_j}{\left|\vec{x}_i - \vec{x}_j\right|} = \frac{1}{8\pi\varepsilon_0} \cdot 2 \cdot \frac{-q^2}{2d}$$

$$= \frac{-q^2}{8\pi\varepsilon_0 d}$$

(f)

2.2

Using the method of images, discuss the problem of a point charge q inside a hollow, grounded, conducting sphere of inner radius a, Find

(a) The potential inside the sphere;

(b) The induced surface-charge density;

(c) The magnitude and direction of the force acting on q.

(d) Is there any change in the solution if the sphere is kept at a fixed potential V? If the sphere has a total charge Q in its inner and outer surfaces?

(a)

$$\Phi_+(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{|\vec{x} - \vec{y}|}$$

$$\Phi_-(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \frac{q'}{|\vec{x} - \vec{y}'|}$$

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \left(\frac{q}{|\vec{x} - \vec{y}|} + \frac{q'}{|\vec{x} - \vec{y}'|} \right)$$

$$y'=\frac{a^2}{y}, q'=-q\frac{a}{y}$$

(b)

$$\sigma=-\varepsilon_0\left.\frac{\partial\Phi}{\partial x}\right|_{x=a}$$

$$=\frac{\varepsilon_0}{4\pi\varepsilon_0}\frac{\partial}{\partial x}\left.\left(\frac{q}{\left(x^2+y^2-2xy\cos r\right)^{\frac{1}{2}}}+\frac{q'}{\left(x^2+y'^2-2xy'\cos r\right)^{\frac{1}{2}}}\right)\right|_{x=a}$$

$$=\frac{-q}{4\pi}\frac{a\left(1-\frac{y^2}{a^2}\right)}{\left(a^2+y^2-2ay\cos r\right)^{\frac{3}{2}}}$$

(c)

$$|F|=\left|\frac{qq'}{4\pi\varepsilon_0\left(\vec{y}'-\vec{y}\right)^2}\right|$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{q^2ay}{\left(a^2-y^2\right)}$$

(d)

2.3

A straight-line charge with constant linear charge density λ is located perpendicular to the x-y plane in the first quadrant at (x_0, y_0). The intersecting planes $x=0$, $y\geq 0$ and $y=0$, $x\geq 0$ are conducting boundary surfaces held at zero potential. Consider the potential, fields, and surface charges in the first quadrant.

(a) The Well-known potential for an isolated line charge at (x_0, y_0) is $\Phi(x,y)=(\lambda/4\pi\varepsilon_0)\ln\left(R^2/r^2\right)$, where $r^2=(x-x_0)^2+(y-y_0)^2$ and R is a constant. Determine the expression for the potential of the line charge in the presence of the intersecting planes. Verify explicitly that the potential and the tangential electric field vanish on the boundary surfaces.

(b) Determine the surface charge density σ on the plane $y=0$, $x\geq 0$. Plot σ/λ versus x for $(x_0=2, y_0=1)$, $(x_0=1, y_0=1)$, and $(x_0=1, y_0=2)$.

(c) Show that the total charge density (pre unit length in z) on the plane $y=0$, $x\geq 0$ is $Q_x=-\frac{2}{\pi}\lambda\tan^{-1}\left(\frac{x_0}{y_0}\right)$. What is the total charge on the plane $x=0$?

(d) Show that far from the origin [$\rho >> \rho_0$, where $\rho=\sqrt{x^2+y^2}$ and

$\rho_0 = \sqrt{{x_0}^2 + {y_0}^2}$] the leading term in the potential is

$$\Phi \to \Phi_{system} = \frac{4\lambda}{\pi\varepsilon_0}\frac{(x_0 y_0)(xy)}{\rho^4}$$ Interpret.

(a)

$$\Phi(x,y) = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\frac{R^2}{(x-x_0)^2+(y-y_0)^2} - \ln\frac{R^2}{(x+x_0)^2+(y-y_0)^2}\right.$$
$$\left. - \ln\frac{R^2}{(x-x_0)^2+(y+y_0)^2} + \ln\frac{R^2}{(x+x_0)^2+(y+y_0)^2}\right]$$

$$when\ y = 0, \Phi(x,0) = 0$$
$$when\ x = 0, \Phi(0,y) = 0$$
$$on\ the\ surface,$$
$$\Phi = 0,\ \delta\Phi = 0,\ \frac{\partial\Phi}{\partial x}\delta x = 0, \Rightarrow E = 0$$

(b)

$$\sigma = -\varepsilon_0 \frac{\partial\Phi}{\partial y}\bigg|_{y=0} = \varepsilon_0 \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{(x-x_0)^2+(y-y_0)^2}{R^2}\cdot\frac{2(y-y_0)R^2}{\left((x-x_0)^2+(y-y_0)^2\right)^2}\right.$$
$$-\frac{(x+x_0)^2+(y-y_0)^2}{R^2}\cdot\frac{2(y-y_0)R^2}{\left((x+x_0)^2+(y-y_0)^2\right)^2} - \frac{(x-x_0)^2+(y+y_0)^2}{R^2}\cdot\frac{2(y+y_0)R^2}{\left((x-x_0)^2+(y+y_0)^2\right)^2}$$
$$\left.+\frac{(x+x_0)^2+(y+y_0)^2}{R^2}\cdot\frac{2(y+y_0)R^2}{\left((x+x_0)^2+(y+y_0)^2\right)^2}\right]\Bigg|_{y=0}$$
$$= \frac{-\lambda}{\pi}\left[\frac{y_0}{(x-x_0)^2+{y_0}^2} - \frac{y_0}{(x+x_0)^2+{y_0}^2}\right]$$

(c)

$$\int_0^\infty dx \frac{1}{(x\mp x_0)^2+{y_0}^2} = \frac{1}{y_0}\tan^{-1}\frac{x\mp x_0}{y_0}\bigg|_0^\infty = \frac{\pi}{2y_0} \pm \frac{1}{y_0}\tan^{-1}\frac{x_0}{y_0}$$
$$\Rightarrow Q_x = -\frac{\lambda y_0}{\pi}\left(\frac{\pi}{2y_0} + \frac{1}{y_0}\tan^{-1}\frac{x_0}{y_0} - \frac{\pi}{2y_0} + \frac{1}{y_0}\tan^{-1}\frac{x_0}{y_0}\right)$$
$$= -\frac{2}{\pi}\lambda\tan^{-1}\left(\frac{x_0}{y_0}\right)$$

(d)

expand following equation

$$\ln\left(\frac{R^2}{(x-x_0)^2+(y-y_0^2)}\right)-\ln\left(\frac{R^2}{(x-x_0)^2+(y+y_0^2)}\right)-$$
$$\ln\left(\frac{R^2}{(x+x_0)^2+(y-y_0^2)}\right)+\ln\left(\frac{R^2}{(x+x_0)^2+(y+y_0^2)}\right)$$

in lowest non - vanishing order in x_0, y_0 gives $16\frac{xy}{(x^2+y^2)^2}x_0y_0$

$$\Rightarrow \Phi = \frac{4\lambda}{\pi\varepsilon_0}\frac{xy}{(x^2+y^2)^2}x_0y_0$$

2.4

A point charge is placed a distance $d > R$ from the center of an equally charged, isolated, conducting sphere of radius R.

(a) Inside of what distance from the surface of the sphere is the point charge attracted rather than repelled by the charged sphere?

(b) What is the limiting value of the force of attraction when the point charge is located a distance $a(=d-R)$ from the surface of the sphere, if $a << R$?

(c) What are the results for parts a and b if the charge on the sphere is twice(half) as large as the point charge, but still the same sign?

(a)

Fig.2.5(P.62)

(b)

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q}{d^2}\left[q-\frac{qR^3(2d^2-R^2)}{d(d^2-r^2)^2}\right]=\frac{q^2}{4\pi\varepsilon_0}\left[\frac{d(d^2-R^2)^2-R^3(2d^2-R^2)}{d^3(d^2-R^2)^2}\right]$$

$\because d = a+R \; and \; a/R \to 0$

$$F = \frac{q^2}{4\pi\varepsilon_0}\left[\frac{(a+R)(a^2+2aR)^2-R^3(R^2+4aR+2a^2)}{(a+R)^3(a^2+2aR)^2}\right]$$

$$=\frac{q^2}{4\pi\varepsilon_0}\left[\frac{\left(\frac{a}{R}+1\right)\left(\frac{a^2}{R^2}+2\frac{a}{R}\right)^2-\left(1+4\frac{a}{R}+2\frac{a^2}{R^2}\right)}{\left(\frac{a}{R}+1\right)^3\left(\frac{a^2}{R}+2a\right)^2}\right]\to\frac{q^2}{4\pi\varepsilon_0}\frac{-1}{(2a)^2}$$

$$=\frac{-q^2}{16\pi\varepsilon_0 a^2}$$

(c)

2.5

(a) Show that the work done to remove the charge q from a distance $r > a$ to infinity against the force, Eq. (2.6), of a grounded conducting sphere is

$$W = \frac{q^2 a}{8\pi\varepsilon_0 (r^2 - a^2)}$$

Relate this result to the electrostatic potential, Eq. (2.3), and the energy discussion of Section 1.11.

(b) Repeat the calculation of the work done to remove the charge q against the force, Eq. (2.9), of an isolated charged conducting sphere. Show that the work done is

$$W = \frac{1}{4\pi\varepsilon_0}\left[\frac{q^2 a}{2(r^2 - a^2)} - \frac{q^2 a}{2r^2} - \frac{qQ}{r}\right]$$

Relate the work to the electrostatic potential, Eq. (2.8), and the energy discussion of Section 1.11.

(d)

$$W = \int_r^\infty |F| dy = \frac{q^2 a}{4\pi\varepsilon_0}\int_r^\infty \frac{dy}{y^3 (1 - a^2/y^2)^2} = \frac{q^2 a}{4\pi\varepsilon_0}\int_r^\infty \frac{y dy}{(y^2 - a^2)^2} = \frac{q^2 a}{4\pi\varepsilon_0}\left(-\frac{1}{2}\frac{1}{(y^2 - a^2)}\right)_r^\infty$$

$$= \frac{q^2 a}{8\pi\varepsilon_0 (r^2 - a^2)}$$

compare to disassemble the charge

$$-W' = -\frac{1}{8\pi\varepsilon_0}\sum_{i\neq j}\frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} = \frac{1}{4\pi\varepsilon_0}\left[\frac{aq^2}{r}\frac{1}{r\left(1 - \frac{a^2}{r^2}\right)}\right] = \frac{q^2 a}{4\pi\varepsilon_0 (r^2 - a^2)} > W$$

The different is because, for W the image charge is moving, what W' is not.

(e)

$$W = \int_r^\infty |F| dy$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\int_r^\infty \frac{Q dy}{y^2} - qa^3 \int_r^\infty \frac{(2y^2 - a^2)}{y(y^2 - a^2)^2} dy\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q^2 a}{2(r^2 - a^2)} - \frac{q^2 a}{2r^2} - \frac{qQ}{r}\right]$$

$$-W' = \frac{1}{4\pi\varepsilon_0}\left[\frac{q^2 a}{(r^2 - a^2)} - \frac{q^2 a}{r^2} - \frac{qQ}{r}\right]$$

2.9

An insulated, spherical, conducting shell of radius a is in a uniform electric field E_0. If the sphere is cut into two hemispheres by a plane perpendicular to the field, find the force required to prevent the hemispheres form separating

(a) If the shell is uncharged;

(b) If the total charge on the shell is Q.

(a)

The charge density induced is

$$\sigma = 3\varepsilon_0 E_0 \cos\theta \text{...(eq. 2.15)}$$

radius force / unit area is $\dfrac{\sigma^2}{2\varepsilon_0}$ (the electrostatic pressure)

Therefore, the force on the right hand hemisphere is

$$F = \frac{1}{2\varepsilon_0}\int \sigma^2 \hat{z}\cdot d\vec{a} = \frac{1}{2\varepsilon_0}(3\varepsilon_0 E_0)^2 2\pi R^2 \int_0^1 x^3 dx \text{ where } x = \cos\theta$$

$$= \frac{9}{4}\pi\varepsilon_0 E_0^2 R^2$$

(b)

The charge density is

$$\sigma = 3\varepsilon_0 E_0 \cos\theta + \frac{Q}{4\pi R^2}$$

$$= 3\varepsilon_0 E_0\left(x + \frac{Q}{12\pi\varepsilon_0 E_0 R^2}\right)$$

$$F_z = \frac{1}{2\varepsilon_0}\int \sigma^2 \hat{z}\cdot d\vec{a}$$

$$= \frac{1}{2\varepsilon_0}(3\varepsilon_0 E_0)^2 2\pi R^2 \int_0^1 x\left(x + \frac{Q}{12\pi\varepsilon_0 E_0 R^2}\right)^2 dx$$

$$= \frac{9}{4}\pi\varepsilon_0 E_0^2 R^2 + \frac{1}{2}QE_0 + \frac{Q^2}{32\pi\varepsilon_0 R^2}$$

2.23

A hollow cube has conducting walls defined by six planes $x = 0$, $y = 0$, $z = 0$, and $x = a$, $y = a$, $z = a$. The walls $z = 0$ and $z = a$ are held at a constant potential V. The other four sides are at zero potential.

(a) Find the potential $\Phi(x, y, z)$ at any point inside the cube.

(b) Evaluate the potential at the center of the cube numerically, accurate to three significant figures. How many terms in the series is it necessary to keep in order to attain this accuracy? Compare your numerical result with the average value of the potential on the walls. See Problem 2.28.

(c) Find the surface-charge density on the surface $z = 0$.

(a)

$$\nabla^2\Phi = 0 \Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

$$\begin{cases} \frac{1}{X}\frac{d^2X}{dx^2} + \alpha^2 = 0 \Rightarrow X = A\cos\alpha x + B\sin\alpha x \\ \frac{1}{Y}\frac{d^2Y}{dy^2} + \beta^2 = 0 \Rightarrow Y = C\cos\beta y + D\sin\beta y \\ \frac{1}{Z}\frac{d^2Z}{dz^2} - \gamma^2 = 0 \Rightarrow Z = E\sinh\gamma z + F\cosh\gamma z \end{cases}$$

where $\gamma^2 = \alpha^2 + \beta^2$

The boundary condition determine the constants:

$\Phi(0, y, z) = 0 \Rightarrow A = 0$

$\Phi(a, y, z) = 0 \Rightarrow \alpha_n = n\pi / a \ (n = 1,2,3...)$

$\Phi(x, a, z) = 0 \Rightarrow C = 0$

$\Phi(x, a, z) = 0 \Rightarrow \beta_m = m\pi / a \ (n = 1,2,3...)$

$\Rightarrow \gamma = \pi\sqrt{n^2 + m^2} / a = \gamma_{nm} / a$

The solution is reduced to

$$\Phi(x, y, z) = \sum_{n,m=1} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[A_{nm}\sinh\frac{\gamma_{nm}z}{a} + B_{nm}\cosh\frac{\gamma_{nm}z}{a}\right]$$

$$\Phi(x, y, 0) = V = \sum_{n,m=1} B_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)$$

$$\Rightarrow B_{nm} = \frac{4V}{a^2}\int_0^a\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)dxdy$$

It can be shown that the individual integrals vanish for even integer values and $2a / n\pi$ for odd integer.

$$\therefore B_{nm} = \frac{4V}{a^2}\frac{2a}{n\pi}\frac{2a}{m\pi} = \frac{16V}{\pi^2 nm}$$

Now since $\Phi(x, y, 0) = \Phi(x, y, a) = V$

$$B_{nm} = A_{nm}\sinh\gamma_{nm} + B_{nm}\cosh\gamma_{nm} \Rightarrow A_{nm} = B_{nm}\frac{1-\cosh\gamma_{nm}}{\sinh\gamma_{nm}}$$

$$\therefore \Phi(x, y, z) = \frac{16V}{\pi^2}\sum_{n,m}^{odd}\frac{1}{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[\frac{1-\cosh\gamma_{nm}}{\sinh\gamma_{nm}}\sinh\frac{\gamma_{nm}z}{a} + \cosh\frac{\gamma_{nm}z}{a}\right]$$

(b)

The potential at the center of the cube is

$$\Phi\left(\frac{a}{2},\frac{a}{2},\frac{a}{2}\right)=\frac{16V}{\pi^2}\sum_{n,m}^{odd}\frac{1}{nm}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{m\pi}{2}\right)\left[\frac{1-\cosh\gamma_{nm}}{\sinh\gamma_{nm}}\sinh\frac{\gamma_{nm}}{2}+\cosh\frac{\gamma_{nm}}{2}\right]$$

with just $n,m=1$

$\Phi\approx 0.347546V$

When we add the two terms $(n=3,m=1)$ and $(n=1,m=3)$

$\Phi\approx 0.332498V$

(c)

$$\frac{\partial\Phi}{\partial z}=\frac{16V}{\pi^2}\sum_{n,m}^{odd}\frac{\gamma_{nm}}{nma}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[\frac{1-\cosh\gamma_{nm}}{\sinh\gamma_{nm}}\cosh\frac{\gamma_{nm}z}{a}+\sinh\frac{\gamma_{nm}z}{a}\right]$$

$$\sigma=-\varepsilon_0\left.\frac{\partial\Phi}{\partial z}\right|_{z=a}=-\frac{16\varepsilon_0V}{\pi^2}\sum_{n,m}^{odd}\frac{\gamma_{nm}}{nma}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[\frac{1-\cosh\gamma_{nm}}{\sinh\gamma_{nm}}\cosh\gamma_{nm}+\sinh\gamma_{nm}\right]$$

$$=-\frac{16\varepsilon_0V}{\pi^2}\sum_{n,m}^{odd}\frac{\gamma_{nm}}{nma}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\frac{\cosh\gamma_{nm}-1}{\sinh\gamma_{nm}}$$

2.26

The two-dimensional region, $\rho\geq a$, $0\leq\phi\leq\beta$, is bounded by conducting surfaces at $\phi=0$, $\rho\geq a$, and $\phi=\beta$ held at zero potential, as indicated in the sketch. At large ρ the potential is determined by some configuration of charges and/or conductors at fixed potentials.

(a) Write down a solution for the potential $\Phi(\rho,\phi)$ that satisfies the boundary conditions for finite ρ.

(b) Keeping only the lowest non-vanishing terms, calculate the electric field components E_ρ and E_ϕ and also the surface-charge densities $\sigma(\rho,0)$, $\sigma(\rho,\beta)$, and $\sigma(\rho,\phi)$ on the three boundary surfaces.

(c) Consider

(a)

$$\Phi(\rho,\phi)=(a_0+b_0\ln\rho)(a_0+b_0\ln\phi)+\sum_\nu\left(a_\nu\rho^\nu+b_\nu\rho^{-\nu}\right)(A_\nu\cos\nu\phi+B_\nu\sin\nu\phi)$$

$$\Phi(\rho,0)=0$$

$$\Rightarrow b_0=0,\ a_0A_0=0,\ A_\nu=0$$

$$\Phi(\rho,\beta)=0$$

$$\Rightarrow \nu=\frac{m\pi}{\beta},\ m=1,2,3\ldots,\ a_0B_0=0$$

$$\Phi(\alpha,\phi)=0=a(A_0+B_0\phi)+\sum_{m=1}^{\infty}\left(a_m\rho^{\frac{m\pi}{\beta}}+b_m\rho^{\frac{-m\pi}{\beta}}\right)\sin\frac{m\pi\phi}{\beta}$$

$$\Rightarrow A_0=B_0=0,\ a_0\ \text{don't care, canbe set to zero,}\ b_m=-a_ma^{\frac{2m\pi}{\beta}}$$

define

$$C_m=a_mB_m$$

$$\Phi(\rho,\phi)=\sum_{m=1}^{\infty}C_m\left(\rho^{\frac{m\pi}{\beta}}-a_ma^{\frac{2m\pi}{\beta}}\rho^{\frac{-m\pi}{\beta}}\right)\sin\frac{m\pi\phi}{\beta}$$

(b)

The lowest non - vanishing term corresponds to m equal to 1

$$\Phi(\rho,\phi)\approx C_1\left(\rho^{\pi/\beta}-a^{\frac{2\pi}{\beta}}\rho^{\frac{-\pi}{\beta}}\right)\sin\frac{\pi\phi}{\beta}$$

$$\mathrm{E}_\rho(\rho,\phi)=-\frac{\partial\Phi}{\partial\rho}=C_1\frac{\pi}{\beta}\left(\rho^{\pi/\beta-1}-a^{\frac{2\pi}{\beta}}\rho^{\frac{-\pi}{\beta}-1}\right)\sin\frac{\pi\phi}{\beta}$$

$$\mathrm{E}_\phi(\rho,\phi)=-\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}=C_1\frac{\pi}{\beta}\left(\rho^{\pi/\beta-1}-a^{\frac{2\pi}{\beta}}\rho^{\frac{-\pi}{\beta}-1}\right)\cos\frac{\pi\phi}{\beta}$$

$$\Rightarrow$$

$$\mathrm{E}_\rho(a,\phi)=C_1\frac{\pi}{\beta}a^{\pi/\beta-1}\sin\frac{\pi\phi}{\beta}$$

$$\mathrm{E}_\phi(a,\phi)=0$$

$$\mathrm{E}_\rho(\rho,0)=0$$

$$\mathrm{E}_\phi(\rho,0)=C_1\frac{\pi}{\beta}\left(\rho^{\pi/\beta-1}-a^{\frac{2\pi}{\beta}}\rho^{\frac{-\pi}{\beta}-1}\right)$$

$$\mathrm{E}_\rho(\rho,\beta)=0$$

$$\mathrm{E}_\phi(\rho,\beta)=C_1\frac{\pi}{\beta}\left(\rho^{\pi/\beta-1}-a^{\frac{2\pi}{\beta}}\rho^{\frac{-\pi}{\beta}-1}\right)$$

Therefore

$$\sigma = -\varepsilon_0 \frac{\partial \Phi}{\partial n}$$

$$\sigma(a,\phi) = \varepsilon_0 E_\rho \big|_{\rho=a} = -2C_1 \varepsilon_0 \frac{\pi}{\beta} a^{\frac{\pi}{\beta}-1} \sin \frac{\pi\phi}{\beta}$$

$$\sigma(\rho,0) = \varepsilon_0 E_\phi \big|_{\phi=0} = -C_1 \varepsilon_0 \frac{\pi}{\beta} \left(\rho^{\frac{\pi}{\beta}-1} - a^{\frac{2\pi}{\beta}} \rho^{-\frac{\pi}{\beta}-1} \right)$$

$$\sigma(\rho,\beta) = \varepsilon_0 E_\phi \big|_{\phi=\beta} = -C_1 \varepsilon_0 \frac{\pi}{\beta} \left(\rho^{\frac{\pi}{\beta}-1} - a^{\frac{2\pi}{\beta}} \rho^{-\frac{\pi}{\beta}-1} \right)$$

(c)

consider

$\beta = \pi$ for $\rho >> a$

$$E_\phi(\rho,0) = E_\phi(\rho,\pi) = -C_1 \left(1 - \frac{a^2}{\rho^2} \right) \approx C_1$$