3.1

Two concentric spheres have radii a, b (b>a) and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential V. The other hemispheres are at zero potential. Determine the potential in the region a ≦ r ≦ b as a series in Legendre polynomials. Include terms at least up to l=4 . Check your solution against known results in the limiting cases b→∞, and a→0 .

The problem is symmertic

$$\Rightarrow \Phi(r,\theta)=\sum_l \left(A_l r^l + B_l r^{-l-1}\right)P_l(\cos\theta)$$

mutiply both sides by $P_{l'}(\cos\theta)$ then integrate

$$\Rightarrow \int_{-1}^{1}\Phi(r,\theta)P_l(\cos\theta)d(\cos\theta)=\frac{2}{2l+1}\left[A_l r^l + B_l r^{-l-1}\right]$$

$r=a$

$$\Rightarrow V\int_0^1 P_l(x)dx=\frac{2}{2l+1}\left[A_l a^l + B_l a^{-l-1}\right]$$

$r=b$

$$\Rightarrow V\int_{-1}^{0} P_l(x)dx=(-1)^l V\int_0^1 P_l(x)dx=\frac{2}{2l+1}\left[A_l b^l + B_l b^{-l-1}\right]$$

$$\left[P_l(x)=\frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l\left(x^2-1\right)^l \Rightarrow P_l(-x)=\frac{1}{2^l l!}\left(-\frac{d}{dx}\right)^l\left(x^2-1\right)^l=(-1)^l P_l(x)\right]$$

$$\int_0^1 P_l(x)dx=\left(\frac{-1}{2}\right)^{\frac{l-1}{2}}\frac{(l-2)!!}{2\left(\frac{l+1}{2}\right)!}=k$$

$$A_l=Vk\left(\frac{2l+1}{2}\right)\frac{(-1)^l\left(1-(-1)^l\left(\frac{a}{b}\right)^{l+1}\right)}{b^l\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)} \quad \& \quad B_l=Vk\left(\frac{2l+1}{2}\right)\frac{a^{l+1}\left(1-(-1)^l\left(\frac{a}{b}\right)^{l}\right)}{\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)}$$

$$\Phi(r,\theta)=V\sum_l\left(\left(\frac{-1}{2}\right)^{\frac{l-1}{2}}\frac{(l-2)!!}{4\left(\frac{l+1}{2}\right)!}\frac{2l+1}{\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)}\left((-1)^l\left(1-(-1)^l\left(\frac{a}{b}\right)^{l+1}\right)\left(\frac{r}{b}\right)^l+\left(1-(-1)^l\left(\frac{a}{b}\right)^l\right)\left(\frac{a}{r}\right)^{l+1}\right)\right)P_l(\cos\theta)$$

$$=\frac{3V}{4}\left[\frac{\left(a^2+b^2\right)r}{a^3-b^3}-\frac{a^2b^2(a+b)}{r^2\left(a^3-b^3\right)}\right]P_1(\cos\theta)-\frac{7V}{16}\left[\frac{\left(a^4+b^4\right)r^3}{a^7-b^7}-\frac{a^4b^4\left(a^3+b^3\right)}{r^4\left(a^7-b^7\right)}\right]P_3(\cos\theta)+\cdots$$

If $b\to\infty$

$$\Phi(r,\theta)=\frac{3V}{4}\left[\frac{a^2}{r^2}\right]P_1(\cos\theta)-\frac{7V}{16}\left[\frac{a^4}{r^4}\right]P_3(\cos\theta)+\cdots$$

If $a\to 0$

$$\Phi(r,\theta)=\frac{3V}{4}\left[\frac{r}{-b}\right]P_1(\cos\theta)-\frac{7V}{16}\left[\frac{r^3}{-b^3}\right]P_3(\cos\theta)+\cdots$$

3.2

A spherical surface of radius R has charge uniformly distributed over its surface with a density $Q/4\pi R^2$, except for a spherical cap at the north pole, defined by the cone $\theta=\alpha$.

(a) Show that the potential inside the spherical surface can be expressed as

$$\Phi=\frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left[P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)\right]\frac{r^l}{R^{l+1}}P_l(\cos\theta)$$

where, for $P_{-1}(\cos\alpha) = -1$. What is the potential outside?

(b) Find the magnitude and the direction of the electric field at the origin.

(c) Discuss the limiting forms of the potential (part a) and electric field (part b) as the spherical cap becomes (1) very small, and (2) so large that the area with charge on it becomes a very small cap at the south pole.

(a)

$$\Phi(r,0)=\frac{1}{4\pi\varepsilon_0}\int\frac{\sigma(r',\theta')}{|\vec{x}-\vec{x}'|}da'$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{Q}{4\pi R^2}\int_0^{2\pi}\int_\alpha^{\pi}4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r^l}{R^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)R^2\sin\theta' d\theta' d\varphi'$$

$$=\frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}P_l(\cos\theta)\int_\alpha^{\pi}P_l(\cos\theta')\sin\theta' d\theta' \underset{\downarrow}{=} \frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}P_l(\cos\theta)\int_{-1}^{\cos^{-1}(\alpha)}P_l(x)dx$$

$$P_l(x)=\frac{1}{2l+1}\left(P'_{l+1}(x)-P'_{l-1}(x)\right)$$

$$=\frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\frac{r^l}{R^{l+1}}P_l(\cos\theta)\left(P_{l+1}(x)-P_{l-1}(x)\right)\Big|_{-1}^{x=\cos^{-1}(\alpha)}$$

$$=\frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left[P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)\right]\frac{r^l}{R^{l+1}}P_l(\cos\theta)$$

(b)

$$\vec{E}=-\nabla\Phi\big|_{r=0}=\frac{\partial}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}$$

$$\frac{\partial\Phi}{\partial r}\bigg|_{r=0}=\left[\frac{Q}{8\pi\varepsilon_0}\sum_{l=1}^{\infty}\frac{1}{(2l+1)}\left[P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)\right]\frac{r^l}{R^{l+1}}P_l(\cos\theta)\right]\Bigg|_{r=0}$$

$$=\frac{Q}{24\pi\varepsilon_0 R^2}\left[P_2(\cos\alpha)-P_0(\cos\alpha)\right]P_1(\cos\theta)=\frac{Q}{24\pi\varepsilon_0 R^2}\left[\frac{3}{2}\cos^2\alpha-\frac{1}{2}-1\right]\cos\theta$$

$$=-\frac{Q}{16\pi\varepsilon_0 R^2}\sin^2\alpha\cos\theta$$

$$\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\bigg|_{r=0}=\left(\frac{Q}{8\pi\varepsilon_0}\sum_{l=1}^{\infty}\frac{1}{(2l+1)}\left[P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)\right]\frac{r^l}{R^{l+1}}\left[\frac{lP_{l-1}(\cos\theta)}{1-\cos^2\theta}-\frac{l\cos\theta P_l(\cos\theta)}{1-\cos^2\theta}\right](-\sin\theta)\right)\Bigg|_{r=0}$$

$$=-\frac{Q}{8\pi\varepsilon_0}\frac{P_2(\cos\alpha)-P_0(\cos\alpha)}{3R^2}\left[\frac{P_0(\cos\theta)}{1-\cos^2\theta}-\frac{\cos\theta P_1(\cos\theta)}{1-\cos^2\theta}\right]\sin\theta$$

$$=-\frac{Q}{16\pi\varepsilon_0 R^2\sin\theta}\left(\cos^2\alpha-1\right)\left[1-\cos^2\theta\right]=\frac{Q}{16\pi\varepsilon_0 R^2}\sin^2\alpha\sin\theta$$

$$\therefore\vec{E}=-\frac{Q}{16\pi\varepsilon_0 R^2}\sin^2\alpha\left(\cos\theta\hat{r}-\sin\theta\hat{\theta}\right)$$

(c)

cap very small

(i) *Cap very small*

$$\Phi=\frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left[P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)\right]\frac{r^l}{R^{l+1}}P_l(\cos\theta)$$

$$\underset{\alpha\to 0}{=}\frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left[P_{l+1}(1)-P_{l-1}(1)\right]\frac{r^l}{R^{l+1}}P_l(\cos\theta)$$

$$\cong\frac{Q}{8\pi\varepsilon_0}\left[P_1(1)-P_{-1}(1)\right]\frac{1}{R}P_0(\cos\theta)=\frac{Q}{4\pi\varepsilon_0 R}$$

$$\vec{E}=-\frac{Q}{16\pi\varepsilon_0 R^2}\sin^2\alpha\left(\cos\theta\hat{r}-\sin\theta\hat{\theta}\right)\underset{\alpha\to 0}{=}0$$

(ii) *Cap very large*

$$\Phi=\frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left[P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)\right]\frac{r^l}{R^{l+1}}P_l(\cos\theta)$$

$$\underset{\alpha\to\pi}{=}\frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left[P_{l+1}(-1)-P_{l-1}(-1)\right]\frac{r^l}{R^{l+1}}P_l(\cos\theta)\cong-\frac{Q}{4\pi\varepsilon_0 R}$$

$$\vec{E}=-\frac{Q}{16\pi\varepsilon_0 R^2}\sin^2\alpha\left(\cos\theta\hat{r}-\sin\theta\hat{\theta}\right)\underset{\alpha\to\pi}{=}0$$

3.3

A thin, flat, conducting, circular disc of radius R is located in the x-y plane with its center at the origin, and is maintained at a fixed potential V. With the information that the charge density on a disc at fixed potential is proportional to $1/\sqrt{R^2-\rho^2}$, where ρ is the distance out from the center of the disc .

(a) Show that for r > R the potential is

$$\Phi(r,\theta,\phi)=\frac{2V}{\pi}\frac{R}{r}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{2l}P_{2l}(\cos\theta)$$

(b) Fond the potential for r < R .

(c) What is the capacitance of the disc?

(a)

$$\rho\left(r',\theta'\right)=k\frac{1}{\sqrt{R^2-\rho^2}}\delta(z)\Theta(\rho-R)=k\frac{\delta(r'\cos\theta')\Theta(r'\sin\theta'-R)}{\sqrt{R^2-\left(r'\sin\theta'\right)^2}}$$

$$=k\frac{\delta(\cos\theta')\Theta(r'\sin\theta'-R)}{r'\sqrt{R^2-\left(r'\sin\theta'\right)^2}}$$

$$Y_{l0}(\theta,\varphi)\equiv\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)\ ,\ \textit{azimuthally symmetric}\ ,\ m=0$$

$$\frac{1}{\left|\vec{x}-\vec{x}'\right|}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r'^l}{r^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)=\sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}}P_l(\cos\theta')P_l(\cos\theta)$$

$$\Phi\left(r,\theta\right)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho\left(r',\theta'\right)}{\left|\vec{x}-\vec{x}'\right|}d^3x'$$

$$\underset{(\rho=r'\sin\theta')}{=}\frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}}P_l(\cos\theta')P_l(\cos\theta)\frac{k\delta(\cos\theta')\Theta(r'\sin\theta'-R)}{r'\sqrt{R^2-\left(r'\sin\theta'\right)^2}}r'^2dr'\sin\theta'd\theta'd\varphi'$$

$$=\frac{k}{2\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}P_l(0)P_l(\cos\theta)\int_0^R\frac{r'^{l+1}}{\sqrt{R^2-r'^2}}dr'=\frac{k}{2\varepsilon_0}\sum_{l=0}^{\infty}\frac{R^l}{r^{l+1}}P_l(0)P_l(\cos\theta)\frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{l+2}{2})}{\Gamma(\frac{l+3}{2})}$$

$$=\frac{k}{2\varepsilon_0R}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{2n+1}P_{2n}(0)P_{2n}(\cos\theta)\frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{2n+2}{2})}{\Gamma(\frac{2n+3}{2})}\ ,\ \left(P_{2n+1}(0)=0\ ,\ P_{2n}(0)=(-1)^n\frac{(2n-1)!!}{(2n)!!}\right)$$

$$=\frac{k}{2\varepsilon_0R}\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}\left(\frac{R}{r}\right)^{2n+1}P_{2n}(\cos\theta)$$

$$=\frac{k}{\varepsilon_0R}\tan^{-1}\left(\frac{R}{r}\right)$$

$$\therefore\Phi\left(r=R,\theta=0\right)=V=\frac{k}{\varepsilon_0R}\tan^{-1}\left(\frac{R}{R}\right)=\frac{k}{\varepsilon_0R}\frac{\pi}{4}\Rightarrow k=\frac{4\varepsilon_0RV}{\pi}$$

$$\Phi\left(r,\theta\right)=\frac{2V}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}\left(\frac{R}{r}\right)^{2n+1}P_{2n}(\cos\theta)$$

(b) for

$$\frac{r}{R}\leq 1$$

$$\Phi(r,\theta)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(r',\theta')}{|\vec{x}-\vec{x}'|}d^3x'$$

$$\underset{(\rho=r'\sin\theta')}{=}\frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta')P_l(\cos\theta)\frac{k\delta(\cos\theta')\Theta(r'\sin\theta'-R)}{r'\sqrt{R^2-(r'\sin\theta')^2}}r'^2dr'\sin\theta'd\theta'd\varphi'$$

$$=\frac{k}{2\varepsilon_0}\int_0^R\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(0)P_l(\cos\theta)\frac{r'}{\sqrt{R^2-(r')^2}}dr'=\frac{k}{2\varepsilon_0}\sum_{l=0}^{\infty}P_l(0)P_l(\cos\theta)\int_0^R\frac{r_<^l}{r_>^{l+1}}\frac{r'}{\sqrt{R^2-(r')^2}}dr'$$

$$=\frac{k}{2\varepsilon_0}\sum_{l=0}^{\infty}P_l(0)P_l(\cos\theta)\left(\int_0^r\frac{r'^l}{r^{l+!}}\frac{r'}{\sqrt{R^2-(r')^2}}dr'+\int_r^R\frac{r^l}{r'^{l+1}}\frac{r'}{\sqrt{R^2-(r')^2}}dr'\right)$$

$$\tan^{-1}\left(\frac{R}{r}\right)=\frac{\pi}{2}-\frac{r}{R}\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l+1)}\left(\frac{r}{R}\right)^{2l}$$

$$\Phi(r,\theta)=V-\frac{2V}{\pi}\frac{r}{R}\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l+1)}\left(\frac{r}{R}\right)^{2l}P_{2l}(\cos\theta)$$

(c)

$$Q=2\pi C\int_0^R\frac{\rho d\rho}{\sqrt{R^2-\rho^2}}=\frac{2VR}{\pi}$$

$$C=\frac{Q}{V}=\frac{2R}{\pi}$$

3.6

Two point charges q and –q are located on the z axis at z=a and z=-a , respectively.

(a) Find the electrostatic potential as an expansion in spherical harmonics and powers of r for both r>a and r<a .

(b) Keeping the product qa ≡ p/2

(c) constant, take the limit of a→0 and find the potential for r≠0. This is by definition a dipole along the z axis and its potential.

(d) Suppose now that the dipole of part b is surrounded by a grounded spherical shell of radius b concentric with the origin. By linear superposition find the potential everywhere inside the shell.

(a)

$$\Phi(\vec{x})=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\vec{x}-\vec{x}_1|}-\frac{q}{|\vec{x}-\vec{x}_2|}\right)$$

$$=\frac{q}{4\pi\varepsilon_0}\left[4\pi\sum_{l,m}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}\left(Y_{lm}^*(0,\phi_1)Y_{lm}(\theta,\phi)-Y_{lm}^*(\pi,\phi_2)Y_{lm}(\theta,\phi)\right)\right]$$

the problem is azimuthally symmetric, only m=0 terms survive.

$$Y_{lm}^{*}(0,\varphi_1)=\sqrt{\frac{2l+1}{4\pi}}P_l(\cos 0)=\sqrt{\frac{2l+1}{4\pi}} \quad \& \quad Y_{lm}^{*}(\pi,\varphi_2)=\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\pi)=(-1)^l\sqrt{\frac{2l+1}{4\pi}}$$

$$\Rightarrow \Phi(\vec{x})=\frac{q}{\varepsilon_0}\sum_l \frac{1}{2l+1}\sqrt{\frac{2l+1}{4\pi}}\frac{r_<^l}{r_>^{l+1}}Y_{l0}(\theta,\varphi)\left[1-(-1)^l\right]=\frac{q}{4\pi\varepsilon_0}\sum_l \frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)\left[1-(-1)^l\right]$$

$$\Phi(\vec{x})=\begin{cases}\dfrac{q}{4\pi\varepsilon_0}\sum_l \dfrac{r^l}{a^{l+1}}P_l(\cos\theta)\left[1-(-1)^l\right], & \text{r < a}\\ \dfrac{q}{4\pi\varepsilon_0}\sum_l \dfrac{a^l}{r^{l+1}}P_l(\cos\theta)\left[1-(-1)^l\right], & \text{r > a}\end{cases}$$

(b)

$$r>a \;\&\; a\to 0$$

$$\Phi(r>a,\theta)=\frac{q}{4\pi\varepsilon_0}\sum_l \frac{a^l}{r^{l+1}}P_l(\cos\theta)\left[1-(-1)^l\right]=\frac{q}{4\pi\varepsilon_0}\sum_{l=odd}\frac{2a^l}{r^{l+1}}P_l(\cos\theta)$$

$$\simeq \frac{q}{4\pi\varepsilon_0}\frac{2a}{r^2}P_1(\cos\theta)=\frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

(c)

$$\Phi=\Phi_s+\Phi_d=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta)+\frac{p}{4\pi\varepsilon_0 r^2}\cos\theta$$

$$\because \Phi(r=b)=0$$

$$\therefore \Phi(r,\theta)=\frac{p}{4\pi\varepsilon_0 b^2}\left(1-\frac{r}{b}\right)\cos\theta$$

3.7

Three point charges (q, -2q, q) are located in a straight line with separation a and with the middle charge (-q) at the origin of a grounded conducting spherical shell of radius b, as indicated in sketch (Jackson p. 137).

(a) Write down the potential of the three charges in the absence of the grounded sphere. Find the limiting form of the potential as a→0 , but the product $qa^2=Q$ remains finite. Write this latter answer in spherical coordinates.

(b) The presence of the grounded sphere of radius b alters the potential for r < b . The added potential can be viewed as caused by the surface-charge density induced on the inner surface at r = b or by image charges located at r > b. Use linear superposition to satisfy the boundary conditions and find the potential everywhere inside the sphere for r < a and r > a. Show that in the limit a→0,

$$\Phi(r,\theta,\phi)\to\frac{Q}{2\pi\varepsilon_0 r^3}\left(1-\frac{r^5}{b^5}\right)P_2(\cos\theta)$$

(a)

this problem is similar to (3.6), we use slightly different method here:

on z-axis the potential is

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{|\vec{x}-\vec{x}_1|} + \frac{q_2}{|\vec{x}-\vec{x}_2|} + \frac{q_3}{|\vec{x}-\vec{x}_3|}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{-2}{r} + \frac{1}{|\vec{x}-\vec{x}_2|} + \frac{1}{|\vec{x}-\vec{x}_3|}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{-2}{r} + 4\pi\sum_{l,m}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}\left(Y_{lm}^*(0,\varphi_2')Y_{lm}(\theta,\varphi) + Y_{lm}^*(\pi,\varphi_3')Y_{lm}(\theta,\varphi)\right)\right]$$

azimuthally symmetric , only $m = 0$ *exist*

$$\Phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{-2}{r} + \sum_l \frac{r_<^l}{r_>^{l+1}}\left(P_l(1) + P_l(-1)\right)P_l(\cos\theta)\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{-2}{r} + \sum_l \frac{r_<^l}{r_>^{l+1}}\left(1 + (-1)^l\right)P_l(\cos\theta)\right]$$

When $a \to 0$

$$\Phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{-2}{r} + \sum_l \frac{r_<^l}{r_>^{l+1}}\left(1 + (-1)^l\right)P_l(\cos\theta)\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{-2}{r} + \sum_{l=even}\frac{2a^l}{r^{l+1}}P_l(\cos\theta)\right] \underset{l=2n}{=} \frac{q}{4\pi\varepsilon_0}\left[\frac{-2}{r} + \sum_n \frac{2a^{2n}}{r^{2n+1}}P_{2n}(\cos\theta)\right]$$

$$\cong \frac{q}{4\pi\varepsilon_0}\left[\frac{2a^2}{r^3}P_2(\cos\theta)\right] = \frac{Q}{2\pi\varepsilon_0 r^3}P_2(\cos\theta) = \frac{Q}{4\pi\varepsilon_0 r^3}\left(3\cos^2\theta - 1\right)$$

(b)

the surface charge on the sphere produce an extra contribution Φ_s to the potential within the sphere.

$$\Phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{-2}{r} + \sum_l \frac{r_<^l}{r_>^{l+1}}\left(1 + (-1)^l\right)P_l(\cos\theta)\right] + \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Rightarrow \Phi(r,\theta) = \begin{cases} \frac{q}{4\pi\varepsilon_0}\left[\frac{-2}{r} + \sum_{l=even}\frac{2r^l}{a^{l+1}}P_l(\cos\theta)\right] + \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)\ ,\ r < a \\ \frac{q}{4\pi\varepsilon_0}\left[\frac{-2}{r} + \sum_{l=even}\frac{2a^l}{r^{l+1}}P_l(\cos\theta)\right] + \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)\ ,\ r > a \end{cases}$$

$$\Phi(b,\theta) = 0 = \frac{q}{4\pi\varepsilon_0}\left[\frac{-2}{b} + \sum_{l=even}\frac{2a^l}{b^{l+1}}P_l(\cos\theta)\right] + \sum_{l=0}^{\infty} A_l b^l P_l(\cos\theta)\ ,\ r = b$$

$$\Rightarrow l = 0\ ,\ A_0 = 0\ ,\ A_{odd} = 0$$

$$\frac{q}{4\pi\varepsilon_0}\left[\sum_{l=1}\frac{2a^{2n}}{b^{2n+1}}P_{2n}(\cos\theta)\right] + \sum_{n=1}^{\infty} A_{2n} b^{2n} P_{2n}(\cos\theta)\ \Rightarrow\ A_{2n} = -\frac{q}{4\pi\varepsilon_0}\frac{2a^{2n}}{b^{4n+1}}$$

$$\Phi(r,\theta)=\begin{cases}\dfrac{q}{4\pi\varepsilon_0}\left[\dfrac{-2}{r}+\dfrac{2}{a}\right]-\sum_{n=1}^{\infty}\left[\dfrac{2}{a^{2n+1}}-\dfrac{2a^{2n}}{b^{4n+1}}\right]r^{2n}P_{2n}(\cos\theta)\ ,\ r<a\\ \dfrac{q}{4\pi\varepsilon_0}\sum_{n=1}^{\infty}\left[\dfrac{2a^{2n}}{r^{2n+1}}-\dfrac{2a^{2n}}{b^{4n+1}}r^{2n}\right]P_{2n}(\cos\theta)\ ,\ r>a\end{cases}$$

$$a\to 0\ ,\ \Phi(r,\theta)=\frac{q}{4\pi\varepsilon_0}\sum_{n=1}^{\infty}\left[\frac{2a^{2n}}{r^{2n+1}}-\frac{2a^{2n}}{b^{4n+1}}r^{2n}\right]P_{2n}(\cos\theta)\approx\frac{Q}{2\pi\varepsilon_0}\left[\frac{1}{r^3}-\frac{r^2}{b^5}\right]P_2(\cos\theta)$$

$$=\frac{Q}{2\pi\varepsilon_0 r^3}\left[1-\left(\frac{r}{b}\right)^5\right]P_2(\cos\theta)$$

3.9

A hollow right circular cylinder of radius b has its axis coincident with the z axis and its ends at 0 and L . The potential on the end faces is zero, while the potential on the cylindrical surface is given as V(φ,z). Using the appropriate separation of variables in cylindrical coordinates, find a series solution for the potential anywhere inside the cylinder.

$\vec{\nabla}^2\phi=0$ with b.c.

$\Phi(\rho,\phi,0)=0$
$\Phi(\rho,\phi,L)=0$
$\Phi(b,\phi,z)=V(\phi,z)$

b.c. Q(φ+2π)= Q(φ)

$$\therefore\Phi(\rho,\phi,z)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}I_m\left(\frac{n\pi}{L}\rho\right)\sin\left(\frac{n\pi}{L}z\right)\{A_{mn}\sin(m\phi)+B_{mn}\cos(m\phi)\}$$

A_{mn} , B_{mn} to be determined from b.c. at ρ=b

$$V(\varphi,z)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}I_m\left(\frac{n\pi}{L}b\right)\sin\left(\frac{n\pi}{L}z\right)\{A_{mn}\sin(m\varphi)+B_{mn}\cos(m\varphi)\}$$

$$\int_0^L dz\int_0^{2\pi}d\varphi V(\varphi,z)\sin(m'\varphi)\sin\left(\frac{n'\pi}{L}z\right)=\pi\sum_{n=0}^{\infty}I_{m'}\left(\frac{\pi nb}{L}\right)A_{m'n}\frac{L}{2}\delta_{n'n}=\pi I_{m'}\left(\frac{\pi n'b}{L}\right)A_{m'n'}\frac{L}{2}$$

$\therefore$

$$A_{mn}=\frac{2}{\pi L I_m\left(\frac{\pi nb}{L}\right)}\int_0^L dz\int_0^{2\pi}d\varphi V(\varphi,z)\sin(m\varphi)\sin\left(\frac{n\pi}{L}z\right)$$

$$B_{mn}=\frac{2}{\pi L I_m\left(\frac{\pi nb}{L}\right)}\int_0^L dz\int_0^{2\pi}d\varphi V(\varphi,z)\cos(m\varphi)\sin\left(\frac{n\pi}{L}z\right)$$

3.17

The Dirichlet Green function for the unbounded space between the planes at z=0 and z=L allows discussion of a point charge or a distribution of charge between parallel conducting planes held at zero potential.

(a) Using cylindrical coordinates show that one form of the Green function is

$$G(x,x')=\frac{4}{L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)I_m\left(\frac{n\pi}{L}\rho_<\right)K_m\left(\frac{n\pi}{L}\rho_>\right)$$

(b) Show that an alternative form of the Green function is

$$G(x,x')=2\sum_{m=-\infty}^{\infty}\int_0^{\infty}dke^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')\frac{\sin(kz_<)\sinh[k(L-z_>)]}{\sinh(kL)}$$

(a)

The differential equation is

$$\nabla^2 G=-\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')$$

eign function in z direction $\sin\left(\frac{n\pi z}{L}\right)$, in ϕ direction $e^{im\phi}$

$$G=\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}A_{mn}(\rho,\rho',z',\phi')\sin\left(\frac{n\pi z}{L}\right)e^{im\phi}$$

plug into differential equation

$$\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}-\left(\frac{n\pi}{L}\right)^2\right)A_{mn}\sin\left(\frac{n\pi z}{L}\right)e^{im\varphi}$$

$$=-\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\varphi-\varphi')\delta(z-z')$$

$$\int_0^L dz\sin\left(\frac{n'\pi z}{L}\right)\int_0^{2\pi}e^{im'\varphi}d\varphi\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}-\frac{m'^2}{\rho^2}-\left(\frac{n'\pi}{L}\right)^2\right)A_{mn}L\pi$$

$$=-\frac{4\pi}{\rho}\delta(\rho-\rho')\sin\left(\frac{n'\pi z'}{L}\right)e^{im'\varphi'}\Rightarrow A_{mn}=g_{mn}(\rho,\rho')\sin\left(\frac{n'\pi z'}{L}\right)e^{im'\varphi'}$$

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}-\frac{m'^2}{\rho^2}-\left(\frac{n'\pi}{L}\right)^2\right)g_{mn}(\rho,\rho')=-\frac{4}{\rho L}\delta(\rho-\rho')$$

$$g_{mn}(\rho,\rho')=\begin{cases}A_mI_m(k\rho)\ ,\ \rho<\rho'\\ B_mK_m(k\rho)\ ,\ \rho>\rho'\end{cases}\Rightarrow When\quad \rho=\rho'\Rightarrow A_mI_m(k\rho')=B_mK_m(k\rho')\ ,\ k=\frac{n\pi}{L}$$

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)-\left(k^2+\frac{m^2}{\rho^2}\right)\right]g_{mn}(\rho,\rho')=-\frac{4}{\rho L}\delta(\rho-\rho')$$

$$\Rightarrow\int_{\rho'^-}^{\rho'^+}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)-\rho\left(k^2+\frac{m^2}{\rho^2}\right)\right]g_{mn}(\rho,\rho')d\rho=-\frac{4}{L}\int_{\rho'^-}^{\rho'^+}\delta(\rho-\rho')d\rho$$

$$\Rightarrow\frac{\partial}{\partial\rho}g_{mn}(\rho,\rho')\Big|_{\rho'^-}^{\rho'^+}=-\frac{4}{L\rho'}\Rightarrow kB_mK_m'(k\rho')-kA_mI_m'(k\rho')=-\frac{4}{L\rho'}$$

$$A_m=-\frac{4}{Lk\rho'}\frac{K_m(k\rho')}{\left(I_m(k\rho')K_m'(k\rho')-I_m'(k\rho')K_m(k\rho')\right)}\quad\&\quad B_m=-\frac{4}{Lk\rho'}\frac{I_m(k\rho')}{\left(I_m(k\rho')K_m'(k\rho')-I_m'(k\rho')K_m(k\rho')\right)}$$

$$\text{Use the relation } : I_m(k\rho')K_m'(k\rho') - I_m'(k\rho')K_m(k\rho') = -\frac{1}{k\rho'}$$

$$\Rightarrow A_m = \frac{4K_m(k\rho')}{L} \quad \& \quad B_m = \frac{4I_m(k\rho')}{L} \Rightarrow g_{mn}(\rho,\rho') = \begin{cases} \frac{4}{L} I_m(k\rho)K_m(k\rho') \ , \ \rho < \rho' \\ \frac{4}{L} I_m(k\rho')K_m(k\rho) \ , \ \rho > \rho' \end{cases} \Rightarrow g_{mn}(\rho,\rho') = \frac{4}{L} I_m(k\rho_<)K_m(k\rho_>)$$

$$G = \frac{4}{L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} I_m\left(\frac{n\pi}{L}\rho_<\right)K_m\left(\frac{n\pi}{L}\rho_>\right)\sin\left(\frac{n\pi z'}{L}\right)\sin\left(\frac{n\pi z}{L}\right)e^{im(\varphi-\varphi')}$$

(b) do expansion in ρ & ϕ directions

$$G = \int_0^\infty dk \sum_{m=-\infty}^{\infty} A_{mk}(z,z',\rho',\phi')e^{im\phi}J_m(k\rho)$$

$$\nabla^2 G = \int_0^\infty dk \sum_{m=-\infty}^{\infty}\left(-k^2 + \frac{\partial^2}{\partial z^2}\right)A_{mk}(z,z',\rho',\phi')e^{im\phi}J_m(k\rho)$$

$$= -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')$$

$$\int_0^\infty dk \sum_{m=-\infty}^{\infty}\left(-k^2 + \frac{\partial^2}{\partial z^2}\right)A_{mk}e^{im\phi}J_m(k\rho) = -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi')\delta(z-z')$$

$$\int_0^\infty dk \sum_{m=-\infty}^{\infty}\left(-k^2 + \frac{\partial^2}{\partial z^2}\right)\left(\int_0^{2\pi} e^{-im'\phi}e^{im\phi}d\phi\right)A_{mk}\int_0^\infty J_m(k\rho)J_{m'}(k\rho)\rho d\rho = -4\pi\int_0^\infty \frac{\delta(\rho-\rho')}{\rho}J_{m'}(k\rho)\rho d\rho\int_0^{2\pi} e^{-im'\phi}\delta(\phi-\phi')d\phi\delta(z-z')$$

$$\int_0^\infty dk\left[\left(-k^2 + \frac{\partial^2}{\partial z^2}\right)A_{mk}\right]\int_0^\infty J_m(k\rho)J_m(k\rho)\rho d\rho = -2J_m(k\rho')e^{-im\phi'}\delta(z-z') \Rightarrow \int_0^\infty \frac{\delta(k-k')}{k}dk\left[\left(-k^2 + \frac{\partial^2}{\partial z^2}\right)A_{mk}\right] = -2J_m(k\rho')e^{-im\phi'}\delta(z-z')$$

$$\Rightarrow \left(-k^2 + \frac{\partial^2}{\partial z^2}\right)A_{mk} = -2kJ_m(k\rho')e^{-im\phi'}\delta(z-z') \Rightarrow A_{mk} = g_{mk}(z)e^{-im\varphi'}J_m(k\rho') \Rightarrow \left(-k^2 + \frac{\partial^2}{\partial z^2}\right)g_{mk}(z) = -2k\delta(z-z')$$

at (Same as (a))

$$\psi_1(0) = 0, \psi_2(L) = 0$$

$$\therefore \psi_1 = \sinh(kz), \psi_2 = \sinh(k(L-z))$$

$$CW\left[\sinh(kz), \sinh(k(L-z))\right] = -2k = -k\sinh(kL)C$$

$$\therefore C = \frac{2}{\sinh(kL)}$$

$$G = \int_0^\infty dk \sum_{m=-\infty}^{\infty} \frac{2\sinh(kz_<)\sinh(k(L-z_>))}{\sinh(kL)}e^{im(\varphi-\varphi')}J_m(k\rho)J_m(k\rho')$$

3.20

(a) From the results of Problem 3.17 or from first principles show that the potential at a point charge q between two infinite parallel conducting planes geld at zero potential can be written as

$$\Phi(z,\rho) = \frac{q}{\pi\varepsilon_0 L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z_0}{L}\right)\sin\left(\frac{n\pi z}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right)$$

where the planes are at z = 0 and z = L and the charge z_0 on the z axis at the point .

(b) Calculate the induced surface-charge densities $\sigma_0(\rho)$ and $\sigma_L(\rho)$ on the lower and upper plates. The result for $\sigma_L(\rho)$ is

$$\sigma_L(\rho)=\frac{q}{L^2}\sum_{n=1}^{\infty}(-1)^n n\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi\rho}{L}\right)$$

Discuss the connection of this expression with that of Problem 3.19b and 3.19c.

(c) From the answer in part (b) , calculate the total charge Q_L on the plate at z = L. By summing the Fourier series or by other means of comparison, check your answer against the known expression of Problem 1.13.

(a)

$$\rho(\vec{x}')=q\frac{1}{\rho'}\delta(\rho'-\rho_0)\delta(\varphi'-\varphi_0)\delta(z'-z_0)$$

$$G(x,x')=\frac{4}{L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}e^{im(\varphi-\varphi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)I_m\left(\frac{n\pi}{L}\rho_<\right)K_m\left(\frac{n\pi}{L}\rho_>\right)$$

$$\Phi=\frac{1}{4\pi\varepsilon_0}\int\rho(\vec{x}')G(\vec{x},\vec{x}')d^3\vec{x}'\ \ (\phi\ is\ symmetry\ ,\ only\ m=0\ term)$$

$$=\frac{q}{\pi\varepsilon_0 L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z_0}{L}\right)\int_0^{\infty}I_0\left(\frac{n\pi}{L}\rho_<\right)K_0\left(\frac{n\pi}{L}\rho_>\right)\delta(\rho'-\rho_0)d\rho'$$

$$=\frac{q}{\pi\varepsilon_0 L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z_0}{L}\right)\int_0^{\infty}I_0\left(\frac{n\pi}{L}\rho_<\right)K_0\left(\frac{n\pi}{L}\rho_>\right)\delta(\rho'-\rho_0)d\rho'$$

$$=\frac{q}{\pi\varepsilon_0 L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z_0}{L}\right)\times$$

$$\left[\int_{\rho'=0}^{\rho}I_0\left(\frac{n\pi}{L}\rho'\right)K_0\left(\frac{n\pi}{L}\rho\right)\delta(\rho'-\rho_0)d\rho'+\int_{\rho'=\rho}^{\infty}I_0\left(\frac{n\pi}{L}\rho\right)K_0\left(\frac{n\pi}{L}\rho'\right)\delta(\rho'-\rho_0)d\rho'\right]$$

$$\underset{\rho_0=0}{=}\frac{q}{\pi\varepsilon_0 L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z_0}{L}\right)\left[I_0(0)K_0\left(\frac{n\pi}{L}\rho\right)+I_0\left(\frac{n\pi}{L}\rho\right)K_0(0)\right]$$

$$=\frac{q}{\pi\varepsilon_0 L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi}{L}\rho\right)$$

(b)

$$\sigma_L=-\hat{n}\cdot\varepsilon_0\nabla\Phi\big|_{z=L}=-(-\hat{z})\cdot\nabla\Phi\big|_{z=L}$$

$$=\frac{q}{\pi L}\sum_{n=1}^{\infty}\frac{n\pi}{L}\cos(n\pi)\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi}{L}\rho\right)=\frac{q}{L^2}\sum_{n=1}^{\infty}(-1)^n n\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi}{L}\rho\right)$$

$$\sigma_0=-\hat{n}\cdot\varepsilon_0\nabla\Phi\big|_{z=0}=-\hat{z}\cdot\nabla\Phi\big|_{z=0}$$

$$=-\frac{q}{\pi L}\sum_{n=1}^{\infty}\frac{n\pi}{L}\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi}{L}\rho\right)=-\frac{q}{L^2}\sum_{n=1}^{\infty}n\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi}{L}\rho\right)$$

(c)

$$Q=\frac{q}{L^2}\int_{\rho=0}^{\infty}\int_{\varphi=0}^{2\pi}\sum_{n=1}^{\infty} n\sin\left(\frac{n\pi z_0}{L}\right)K_0\left(\frac{n\pi}{L}\rho\right)\rho d\rho d\varphi=\frac{2\pi q}{L^2}\sum_{n=1}^{\infty} n(-1)^n\sin\left(\frac{n\pi z_0}{L}\right)\int_0^{\infty}K_0\left(\frac{n\pi}{L}\rho\right)\rho d\rho$$

$$=\frac{2\pi q}{L^2}\frac{L^2}{\pi^2}\sum_{n=1}^{\infty}(-1)^n\sin\left(\frac{n\pi z_0}{L}\right)\frac{1}{n}=\frac{2q}{\pi}I_m\sum_{n=1}^{\infty}(-1)^n\frac{e^{\frac{n\pi z_0}{L}i}}{n}=-\frac{2q}{\pi}I_m\ln\left(1+e^{\frac{\pi z_0}{L}i}\right)$$

$$=-\frac{2q}{\pi}I_m\ln\left(e^{-\frac{\pi z_0}{L}i}\right)\left(e^{-\frac{\pi z_0}{2L}i}+e^{\frac{\pi z_0}{2L}i}\right)=\frac{qz_0}{L}$$

3.22

The geometry of a two-dimensional potential problem is defined in polar coordinates by the surfaces $\phi=0$, $\phi=\beta$ and ρ=a , as indicated in the sketch. Using separation of variables in polar coordinates, show that the Green function can be written as

$$G(\rho,\phi,\rho',\phi')=\sum_{m=1}^{\infty}\frac{4}{m}\rho_<^{\frac{m\pi}{\beta}}\left(\frac{1}{\rho_>^{\frac{m\pi}{\beta}}}-\frac{\rho_>^{\frac{m\pi}{\beta}}}{\rho^{2\frac{m\pi}{\beta}}}\right)\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right)$$

Problem 2.25 may be of use.

From 2.24, for , the angular solution is $Q_m(\phi)\sim\sin\left(\frac{m\pi\phi}{\beta}\right)$

$$\delta(\phi-\phi')=\frac{2}{\beta}\sum_{m=1}^{\infty}\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right)$$

$$\vec{\nabla}^2G(\rho,\phi,\rho',\phi')=-\frac{8\pi}{\rho\beta}\delta(\rho-\rho')\sum_{m=1}^{\infty}\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right)$$

Expand G

$$\delta(\phi-\phi')=\frac{2}{\beta}\sum_{m=1}^{\infty}\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right)$$

$$G(\rho,\phi,\rho',\phi')=\sum_{m=1}^{\infty}g_m(\rho,\rho')\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right)$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g_m}{\partial\rho}\right)-\frac{1}{\rho^2}\left(\frac{m\pi}{\beta}\right)^2=-\frac{8\pi}{\rho\beta}\delta(\rho-\rho')$$

For ,

$$\begin{cases}\rho<\rho', g_m(\rho\to 0,\rho')=0\\ \rho>\rho', g_m(a,\rho')=0\end{cases}\Rightarrow\begin{cases}B_m=0, g_m(\rho,\rho')=A_m(\rho')\rho^{m\pi/\beta}\\ A_m=-B_m\rho^{-2m\pi/\beta}, g_m(\rho,\rho')=B_m(\rho')\left(\rho^{m\pi/\beta}-a^{-2m\pi/\beta}\rho^{m\pi/\beta}\right)\end{cases}$$ is

invariant under ρ,ρ' exchange

$$\therefore g_m = C_m \rho_<^{m\pi/\beta} \left[\rho_>^{\ m\pi/\beta} - a^{-2m\pi/\beta} \rho_>^{\ m\pi/\beta} \right]$$

integrate across the jump

$$\int_{\rho'-\varepsilon}^{\rho'+\varepsilon} d\rho \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial g_m}{\partial \rho} \right) - \frac{1}{\rho^2} \left(\frac{m\pi}{\beta} \right)^2 \right\} = -\int_{\rho'-\varepsilon}^{\rho'+\varepsilon} d\rho \frac{8\pi}{\beta\rho} \delta(\rho - \rho') d\rho$$

as $\left.\frac{\partial g_m}{\partial \rho}\right|_{\rho'_+} - \left.\frac{\partial g_m}{\partial \rho}\right|_{\rho'_-} = -\frac{8\pi}{\beta\rho'}$

$$-C_m \frac{1}{\rho'} \frac{m\pi}{\beta} \left\{ 1 + \left(\frac{\rho'}{a} \right)^{2m\pi/\beta} \right\} - C_m \frac{1}{\rho'} \frac{m\pi}{\beta} \left\{ 1 - \left(\frac{\rho'}{a} \right)^{2m\pi/\beta} \right\} = -\frac{8\pi}{\beta\rho'}$$

$$\therefore C_m = \frac{4}{m}$$

$$g_m = \frac{4}{m} \rho_<^{m\pi/\beta} \left[\rho_>^{\ m\pi/\beta} - a^{-2m\pi/\beta} \rho_>^{\ m\pi/\beta} \right]$$

$$G(\rho, \phi, \rho', \phi') = \sum_{m=1}^{\infty} \frac{4}{m} \rho_<^{\frac{m\pi}{\beta}} \left(\frac{1}{\rho_>^{\frac{m\pi}{\beta}}} - \frac{\rho_>^{\frac{m\pi}{\beta}}}{\rho^{2\frac{m\pi}{\beta}}} \right) \sin\left(\frac{m\pi\phi}{\beta} \right) \sin\left(\frac{m\pi\phi'}{\beta} \right)$$