4.1

Calculate the multipole moments q_{lm} of the charge distributions shown as parts a and b. Try to obtain results for the nonvanishing moments valid for all l, but in each case find the first two sets of nonvanishing moments at the very least.

(a) Page 170

(b) Page 170

(c) For the charge distribution of the second set b write down the multi-pole expansion for the potential. Keeping only the lowest-order term in the expansion, plot the potential in the x-y plane as a function of distance from the origin for distances greater than a.

(d) Calculate directly from Coulomb's law the exact potential for b in the x-y plane. Plot it as a function of distance and compare with the result found in part a.

(a)

$$\rho(\vec{x}) = \frac{q}{r^2}\delta(r-a)\delta(\cos\theta)\left[\delta(\phi)+\delta\left(\phi+\frac{\pi}{2}\right)-\delta(\phi-\pi)-\delta\left(\phi+\frac{3\pi}{2}\right)\right]$$

$$q_{lm} = \int r^l Y_l^{m*}(\theta,\phi)\rho(\vec{x})d^3x$$

$$= \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}qa^l P_l^m(0)\left\{1+e^{-im\frac{\pi}{2}}-e^{-im\pi}-e^{-im\frac{3\pi}{2}}\right\}$$

$$\therefore\ q_{lm}\neq 0 \text{ only for both } l, m \text{ odd}$$

$$q_{lm} = 2\left\{1+i(-1)^{k+1}\right\}\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}qa^l P_l^m(0)$$

$$q_{1,\pm1} = \mp(1\mp i)\sqrt{\frac{3}{2\pi}}qa$$

$$q_{3,\pm1} = \pm(1\mp i)\sqrt{\frac{21}{16\pi}}qa^3$$

(b)

$$\rho = \frac{q}{2\pi r^2}\left\{\delta(r-a)\delta(\cos\theta-1)+\delta(r-a)\delta(\cos\theta+1)-2\delta(r)\right\}$$

The system is symmetry for the z axis , only exist m=0

$$q_{l0} = \int r^l Y_l^{0*}(\theta,\varphi)\rho(\vec{x})d^3x = \int r^l\sqrt{\frac{(2l+1)}{4\pi}}P_l(\cos\theta)\rho(\vec{x})d^3x$$

$$= q\int r^l\sqrt{\frac{(2l+1)}{4\pi}}P_l(\cos\theta)\left\{\delta(r-a)\delta(\cos\theta-1)+\delta(r-a)\delta(\cos\theta+1)-2\delta(r)\right\}drd(\cos\theta)$$

$$= qa^l\sqrt{\frac{(2l+1)}{4\pi}}\left[P_l(1)+P_l(-1)-2\delta_{l0}\right]$$

first two nonvanishing moment :

$$q_{2,0} = \sqrt{\frac{5}{\pi}}qa^2 \ \ \& \ \ q_{4,0} = \sqrt{\frac{9}{\pi}}qa^4$$

(c)

$$\Phi(\vec{r})=\frac{1}{4\pi\varepsilon_0}\sum_{lm}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}=\frac{1}{4\pi\varepsilon_0}\left(\frac{4\pi}{5}\sqrt{\frac{5}{\pi}}qa^2\sqrt{\frac{5}{4\pi}}\frac{P_2(\cos\theta)}{r^3}+......\right)$$

$$=\frac{q}{4\pi\varepsilon_0}\frac{a^2}{r^3}\left(2P_2(\cos\theta)+......\right)=\frac{q}{4\pi\varepsilon_0}\frac{a^2}{r^3}\left(3\cos^2\theta-1\right)+.......$$

(d)

the exact potential in the x-y plane

$$\Phi\left(r,\theta=\frac{\pi}{2}\right)=\frac{1}{4\pi\varepsilon_0}\left\{\frac{2q}{\sqrt{r^2+a^2}}-\frac{2q}{r}\right\}=\frac{q}{2\pi\varepsilon_0}\frac{1}{r}\left\{\frac{1}{\sqrt{1+\left(\frac{a}{r}\right)^2}}-1\right\}=-\frac{q}{4\pi\varepsilon_0}\frac{a^2}{r^3}+...$$

4.2

A point dipole with dipole moment p is located at the point $\mathbf{x_0}$, From the properties of the derivative of a Dirac delta function, show that for calculation of the potential Φ of the energy of a dipole in an external field, the dipole can be described by an effective charge density $\rho_{eff}(x)=-p\cdot\nabla\delta(x-x_0)$

(i)

$$\rho_{eff}(\vec{x})=-\vec{p}\cdot\vec{\nabla}\delta(\vec{x}-\vec{x}_0)$$

$$\Phi(\vec{x})=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3\vec{x}'=-\frac{1}{4\pi\varepsilon_0}\int\frac{\vec{p}}{|\vec{x}-\vec{x}'|}\cdot\nabla'\delta(\vec{x}'-\vec{x}_0)\,d^3\vec{x}'$$

$$=-\frac{1}{4\pi\varepsilon_0}\int\nabla'\cdot\left(\frac{\vec{p}}{|\vec{x}-\vec{x}'|}\delta(\vec{x}'-\vec{x}_0)\right)d^3\vec{x}'+\frac{1}{4\pi\varepsilon_0}\int\delta(\vec{x}'-\vec{x}_0)\nabla'\cdot\frac{\vec{p}}{|\vec{x}-\vec{x}'|}d^3\vec{x}'$$

$$=-\frac{1}{4\pi\varepsilon_0}\oint\left(\frac{\vec{p}}{|\vec{x}-\vec{x}'|}\delta(\vec{x}'-\vec{x}_0)\right)\cdot\hat{n}'da'+\frac{1}{4\pi\varepsilon_0}\int\delta(\vec{x}'-\vec{x}_0)\nabla'\cdot\frac{\vec{p}}{|\vec{x}-\vec{x}'|}d^3\vec{x}'$$

$$=\frac{\vec{p}}{4\pi\varepsilon_0}\cdot\int\delta(\vec{x}'-\vec{x}_0)\nabla'\frac{1}{|\vec{x}-\vec{x}'|}d^3\vec{x}'=\frac{\vec{p}}{4\pi\varepsilon_0}\cdot\int\delta(\vec{x}'-\vec{x}_0)\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}d^3\vec{x}'=\frac{\vec{p}}{4\pi\varepsilon_0}\cdot\frac{(\vec{x}-\vec{x}_0)}{|\vec{x}-\vec{x}_0|^3}$$

(ii)

Φ_{ex}: External potential

$\vec{E}_{ex}$: External field

$$W=\int\rho(\vec{x})\Phi_{ex}(\vec{x})d^3x=-\vec{p}\cdot\int\Phi_{ex}(\vec{x})\vec{\nabla}\delta(\vec{x}-\vec{x}_0)d^3x$$

$$=-\int\vec{\nabla}\cdot\left(\Phi_{ex}(\vec{x})\vec{p}\delta(\vec{x}-\vec{x}_0)\right)d^3x+\int\vec{p}\cdot\delta(\vec{x}-\vec{x}_0)\vec{\nabla}\Phi_{ex}(\vec{x})d^3x$$

$$=-\oint\left(\Phi_{ex}(\vec{x})\vec{p}\delta(\vec{x}-\vec{x}_0)\right)\cdot\hat{n}da+\int\vec{p}\cdot\delta(\vec{x}-\vec{x}_0)\vec{\nabla}\Phi_{ex}(\vec{x})d^3x$$

$$=\int\vec{p}\cdot\delta(\vec{x}-\vec{x}_0)\vec{\nabla}\Phi_{ex}(\vec{x})d^3x=-\int\vec{p}\cdot\vec{E}_{ex}(\vec{x})\delta(\vec{x}-\vec{x}_0)d^3x=-\vec{p}\cdot\vec{E}_{ex}(\vec{x}_0)$$

4.7

A localized distribution of charge has a charge density $\rho(r)=\frac{1}{64\pi}r^2e^{-r}\sin^2\theta$,

(a) Make a multipole expansion of the potential due to this charge density and determine all the nonvanishing multipole moments. Write down the potential at large distances as a finite expansion in Legendre polynomials.

(b) Determine the potential explicitly at any point in space, and show that near the origin, correct to $\mathbf{r}^2$ inclusive, $\Phi(r)\approx\frac{1}{4\pi\varepsilon_0}\left[\frac{1}{4}-\frac{r^2}{120}P_2(\cos\theta)\right]$

(c) If there exists at the origin a nucleus with a quadrupole moment $\mathbf{Q=10^{l28}m^2}$, determine the magnitude of the interaction energy, assuming that the unit of charge in $\boldsymbol{\rho(\mathbf{r})}$ above is the electronic charge and the inut of length is the hydrogen Bohr radius $\mathbf{a_0={}^{4\pi\varepsilon_0 l}/_{me^2}=0.59\times10^{10}m}$. Express your answer as a frequency by dividing by Planck's constant h.

The charge density in this problem is that for the $\mathbf{m=\pm1}$ states of the 2p level in hydrogen, while the quadrupole interaction is of the same order as found in molecules.

(a)

since ρ is independent on φ, and r factor made the charge distribution local, we can write it in terms of spherical harmonics with

$$q_{lm}=\int\rho(r',\theta,'\varphi')r'^{l}Y_{lm}^{*}(\theta,'\varphi')d^3x'$$

$$Y_2^0=\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}(1-\sin^2\theta)-\frac{1}{2}\right)$$

$$\therefore\sin^2\theta=-\frac{2}{3}\sqrt{\frac{4\pi}{5}}Y_2^0+\sqrt{4\pi}\frac{2}{3}Y_0^0$$

$$\textit{only } l=0,2 \textit{ multipole contribute}$$

$$q_{00}=\frac{2\sqrt{4\pi}}{3}\int_0^\infty r^2\left(\frac{1}{64\pi}r^2e^{-r}\right)dr=\frac{1}{2\sqrt{\pi}}$$

$$q_{20}=\frac{-2}{3}\sqrt{\frac{4\pi}{5}}\int_0^\infty r^4\left(\frac{1}{64\pi}r^2e^{-r}\right)dr=-3\sqrt{\frac{5}{\pi}}$$

$$\Phi(\vec{x})=\frac{1}{4\pi\varepsilon_0}\left[4\pi q_{00}\frac{Y_0^0}{r}+4\pi q_{20}\frac{Y_2^0}{5r^3}\right]=\frac{1}{4\pi\varepsilon_0}\left[\sqrt{4\pi}q_{00}\frac{P_0}{r}+\sqrt{\frac{4\pi}{5}}q_{20}\frac{P_2}{5r^3}\right]$$

$$=\frac{1}{4\pi\varepsilon_0}\left[\frac{P_0}{r}-6\frac{P_2}{r^3}\right]=\frac{1}{4\pi\varepsilon_0 r}\left[1-\frac{6}{r^2}P_2(\cos\theta)\right]=\frac{1}{4\pi\varepsilon_0 r}\left[1-\frac{3}{r^2}(3\cos^2\theta-1)\right]$$

(b)

$$only\ l=0,2\ multipole\ contribute$$

$$\Phi(\vec{x})=\frac{4\pi}{4\pi\varepsilon_0}\sum_{lm}\frac{1}{2l+1}r^l Y_l^m(\theta,\varphi)\int Y_l^{m*}(\theta',\varphi')r'^2 d\Omega'\frac{\rho(\vec{x}')}{r'^{l+1}}dr'$$

$$=\frac{1}{\varepsilon_0}\left[Y_0^0\sqrt{4\pi}\frac{2}{3}\int_0^\infty\left(\frac{1}{64\pi}r^2e^{-r}\right)rdr+\frac{Y_2^0}{5}r^2\left(-\frac{2}{3}\sqrt{\frac{4\pi}{5}}\right)\int_0^\infty\left(\frac{1}{64\pi}r^2e^{-r}\right)\frac{1}{r}dr\right]$$

$$=\frac{1}{\varepsilon_0}\left[Y_0^0\frac{2}{3}\frac{3}{32\pi}+\frac{Y_2^0}{5}r^2\left(-\frac{2}{3}\sqrt{\frac{4\pi}{5}}\right)\frac{1}{64\pi}\right]=\frac{1}{4\pi\varepsilon_0}\left[\frac{1}{4}-\frac{r^2}{120}P_2(\cos\theta)\right]$$

(c)

$$W=-\frac{1}{6}\sum_{ij}Q_{ij}\frac{\partial E_j}{\partial x_i}\bigg|_{x=0}$$

$$-2Q_{11}=-2Q_{22}=Q_{33}=eQ$$

$$W=\frac{eQ}{6}\left|\frac{1}{2}\frac{\partial E_x}{\partial x}+\frac{1}{2}\frac{\partial E_y}{\partial y}-\frac{\partial E_z}{\partial z}\right|_{x=0}=\frac{eQ}{6}\left|\frac{1}{2}\vec{\nabla}\cdot\vec{E}-\frac{3}{2}\frac{\partial E_z}{\partial z}\right|_{x=0}=\frac{eQ}{6}\left|\frac{1}{2}\frac{\rho}{\varepsilon_0}-\frac{3}{2}\frac{\partial E_z}{\partial z}\right|_{x=0}$$

$$=-\frac{eQ}{4}\left|\frac{\partial E_z}{\partial z}\right|_{x=0}=\frac{eQ}{4}\left|\frac{\partial^2\Phi}{\partial z^2}\right|_{x=0}$$

$$\frac{\partial^2\Phi}{\partial z^2}=\frac{1}{240\pi\varepsilon_0}$$

$$W=\frac{eQ}{960\pi\varepsilon_0}$$

$$\frac{W}{\hbar}=\frac{1}{240}\frac{e^2Q}{4\pi\varepsilon_0\hbar a_0^3}=\frac{1}{240}\frac{acQ}{a_0^3}=6.16\times10^6\ rad/s$$

$$\cong 1MHz$$

4.8(a)

$$\Phi(\vec{x})=\sum_m(A_m\rho^m+\frac{B_m}{\rho^m})(A_m'\cos m\phi+B_m'\sin m\phi)\ \ ,\ \rho>b$$

$$=\sum_m(C_m\rho^m+\frac{D_m}{\rho^m})(E_m\cos m\phi+F_m\sin m\phi)\ ,\ a<\rho<b$$

$$=\sum_m\left(G_m\rho^m\cos m\phi+H_m\rho^m\sin m\phi\right)\ ,\ a>\rho$$

$$B.C.\Rightarrow\begin{cases}\rho\to\infty\Rightarrow\Phi(\vec{x})=-E_0\rho\cos\phi\approx A_m\rho^m\left(A_m'\cos m\phi+B_m'\sin m\phi\right)\\ \Phi(a)\ \&\ \Phi(b)\end{cases}$$

$$\Rightarrow\begin{cases}B_m'=F_m=H_m=0,\ all\ m\\ A_m'=E_m=G_m=0,\ except\ m=1\end{cases}$$

$$\begin{cases} \Phi(\rho,\phi) = \left(-E_0\rho + \dfrac{B_1}{\rho}\right)\cos\phi \ , \ \rho > b \\ \qquad\quad = \left(C_1\rho + \dfrac{D_1}{\rho}\right)\cos\phi \ , \ a < \rho < b \\ \qquad\quad = G_1\rho\cos\phi \ , \ a > \rho \end{cases}$$

$$\begin{cases} E_\rho(\rho,\phi) = \left(E_0 + \dfrac{B_1}{\rho^2}\right)\cos\phi \ , \ \rho > b \\ \qquad\quad = \left(-C_1 + \dfrac{D_1}{\rho^2}\right)\cos\phi \ , \ a < \rho < b \\ \qquad\quad = -G_1\cos\phi \ , \ a > \rho \end{cases} \quad \& \quad \begin{cases} E_\phi(\rho,\phi) = \left(-E_0 + \dfrac{B_1}{\rho^2}\right)\sin\phi \ , \ \rho > b \\ \qquad\quad = \left(C_1 + \dfrac{D_1}{\rho^2}\right)\sin\phi \ , \ a < \rho < b \\ \qquad\quad = G_1\sin\phi \ , \ a > \rho \end{cases}$$

$$\varepsilon' = \frac{\varepsilon}{\varepsilon_0} \Rightarrow \begin{cases} -E_0 + \dfrac{B_1}{b^2} = C_1 + \dfrac{D_1}{b^2} \\ C_1 + \dfrac{D_1}{a^2} = G_1 \\ \left(E_0 + \dfrac{B_1}{b^2}\right) = \varepsilon'\left(-C_1 + \dfrac{D_1}{b^2}\right) \\ \varepsilon'\left(-C_1 + \dfrac{D_1}{a^2}\right) = -G_1 \end{cases} \Rightarrow \begin{cases} C_1 = \dfrac{2E_0b^2(\varepsilon'+1)}{(\varepsilon'-1)^2a^2 - (\varepsilon'+1)^2b^2} \\ D_1 = \dfrac{2E_0a^2b^2(\varepsilon'-1)}{(\varepsilon'-1)^2a^2 - (\varepsilon'+1)^2b^2} \\ G_1 = \dfrac{4\varepsilon'E_0b^2}{(\varepsilon'-1)^2a^2 - (\varepsilon'+1)^2b^2} \\ B_1 = \dfrac{(\varepsilon'^2-1)(a^2-b^2)E_0b^2}{(\varepsilon'-1)^2a^2 - (\varepsilon'+1)^2b^2} \end{cases}$$

(b)

$$b = 2a \Rightarrow \begin{cases} C_1 = \dfrac{8E_0(\varepsilon'+1)}{(\varepsilon'-1)^2 - 4(\varepsilon'+1)^2} \\ D_1 = \dfrac{8E_0a^2(\varepsilon'-1)}{(\varepsilon'-1)^2 - 4(\varepsilon'+1)^2} \\ G_1 = \dfrac{16\varepsilon E_0}{(\varepsilon'-1)^2 - 4(\varepsilon'+1)^2} \\ B_1 = \dfrac{-12(\varepsilon'^2-1)a^2E_0}{(\varepsilon'-1)^2 - 4(\varepsilon'+1)^2} \end{cases}$$

(c)

$$a=0 \Rightarrow \begin{cases} C_1 = -\dfrac{2E_0}{(\varepsilon'+1)} \\ D_1 = 0 \\ G_1 = -\dfrac{4\varepsilon'E_0}{(\varepsilon'+1)^2} \\ B_1 = \dfrac{(\varepsilon'^2-1)E_0b^2}{(\varepsilon'+1)^2} \end{cases} \quad \& \quad b >> a \Rightarrow \begin{cases} C_1 = -\dfrac{2E_0}{(\varepsilon'+1)} \\ D_1 = -\dfrac{2E_0a^2(\varepsilon'-1)}{(\varepsilon'+1)^2} \\ G_1 = -\dfrac{4\varepsilon'E_0}{(\varepsilon'+1)^2} \\ B_1 = \dfrac{(\varepsilon'^2-1)E_0b^2}{(\varepsilon'+1)^2} \end{cases}$$

4.10

Two concentric conduction spheres of inner and outer radii a and b, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemispherical shell of dielectric (of dielectric constant $\varepsilon/\varepsilon_0$), as shown in page 172.

(a) Find the electric field everywhere between the spheres.

(b) Calculate the surface-charge distribution on the inner sphere.

(c) Calculate the polarization-charge density induced on the surface of the dielectric at .

(a)

There cannot be any potential difference on a conductor surface. Therefore the electric field on these two region must be the same. Applying Gauss's law with Gaussian surface of radius r between a,b.

$$at\ a < r < b$$

$$\oint \vec{D} \cdot d\vec{A} = Q$$

$$(D_1 + D_2) 2\pi r^2 = Q = q_1 + q_2$$

$$D_1 = \varepsilon_0 E_1 = \frac{q_1}{2\pi r^2} \ \&\ D_2 = \varepsilon E_2 = \frac{q_2}{2\pi r^2}$$

$$\hat{n}_2 \times \left(\vec{E}_2 - \vec{E}_1 \right) = 0 \text{ , at the plane of } z = 0 \ \Rightarrow E_1 = E_2 \Rightarrow \frac{q_1}{\varepsilon_0} = \frac{q_2}{\varepsilon}$$

$$q_1 = \frac{\varepsilon_0}{\varepsilon + \varepsilon_0} Q \ \&\ q_2 = \frac{\varepsilon}{\varepsilon + \varepsilon_0} Q \ \Rightarrow E_1 = E_2 = \frac{Q}{2\pi (\varepsilon + \varepsilon_0) r^2}$$

$$\begin{cases} \vec{E} = \vec{E}_1 = \vec{E}_2 = \dfrac{Q}{2\pi (\varepsilon + \varepsilon_0) r^2} \hat{e}_r \ , \ a < r < b \\ \vec{E} = 0 \ , \ other \end{cases}$$

(b)

$$\begin{cases} \sigma_{upper} = \varepsilon_0 \hat{e}_r \cdot \vec{E}(r = a) = \dfrac{Q}{2\pi a^2} \dfrac{\varepsilon_0}{\varepsilon + \varepsilon_0}, \text{ vacuum space} \\ \sigma_{down} = \varepsilon \hat{e}_r \cdot \vec{E}(r = a) = \dfrac{Q}{2\pi a^2} \dfrac{\varepsilon}{\varepsilon + \varepsilon_0}, \text{ region with dielectric} \end{cases}$$

(c)

$$\vec{D}_1 = \varepsilon_0 \vec{E}_1 = \varepsilon_0 \vec{E} = \varepsilon_0 \vec{E} + \vec{P}_1 \Rightarrow \vec{P}_1 = (\varepsilon_0 - \varepsilon_0) \vec{E} = 0, \text{ vacuum space}$$

$$\vec{D}_2 = \varepsilon \vec{E}_2 = \varepsilon \vec{E} = \varepsilon_0 \vec{E} + \vec{P}_2 \Rightarrow \vec{P}_2 = (\varepsilon - \varepsilon_0) \vec{E} = \frac{Q}{2\pi a^2} \frac{\varepsilon - \varepsilon_0}{\varepsilon + \varepsilon_0} \hat{e}_r, \text{ region with dielectric}$$