5.1

Starting with the differential expression

$$dB = \frac{\mu_0 I}{4\pi} dl' \times \frac{x-x'}{|x-x'|^3}$$

for the magnetic induction at the point P with coordinate x produced by an increment of current $\mathbf{dl'}$ at $\mathbf{x'}$, show explicitly that for a closed loop carrying a current I the magnetic induction at P is $B = \frac{\mu_0 I}{4\pi} \nabla\Omega$ where Ω is the solid angle subtended by the loop at the point P. This corresponds to a magnetic scalar potential, $\Phi_M = \mu_0 I\Omega / 4\pi$. The sign convention for the solid angle is that Ω is positive if the point P views the "inner" side of the surface spanning the loop, that is, if a unit normal n to the surface is defined by the direction of current flow via the right-hand rile, Ω is positive if n points away from the point P, and negative otherwise. This is the same convention as in Section 1.6 for the electric dipole layer.

$$\vec{B}(\vec{x})\cdot\hat{x}_i = \frac{\mu_0 I}{4\pi}\oint_c \hat{x}_i \cdot \left[d\vec{l}' \times \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\right] = \frac{\mu_0 I}{4\pi}\oint_c \hat{x}_i \cdot \left[d\vec{l}' \times \vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|}\right]$$

$$= \oint_c d\vec{l}' \cdot \left[\vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|} \times \hat{x}_i\right] = \int_S d\vec{a}' \cdot \left\{\vec{\nabla}' \times \left[\vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|} \times \hat{x}_i\right]\right\}$$

$$= \int_S d\vec{a}' \cdot \left\{\left[\vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|}\right](\vec{\nabla}'\cdot\hat{x}_i) - \hat{x}_i\left[\vec{\nabla}'^2 \frac{1}{|\vec{x}-\vec{x}'|}\right] + (\hat{x}_i\cdot\vec{\nabla}')\left[\vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|}\right] - \left(\left[\vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|}\right]\cdot\vec{\nabla}'\right)\hat{x}_i\right\}$$

$$= \int_S d\vec{a}' \cdot \left\{0 - 0 + (\hat{x}_i\cdot\vec{\nabla}')\left[\vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|}\right] - 0\right\} = \int_S d\vec{a}' \cdot \left\{\frac{\partial}{\partial x_i'}\left[\vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|}\right]\right\}$$

$$= -\int_S d\vec{a}' \cdot \left\{\frac{\partial}{\partial x_i}\left[\vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|}\right]\right\} = -\frac{\partial}{\partial x_i}\int_S d\vec{a}' \cdot \left[\vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|}\right] = \frac{\partial}{\partial x_i}\int_S d\Omega' = \frac{\partial}{\partial x_i}\Omega(\vec{x})$$

$$\Rightarrow B_i = \frac{\mu_0 I}{4\pi}\frac{\partial}{\partial x_i}\Omega(\vec{x}) \;\Rightarrow\; \vec{B}(\vec{x}) = \frac{\mu_0 I}{4\pi}\vec{\nabla}\Omega(\vec{x})$$

5.3

A right-circular solenoid of finite length L and radius a has N turns per unit length and carries a current I. Show that the magnetic induction on the cylinder axis in the limit is

$$B_z = \frac{\mu_0 NI}{2}(\cos\theta_1 + \cos\theta_2)$$

where the angles are defined in the figure at page 225.

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l} \times \hat{r}}{r^2}$$

for one loop

$$B_z = \frac{\mu_0}{4\pi}\frac{I2\pi a\sin\theta}{d^2} = \frac{\mu_0}{4\pi}\frac{2\pi \sin^3\theta}{a}$$

$$NL \to \infty \;\Rightarrow\; dN = Ndz$$

$$\frac{d\theta}{dz} = \frac{\sin\theta}{d} \Rightarrow dN = N\frac{d\theta a}{\sin^2\theta}$$

$$\Rightarrow B = \int B_z dN = \frac{\mu_0 I}{4\pi}\cdot 2\pi N\int_{\theta_2}^{\pi-\theta_1}\sin\theta d\theta = \frac{\mu_0 IN}{2}\left[\cos\theta_2 - \cos\left(\pi-\theta_1\right)\right] = \frac{\mu_0 IN}{2}\left[\cos\theta_2 + \cos\theta_1\right]$$

5.6

A cylindrical conductor of radius a has a hole of radius b bored parallel to, and centered a distance d from, the cylinder axis d+b<a . The current density is uniform throughout the remaining metal of the cylinder and is parallel to the axis. Use Ampere's law and principle of linear superposition to find the magnitude and the direction of the magnetic-flux density in the hole.

$$\vec{B}_{cavity} = \vec{B}_{nohole} + \vec{B}_{hole} = \frac{\mu_0 I}{2\pi\rho}\hat{\varphi} + \frac{\mu_0 I'}{2\pi\rho'}\hat{\varphi}'$$

$$I = \pi\rho^2 J$$

$$I' = -\pi\rho'^2 J$$

$$\vec{B}_{cavity} = \frac{\mu_0 J\rho}{2}\hat{\varphi} - \frac{\mu_0 J\rho'}{2}\hat{\varphi}'$$

$$\vec{\rho} - \vec{\rho}' = \vec{d} \Rightarrow \hat{z}\times\left(\vec{\rho}-\vec{\rho}'\right) = \hat{z}\times\vec{d} \Rightarrow \rho\hat{\varphi} - \rho'\hat{\varphi}' = \hat{z}\times\vec{d} \Rightarrow \vec{B}_{cavity} = \frac{1}{2}\mu_0 J\hat{z}\times\vec{d}$$

$$J = \frac{I}{\pi\left(a^2-b^2\right)} \Rightarrow \vec{B}_{cavity} = \frac{\mu_0 I}{\pi\left(a^2-b^2\right)}\hat{z}\times\vec{d}$$

5.11

A circular loop of wire carrying a current I is located with its center at the origin of coordinates and the normal to its plane having spherical angles θ_0, ϕ_0. There is an applied magnetic field, $B_X = B_0(1+\beta y)$ and $B_y = B_0(1+\beta x)$.

(a) Calculate the force acting on the loop without making any approximation. Compare your result with the approximate result (5.69) . Comment.

(b) Calculate the torque in lowest order. Can you deduce anything about the higher order contributions? Do the vanish for the circular loop? What about for other shapes?

(a)

$$\vec{B} = B_0(1+\beta y)\hat{x} + B_0(1+\beta x)\hat{y}$$
$$\hat{n} = \sin\theta_0 \cos\varphi_0 \hat{x} + \sin\theta_0 \sin\varphi_0 \hat{y} + \cos\theta_0 \hat{z}$$
$$d\vec{F} = I_1\left(d\vec{l}_1 \times \vec{B}\right) \Rightarrow \vec{F} = I\int_C d\vec{l} \times \vec{B}$$

$$\left|\vec{F}_x\right| = \vec{F}\cdot\hat{x} = I\int_C \hat{x}\cdot\left(d\vec{l}\times\vec{B}\right) = I\int_C d\vec{l}\cdot\left(\vec{B}\times\hat{x}\right) = I\int_C d\vec{l}\cdot\left(-B_y\hat{z}\right)$$
$$= -IB_0\int_C d\vec{l}\cdot(1+\beta x)\hat{z} = -IB_0\int_s \vec{\nabla}\times(1+\beta x)\hat{z}\cdot d\vec{A}$$
$$\vec{\nabla}\times(1+\beta x)\hat{z} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & 1+\beta x \end{vmatrix} = -\beta\hat{y}$$

$$\left|\vec{F}_x\right| = -IB_0\int_s(-\beta\hat{y})\cdot(\hat{n}dA) = IB_0\beta\int_s(\hat{y}\cdot\hat{n})dA = IB_0\beta\sin\theta_0\sin\varphi_0\int_S dA$$
$$\Rightarrow \left|\vec{F}_x\right| = IB_0\beta\sin\theta_0\sin\varphi_0\pi a^2$$
$$\left|\vec{F}_y\right| = IB_0\beta\int_s(\hat{x}\cdot\hat{n})dA = IB_0\beta\sin\theta_0\cos\varphi_0\pi a^2$$
$$\left|\vec{F}_z\right| = \vec{F}\cdot\hat{z} = I\int_C d\hat{l}\cdot\left(\vec{B}\times\hat{z}\right) = I\int_C d\hat{l}\cdot\left(B_y\hat{x}\times B_x\hat{y}\right) = I\int_S d\hat{l}\cdot\left\{\vec{\nabla}\times\left[B_y\hat{x} - B_x\hat{y}\right]\right\} d\vec{A}$$
$$\vec{\nabla}\times\left[B_y\hat{x} - B_x\hat{y}\right] = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_0(1+\beta x) & B_0(1+\beta y) & 0 \end{vmatrix} = 0$$

$$\Rightarrow \left|\vec{F}_z\right| = 0$$
$$m = \left|\vec{m}\right| = IA$$
$\left|\vec{F}_x\right| = IB_0\beta\sin\theta_0\sin\phi_0\pi a^2 = B_0\beta m_y \because \sin\theta_0\sin\phi_0\pi a^2 =$ y projection of loop area

$$\Rightarrow \left|\vec{F}_y\right| = IB_0\beta\sin\theta_0\cos\phi_0\pi a^2 = B_0\beta m_x$$
from 5.69 $\vec{F} = \vec{\nabla}\left(\vec{m}\cdot\vec{B}\right)$

$$\vec{F} = B_0\beta m_y\hat{x} + B_0\beta m_x\hat{y}$$

$$\vec{N} = \vec{m}\times\vec{B}(0) = -m_z B_0\hat{x} + m_z B_0\hat{y} + B_0\left(m_x - m_y\right)\hat{z}$$

Exact torque

$$\vec{N} = \int \vec{x}\times\left(\vec{J}\times\vec{B}\right)d^3x = \int\left[\vec{J}\left(\vec{x}\cdot\vec{B}\right) - \vec{B}\left(\vec{x}\cdot\vec{J}\right)\right]d^3x$$
$$= \int\left\{\vec{J}\left[B_0 x(1+\beta y) + B_0 y(1+\beta x)\right] - \left[B_0(1+\beta y)\hat{x} + B_0(1+\beta x)\hat{y}\right]\left(\vec{x}\cdot\vec{J}\right)\right\}d^3x$$

If we can isolate the lowest order term , all others are higher order contributions. Note that 5.54

$$\vec{m} = \frac{1}{2}\int \vec{x}'\times\vec{J}\left(\vec{x}'\right)d^3x'$$
$$\Rightarrow \left[\frac{1}{2}\int\left(yJ_z - zJ_y\right)d^3x\right]\hat{x} + \left[\frac{1}{2}\int\left(zJ_x - xJ_z\right)d^3x\right]\hat{y} + \left[\frac{1}{2}\int\left(xJ_y - yJ_x\right)d^3x\right]\hat{z}$$
$$= m_x\hat{x} + m_y\hat{y} + m_z\hat{z}$$

5.13

A sphere of radius a carries a uniform surface-charge distribution σ. The sphere is rotated about a diameter with constant angular velocity ω. Find the vector potential and magnetic-flux density both inside and outside the sphere.

Choose the $\vec{\omega}$ along the $\hat{\mathbf{z}}$ axis

$$\vec{K}=\sigma\vec{v}=\sigma\left(\vec{\omega}\times\vec{x}'\right)\Rightarrow\vec{\omega}=\omega\hat{k}\ \&\ \vec{x}'=a\sin\theta'\cos\varphi'\hat{i}+a\sin\theta'\sin\varphi'\hat{j}+a\cos\theta'\hat{k}$$

$$=\sigma\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & \omega\\ a\sin\theta'\cos\varphi' & a\sin\theta'\sin\varphi' & a\cos\theta'\end{vmatrix}=\sigma\left\{\left(-\omega a\sin\theta'\sin\varphi'\right)\hat{i}+\left(\omega a\sin\theta'\cos\varphi'\right)\hat{j}\right\}$$

$$\vec{A}(\vec{x})=\frac{\mu_0}{4\pi}\int\frac{\vec{K}(\vec{x}')}{\left|\vec{x}-\vec{x}'\right|}da'=\mu_0\int\vec{K}(\vec{x}')\sum_{lm}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)da'$$

$$=\sum_{lm}\frac{\mu_0 a^2}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)\int\vec{K}(\vec{x}')Y_{lm}^*(\theta',\varphi')d\Omega'$$

$$=\sum_{lm}\frac{\mu_0\sigma\omega a^3}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)\int\left\{\left(-\sin\theta'\sin\varphi'\right)\hat{i}+\left(\sin\theta'\cos\varphi'\right)\hat{j}\right\}Y_{lm}^*(\theta',\varphi')\,d\Omega'$$

$$=\sum_{lm}\frac{\mu_0\sigma\omega a^3}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)\int\left\{\left(-\frac{i}{2}\sqrt{\frac{8\pi}{3}}\left(Y_{11}+Y_{1-1}\right)\right)\hat{i}+\left(\frac{1}{2}\sqrt{\frac{8\pi}{3}}\left(-Y_{11}+Y_{1-1}\right)\right)\hat{j}\right\}Y_{lm}^*(\theta',\varphi')\,d\Omega'$$

$$=\frac{\mu_0\sigma\omega a^3}{3}\frac{r_<}{r_>^2}\left\{\left(-\frac{i}{2}\sqrt{\frac{8\pi}{3}}\left(Y_{11}(\theta,\varphi)+Y_{1-1}(\theta,\varphi)\right)\right)\hat{i}+\left(\frac{1}{2}\sqrt{\frac{8\pi}{3}}\left(-Y_{11}(\theta,\varphi)+Y_{1-1}(\theta,\varphi)\right)\right)\hat{j}\right\}$$

$$=\frac{\mu_0\sigma\omega a^3}{3}\frac{r_<}{r_>^2}\left\{\left(-\sin\theta\sin\varphi\right)\hat{i}+\left(\sin\theta\cos\varphi\right)\hat{j}\right\}=\frac{\mu_0\sigma\omega a^3\sin\theta}{3}\frac{r_<}{r_>^2}\hat{e}_\phi$$

Because orthogonal, the integration becomes

$$\vec{A}=\frac{\mu_0\sigma a^3\omega}{3}\frac{r_<}{r_>^2}\sin\theta\left[-\sin\varphi\hat{x}+\cos\varphi\hat{y}\right]=\frac{\mu_0\sigma a^3\omega}{3}\frac{r_<}{r_>^2}\sin\theta\hat{\varphi}$$

$$\Rightarrow\vec{A}_{in}=\frac{\mu_0\sigma a\omega}{3}r\sin\theta\hat{\varphi}\Rightarrow\vec{B}_{in}=\nabla\times\vec{A}_{in}=\frac{2}{3}\mu_0\sigma a\omega\hat{z}$$

$$\vec{A}_{out}=\frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{x}}{|x|^3}\Rightarrow\vec{B}_{out}=\frac{\mu_0}{4\pi}\frac{3(\vec{m}\cdot\vec{n})\vec{n}-\vec{m}}{|x|^3}\ ,\ \left(\text{hint}:\vec{m}=\frac{4\pi a^3}{3}\sigma\omega a\hat{z}\right)$$

5.18

A circular loop of wire having a radius a and carrying a current I is located in vacuum with its center a distance d away from a semi-infinite slab of permeability μ. Find the force acting on the loop when

(a) The plane of the loop is parallel to the face of the slab,

(b) The plane of the loop is perpendicular to the face of the slab.

(c) Determine the limiting form of your answer to parts a and b when d>>a. Can you obtain these limiting values on some simple and direct way?

(a)

$$I'=\left(\frac{\mu_r-1}{\mu_r+1}\right)I \;\&\; \mu_r=\frac{\mu}{\mu_0} \;\Rightarrow\; J_\phi=\frac{I'}{\sqrt{\left(a^2+4d^2\right)}}\delta\left(r-\sqrt{a^2+4d^2}\right)\delta\left(\cos\theta-\frac{2d}{\sqrt{a^2+4d^2}}\right)$$

From (5.48) & (5.49)

$$B_r=\frac{\mu_0 Ia}{2r}\sum_l(-1)^l\frac{(2l+1)!!}{2^l l!}\frac{r_<^{2l+1}}{r_>^{2l+2}}P_{2l+1}(\cos\theta)$$

$$B_\theta=-\frac{\mu_0 Ia^2}{4}\sum_l(-1)^l\frac{(2l+1)!!}{2^l(l+1)!}\begin{pmatrix}-\left(\frac{2l+2}{2l+1}\right)\frac{1}{a^3}\left(\frac{r}{a}\right)^{2l}\\ \frac{1}{r^3}\left(\frac{a}{r}\right)^{2l}\end{pmatrix}P_{2l+1}^1(\cos\theta)$$

$$\vec{F}=\int\vec{J}(\vec{x})\times\vec{B}(\vec{x})d^3x$$

$$=\int\hat{e}_\theta\frac{I'}{\sqrt{\left(a^2+4d^2\right)}}\delta\left(r-\sqrt{a^2+4d^2}\right)\delta\left(\cos\theta-\frac{2d}{\sqrt{a^2+4d^2}}\right)\frac{\mu_0 Ia}{2r}\sum_l(-1)^l\frac{(2l+1)!!}{2^l l!}\frac{r_<^{2l+1}}{r_>^{2l+2}}P_{2l+1}(\cos\theta)d^3x$$

$$+\int\hat{e}_r\frac{I'}{\sqrt{\left(a^2+4d^2\right)}}\delta\left(r-\sqrt{a^2+4d^2}\right)\delta\left(\cos\theta-\frac{2d}{\sqrt{a^2+4d^2}}\right)\frac{\mu_0 Ia^2}{4}\sum_l(-1)^l\frac{(2l+1)!!}{2^l(l+1)!}\begin{pmatrix}-\left(\frac{2l+2}{2l+1}\right)\frac{1}{a^3}\left(\frac{r}{a}\right)^{2l}\\ \frac{1}{r^3}\left(\frac{a}{r}\right)^{2l}\end{pmatrix}P_{2l+1}^1(\cos\theta)d^3x$$

$$=\hat{e}_z\frac{\mu_0 I'Ia^2}{2}\sum_l(-1)^l\frac{(2l+1)!!}{2^l l!}\left(\frac{1}{(l+1)}\begin{pmatrix}-\left(\frac{2l+2}{2l+1}\right)\frac{d}{a^3}\left(\frac{a^2+4d^2}{a^2}\right)^l\\ \frac{d}{\left(a^2+4d^2\right)^{\frac{3}{2}}}\left(\frac{a^2}{a^2+4d^2}\right)^l\end{pmatrix}P_{2l+1}^1\left(\frac{2d}{\sqrt{a^2+4d^2}}\right)-\begin{pmatrix}\frac{\left(a^2+4d^2\right)^l}{a^{2l+2}}\\ \frac{a^{2l+1}}{\left(\sqrt{a^2+4d^2}\right)^{2l+2}}\end{pmatrix}P_{2l+1}\left(\frac{2d}{\sqrt{a^2+4d^2}}\right)\right)$$

upper : $r<a$
lower : $r>a$

(b)

$$J_\phi=I'\delta\left(\rho-2d\cos\phi\pm\sqrt{4d^2\cos^2\phi-\left(4d^2-a^2\right)}\right)\delta(z)$$

$$\textit{Form problem } (5-10(b))\Rightarrow\vec{A}(\rho,z)=\hat{e}_\phi\frac{\mu_0 Ia}{2}\int_0^\infty dk e^{-k|z|}J_1(ka)J_1(k\rho)$$

$$B_\rho=-\frac{\partial}{\partial z}A_\phi\;,\;B_z=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho A_\phi\right)$$

$$\vec{F}=\int\vec{J}(\vec{x})\times\vec{B}(\vec{x})d^3x=\int\left(J_\phi B_z\hat{e}_\rho-J_\phi B_\rho\hat{e}_z\right)\rho d\rho d\phi dz$$

$$B_\rho = \frac{\mu_0 Ia}{2}\int_0^\infty dk\ ke^{-k|z|}J_1(ka)J_1(k\rho)\frac{\partial}{\partial z}|z|$$
$$B_z = \frac{\mu_0 Ia}{2}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\int_0^\infty dke^{-k|z|}J_1(ka)J_1(k\rho)\right)$$
$$= \left[\frac{\mu_0 Ia}{2}\frac{1}{\rho}\int_0^\infty dke^{-k|z|}J_1(ka)J_1(k\rho) + \frac{\mu_0 Ia}{2}\int_0^\infty dk\ ke^{-k|z|}J_1(ka)J_1'(k\rho)\right]$$

$$\vec{F} = \int \vec{J}(\vec{x})\times\vec{B}(\vec{x})d^3x = \int\left(J_\phi B_z\hat{e}_\rho - J_\phi B_\rho\hat{e}_z\right)\rho d\rho d\phi dz$$
$$\int\left(J_\phi B_z\hat{e}_\rho\right)\rho d\rho d\phi dz$$
$$= \frac{\mu_0 II'a}{2}\int d\phi\int_0^\infty dk\ \hat{e}_\rho J_1(ka)\left(J_1'\left(k\left(2d\cos\phi \pm \sqrt{4d^2\cos^2\phi - \left(4d^2 - a^2\right)}\right)\right)\right.$$
$$\left.\times\left(k\left(2d\cos\phi \pm \sqrt{4d^2\cos^2\phi - \left(4d^2 - a^2\right)}\right)\right) + J_1\left(k\left(2d\cos\phi \pm \sqrt{4d^2\cos^2\phi - \left(4d^2 - a^2\right)}\right)\right)\right)$$
$$\int\left(J_\phi B_\rho\hat{e}_z\right)\rho d\rho d\phi dz$$
$$= \hat{e}_z\frac{\mu_0 II'a}{2}\int d\phi\int_0^\infty dk\ J_1(ka)J_1\left(k\left(2d\cos\phi \pm \sqrt{4d^2\cos^2\phi - \left(4d^2 - a^2\right)}\right)\right)\left(k\left(2d\cos\phi \pm \sqrt{4d^2\cos^2\phi - \left(4d^2 - a^2\right)}\right)\right)$$

5.19 A magnetically "hard" material is in the shape of a right circular cylinder of length L and radius a. The cylinder has a permanent magnetization $\mathbf{M_0}$, uniform throughout its volume and parallel to its axis.

(a) Determine the magnetic field H and magnetic induction B at all points on the axis of the cylinder, both inside and outside.

(b) Plot the ratios $\mathbf{B}/\mu_0\mathbf{M_0}$ on the axis as functions of z for $\mathbf{L}/\mathbf{a} = 5$.

(a) Magnetic hard material, see 5.9(c),

scalar potential

$$\vec{H}(\vec{x}) = -\nabla\Phi_M(\vec{x})$$
$$\Phi_M(\vec{x}) = -\frac{1}{4\pi}\int_V \frac{\nabla'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' + \frac{1}{4\pi}\oint_s \frac{\hat{n}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}da'$$
$$\vec{M}(\vec{x}') = M_0\hat{e}_z \ \Rightarrow\ \nabla'\cdot\vec{M}(\vec{x}') = 0\ \ \&\ \ \hat{n}'\cdot\vec{M}(\vec{x}') = \begin{cases} M_0, on\ the\ top \\ -M_0, on\ the\ bottom \\ 0\ ,\ other \end{cases}$$
$$\Phi_M(\vec{x}) = \frac{1}{4\pi}\oint_s \frac{\hat{n}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}da' = \frac{M_0}{2}\int_0^a \frac{1}{\sqrt{\rho'^2 + \left(\frac{L}{2} - z\right)^2}}\rho' d\rho' - \frac{M_0}{2}\int_0^a \frac{1}{\sqrt{\rho'^2 + \left(\frac{L}{2} + z\right)^2}}\rho' d\rho'$$
$$= \frac{M_0}{2}\left(\sqrt{a^2 + \left(\frac{L}{2} - z\right)^2} - \left|\frac{L}{2} - z\right| - \sqrt{a^2 + \left(\frac{L}{2} + z\right)^2} + \left|\frac{L}{2} + z\right|\right)$$

$(i)\ inside\ |z|<\frac{L}{2}$

$$\Phi_M(\vec{x})=\frac{M_0}{2}\left(\sqrt{a^2+\left(\frac{L}{2}-z\right)^2}-\sqrt{a^2+\left(\frac{L}{2}+z\right)^2}+2z\right)$$

$$\vec{H}(\vec{x})=-\nabla\Phi_M(\vec{x})=\hat{e}_z\frac{M_0}{2}\left(\frac{\left(\frac{L}{2}-z\right)}{\sqrt{a^2+\left(\frac{L}{2}-z\right)^2}}+\frac{\left(\frac{L}{2}+z\right)}{\sqrt{a^2+\left(\frac{L}{2}+z\right)^2}}-2\right)$$

$$\vec{B}=\mu_0(\vec{H}+\vec{M})=\hat{e}_z\frac{\mu_0 M_0}{2}\left(\frac{\left(\frac{L}{2}-z\right)}{\sqrt{a^2+\left(\frac{L}{2}-z\right)^2}}+\frac{\left(\frac{L}{2}+z\right)}{\sqrt{a^2+\left(\frac{L}{2}+z\right)^2}}\right)$$

$(ii)\ outside\ |z|>\frac{L}{2}$

$$\Phi_M(\vec{x})=\frac{M_0}{2}\left(\sqrt{a^2+\left(z-\frac{L}{2}\right)^2}-\sqrt{a^2+\left(z+\frac{L}{2}\right)^2}+L\right)$$

$$\vec{H}(\vec{x})=-\nabla\Phi_M(\vec{x})=\hat{e}_z\frac{M_0}{2}\left(-\frac{\left(z-\frac{L}{2}\right)}{\sqrt{a^2+\left(z-\frac{L}{2}\right)^2}}+\frac{\left(z+\frac{L}{2}\right)}{\sqrt{a^2+\left(z+\frac{L}{2}\right)^2}}\right)$$

$$\vec{B}=\mu_0(\vec{H}+\vec{M})=\mu_0\vec{H}=\hat{e}_z\frac{\mu_0 M_0}{2}\left(-\frac{\left(z-\frac{L}{2}\right)}{\sqrt{a^2+\left(z-\frac{L}{2}\right)^2}}+\frac{\left(z+\frac{L}{2}\right)}{\sqrt{a^2+\left(z+\frac{L}{2}\right)^2}}\right)$$

5.20

(a) Starting from the force equation (5.12) and the fact that a magnetization M inside a volume V bounded by a surface S is equivalent to a volume current density $\mathbf{J_M}=(\nabla\times\mathbf{M})$ and a surface current density $(\mathbf{M}\times\mathbf{n})$, show that in the absence of macroscopic conduction currents the total magnetic force one the body can be written

$$F=-\int_V(\nabla\cdot M)B_e d^3x+\int_S(M\cdot n)B_e da$$

where $\mathbf{B_e}$ is the applied magnetic induction (not including that of the body in question). The force is now expressed in terms of the effective charge densities $\rho_\mathbf{M}$ and $\sigma_\mathbf{M}$. If the distribution of magnetization is not discontinuous, the surface can be at infinity and the force given by just the volume integral.

(b) A sphere of radius R with uniform magnetization ha its center at the origin of coordinates and its direction of magnetization making spherical angles θ_0, ϕ_0. If the external magnetic field is the same as in Problem 5.11, use the expression of part a to evaluate the components of the force acting on the sphere.

(a)

$$\vec{F}=\int_V \vec{J}\times\vec{B}d^3x+\int_S \vec{K}\times\vec{B}da$$
$$\vec{J}_m=\vec{\nabla}\times\vec{M}$$
$$\vec{K}_m=\vec{M}\times\hat{n}$$
$$\Rightarrow \vec{F}=\int_V\left(\vec{\nabla}\times\vec{M}\right)\times\vec{B}_e d^3x+\int_S\left(\vec{M}\times\hat{n}\right)\times\vec{B}_e da=-\int_V \vec{B}_e\times\left(\vec{\nabla}\times\vec{M}\right)d^3x+\int_S\left(\vec{M}\times\hat{n}\right)\times\vec{B}_e da$$

$$\vec{\nabla}\left(\vec{M}\cdot\vec{B}_e\right)=\left(\vec{M}\cdot\vec{\nabla}\right)\vec{B}_e+\left(\vec{B}_e\cdot\vec{\nabla}\right)\vec{M}+\vec{M}\times\left(\vec{\nabla}\times\vec{B}_e\right)+\vec{B}_e\times\left(\vec{\nabla}\times\vec{M}\right)$$
$$=\left(\vec{M}\cdot\vec{\nabla}\right)\vec{B}_e+\left(\vec{B}_e\cdot\vec{\nabla}\right)\vec{M}+\vec{B}_e\times\left(\vec{\nabla}\times\vec{M}\right)\because \vec{\nabla}\times\vec{B}_e=0$$
$$\int_S\left(\vec{M}\times\hat{n}\right)\times\vec{B}_e da=-\int_S\vec{B}_e\times\left(\vec{M}\times\hat{n}\right)da=\int_S\left[-\left(\vec{B}_e\cdot\hat{n}\right)\vec{M}+\left(\vec{B}_e\cdot\vec{M}\right)\hat{n}\right]da$$
$$\Rightarrow \vec{F}=\int_V\left[\left(\vec{M}\cdot\vec{\nabla}\right)\vec{B}_e+\left(\vec{B}_e\cdot\vec{\nabla}\right)\vec{M}-\vec{\nabla}\left(\vec{M}\cdot\vec{B}_e\right)\right]d^3x+\int_S\left[-\left(\vec{B}_e\cdot\hat{n}\right)\vec{M}+\left(\vec{B}_e\cdot\vec{M}\right)\hat{n}\right]da$$

$$\text{use}\int_V\left(\vec{C}\cdot\vec{\nabla}\right)\vec{D}d^3x=-\int_V\left(\vec{\nabla}\cdot\vec{C}\right)\vec{D}d^3x+\int_S\left(\hat{n}\cdot\vec{C}\right)\vec{D}da$$
$$\int_V\left(\vec{B}_e\cdot\vec{\nabla}\right)\vec{M}d^3x=-\int_V\left(\vec{\nabla}\cdot\vec{B}_e\right)\vec{M}d^3x+\int_S\left(\hat{n}\cdot\vec{B}_e\right)\vec{M}da=\int_S\left(\hat{n}\cdot\vec{B}_e\right)\vec{M}da$$
$$\int_V\left(\vec{M}\cdot\vec{\nabla}\right)\vec{B}_e d^3x=-\int_V\left(\vec{\nabla}\cdot\vec{M}\right)\vec{B}_e d^3x+\int_S\left(\hat{n}\cdot\vec{M}\right)\vec{B}_e da$$
$$\vec{F}=\int_V\left[-\left(\vec{\nabla}\cdot\vec{M}\right)\vec{B}_e-\vec{\nabla}\left(\vec{M}\cdot\vec{B}_e\right)\right]d^3x+\int_S\left[\left(\hat{n}\cdot\vec{M}\right)\vec{B}_e+\left(\vec{B}_e\cdot\vec{M}\right)\hat{n}\right]da$$
$$=-\int_V\left(\vec{\nabla}\cdot\vec{M}\right)\vec{B}_e d^3x+\int_S\left(\hat{n}\cdot\vec{M}\right)\vec{B}_e da$$

(b)

$$\vec{M}=M_0\left(\sin\theta_0\cos\varphi_0,\sin\theta_0\sin\varphi_0,\cos\theta_0\right)$$
$$\vec{B}_e=B_0\left(1+\beta y,1+\beta x,0\right)=B_0\left(1+\beta r\sin\theta\sin\varphi,1+\beta r\sin\theta\cos\theta,0\right)$$
$$\hat{n}=\left(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta\right)$$

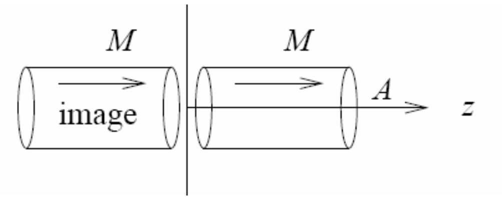
Then

$$\vec{F}=\int_S\left(\vec{M}\cdot\hat{n}\right)\vec{B}_e da$$
$$=R^2M_0B_0\int d\Omega\left(\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos\left(\varphi-\varphi_0\right)\right)\left(1+\beta R\sin\theta\sin\varphi,1+\beta R\sin\theta\cos\theta,0\right)$$
$$=\frac{2\pi}{3}M_0B_0R^3\beta\left(\sin\theta_0\sin\varphi_0,\sin\theta_0\cos\varphi_0,0\right)$$

5.22

Show that in general a long, straight bar of uniform cross-sectional area A with uniform lengthwise magnetization M, when placed with its flat end against an infinitely permeable flat surface, adheres with a force given approximately by $F\cong\frac{\mu_0}{2}AM^2$. Relate your discussion to the electrostatic considerations in Section 1.11.

This problem is best solved by considering an image magnet. The infinite permeability of the at surface ensures that the magnetic field must be perpendicular to the surface. As a result, this is similar to the electrostatic case of electric field lines being perpendicular to the surface of a perfect conductor. For magnetostatics, this means that we may use a magnetic scalar potential _M (since there are no free currents) subject to the condition _M = 0 at z = 0 (taking the surface to lie in the x-y plane at z = 0). The image problem is then set up as follows



$$W = -\frac{1}{2}\int \vec{M}\cdot\vec{B} d^3x$$

Here it is important to note that, while we solve this problem using an image magnet, the only quantities that show up in this energy integral are the actual sources of magnetization ~M and the actual magnetic induction ~B. We place the magnet at a distance z_0 from the z = 0 surface so that

$$\vec{M}_{real} = \begin{cases} M\hat{z} & , \ z_0 < z < z_0 + L \\ 0 & , \ \text{otherwise} \end{cases}$$

As a result

$$W(z_0) = -\frac{1}{2}M\int_{Z_0}^{Z_0+L} dz \int da\cdot B_z(\vec{x}) \approx -\frac{1}{2}MA\int_{z_0}^{z_0+L} dz\ B(z)$$

where we have approximated that the magnetic induction is roughly uniform across the face of the magnet. Using the image magnet setup, there are two sources of magnetic induction

$$\vec{B}(z) = \vec{B}_{real}(z) + \vec{B}_{image}(z)$$

Using (1) we see that

$$\vec{B}_{real}(z) = -\frac{1}{2}u_0 M\left[\frac{z-z_0-L}{\sqrt{a^2+(z-z_0-L)^2}} - \frac{z-z_0}{\sqrt{a^2+(z-z_0)^2}}\right]\hat{z}$$

$$\vec{B}_{image}(z) = -\frac{1}{2}u_0 M\left[\frac{z-z_0+L}{\sqrt{a^2+(z-z_0+L)^2}} - \frac{z+z_0+L}{\sqrt{a^2+(z+z_0+L)^2}}\right]\hat{z}$$

Here we have shifted the coordinates such that the real magnet lies between z_0 and z_0+L and the image magnet lies between z = -z_0-L and z = -z_0. In principle, we may insert these expressions into (5) to compute the magnetostatic energy.

However, as a simplification, we note that the integral of ~M .B_{real} gives a position independent (ie z_0 independent) self energy. Hence this will not contribute to the force. As a result, we only need to insert B_{image} into (5). This gives us

$$W(z_0) \approx \frac{1}{4}u_0 M^2 A \int_{Z_0}^{Z_0+L} \left[\frac{z+z_0}{\sqrt{a^2+(z+z_0)^2}} - \frac{z+z_0+L}{\sqrt{a^2+(z+z_0+L)^2}} \right]$$

$$= \frac{1}{4}u_0 M^2 A \left[\sqrt{a^2+(z+z_0)^2} - \sqrt{a^2+(z+z_0+L)^2} \right]_{Z_0}^{Z_0+L}$$

$$= \frac{1}{4}u_0 M^2 A \left[2\sqrt{a^2+4(z_0+\frac{1}{2}L)^2} - \sqrt{a^2+4(z_0)^2} - \sqrt{a^2+4(z_0+L)^2} \right]$$

The force is then

$$F_z = -\frac{\partial W}{\partial z_0}\Big|_{z_0=0} \approx -u_0 M^2 A \left[\frac{2z_0+L}{\sqrt{a^2+4(z_0+\frac{1}{2}L)^2}} - \frac{z_0}{\sqrt{a^2+4(z_0)^2}} - \frac{z_0+L}{\sqrt{a^2+4(z_0+L)^2}} \right] |z_0 = 0$$

$$= -u_0 M^2 A \left[\frac{L}{\sqrt{a^2+L^2}} - \frac{L}{\sqrt{a^2+4L^2}} \right]$$

$$\approx -\frac{1}{2}u_0 M^2 A$$

where in the last line we used L >> a (a condition that we needed anyway to ensure that B_z is nearly uniform on the end caps). Note that we could have alternatively used the result of Problem 5.20

$$\vec{F} = -\int_V (\vec{\nabla}\cdot\vec{M})\vec{B}_e d^3x + \int_s (\vec{M}\cdot\vec{n})\vec{B}_e da$$

where the applied magnetic induction ~B_e is given by B_{image} in (6) with $z_0 = 0$. Since the magnetization is uniform, the force arises entirely from the surface term

$$\vec{F} = \int_s (\vec{M}\cdot\vec{n})\vec{B}_e da = \hat{z}M \int [-B_z(0) + B_z(L)] da$$

$$\approx \hat{z}MA[B_z(L) - B_z(0)] = \hat{z}\frac{1}{2}u_0 M^2 A \left[\frac{2L}{\sqrt{a^2+4L^2}} - \frac{L}{\sqrt{a^2+L^2}} - \frac{L}{\sqrt{a^2+L^2}} \right]$$

$$\approx -\hat{z}\frac{1}{2}u_0 M^2 A$$

What we have done here is to calculate the force through the magnetostatic energy

$$F = -\nabla W(\vec{x})$$

where ~x denotes the position of the bar magnet. This is the magnetostatic equivalent of the force discussion in Section 1.11, which states that \Forces acting between charged bodies can be obtained by calculating the change in the total electrostatic energy of the system under small virtual displacements." In fact, this statement is true in general, provided we use the complete (electrostatic plus magnetostatic) energy of the system. Curiously, a conductor with surface-charge density _ feels an outward force of the form

$$F \approx \frac{\sigma^2 A}{2\varepsilon_0}$$

which is roughly the electrostatic equivalent of

$$F \approx -\frac{1}{2} u_0 M^2 A$$

5.30

(a) Show that a surface current density $K(\phi) = I\cos\phi / 2R$ flowing in the axial direction on a right circular cylindrical surface of radius R produces inside the cylinder a uniform magnetic induction in a direction perpendicular to the cylinder axis. Show that the field outside is that of a two-dimensional dipole.

(b) Calculate the total magnet-static field energy per unit length. How is it divided inside and outside the cylinder?

(c) What is the inductance per unit length of the system, viewed as a ling circuit with current flowing up one side of the cylinder and back the other?

(a)

$$\vec{J}(\vec{x}') = K(\phi)\delta(\rho - R)\hat{z}$$

$$K(\phi) = \frac{I\cos\phi}{2R}$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} = \frac{\mu_0}{4\pi}\int d^3x' \frac{I\cos\phi'}{2R|\vec{x}-\vec{x}'|}$$

$$\vec{x}' = R\cos\phi'\hat{x} + R\sin\phi'\hat{y} + z'\hat{z}$$

by 3.149

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{\pi}{2}\sum_{m=-\infty}^{\infty}\int_0^\infty dk e^{im(\phi-\phi')}\cos[k(z-z')]I_m(k\rho_<)K_m(k\rho_>)$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{I\cos\phi'}{2R}\frac{\pi}{2}\sum_{m=-\infty}^{\infty}\int_0^\infty dk e^{im(\phi-\phi')}\cos[k(z-z')]I_m(k\rho_<)K_m(k\rho_>)$$

the integeral pick up $m = 1$ by orthogonality of cosines.

Using $\cos(\phi-\phi') = \cos\phi\cos\phi' + \sin\phi\sin\phi'$

and $\int_0^{2\pi} \mathrm{d}\phi'\cos\phi\cos(\phi-\phi') = \cos\phi\int_0^{2\pi}\mathrm{d}\phi'\cos^2\phi' = \pi\cos\phi$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\hat{z}\int_0^{2\pi} Rd\phi' \int_{-\infty}^{\infty} dz' \frac{I\cos\phi'}{2R}\frac{4}{\pi}\int_0^{\infty} dk \cos[k(z-z')]\cos(\phi-\phi')I_1(k\rho_<)K_1(k\rho_>)$$

$$= \frac{\mu_0 I}{2\pi}\hat{z}\cos\phi \int_{-\infty}^{\infty} dz' \int_0^{\infty} dk \cos[k(z-z')]I_1(k\rho_<)K_1(k\rho_>)$$

$$\because 2\pi\delta(k) = \int_{-\infty}^{\infty} dx e^{-ikx}$$

$$\Rightarrow \int_{-\infty}^{\infty} dz' \cos[k(z-z')] = \mathrm{Re}\left(e^{ikz}\int_{-\infty}^{\infty} dz' e^{-ikz'}\right) = \mathrm{Re}\left(e^{ikz}2\pi\delta(k)\right)$$

applying the asymptotic form 3.102

$$\vec{A}(\vec{x}) = \mu_0 I\hat{z}\cos\phi \int_0^{\infty} dk\, \mathrm{Re}\left[e^{ikz}\delta(k)\frac{1}{\Gamma(2)}\left(\frac{k\rho_<}{2}\right)\frac{\Gamma(1)}{2}\left(\frac{2}{k\rho_>}\right)\right]$$

$$= \frac{1}{2}\mu_0 I\hat{z}\cos\phi \int_0^{\infty} dk\, \mathrm{Re}\left[\delta(k)\frac{\rho_<}{\rho_>}\right]$$

$$= \frac{1}{4}\mu_0 I\hat{z}\cos\phi \int_{-\infty}^{\infty} dk\, \mathrm{Re}\left[\delta(k)\frac{\rho_<}{\rho_>}\right]$$

$$= \frac{1}{4}\mu_0 I\cos\phi\frac{\rho_<}{\rho_>}\hat{z}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \hat{\rho}\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \hat{\phi}\frac{\partial A_z}{\partial\rho}$$

$$= \frac{1}{4}\mu_0 I\left(-\hat{\rho}\frac{1}{\rho}\sin\phi\frac{\rho_<}{\rho_>} - \hat{\phi}\cos\phi\begin{Bmatrix}\frac{1}{R}, \rho < R \\ -\frac{R}{\rho^2}, \rho > R\end{Bmatrix}\right)$$

$$= -\frac{1}{4}\mu_0 I\begin{cases}\hat{\rho}\frac{\sin\phi}{R} + \hat{\phi}\frac{\cos\phi}{R} = \frac{1}{R}\hat{y}, \rho < R \\ \frac{R}{\rho^2}\left(\hat{\rho}\sin\phi - \hat{\phi}\cos\phi\right), \rho > R\end{cases}$$

$$\vec{A}_> = \frac{1}{4}\mu_0 I\frac{R}{\rho}\cos\phi\hat{z}$$

$$= \frac{1}{4}\mu_0 I\frac{R}{\rho}\hat{\rho}\times\hat{y}$$

$$= \frac{\mu_0}{4\pi}\frac{-IR\hat{y}\times\vec{\rho}}{\rho^2}$$

(b)

$$W = \frac{1}{2\mu_0}\int_{\rho<R} d^3x\vec{B}^2 + \frac{1}{2\mu_0}\int_{\rho>R} d^3x\vec{B}^2$$

$$= \frac{1}{2\mu_0}\int_{\rho<R} d^3x\left(-\frac{\mu_0 I}{4R}\right)^2 + \frac{1}{2\mu_0}\int_{\rho>R} d^3x\left(-\frac{\mu_0 I}{4}\frac{R}{\upsilon^2}\right)^2\left(\sin^2\phi + \cos^2\phi\right)$$

$$= \frac{\mu_0 I^2}{32R^2}\int_{\rho<R} d^3x + \frac{\mu_0 I^2 R^2}{32}\int_0^{2\pi}\rho d\phi\int_0^l dz\int_R^{\infty} d\rho\frac{1}{\rho^4}$$

$$= \frac{\mu_0 I^2}{32R^2}\pi R^2 l + \frac{\mu_0 I^2 R^2}{32}2\pi l\left[-\frac{1}{2\rho^2}\right]_R^{\infty}$$

$$= \frac{\mu_0\pi l I^2}{32} + \frac{\mu_0\pi l I^2}{32}$$

(c) Since we have only one circuit . Comparing to the above we read off $L/I = \pi \mu_0/8$.