6.8

A dielectric sphere of dielectric constant ε and radius a is located at the origin. There is a uniform applied electric field E_0 in the x direction. The sphere rotates with an angular velocity ω about the z axis. Show that there is a magnetic field

$\vec{H} = -\nabla\Phi_M$, where $\Phi_M = \frac{3}{5}\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right)\varepsilon_0 E_0 \omega \left(\frac{a}{r_>}\right)^5 \cdot xz$

Where $r_>$ is thr larger of r and a. The motion is nonrelativistic.

You may use thr results of Section 4.4 for the dielectric sphere in an applied field.

$$\vec{P} = 3\varepsilon_0 \frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\vec{E}_0 \quad \text{by Eq.(4.57)}$$

Therefore, the boud volume and surface charge densities are:

$$\rho_b = -\nabla\cdot\vec{P} = 0, \sigma_b = \vec{P}\cdot\vec{n}$$

$\vec{n}$ is the normal vector on the sphere surface.

Since the sphere is rotating, the bound surface charge result an effective surface current with density:

$$\vec{K}_M = \sigma_b\vec{\upsilon} = \left(\vec{P}\cdot\vec{n}\right)\left(\vec{\omega}\times\vec{r}\right)\Big|_{r=a} = a(\vec{P}\cdot\vec{n})(\vec{\omega}\times\vec{n}) = \vec{M}\times\vec{n}$$

we identify $\left(\vec{P}\cdot\vec{r}\right)\vec{\omega} = a\left(\vec{P}\cdot\vec{n}\right)\vec{\omega}$ as an effective magnetization($\vec{M}$).

Therefore, the effective magnetic surface charge density

$$\sigma_M(\theta,\phi) = \vec{M}\cdot\vec{n} = a\left(\vec{P}\cdot\vec{n}\right)\left(\vec{\omega}\cdot\vec{n}\right)$$

$$= 3\varepsilon_0 a\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\left(E_0\hat{z}\cdot\vec{n}\right)\left(\omega\cos\theta\right) = 3\varepsilon_0\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}a\omega E_0\sin\theta\cos\theta\cos\phi$$

$$\Phi_M\left(\vec{r}\right) = \frac{1}{4\pi}\oint\frac{\sigma_M}{\left|\vec{r}-\vec{r}'\right|}da' = \frac{3}{4\pi}\varepsilon_0\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}a^3\omega E_0\int\frac{\sin\theta'\cos\theta'\cos\phi'}{\left|\vec{r}-\vec{r}'\right|}d\Omega'$$

$$\sin\theta'\cos\theta'\cos\phi' = -\sqrt{\frac{8\pi}{15}}\frac{1}{2}\left(Y_{21}(\theta',\phi') + Y_{21}^*(\theta',\phi')\right)$$

$$= -\sqrt{\frac{8\pi}{15}}\frac{1}{2}\left(Y_{21}(\theta',\phi') - Y_{2-1}(\theta',\phi')\right) = -\sqrt{\frac{8\pi}{15}}\operatorname{Re}\{Y_{21}(\theta',\phi')\}$$

$$\int\frac{\sin\theta'\cos\theta'\cos\phi'}{\left|\vec{r}-\vec{r}'\right|}d\Omega' = -\sqrt{\frac{8\pi}{15}}\operatorname{Re}\{\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\phi)\int Y_{lm}^*(\theta',\phi')Y_{21}(\theta',\phi')d\Omega'\}$$

$$= -\sqrt{\frac{8\pi}{15}}\operatorname{Re}\{\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\phi)\delta_{l,2}\delta_{m,1}\}$$

$$= \frac{4\pi}{5}\frac{r_<^2}{r_>^3}\{-\sqrt{\frac{8\pi}{15}}\operatorname{Re}\{Y_{21}(\theta,\phi)\}\} = \frac{4\pi}{5}\frac{r_<^2}{r_>^3}\sin\theta\cos\theta\cos\phi$$

where $r_< = \min(r,a)$ and $r_> = \max(r,a)$.

Therefore the scalar potential

$$\Phi_M(\vec{r}) = \frac{1}{4\pi}\oint \frac{\sigma_M}{|\vec{r}-\vec{r}'|}da' = \frac{3}{4\pi}\varepsilon_0\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}a^3\omega E_0\{\frac{4\pi}{5}\frac{r_<^2}{r_>^3}\sin\theta\cos\theta\cos\phi\}$$

$$= \frac{3}{5}\varepsilon_0\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\omega E_0\frac{a^3 r_<^2}{r^2 r_>^3}(r\sin\theta\cos\phi)(r\cos\theta) = \frac{3}{5}\varepsilon_0\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\omega E_0\{\frac{a}{r_>}\}^5\cdot xz = \frac{1}{5}P\omega\{\frac{a}{r_>}\}^5\cdot xz$$

6.11

A transverse plane wave is incident normally in vacuum on a perfectly absorbing flat screen.

(a) From the law of conservation of linear momentum, show that the pressure(called radiation pressure) exerted on the screen is equal to the field energy per unit volume in the wave.

(b) In the neighborhood of the earth the flux of electromagnetic energy from the sun is approximately 1.4kW/m^2. If an interplanetary "sailplane" had a sail of mass 1g/m^2 of area and negligible other weight, what would be its maximum acceleration in meters per second squared due to the solar "wind" (corpuscular radiation)?

(a) Eq(6.22)

$$\frac{d}{dt}\left(\vec{P}_{mech}+\vec{P}_{field}\right)_\alpha = \sum_\beta\int_V \frac{\partial}{\partial x_\beta}T_{\alpha\beta}d^3x = \oint_s\sum_\beta T_{\alpha\beta}n_\beta da = \oint_s \vec{T}\cdot\vec{n}da$$

$$T_{\alpha\beta} = \varepsilon_0\left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}\left(\vec{E}\cdot\vec{E}+c^2\vec{B}\cdot\vec{B}\right)\delta_{\alpha\beta}\right]$$

In Cartesian coordinate system with the z-axis along the wave propagation direction $\vec{n} = -\hat{e}_z = (0,0,-1)$ $\vec{E} = (E_x, E_y, 0)$ $\vec{B} = (B_x, B_y, 0)$

$$\vec{T}\cdot\vec{n} = \varepsilon_0\begin{bmatrix} E_x^2+c^2B_x^2-\frac{1}{2}\left(E^2+c^2B^2\right) & E_xE_y+c^2B_xB_y & 0 \\ E_yE_x+c^2B_yB_x & E_y^2+B_y^2-\frac{1}{2}\left(E^2+c^2B^2\right) & 0 \\ 0 & 0 & -\frac{1}{2}\left(E^2+c^2B^2\right)\end{bmatrix}\begin{bmatrix}0\\0\\-1\end{bmatrix}$$

$$= \frac{1}{2}\left(E^2+c^2B^2\right)\hat{e}_z$$

$$\frac{d}{dt}\left(P_{mech}+P_{field}\right) = \oint_s \vec{T}\cdot\vec{n}da \Rightarrow \vec{F} = \frac{d}{dt}p_{mech} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)A\hat{e}_z$$

$$\frac{\vec{F}}{A} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)\hat{e}_z = \frac{P}{c}\hat{e}_z$$

(b)

$$P_z = \frac{F}{A} = \frac{1.4\times10^3 w/m^2}{3\times10^8 m/s}$$

$$a = \frac{F}{A} = \frac{F/A}{m/A} = \frac{1.4\times10^3 w/m^2}{3\times10^8 m/s \times 1\times10^{-3} kg/m^2}$$

$$= 4.66\times10^{-3} m/s^2$$

In the solar wind, there are approximately 10×10^4 protons$/\mathrm{m}^2\cdot\sec$,

with average velocity $\mathrm{v} = 4\times10^5\ m/s$

$$\frac{\Delta \mathrm{P_z}}{\Delta \mathrm{tA}} = P_z = 10^5 \times 4\times10^5 \times 1.67\times10^{-27} = 6.68\times10^{-17}\ N/m^2$$

$$a = \frac{6.68\times10^{-17}\ N/m^2}{10^{-3}\ kg/m^2} = 6.68\times10^{-14}\ m/s^2$$

6.18

Consider the Dirac expression $\frac{g}{4\pi}\int_L \frac{d\vec{l}'\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$

For the vector potential of a magnetic monopole and its associated string L. Suppose for definiteness that the monopole is located at the origin and the string along the negative z axis.

(a) Calculate A explicitly and show that in spherical coordinates it has components

$$\mathrm{A_r} = 0,\ \mathrm{A}_\theta = 0,\ \mathrm{A}_\phi = \frac{g(1-\cos\theta)}{4\pi r\sin\theta} = \left(\frac{g}{4\pi r}\right)\tan\left(\frac{\theta}{2}\right)$$

(b) Verify that $\vec{B} = \nabla\times\vec{A}$ is the Coulomb-like field of a point charge, except perhaps at θ=π.

(c) With the $\vec{B}$ determined in part b, evaluate the total magnetic flux passing through the circular loop of radius R sinθshown in the figure. Consider θ<π/2 and θ>π/2 separately, but always calculate the upward flux.

(d) From $\oint\vec{A}\cdot d\vec{l}$ around the loop, determine the total magnetic flux through the loop. Compare the result with that found in part c. Show that they are equal for 0< θ <π/2, but have a constant difference for π/2< θ <π. Interpret this difference.

(a)

$$A(\vec{x}) = \frac{g}{4\pi} \int_{z'=-\infty}^{z'=0} \frac{\hat{z}dz' \times (\vec{x} - \hat{z}z')}{|\vec{x} - \hat{z}z'|^3}$$

$$= \frac{g}{4\pi} \int_{z'=-\infty}^{z'=0} dz' \frac{\hat{z} \times \vec{x}}{\left|\rho^2 + (z - z')^2\right|^{3/2}} = \frac{g}{4\pi} (\hat{z} \times \vec{x}) \int_{\infty}^{-z} \frac{du}{(\rho^2 + u^2)^{3/2}}$$

let $u = \rho \tan\theta, du = \rho \sec^2\theta$

$$= \frac{g}{4\pi} (\hat{z} \times \vec{x}) \int \frac{\rho \sec^2\theta}{\rho^3 \sec^3\theta} d\theta = \frac{g}{4\pi} (\hat{z} \times \vec{x}) \frac{1}{\rho^2} \int \cos\theta d\theta = \frac{1}{\rho^2} \sin\theta \Big|_{u=\infty}^{u=-z}$$

$$= \frac{g}{4\pi} (\hat{z} \times \vec{x}) \frac{1}{\rho^2} \left[\frac{u}{\sqrt{u^2 + \rho^2}} \right]_{\infty}^{-z} = \frac{g}{4\pi} (\hat{z} \times \vec{x}) \frac{1}{\rho^2} \left[\frac{-z}{\sqrt{z^2 + \rho^2}} + 1 \right]$$

$$= \frac{g}{4\pi} (\hat{z} \times \vec{x}) \frac{1}{\rho^2} \left[1 - \frac{z}{r} \right]$$

where $\rho^2 = x^2 + y^2, r^2 = x^2 + y^2 + z^2, \hat{z} \times \vec{x} = \rho\hat{\phi} = r\sin\theta\hat{\phi}$

$$A(\vec{x}) = \frac{g}{4\pi} \frac{r\sin\theta}{\rho^2 r} [r - z]\hat{\phi}$$

$$= \frac{g}{4\pi} \frac{r\sin\theta}{r^2 r \sin\theta} r \left(1 - \frac{z}{r} \right) \hat{\phi}$$

$$= \frac{g}{4\pi} \frac{(1 - \cos\theta)}{r\sin\theta} \hat{\phi}$$

(b)

$$\vec{B} = \nabla \times \vec{A} = \hat{r} \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \, A_\phi + \hat{\theta} \frac{-1}{r} r \, A_\phi = \hat{r} \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} \left(\frac{g}{4\pi} \frac{1 - \cos\theta}{r} \right)$$

$$= \hat{r} \frac{g}{4\pi r^2}$$

(c)

$\theta < \pi/2$ $$\int \vec{B}\cdot\vec{n}da = \int_{\cos\theta}^{1}\int_{\phi=0}^{2\pi} \vec{B}\cdot\hat{r}r^2\, d\cos\theta\, d\phi = \frac{g}{4\pi}2\pi[\cos\theta]_{\cos\theta}^{1} = \frac{g}{2}(1-\cos\theta)$$

$\theta > \pi/2$ $$\int \vec{B}\cdot\vec{n}da = \int_{-1}^{\cos\theta}\int_{\phi=0}^{2\pi} \vec{B}\cdot(-\hat{r})r^2 d\cos\theta\, d\phi = \frac{g}{2}[\cos\theta]_{-1}^{\cos\theta} = -\frac{g}{2}(\cos\theta+1)$$

(d)

$$\oint \vec{A}\cdot d\mathrm{l} = \int_{\phi=0}^{2\pi} \frac{g}{4\pi}\frac{1-\cos\theta}{\sin\theta} r\sin\theta d\phi = \frac{g}{2}(1-\cos\theta)$$

This is same to part (c) in region θ <π/2,

for θ >π/2 there is a constant difference of $\oint \vec{A}\cdot d\vec{l} - \int \vec{B}\cdot\hat{n}da = g$

Obviously, the difference g is the (upward) magnetic flux through the string, which is included in part (d) but has been neglected in part(c).