7.1

For each set of Stokes parameters given below deduce the amplitude of the electric field, up to an overall phase, in both linear polarization and circular polarization bases and make an accurate drawing similar to Fig. 7.4 showing the lengths of the axes of one of the ellipses and its orientation.

(a) $s_0 = 3, \quad s_1 = -1, \quad s_2 = 2, \quad s_3 = -2;$

(b) $s_0 = 23, \quad s_1 = 0, \quad s_2 = 24, \quad s_3 = 7.$

Apply Eq. (7.26)(7.27)and(7.28)

$$\sqrt{\frac{s_0+s_1}{2}} = a_1 \qquad \sqrt{\frac{s_0-s_1}{2}} = a_2 \qquad \delta_l = \delta_2 - \delta_1 = \sin^{-1}\left(\frac{s_3}{2a_1a_2}\right)$$

$$\sqrt{\frac{s_0+s_3}{2}} = a_+ \qquad \sqrt{\frac{s_0-s_3}{2}} = a_- \qquad \delta_c = \delta_- - \delta_+ = \sin^{-1}\left(\frac{s_2}{2a_+a_-}\right)$$

(a)

$$s_0 = 3, s_1 = -1, s_2 = 2, s_3 = -2$$

$$a_1 = 1, a_2 = \sqrt{2}$$

$$\delta_l = \sin^{-1}\left(\frac{-2}{2\sqrt{2}}\right) = -\frac{1}{4}\pi(rad)$$

$$a_+ = \frac{1}{\sqrt{2}}, a_- = \sqrt{\frac{5}{2}}$$

$$\delta_c = \sin^{-1}\left(\frac{2}{2\left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{\frac{5}{2}}\right)}\right) = 1.1071(rad)$$

(b)

$$s_0 = 23, s_1 = 0, s_2 = 24, s_3 = 7$$

$$\delta_l = \sin^{-1}\left(\frac{s_3}{2a_1a_2}\right) = \sin\left(\frac{7}{2\sqrt{\frac{25}{2}}\sqrt{\frac{25}{2}}}\right) = 0.28379(rad)$$

$$a_+ = \sqrt{\frac{32}{2}} = 4$$

$$a_- = \sqrt{\frac{s_0 - s_3}{2}} = 3$$

$$\delta_c = \delta_- - \delta_+ = \sin^{-1}\left(\frac{24}{2(4\times 3)}\right) = \frac{1}{2}\pi(rad)$$

7.2

A plane wave is incident on a layered interface as shown in the figure. The indices of refraction of the three nonpermeable media are n_1, n_2, n_3. The thickness of the intermediate layer is d. Each of the other media is semi-infinite.

(a) Calculate the transmission and reflection coefficients (ratios of transmitted and reflected Poynting's flux to the incident flux), and sketch their behavior as a function of frequency for $n_1 = 1, n_2 = 2, n_3 = 3; n_1 = 3, n_2 = 2, n_3 = 1;$ and $n_1 = 2, n_2 = 4, n_3 = 1.$

(b) The medium n_1 is part of an optical system (e.g., a lens); medium n_3 is air ($n_3 = 1$). It is desired to put an optical coating (medium n_2) on the surface so that there is no reflected wave for a frequency ω_0. What thickness a and index of refraction n_2 are necessary?

(a) Choose a coordinate system such that the electric field is along x-axis, the magnetic field along the y-axis and the wave propagates in z-direction. In medium n_1, the incident and reflected waves are described by:

$$\vec{E}^i = E^i e^{i(k_1 z - \omega t)}\hat{x}, \quad \vec{B}^i = \frac{E^i}{\upsilon_1} e^{i(k_1 z - \omega t)}\hat{y} \;; \vec{E}^r = E^r e^{i(-k_1 z - \omega t)}\hat{x}, \quad \vec{B}^r = -\frac{E^r}{\upsilon_1} e^{-i(k_1 z - \omega t)}\hat{y}$$

In medium n_2, there are both forward (denoted as +) and backward (-) propagating waves and are described by:

$$\vec{E}^+ = E^+ e^{i(k_2 z - \omega t)}\hat{x}, \quad \vec{B}^+ = \frac{E^+}{\upsilon_2} e^{i(k_2 z - \omega t)}\hat{y} \;; \vec{E}^- = E^- e^{i(k_2 z - \omega t)}\hat{x}, \quad \vec{B}^- = -\frac{E^-}{\upsilon_2} e^{i(k_2 z - \omega t)}\hat{y}$$

In medium n_3, there is only transmitted wave:

$$\vec{E}^t = E^t e^{i(k_3 z - \omega t)}\hat{x}, \quad \vec{B}^t = \frac{E^t}{\upsilon_3} e^{i(k_3 z - \omega t)}\hat{y}$$

Where $k_1 = \omega/\upsilon_1$, $k_2 = \omega/\upsilon_2$, $k_3 = \omega/\upsilon_3$ are wave numbers in the three media. For nonpermeable media $(\mu_1 \approx \mu_2 \approx \mu_3 \approx \mu_0)$, $\vec{E}_\parallel$ and $\vec{B}_\parallel$ are continuous at each interface (x=0, d). At x=0, one has:

$$E^t + E^r = E^+ + E^- ; \quad \frac{E^i - E^r}{\tau_1} = \frac{E^+ - E^-}{\tau_2}$$

At x=d, one has: $E^+ e^{ik_2 d} + E^- e^{-ik_2 d} = E^t e^{ik_3 d}$; $\quad \dfrac{E^+ e^{ik_2 d} - E^- e^{-ik_2 d}}{\upsilon_2} = \dfrac{E^t}{\upsilon_3} e^{ik_3 d}$

Let $\quad \alpha = \dfrac{\upsilon_1}{\upsilon_2} = \dfrac{n_2}{n_1}; \quad \beta = \dfrac{\upsilon_2}{\upsilon_3} = \dfrac{n_3}{n_2}$

The four equations are then

$$E^i + E^r = E^+ + E^-; \quad E^i - E^r = \alpha\left(E^+ - E^-\right)$$
$$E^+ e^{ik_2 d} + E^- e^{-ik_2 d} = E^t e^{ik_3 d}; \qquad E^+ e^{ik_2 d} - E^- e^{-ik_2 d} = \beta E^t e^{ik_3 d}$$

Solving for $E^+ e^{ik_2 d}$ and $E^- e^{-ik_2 d}$ from the last two equations :

$$E^+ e^{ik_2 d} = \frac{1}{2}(1+\beta) E^t e^{ik_3 d}, \quad E^- e^{-ik_2 d} = \frac{1}{2}(1-\beta) E^t e^{ik_3 d}$$

Add the first two equations to eliminate E^r :

$$2E^i = (1+\alpha)E^+ + (1+\alpha)E^- = \frac{1}{2} E^t e^{ik_3 d}\left[(1+\alpha)(1+\beta)e^{-ik_2 d} + (1-\alpha)(1-\beta)e^{ik_2 d}\right]$$

Solving for E^t in terms of E^i :

$$\frac{E^i}{E^t} = \frac{1}{2} e^{ik_3 d}\left[(1+\alpha\beta)\cos(k_2 d) - 2i(\alpha+\beta)\sin(k_2 d)\right]$$

Therefore,

$$4\left|\frac{E^i}{E^t}\right|^2 = (1+\alpha\beta)^2\cos^2(k_2 d) + (\alpha+\beta)^2\sin^2(k_2 d) = (1+\alpha\beta)^2 - (1-\alpha^2)(1-\beta^2)\sin^2(k_2 d)$$

The transmission coefficient

$$T = \frac{I^t}{I^i} = \frac{\varepsilon_3\upsilon_3\left|E^t\right|^2}{\varepsilon_1\upsilon_1\left|E^i\right|^2} = \frac{n_3}{n_1}\left|\frac{E^t}{E^i}\right|^2 = \frac{4\alpha\beta}{(1+\alpha\beta)^2 - (1-\alpha^2)(1-\beta^2)\sin^2(k_2 d)}$$
$$= \frac{4 n_1 n_2^2 n_3}{n_2^2(n_1+n_3)^2 + (n_2^2 - n_3^2)(n_2^2 - n_1^2)\sin^2\left(n_2 \frac{d\omega}{c}\right)}$$

It varies between the two extremism values $T_1 = \dfrac{4 n_1 n_2^2 n_3}{\left(n_2^2 + n_1 n_3\right)^2}$, $T_2 = \dfrac{4 n_1 n_3}{(n_1+n_3)^2}$

As a function of ω for a fixed d or as a function of d for fixed ω. From the energy conservation,

$$R = 1 - T = \frac{n_2^2(n_1-n_3)^2 + (n_2^2 - n_3^2)(n_2^2 - n_1^2)\sin^2\left(n_2 \frac{d\omega}{c}\right)}{n_2^2(n_1-n_3)^2 + (n_2^2 - n_3^2)(n_2^2 - n_1^2)\sin^2\left(n_2 \frac{d\omega}{c}\right)}$$

In the special case of d=0, the coefficient reduce to the familiar forms of two media.

(b) For $n_3 = 1$, the reflection coefficient

$$R = \frac{n_2^2(n_1-1)^2 + (n_2^2 - 1)(n_2^2 - n_1^2)\sin^2\left(n_2 \frac{d\omega}{c}\right)}{n_2^2(n_1+1)^2 + (n_2^2 - 1)(n_2^2 - n_1^2)\sin^2\left(n_2 \frac{d\omega}{c}\right)}$$

To have zero reflection at $\omega = \omega_0$, the following condition must be satisfied:

$$n_2^2(n_1-1)^2 + (n_2^2 - 1)(n_2^2 - n_1^2)\sin^2\left(n_2 \frac{d\omega_0}{c}\right) = 0$$

Since $n_1 > 1, n_2 > 1$, this is only possible if $n_2 < n_1$. One set of possible solytion is given by $\sin^2\left(n_2 \frac{d\omega}{c}\right) = 1$, and $n_2^2(n_1-1)^2 + (n_2^2-1)(n_2^2-n_1^2) = 0$

This leads to $n_2 = \sqrt{n_1}$ and $d = \left(l + \frac{1}{2}\right)\pi \dfrac{c}{\sqrt{n_1}\,\omega_0}$ where l is a non-zero integer.

7.3

Two plane semi-infinite slabs of the same uniform, isotropic, nonpermeable, lossless dielectric with index of refraction n are parallel and separated by an air gap (n=1) of width d. A plane electromagnetic wave of frequency ω is incident on the gap from one of the slabs with angle of incidence i. For linear polarization both parallel to and perpendicular to the plane of incidence,

(a) Calculate the ratio of power transmitted into the second slab to the incident power and the ratio of reflected to incident power;

(b) For I greater than the critical angle for total internal reflection, sketch the ratio of transmitted power to incident power as a function of d measured in units of wavelength in the gap.

(a) We may take the results from problem 7.2, but we need to redefine the reflection and transmission amplitudes and the phase ϕ, gotten from the path difference for waves reflected and the first and second interface. To make the connection with 7.2, I will say region 1 is the left slab, region 2 is the gap, and region 3 is the right slab.

For the ij interface $r_{ij} = \frac{E'_{0j}}{E_{oi}} \quad R_{ij} = \frac{E''_{0j}}{E_{0i}}$

Thus from the figure $E''_0 = E_0 R_{12} + r_{12} E_0 R_{23} r_{21} e^{i\phi} + r_{12} E_0 R_{21} R_{23} r_{21} e^{i2\phi} + ...$

$$E''_0 = E_0 R_{12} + r_{12} E_0 R_{23} r_{21} e^{i\phi} \sum_{n=0}^{\infty} \left(R_{21} R_{23} e^{i\phi}\right)^n$$

$$E''_0 = E_0 \left(R_{12} + \frac{r_{12} r_{21} R_{23}}{\left(e^{-i\phi} - R_{21} R_{23}\right)} \right)$$

Similarly

$$E'_0 = e^{i\phi/2} \left(E_0 r_{12} r_{23} + E_0 r_{12} R_{23} R_{21} r_{23} e^{i\phi} + ...\right)$$

$$E'_0 = E_0 e^{i\phi/2} \frac{r_{12} r_{23}}{1 - R_{21} R_{23} e^{i\phi}}$$

Where the phase shift for the internally reflected wave is given by

$$\phi = \frac{2\pi \left(2d/\cos r\right)}{\lambda_2} = \frac{\omega \left(2d/\cos r\right)}{c} = \frac{2d\omega}{c\cos r} = \frac{2d\omega}{c\sqrt{1 - n^2 \sin^2 i}}$$

Note that the angle r is related to I by Snell's law: $\sin r = n \sin i$. Also note that the overall phase $e^{i\phi/2}$, neglected in the transmitted wave because, if ϕ is real as in problem 7.2, its effect cancels when calculating T, cannot be neglected if ϕ is complex. I will assume $\mu \cong \mu_0$ is what follows.

$\vec{E}$ polarized perpendicular to the plane of incidence.

$$r_{12} = \frac{2n\cos i}{n\cos i + \sqrt{1 - n^2 \sin^2 i}} \qquad R_{12} = \frac{n\cos i - \sqrt{1 - n^2 \sin^2 i}}{n\cos i + \sqrt{1 - n^2 \sin^2 i}}$$

And

$$r_{21} = \frac{2\cos r}{\cos r + \sqrt{n^2 - \sin^2 r}} = \frac{2\cos r}{\cos r + n\cos i}$$

$$R_{21} = \frac{\cos r - \sqrt{n^2 - \sin^2 r}}{n\cos r + \sqrt{n^2 - \sin^2 r}} = \frac{\cos r - n\cos i}{n\cos i + n\cos i}$$

From symmetry, $r_{23} = r_{12}$ and $R_{23} = R_{21}$

Now for a plane wave

$$S_i = \frac{1}{2}\sqrt{\frac{\varepsilon_3}{\varepsilon_1}} \frac{|E'_0|^2}{|E_0|^2} = \frac{n}{n} \frac{|E'_0|^2}{|E_0|^2} = \frac{|E'_0|^2}{|E_0|^2}$$

if $\mu_1 \cong \mu_3$, which should hold for optical frequencies.

(b) If $i > i_c$, when $\cos r = \pm\sqrt{1 - \sin^2 r} = \pm\sqrt{1 - n^2 \sin^2 i}$ is purely imaginary, say $\cos r = -i\alpha$, where α is real and I have chosen the sign in order that the results found later are physical. Then we can write

$$E'_0 = E_0 e^{-\frac{2\pi d}{\lambda_2 \alpha}} \frac{r_{12} r_{23}}{1 - \left(R_{21}\right)^2 e^{-\frac{4\pi d}{\lambda_2 \alpha}}}$$

$$E''_0 = E_0 \left[R_{12} + \frac{r_{12} r_{23} R_{23}}{e^{\frac{4\pi d}{\lambda_2 \alpha}} - \left(R_{21}\right)^2} \right]$$

Qualitatively, we see the effect of the gap is greatest when d is small. When d is large, we can write $E'_0 \to 0 \quad E''_0 \to E_0 R_{21}$

and multiple scattering effects may be neglected. Plugging the above result into T and R gives you these quantities in term of d, which of d, which may be plotted.

7.4

A plan-polarized electromagnetic wave of frequency ω in free space is incident normally on the flat surface of a nonpermeable medium of conductivity σ and dielectric constant ε.

(a) Calculate the amplitude and phase of the reflected wave relative to the incident wave for arbitrary σ and ε.

(b) Discuss the limiting cases of a very poor and a very good conductor, and show that for a good conductor the reflection coefficient (ratio of reflected to incident intensity) is approximately $R \cong 1 - 2\frac{\omega}{c}\delta$ where δ is the skin depth.

(a) At normal incident, the reflected wave E_0'' is given by $\frac{E_0''}{E_0}=\frac{1-n}{1+n}$

where $n=c\sqrt{\mu\varepsilon}=c\sqrt{i\frac{\mu\sigma}{\omega}}=(1+i)\frac{c}{\omega}\sqrt{\frac{\mu\sigma\omega}{2}}=(1+i)\frac{c}{\omega\delta}$

where $\sigma=\sqrt{\frac{2}{\sigma\mu\omega}}$ is the skin depth. Therefore,

$$r=\frac{\sqrt{1+4c^4/\omega^4\delta^4}}{1+2c/\omega\delta+2c^2/\omega^2\delta^2}=-\frac{\sqrt{\omega^4\delta^4+4c^4}}{2c^2+2c\omega\delta+\omega^2\delta^2}$$

$$\tan\phi=-\frac{2c/\omega\delta}{1-2c^2/\omega^2\delta^2}=-\frac{2c\omega\delta}{\omega^2\delta^2-2c^2}$$

For a perfect conductor, $\sigma\to\infty\Rightarrow\delta\to0$, the amplitude and the phase $r\to1$ and $\tan\phi\to0^-(\phi\to\pi)$ As expected the reflected wave has a 180^o phase change with respect to the incident wave.

(b) The reflection coefficient

$$R=r^2=\frac{\omega^4\delta^4+4c^4}{\left(2c^2+2c\omega\delta+\omega^2\delta^2\right)^2}\approx\frac{1+\frac{(\omega\delta)^4/c}{4}}{\left(1+\omega\delta/c\right)^2}\approx1-2\frac{\omega\delta}{c}$$

7.6

A plane wave of frequency ω is incident normally from vacuum on a semi-infinite slab of material with a complex index of refraction $n(\omega)[n^2(\omega)=\varepsilon(\omega)/\varepsilon_0]$

(a) Show that the ratio of reflected power to incident power is $R=\left|\frac{1-n(\omega)}{1+n(\omega)}\right|^2$

while the ratio of power transmitted into the medium to the incident power is

$T=\frac{4\mathrm{Re}(\omega)}{|1+n(\omega)|^2}$

(b) Evaluate $\mathrm{Re}\left[i\omega\left(E\cdot D^*-B\cdot H^*\right)/2\right]$ as a function of (x, y, z). Show that this rate of change of energy per unit volume accounts for the relative transmitted power T.

(c) For a conductor, with $n^2=1+i\left(\sigma/\omega\varepsilon_0\right)$, σ real, write out the results of parts a and b in the limit $\varepsilon_0\omega<<\sigma$. Express your answer in terms of δ as much as enter the complex form of Poynting's theorem?

(a) For incident (i=r=0), (7.39)reduces to

$$E_t=\frac{2}{1+\sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}}}E_i=\frac{2}{1+n(\omega)}E_i$$

$$E_r=\frac{1-\sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}}}{\sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}}+1}E_i=\frac{1-n(\omega)}{n(\omega)+1}E_i$$

$$I=\langle S\rangle_t=\mathrm{Re}\frac{1}{2}\left(\bar{E}\times\bar{H}^*\right)$$

$$B_i=H_i=\sqrt{\varepsilon_0}\frac{\bar{k}_i\times\bar{E}_i}{k_i}=\sqrt{\varepsilon_0}\frac{(k\sin i\hat{e}_x+k\cos i\hat{e}_z)\times E_i\hat{e}_y}{k}=-\sqrt{\varepsilon_0}E_i\hat{e}_x$$

$$B_r=H_r=\sqrt{\varepsilon_0}\frac{k_r\times\bar{E}_r}{k_r}=\sqrt{\varepsilon_0}\frac{(k\sin i\hat{e}_x-k\cos i\hat{e}_z)\times E_r\hat{e}_y}{k}=\sqrt{\varepsilon_0}E_r\hat{e}_x$$

$$I_i=\mathrm{Re}\frac{1}{2}\left(\bar{E}_i\times\bar{H}_i^*\right)=\frac{1}{2}E_i\hat{e}_y\times\left(-\sqrt{\varepsilon_0}E_i\hat{e}_x\right)=\frac{1}{2}\sqrt{\varepsilon_0}|E_i|^2\hat{e}_z$$

$$I_r=\mathrm{Re}\frac{1}{2}\left(\bar{E}_r\times\bar{H}_r^*\right)=\frac{1}{2}E_i\hat{e}_y\times\sqrt{\varepsilon_0}E_r\hat{e}_x=\frac{1}{2}\sqrt{\varepsilon_0}|E_r|^2(-\hat{e}_z)$$

$$R=\frac{|I_r|}{|I_i|}=\left|\frac{1-n(\omega)}{n(\omega)+1}\right|^2$$

$$B_t=H_t=\sqrt{\varepsilon(\omega)}\frac{k_t\times\bar{E}_t}{k_r}=\sqrt{\varepsilon(\omega)}\frac{(k\sin i\hat{e}_x+k\cos i\hat{e}_z)\times E_t\hat{e}_y}{k}=-\sqrt{\varepsilon(\omega)}E_t\hat{e}_x$$

$$I_t=\mathrm{Re}\frac{1}{2}\left(\bar{E}_t\times\bar{H}_t\right)=\frac{1}{2}E_t\hat{e}_y\times\left(-\mathrm{Re}\left[\sqrt{\varepsilon(\omega)}\right]E_t\hat{e}_x\right)=\frac{1}{2}\mathrm{Re}\left[\sqrt{\varepsilon(\omega)}\right]|E_t|^2(-\hat{e}_z)$$

$$T=\frac{|I_t|}{|I_i|}=\frac{\mathrm{Re}\left[\sqrt{\varepsilon(\omega)}\right]|E_t|^2}{\sqrt{\varepsilon_0}|E_i|^2}=\mathrm{Re}\left[\frac{\sqrt{\varepsilon(\omega)}}{\sqrt{\varepsilon_0}}\right]\left|\frac{2}{1+n(\omega)}\right|^2=\frac{4\mathrm{Re}[n(\omega)]}{|1+n(\omega)|^2}$$

Conservation of energy R+T=1

$$R+T=\frac{|1+n(\omega)|^2+4\mathrm{Re}[n(\omega)]}{|1+n(\omega)|^2}$$

$$=\frac{[n(\omega)-1][n^*(\omega)-1]+2[n+n^*(\omega)]}{[1+n(\omega)][1+n^*(\omega)]}$$

$$=\frac{nn^*+n+n^*+1}{(1+n)(1+n^*)}=1$$

(b)

$$\vec{E} = \vec{E}_t e^{ik\vec{n}\cdot\vec{x}},\ \vec{B}_t = \frac{1}{\omega}\vec{k}\times\vec{E}_t e^{ik\vec{n}\cdot\vec{z}} = \frac{1}{\omega}k\vec{n}\times\vec{E}_t e^{ik\vec{n}\cdot\vec{z}} = \frac{1}{v}\vec{n}\times\vec{E}_t e^{ik\vec{n}\cdot\vec{z}} = \sqrt{\mu\varepsilon}\,\vec{n}\times\vec{E}_t e^{ik\vec{n}\cdot\vec{z}}$$
$$= \sqrt{\mu_0\varepsilon_0}\,n(\omega)\vec{n}\times\vec{E}_t e^{ik\vec{n}\cdot\vec{z}}$$

$$n(\omega) = \frac{c}{v} = \frac{\sqrt{\mu\varepsilon}}{\sqrt{\mu_0\varepsilon_0}}$$

$$k(\omega) = \frac{\omega n(\omega)}{c}$$

$$\mathrm{Re}\left[i\omega\left(\vec{E}\cdot\vec{D}^* - \vec{B}\cdot\vec{H}^*\right)/2\right]$$
$$= \mathrm{Re}\left[\frac{i\omega}{2}\left(\vec{E}_t e^{ikz}\cdot\varepsilon^*(\omega)\vec{E}_t e^{-ik\vec{n}\cdot\vec{z}} - \sqrt{\mu_0\varepsilon_0}\,n(\omega)\vec{n}\times\vec{E}_{tt}e^{ik\vec{n}\cdot\vec{z}}\cdot\frac{1}{\mu}\sqrt{\mu_0\varepsilon_0}\,n^*(\omega)\vec{n}\times\vec{E}_{tt}e^{-ik\vec{n}\cdot\vec{z}}\right)\right]$$
$$= \mathrm{Re}\left[\frac{i\omega}{2}\left(\varepsilon^*(\omega) - \varepsilon_0 n(\omega)n^*(\omega)\right)\left|\vec{E}_t\right|^2 e^{-2\mathrm{Im}[k]z}\right]$$
$$= \mathrm{Re}\left[\frac{i\omega\varepsilon_0}{2}\left(\frac{\varepsilon^*(\omega)}{\varepsilon_0} - n(\omega)n^*(\omega)\right)\left|\vec{E}_t\right|^2 e^{-2\mathrm{Im}[k]z}\right]$$
$$= \mathrm{Re}\left[\frac{i\omega\varepsilon_0}{2}\left(n^{2*}(\omega) - \left|n(\omega)\right|^2\right)\left|\vec{E}_t\right|^2 e^{-2\mathrm{Im}[k]z}\right]$$
$$= \mathrm{Re}\left[\frac{i\omega\varepsilon_0}{2}\left\{\left[(n_R + in_I)(n_R + in_I)\right]^* - (n_R + in_T)(n_R + n_I)^*\right\}\left|\vec{E}_t\right|^2 e^{-2\mathrm{Im}[k]z}\right]$$
$$= \mathrm{Re}\left[\frac{i\omega\varepsilon_0}{2}\left(-2in_R n_I - 2n_I\right)\right]\omega\varepsilon_0\left|\vec{E}_t\right|^2 e^{-2\mathrm{Im}[k]z}$$
$$= \mathrm{Im}[n(\omega)]\mathrm{Re}[n(\omega)]\omega\varepsilon_0\left|\vec{E}_t\right|^2 e^{-2\mathrm{Im}[k]z}$$
$$= \mathrm{Im}[n(\omega)]\sqrt{\frac{\varepsilon_0}{\mu_0}}\,\mathrm{Re}[k(\omega)]\omega\varepsilon_0\left|\vec{E}_t\right|^2 e^{-2\mathrm{Im}[k]z}$$

7.13

A stylized model of the ionosphere is a medium described by the dielectric constant (7.59). Consider the earth with such a medium beginning suddenly at a height h and extending to infinity. For waves with polarization both perpendicular to the plane of incident (from a horizontal antenna) and in the plane of incidence (from a vertical antenna),

(a) Show from Fresnel's equation for reflection and refraction that for $\omega > \omega_p$ there is a range of angles of incidence for which reflection is not total, but for larger angles there is total reflection back toward the earth.

(b) A radio amateur operating at a wavelength of 21 meters in the early evening finds that she can receive distant stations located more than 1000km away, but none closer. Assuming that the signals are being reflected from the F layer of the ionosphere at an effective height of 300km, calculate the electron the electron density. Compare with the know maximum and minimum F layer densities of $\sim 2\times10^{12}\,\mathrm{m}^{-3}$ in the daytime and $\sim(2-4)\times10^{11}\,\mathrm{m}^{11}$ a night.

(a) The index of refraction of the ionosphere is $n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{1-\frac{\omega_p^2}{\omega^2}} = \frac{1}{\omega}\sqrt{\omega^2-\omega_p^2}$

The ratios between the amplitudes of reflected and incident wave are given by Eqs. (7.39) and (7.41) for the two polarizations. Note the Eqs. (7.41) and (7.39) have different sign conventions for E_0''.

$$\begin{cases}\dfrac{E_0''}{E_0} = \dfrac{n\cos\theta - \sqrt{n'^2 - n^2\sin^2\theta}}{n\cos\theta + \sqrt{n'^2 - n^2\sin^2\theta}} & \text{for } \vec{E}\perp \text{plane of incidence (7.39)}\\[2ex] \dfrac{E_0''}{E_0} = \dfrac{n'^2\cos\theta - n\sqrt{n'^2 - n^2\sin^2\theta}}{n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}} & \text{for } \vec{E}\parallel \text{plane of incidence (7.41)}\end{cases}$$

earth $n = 1$, medium of ionosphere n'

$$\begin{cases}\dfrac{E_0''}{E_0} = \dfrac{\cos\theta - \sqrt{n'^2 - \sin^2\theta}}{\cos\theta + \sqrt{n'^2 - \sin^2\theta}} = 1(\because \text{total reflection}) & \text{for } \vec{E}\perp \text{plane of incidence}\\[2ex] \dfrac{E_0''}{E_0} = \dfrac{n'^2\cos\theta - \sqrt{n'^2 - \sin^2\theta}}{n'^2\cos\theta + \sqrt{n'^2 - \sin^2\theta}} = 1 & \text{for } \vec{E}\parallel \text{plane of incidence}\end{cases}$$

$$\begin{cases}\cos\theta - \sqrt{n'^2 - \sin^2\theta} = \cos\theta + \sqrt{n'^2 - \sin^2\theta} & \text{for } \vec{E}\perp \text{plane of incidence}\\ n'^2\cos\theta - \sqrt{n'^2 - \sin^2\theta} = n'^2\cos\theta + \sqrt{n'^2 - \sin^2\theta} & \text{for } \vec{E}\parallel \text{plane of incidence}\end{cases}$$

$$\sqrt{n'^2 - \sin^2\theta} = 0 \Rightarrow \quad \sin\theta = n \quad \theta_c = \sin^{-1} n$$

In both cases, the amplitude of the ratio is unity when $\sin\theta'$ is imaginary. This corresponds cases that the incidence angle θ is greater than the critical angle θ_c:

$$\theta_c = \sin^{-1} n = \sin^{-1}\left\{\frac{\sqrt{\omega^2 - \omega_p^2}}{\omega}\right\}$$

Therefore, the reflection is partial if $\theta < \theta_c$ and is total if $\theta > \theta_c$ for $\omega > \omega_p$

(b) For simplicity, treat the ionosphere and the earth as flat surfaces and assume that the amateur can only receive distant stations when the wave is totally reflected. In this case,

$$\sin\theta_c = \frac{d}{\sqrt{4h^2 + d^2}} \Rightarrow \frac{\sqrt{\omega^2 - \omega_p^2}}{\omega} = \frac{d}{\sqrt{4h^2 + d^2}} \Rightarrow \omega_p = 2\pi\frac{c}{\lambda}\sqrt{\frac{4h^2}{d^2 + 4h^2}}$$

Where h=300km is the effective height of the F layer, d=10000km is the distance between the station and the receiver and $\lambda = 21\text{m}$ is the wavelength. Plugging in the numbers, we get the plasma frequency

$$\omega_p = 2\pi\frac{3\times10^8}{21}\sqrt{\frac{4\times(300)^2}{(1000)^2 + 4\times(300)^2}} = 4.6\times10^7\,Hz$$

which corresponds to an electron density $n = \frac{m\varepsilon_0\omega_p^2}{c^2} = 6.6\times10^{11}/\text{m}^3$

Note the day-night difference is due to the sunlight.

7.19

An approximately monochromatic plane wave packet in one dimension has the instantaneous form, $u(x,0) = f(x)e^{ik_0x}$, with $f(x)$ the modulation envelope. For each of the forms $f(x)$ below, calculate the wave-number spectrum $|A(k)|^2$ of the packet, sketch $|u(x,0)|^2$ and $|A(k)|^2$, evaluate explicitly the rms deviations from the means Δx and Δk (defined in terms of the intensities $|u(x,0)|^2$ and $|A(k)|^2$), and test inequality(7.82).

(a) $f(x) = Ne^{-\alpha|x|/2}$

(b) $f(x) = Ne^{-\alpha|x|/4}$

(c) $f(x) = \begin{cases} N(1-\alpha|x|) & \text{for } \alpha|x| < 1 \\ 0 & \text{for } \alpha|x| > 1 \end{cases}$

(d) $f(x) = \begin{cases} N & \text{for } |x| > a \\ 0 & \text{for } |x| < a \end{cases}$

(a)

$$u(x,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{ikx}dk$$

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,0)e^{-ikx}dx \qquad u(x,0) = f(x)e^{ik_0x}$$

$$f(x) = Ne^{-\alpha|x|/2}$$

$$A(k) = \frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha|x|/2}e^{i(k-k_0)}dx = \frac{N}{\sqrt{2\pi}}\left(\int_{-\infty}^{0} e^{-\alpha|x|/2}e^{i(k-k_0)}dx + \int_{0}^{\infty} e^{-\alpha|x|/2}e^{i(k-k_0)}dx\right)$$

$$= \frac{N}{\sqrt{2\pi}}2\,\text{Re}\left(\int_0^{\infty} e^{-\alpha|x|/2}e^{i(k-k_0)}dx\right) = \frac{N}{\sqrt{2\pi}}2\cdot2\frac{\alpha}{4\left((k-k_0)^2 + \alpha^2\right)}$$

Let us take N=1 and measure x in units of α^{-1}, then

$$|u(x,0)|^2 = e^{-|x|} \qquad |A(k)|^2 = \frac{1}{2\pi}\left(\frac{1}{k^2 + \frac{1}{4}}\right)^2$$

(b)

$$f(x) = Ne^{-\alpha^2x^2} \qquad A(k) = \frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha^2x^2/4}e^{-i(k-k_0)x}dx = \frac{N}{\sqrt{2\pi}}2\,\text{Re}\left(\int_0^{\infty} e^{-\alpha^2x^2/4}e^{-i(k-k_0)x}dx\right)$$

$$= \frac{N}{\sqrt{2\pi}}2\int e^{-\alpha^2x^2/4}\cos((k-k_0)x)dx = \frac{N}{\sqrt{2\pi}}2\frac{\sqrt{\pi}}{\alpha}e^{-(k-k_0)^2/\alpha^2}$$

Taking N=1, α =1

(c)

$$f(x)=N(1-\alpha|x|),\ |x|>1,\ 0\text{ otherwise}$$

$$A(k)=\frac{N}{\sqrt{2\pi}}\int_{-1}^{1}(1-\alpha|x|)e^{-i(k-k_0)x}dx=\frac{2N}{\sqrt{2\pi}}\int_{0}^{1}(1-\alpha|x|)\cos((k-k_0)x)dx$$

$$=\frac{2N}{\sqrt{2\pi}}\left(\frac{(\sin(k-k_0))(k-k_0)-\alpha\cos(k-k_0)-\alpha(k-k_0)\sin(k-k_0)+\alpha}{(k-k_0)^2}\right)$$

(d)

$$f(x)=n,\ |x|<a,0\text{ otherwise}$$

$$A(k)=\frac{N}{\sqrt{2\pi}}2\,\mathrm{Re}\left(\int_0^a e^{-i(k-k_0)x}dx\right)=\frac{N}{\sqrt{2\pi}}2\int_0^a\cos((k-k_0)x)dx=N\sqrt{\frac{2}{\pi}}\frac{\sin(k-k_0)a}{(k-k_0)}$$

Choosing N=1, a=1

7.20

A homogeneous, isotropic, nonpermeable dielectric is characterized by an index of refraction $n(\omega)$, which is in general complex in order to describe absorptive processes.

(a) show that the general solution for plane waves in one dimension can be written

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega e^{-i\omega t}\left[A(\omega)e^{i(\omega/c)n(\omega)x}+B(\omega)e^{-i(\omega/c)n(\omega)x}\right]$$

where $u(x,t)$ is a component of E or B.

(b) if $u(x,t)$ is real, show that $n(-\omega)=n^*(\omega)$.

(c) Show that, if $u(0,t)$ and $\partial u(0,t)/\partial x$ are the boundary values of u and its derivative at $x=0$, the coefficients $A(\omega)$ and $B(\omega)$ are

$$\begin{Bmatrix}A(\omega)\\B(\omega)\end{Bmatrix}=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dte^{i\omega t}\left[u(0,t)\mp\frac{ic}{\omega n(\omega)}\frac{\partial u}{\partial x}(0,t)\right]$$

(a) Just Fourier transform

(b) $u(x,t)$ real, then $u(x,t)-u^*(x,t)=0$

$$u(x,t)-u^*(x,t)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega\left\{e^{-i\omega t}\left[A(\omega)e^{i(\omega/c)n(\omega)x}+B(\omega)e^{-i(\omega/c)n(\omega)x}\right]-e^{i\omega t}\left[A(\omega)e^{-i(\omega/c)n^*(\omega)x}+B(\omega)e^{i(\omega/c)n^*(\omega)x}\right]\right\}$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega e^{-i\omega t}\left\{\left[A(\omega)e^{i(\omega/c)n(\omega)x}+B(\omega)e^{-i(\omega/c)n(\omega)x}\right]-\left[A(\omega)e^{i(\omega/c)n^*(\omega)x}+B(\omega)e^{-i(\omega/c)n^*(\omega)x}\right]\right\}$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega e^{-i\omega t}\left\{A(\omega)e^{i(\omega/c)x}\left[e^{n(\omega)}-e^{n^*(\omega)}\right]+B(\omega)e^{-i(\omega/c)x}\left[e^{n(\omega)}-e^{n^*(\omega)}\right]\right\}$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega e^{-i\omega t}\left\{\left[A(\omega)e^{i(\omega/c)x}+B(\omega)e^{-i(\omega/c)x}\right]\left(e^{n(\omega)}-e^{n^*(\omega)}\right)\right\}$$

$$=0$$

$$\Rightarrow e^{n(\omega)}-e^{n^*(\omega)}=0$$

$$\Rightarrow n(\omega)=n^*(\omega)$$

(c)

$$u(0,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega e^{-i\omega t}\left[A(\omega)+B(\omega)\right]$$

$$\Rightarrow\left[A(\omega)+B(\omega)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dte^{i\omega t}u(0,t)$$

$$\frac{\partial u(0,t)}{\partial x}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega e^{-i\omega t}\left[i\frac{\omega}{c}n(\omega)A(\omega)e^{i(\omega/c)n(\omega)x}-i\frac{\omega}{c}n(\omega)B(\omega)e^{-i(\omega/c)n(\omega)x}\right]\Bigg|_{x=0}$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega e^{-i\omega t}i\frac{\omega}{c}n(\omega)\left[A(\omega)-B(\omega)\right]$$

$$\Rightarrow\left[A(\omega)-B(\omega)\right]=-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega e^{i\omega t}\frac{ic}{\omega n(\omega)}\frac{\partial u(0,t)}{\partial x}$$

$$\Rightarrow\begin{Bmatrix}A(\omega)\\B(\omega)\end{Bmatrix}=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dte^{i\omega t}\left[u(0,t)\mp\frac{ic}{\omega n(\omega)}\frac{\partial u}{\partial x}(0,t)\right]$$

7.22

Use the Kramers-Kronig relation (7.120) to calculate the real part of $\varepsilon(\omega)$, given the imaginary part of $\varepsilon(\omega)$ for positive ω as

(a) $\mathrm{Im}\,\varepsilon/\varepsilon_0=\lambda\left[\theta(\omega-\omega_1)-\theta(\omega-\omega_2)\right]$

$\omega_2>\omega_1>0$

(b) $\mathrm{Im}\,\varepsilon/\varepsilon_0=\dfrac{\lambda\gamma\omega}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}$

(a)

KKR $\mathrm{Re}\frac{\varepsilon(\omega)}{\varepsilon_0}=1+\frac{2}{\pi}P\int_0^\infty dx\frac{x\,\mathrm{Im}\,\varepsilon(x)/\varepsilon_0}{x^2-\omega^2}$

$\mathrm{Im}\,\varepsilon/\varepsilon_0=\lambda[\theta(\omega-\omega_1)-\theta(\omega-\omega_2)]$

take $\lambda=1$, $\omega_1=1$, $\omega_2=2$

$$\mathrm{Re}\frac{\varepsilon(\omega)}{\varepsilon_0}=1+\frac{2}{\pi}P\int_{\omega_1}^{\omega_2}dx\frac{x}{x^2-\omega^2}$$

for $\omega<\omega_1$ or for $\omega>\omega_2$

$$\mathrm{Re}\frac{\varepsilon(\omega)}{\varepsilon_0}=1+\frac{\lambda}{\pi}(\ln(\omega_2-\omega)(\omega_2+\omega)-\ln(\omega_1-\omega)(\omega_1+\omega))$$

$$g(\omega)=1+\frac{1}{\pi}(\ln(2-\omega)(2+\omega)-\ln(1-\omega)(1+\omega))$$

for $\omega_1<\omega<\omega_2$

$$\mathrm{Re}\frac{\varepsilon(\omega)}{\varepsilon_0}=1+\frac{2\lambda}{\pi}\left[\int_{\omega_1}^{\omega-\delta}dx\frac{x}{x^2-\omega^2}+\int_{\omega+\delta}^{\omega_2}dx\frac{x}{x^2-\omega^2}\right]$$

$$\mathrm{Re}\frac{\varepsilon(\omega)}{\varepsilon_0}=1+\frac{\lambda}{\pi}(\ln(\omega_2-\omega)(\omega+\omega_2)-\ln(\omega-\omega_1)(\omega+\omega_1))$$

$$h(\omega)=1+\frac{1}{\pi}(\ln(2-\omega)(\omega+2)-\ln(\omega-1)(\omega+1))$$

can be plotted.

(b) take $\lambda=1$, $\gamma=0.25$, $\omega_0=1$

$$k(\omega)=\frac{\omega/4}{(1-\omega^2)^2+\omega^2/16}$$

$$\mathrm{Re}\frac{\varepsilon(\omega)}{\varepsilon_0}=1+\frac{2\lambda\gamma}{\pi}P\int_0^\infty dx\frac{x^2}{\left((\omega_0^2-x^2)^2+\gamma^2x^2\right)(x^2-\omega^2)}$$

defining $y=x/\omega_0$, $r=\omega/\omega_0$, $\alpha=\gamma/\omega_0$

$$\mathrm{Re}\frac{\varepsilon(\omega)}{\varepsilon_0}=1+\frac{2\lambda\alpha}{\pi\omega_0}P\int_0^\infty dy\frac{y^2}{\left((1-y^2)^2+\alpha^2y^2\right)(y^2-r^2)}$$

Take case $\alpha=\frac{1}{4}$, $\lambda=1$, $\omega_0=1$

$$\mathrm{Re}\frac{\varepsilon(\omega)}{\varepsilon_0}=1+\frac{2\lambda\alpha}{\pi\omega_0}P\int_0^\infty dy\frac{y^2}{\left((1-y^2)^2+\alpha^2y^2\right)(y^2-r^2)}$$

$$=1-4\frac{4\omega^2-4}{16-31\omega^2+16\omega^4}$$

can be plotted.

7.28

A circularly polarized plane wave moving in the z direction has a finite extent in the x and y directions. Assuming that the amplitude modulation is slowly varying (the wave is many wavelengths broad), show that the electric and magnetic fields are given approximately by

$$E(x,y,z,t)\cong\left[E_0(x,y)(e_1\pm ie_2)+\frac{i}{k}\left(\frac{\partial E_0}{\partial x}\pm i\frac{\partial E_0}{\partial y}\right)e_3\right]e^{ikz-i\omega t}$$

$$B\cong\mp i\sqrt{\mu\varepsilon}E$$

where e_1, e_2, e_3 are unit vectors in the x, y, z directions.

Since the wave has a finite extent in x and y dimensions, the wave is not a plane wave. Assuming the wave is dominated by the transverse polarization, but have a small longitudinal part, the wave can be written as

$$E(x,y,z,t)=[E_0(x,y)(e_1\pm ie_2)+F(x,y)e_3]e^{ikz-i\omega t}$$

It must be satisfy Maxwell's equation

$$\nabla\cdot E(x,y,z,t)$$
$$=\left\{\frac{\partial E_0(x,y)}{\partial x}\pm i\frac{\partial E_0(x,y)}{\partial y}+F(x,y)ik\right\}e^{i(kz-\omega t)}$$
$$=0$$
$$\Rightarrow F(x,y)=\frac{i}{k}\left\{\frac{\partial E_0(x,y)}{\partial x}\pm i\frac{\partial E_0(x,y)}{\partial y}\right\}$$
$$\Rightarrow E(x,y,z,t)\cong\left[E_0(x,y)(e_1\pm ie_2)+\frac{i}{k}\left(\frac{\partial E_0}{\partial x}\pm i\frac{\partial E_0}{\partial y}\right)e_3\right]e^{ikz-i\omega t}$$

The magnetic field

$$-\frac{\partial B}{\partial t}=\nabla\times E=\nabla\times\left\{\left[E_0(x,y)(e_1\pm ie_2)+\frac{i}{k}\left(\frac{\partial E_0}{\partial x}\pm i\frac{\partial E_0}{\partial y}\right)e_3\right]e^{ikz-i\omega t}\right\}$$

Since the amplitude modulation is slowly varying, $\partial E_0/\partial x$ and $\partial E_0/\partial y$ are generally small. Neglecting terms of $\partial^2E_0/\partial x^2$, $\partial^2E_0/\partial y^2$.

$$-\frac{\partial B}{\partial t}=\nabla\times\left\{E_0(x,y)(e_1\pm ie_2)e^{ikz-i\omega t}\right\}$$

$$\Rightarrow B=-\frac{i}{\omega}\left\{-\frac{\partial E_2}{\partial z}e_1+\frac{\partial E_1}{\partial z}e_2+\left(\frac{\partial E_2}{\partial x}-\frac{\partial E_1}{\partial y}\right)e_3\right\}$$

$$=-\frac{i}{\omega}\left\{\mp iE_0(x,y)(ik)e_1+E_0(x,y)(ik)e_2+\left(\pm\frac{\partial E_0}{\partial x}-\frac{\partial E_0}{\partial y}\right)e_3\right\}$$

$$=-\frac{i}{\omega}(\pm k)\left\{E_0(x,y)(e_1\pm ie_2)+\left(\frac{\partial E_0}{\partial x}\pm\frac{\partial E_0}{\partial y}\right)e_3\right\}$$

$$=\mp\frac{ik}{\omega}E=\mp i\sqrt{\mu\varepsilon}E$$