8.2

A transmission line consisting of

Two concentric circular cylinders of metal with conductivity σ and skin depth δ, as shown, is filled with a uniform lossless dielectric (μ,ε). A TEM mode is propagated along this line. Section 8.1 applies.

(a) Show that the time averaged power flow along the line is

$$P=\sqrt{\frac{\mu}{\varepsilon}}\pi a^2|H_0|^2\ln\left(\frac{b}{a}\right)$$

where H_0 is the peak value of the azimuthal magnetic field at the surface of the inner conductor.

(b) Show that the transmitted power is attenuated along the line as

$$P(z)=P_0e^{-2\gamma z}$$

where

$$\gamma=\frac{1}{2\sigma\delta}\sqrt{\frac{\mu}{\varepsilon}}\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln\left(\frac{a}{b}\right)}$$

(c) The characteristic impedance Z_0 of the line is defined as the ratio of the voltage between the cylinders to the axial current flowing in one of them at any position Z. Show that for this line

$$Z_0=\frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\ln\left(\frac{b}{a}\right)$$

(d) Show that the series resistance and inductance per unit length of the line are

$$R=\frac{1}{2\pi\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right)$$

$$L=\left\{\frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)+\frac{\mu_c\delta}{4\pi}\left(\frac{1}{a}+\frac{1}{b}\right)\right\}$$

where μ_c is the permeability of the conductor. The correction to the inductance comes from the penetration of the flux into the conductors by a distance of order δ.

Sol:

(a) By symmetry, $\Phi=\Phi(\rho)$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)\Phi=\frac{1}{\rho}\frac{\partial\Phi}{\partial\rho}+\frac{\partial^2\Phi}{\partial\rho^2}=0 \text{ and } \vec{E}=-\nabla\Phi$$

$$\frac{\partial\left(\frac{\partial\Phi}{\partial\rho}\right)}{\partial\rho} = -\frac{1}{\rho}\left(\frac{\partial\Phi}{\partial\rho}\right)$$

$$\ln\left(\frac{\partial\Phi}{\partial\rho}\right) = -\ln\rho + \ln C_1 \Rightarrow \frac{\partial\Phi}{\partial\rho} = \frac{C_1}{\rho}$$

$$\Rightarrow \Phi = C_1 \ln\rho + C_2$$

$$\vec{E} = -\nabla\Phi = -\hat{\rho}\frac{\partial}{\partial\rho}\left(C_1\ln\rho + C_2\right) = \hat{\rho}\frac{const.}{\rho}$$

For TEM wave propagating in z-direction,

$$\vec{H} = \frac{1}{\mu}\vec{B} = \frac{1}{\mu}\sqrt{\varepsilon\mu}\hat{z}\times\vec{E} = \sqrt{\frac{\varepsilon}{\mu}}\frac{const.}{\rho}\hat{\phi}$$

$$H(\rho = a) = H_0 = \sqrt{\frac{\varepsilon}{\mu}}\frac{const.}{a}$$

Let $\Rightarrow const. = H_0 a\sqrt{\frac{\mu}{\varepsilon}}$

$$\Rightarrow \begin{cases} \vec{E} = \hat{\rho}\sqrt{\frac{\mu}{\varepsilon}}H_0\frac{a}{\rho} \\ \vec{H} = \hat{\phi}\frac{H_0 a}{\rho} \end{cases}$$

$$\vec{S} = \frac{1}{2}\vec{E}\times\vec{H}^* = \frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}}|H_0|^2\frac{a^2}{\rho^2}\hat{z}$$

$$P = \int_a^b\int_0^{2\pi}\mathrm{Re}\{\hat{z}\cdot\vec{S}\}\rho d\rho d\phi = 2\pi\int_a^b\frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}}|H_0|^2\frac{a^2}{\rho}d\rho$$

$$= \sqrt{\frac{\mu}{\varepsilon}}\pi a^2|H_0|^2\ln\left(\frac{b}{a}\right)$$

(b) Form Jackson (8.57)

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\oint_C\left|\hat{n}\times\vec{H}\right|^2 dl$$

On the inner surface

$$-\left.\frac{dP}{dz}\right|_{in} = \frac{1}{2\sigma\delta}|H_0|^2 a^2\frac{1}{a}\oint_C dl = \frac{\pi a}{\sigma\delta}|H_0|^2$$

On the outer surface

$$-\left.\frac{dP}{dz}\right|_{in} = \frac{1}{2\sigma\delta}|H_0|^2 a^2\frac{1}{b}\oint_C dl = \frac{\pi}{\sigma\delta}\frac{a^2}{b}|H_0|^2$$

Therefore, power loss

$$\frac{dP}{dz}=\left.\frac{dP}{dz}\right|_{in}+\left.\frac{dP}{dz}\right|_{out}=-\frac{\pi a^2}{\sigma\delta}|H_0|^2\left(\frac{1}{a}+\frac{1}{b}\right)$$

According to $P(z)=P_0e^{-2\gamma z}$

$$\frac{dP}{dz}=P_0\cdot -2\gamma e^{-2\gamma z}=-2\gamma P(z)$$

$$\Rightarrow \gamma=\frac{1}{-2P(z)}\frac{dP}{dz}$$

$$=\frac{1}{2\sigma\delta z}\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln\left(\frac{b}{a}\right)}$$

(c) $Z_0=\frac{|\Delta V|}{I}$

$$\Delta V=\left|\int \vec{E}\cdot d\vec{l}\right|=\int_a^b\sqrt{\frac{\mu}{\varepsilon}}H_0\frac{a}{\rho}d\rho=\sqrt{\frac{\mu}{\varepsilon}}H_0 a\ln\left(\frac{b}{a}\right)$$

$$I=\oint\vec{K}\cdot d\vec{l}=\oint\hat{n}\times\vec{H}\cdot d\vec{l}=2\pi a\frac{H_0a}{a}=2\pi aH_0$$

$$Z_0=\frac{|\Delta V|}{I}=\frac{\sqrt{\frac{\mu}{\varepsilon}}H_0a\ln\left(\frac{b}{a}\right)}{2\pi aH_0}=\frac{\sqrt{\frac{\mu}{\varepsilon}}\ln\left(\frac{b}{a}\right)}{2\pi}$$

(d) Because complex condition,

$$\frac{1}{2}I^2R=\left|\frac{dP}{dz}\right|$$

$$\Rightarrow R=2\frac{\frac{\pi a^2}{\sigma\delta}|H_0|^2\left(\frac{1}{a}+\frac{1}{b}\right)}{(2\pi a)^2|H_0|^2}=\frac{1}{2\pi\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right)$$

Consider the energy in volume,

$$U_{vol}=\int\frac{1}{4}\mu|\vec{H}|^2da=2\pi\frac{1}{4}\mu\int_a^b\frac{|H_0|^2a^2}{\rho^2}\rho d\rho=2\pi\frac{|H_0|^2a^2}{4}\mu\ln\left(\frac{b}{a}\right)$$

Therefore, consider the energy in wall, we known that

$$H(\xi)=H_{//}e^{-\xi/\delta}=H_0e^{-\xi/\delta}$$

where ξ indicated the distance into the conductor.

Assume $a>>\delta$

$$U_{inner}=\int\frac{1}{4}\mu|\vec{H}|^2da=2\pi\frac{1}{4}\mu_c\int_0^\infty|H_0|^2e^{-2\xi/\delta}ad\xi=\pi\frac{|H_0|^2a\delta}{4}\mu_c$$

$$U_{outter} = \int \frac{1}{4}\mu\left|\vec{H}\right|^2 da = 2\pi\frac{1}{4}\mu_c\int_0^\infty \left|H_0\right|^2\left(\frac{a}{b}\right)^2 e^{-2\xi/\delta} bd\xi = \pi\frac{\left|H_0\right|^2 a^2\delta}{4b}\mu_c$$

$$U_{total} = \frac{\pi a^2}{2}\left|H_0\right|^2\left[\mu\ln\left(\frac{b}{a}\right)+\frac{\delta\mu_c}{2}\left(\frac{1}{a}+\frac{1}{b}\right)\right]$$

$$L = \frac{4U_{total}}{\left|I\right|^2} = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)+\frac{\delta\mu_c}{4\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$$

8.3

(a) A transmission line consists of two identical thin strips of metal, shown in cross section in the sketch. Assuming that $b >> a$, discuss the propagation of a TEM mode on this line, repeating the derivations of Problem 8.2. Show that

$$P = \frac{ab}{2}\sqrt{\frac{\mu}{\varepsilon}}\left|H_0\right|^2$$

$$\gamma = \frac{1}{a\sigma\delta}\sqrt{\frac{\varepsilon}{\mu}}$$

$$Z_0 = \sqrt{\frac{\mu}{\varepsilon}}\left(\frac{a}{b}\right)$$

$$R = \frac{2}{a\delta b}$$

$$L = \left(\frac{\mu a + \mu_c\delta}{b}\right)$$

where the symbols on the left have the same meanings as in Problem 8.2.

Choose a rectangular coordinate system with x parallel to the strip along side b, y perpendicular to the strip and z along the line. Let $K(\bar{z},t) = K_0 e^{i(kz-\omega t)}\hat{z}$ be the surface current density of the top strip. Thus, the magnetic field in between the two strips is given by

$$\vec{B} = \mu K\hat{x} = \mu K_0 e^{i(kz-\omega t)}\hat{x}, \quad \vec{H} = \frac{\vec{B}}{\mu} = K_0 e^{i(kz-\omega t)}\hat{x}$$

Therefore, $K_0 = H_0$. The electric field can be derived from the Maxwell's equation:

$$\nabla\times\vec{B} = \mu\varepsilon\frac{\partial\vec{E}}{\partial t} = -i\mu\varepsilon\omega\vec{E} \Rightarrow \vec{E} = -\frac{\nabla\times\vec{B}}{i\mu\varepsilon\omega} = -\frac{\vec{k}\times\vec{B}}{\mu\varepsilon\omega} = -\frac{kH_0}{\varepsilon\omega}e^{i(kz-\omega t)}\hat{y}$$

The average Poynting vector

$$\bar{S} = \frac{1}{2}\vec{E}\times\vec{H}^* = \frac{1}{2}\left\{-\frac{kH_0}{\varepsilon\omega}\right\}\hat{y}\times\left(H_0^*\hat{x}\right) = \frac{k\left|H_0\right|^2}{2\varepsilon\omega}\hat{z} = \frac{k\left|H_0\right|^2}{2\varepsilon\omega}\hat{z} = \frac{\sqrt{\mu\varepsilon}\left|H_0\right|^2}{2\varepsilon}\hat{z}$$

The average power transmitted along the line

$$P = \int \vec{S} \cdot \hat{z} da = \frac{ab}{2}\sqrt{\frac{\mu}{\varepsilon}}\left|H_0\right|^2$$

In terms of power P, $\left|H_0\right|^2 = \frac{2P}{ab}\sqrt{\frac{\varepsilon}{\mu}}$

The power loss per unit area, $\frac{dP}{da} = -\frac{1}{2\sigma\delta}\left|\vec{K}_{eff}\right|^2 = -\frac{1}{2\sigma\delta}\left|\vec{H}_{\parallel}\right|^2 = -\frac{1}{2\sigma\delta}\left|H_0\right|^2$

The power loss per unit length along the *z*,

$$\frac{dP}{dz} = 2b\frac{dP}{da} = -\frac{b}{\sigma\delta}\left|H_0\right|^2 = -2\left\{\frac{1}{a\sigma\delta}\sqrt{\frac{\varepsilon}{\mu}}\right\}P = -2\gamma P$$

Thus, $P(z) = P_0 e^{-2\gamma z}$ with $\gamma = \frac{1}{a\sigma\delta}\sqrt{\frac{\varepsilon}{\mu}}$

The potential difference between the two strips $V = \int \vec{E} \cdot d\vec{l} = \frac{kH_0}{\varepsilon\omega} a e^{i(kz-\omega t)}$

The wave impedance $Z = \frac{V}{I} = \frac{V}{Kb} = \frac{ka}{\varepsilon\omega} = \frac{a}{b}\sqrt{\frac{\mu}{\varepsilon}}$

The series resistance per unit length $R = -\frac{2}{\left|I\right|^2}\frac{dP}{dz} = -\frac{2}{\left|H_0\right|^2 b^2}\left(-\frac{b}{\sigma\delta}\left|H_0\right|^2\right) = \frac{2}{a\delta b}$

The inductance per unit length

$$L = \frac{1}{\left|I\right|^2}\int \vec{B} \cdot \vec{H}^* da = \frac{1}{\left|H_0\right|^2 b^2}\left\{ab\mu\left|H_0\right|^2 + 2\int_{conductor} \vec{B} \cdot \vec{H}^* da\right\}$$

where the integration of the second term taking into account the magnetic energy stored inside the conductors. Note that inside the conductors,

$$\vec{H}(\xi, t) = H_0 e^{-(1-i)\xi/\delta} e^{-i\omega t}\hat{x}$$

Thus, $\int_{conductor} \vec{B} \cdot \vec{H}^* da = \mu_c \int_0^\infty \left|H_0\right|^2 e^{-2\xi/\delta}(bd\xi) = \frac{1}{2}\mu_c\delta\left|H_0\right|^2 b$

Therefore, $L = \frac{1}{\left|H_0\right|^2 b^2}\left(ab\mu\left|H_0\right|^2 + 2\cdot\frac{1}{2}\mu_c\delta\left|H_0\right|^2 b\right) = \frac{\mu a + \mu_c\delta}{b}$

(b) The lower half of the figure shows the cross section of a microstrip line with a strip of width b mounted on a dielectric substrate of thickness h and dielectric constant ε, all on a ground plane. What differences occur here compared to part a if $b >> h$? If $b << h$?

For the case $b>>h$, the electric and magnetic fields are mostly confined in the region between the strip and the ground plane and are uniform within the region. Therefore, this case is very similar to part (a) with the slab and its mirror image. However the case $b<<h$ is very different from (a). This case can be approximated by a wire above a grounding plane. The dielectric substrate should have little effect on the quantities calculated in (a) since both electric and magnetic fields extend mostly in the region without the substrate.

8.4

Transverse electric and magnetic waves are propagated along a hollow, right circular cylinder with inner radius R and conductivity σ.

(a) Find the cutoff frequencies of the various TE and TM modes. Determine numerically the lowest cutoff frequency (the dominant mode) in terms of the tube radius and the ratio of cutoff frequencies of the next four higher modes to that of the dominant mode. For this pat assume that the conductivity of the cylinder is infinite.

(b) Calculate the attenuation constants of the waveguide as a function of frequency for the lowest two distinct modes and plot them as a function of frequency.

Sol:

(a) Following the analysis on page 369 with the replacement $\frac{p\pi}{d} \to k$, it is seen that the cutoff frequencies are

$\omega_{M,mn} = \frac{x_{mn}c}{R}$ for mode TM_{mn} and $\omega_{E,mn} = \frac{x'_{mn}c}{R}$ for mode TE_{mn}.

There x_{mn} is the n-th zero of the Bessel function $J_m(x)$ and x'_{mn} is the n-th zero $\frac{d}{dx}J_m(x)$. The fundamental mode is TE_{11} with $\omega_{E,11} = \frac{1.841c}{R} \equiv \omega_0$ with $c = \frac{1}{\sqrt{\mu\varepsilon}}$.

The next four higher modes are:

TM_{01} with $\frac{\omega_{M,01}}{\omega_0} = 1.306$

TE_{21} with $\dfrac{\omega_{E,21}}{\omega_0} = 1.659$

TE_{01} with $\dfrac{\omega_{E,01}}{\omega_0} = 2.081$

TM_{11} with $\dfrac{\omega_{M,11}}{\omega_0} = 2.081$

(b) TE_{11}: We require ψ to calculate the power P from Eq. 8.51 in Jackson, and all magnetic fields to calculate

$$\frac{dP}{dz} = -\frac{1}{2\sigma\delta}\int \left|\bar{H}\right|^2 dl$$

$$H_z = \psi = H_0 J_1\left(\frac{x'_{11}}{R}\rho\right)e^{i\phi}$$

$$H_t = \frac{ik}{\gamma^2}\nabla_t\psi = \frac{ikH_0}{\gamma^2}e^{i\phi}\left[\hat{\rho}\frac{x'_{11}}{R}J_1\left(\frac{x'_{11}}{R}\rho\right) + \hat{\phi}\frac{i}{\rho}J_1\left(\frac{x'_{11}}{R}\rho\right)\right]$$

with $\gamma^2 = \mu\varepsilon\omega_{E,11}^2$ and $k^2 = \mu\varepsilon\left(\omega^2 - \omega_{E,11}^2\right)$ and $\delta = \sqrt{\dfrac{2}{\mu_c\sigma\omega}}$.

Using Eq. 8.51

$$P = \frac{\pi\left|H_0\right|^2\omega^2}{\omega_{E,11}^2}\sqrt{\frac{\mu}{\varepsilon}}\sqrt{1 - \frac{\omega_{E,11}^2}{\omega^2}}\int_0^R \rho J_1^2\left(\frac{x'_{11}}{R}\rho\right)d\rho$$

$$\int_0^R \rho J_1^2\left(\frac{x'_{11}}{R}\rho\right)d\rho = \frac{R^2}{2}\left(1 - \frac{1}{x_{11}'^{\,2}}\right)J_1^2\left(x'_{11}\right)$$

$$\left|H_z\right| = \left|H_0\right|J_1\left(x'_{11}\right)$$

$$\left|H_t\right| = \frac{k\left|H_0\right|}{\gamma^2 R}J_1\left(x'_{11}\right)$$

$$\left|\frac{dP}{dz}\right| = \frac{1}{2\sigma\delta}2\pi R\left(\left|H_z\right|^2 + \left|H_t\right|^2\right) = \frac{\pi R}{\sigma\delta}J_1^2\left(x'_{11}\right)\left(1 + \frac{k^2}{\gamma^4 R^2}\right)$$

The attenuation constant for a hollow brass guide, for which $\mu = \mu_c = \mu_0$ and $\varepsilon = \varepsilon_0$ is then found to be

$$\beta_{E,11}(\omega) = \frac{1}{2P}\left|\frac{dP}{dz}\right| = \frac{1}{R}\sqrt{\frac{\varepsilon_0}{2\sigma}}\frac{\mu_0\varepsilon_0\omega_{E,11}^4 R^2 + \omega^2 - \omega_{E,11}^2}{\sqrt{\omega}\sqrt{\omega^2 - \omega_{E,11}^2}\left(\mu_0\varepsilon_0\omega_{E,11}^2 R^2 - 1\right)}$$

TM_{01}: The required fields are, using $\gamma = \dfrac{x_{01}}{R}$ and $Z = \dfrac{k}{\varepsilon\omega}$

$$E_z = \psi = E_0 J_0\left(\frac{x_{01}}{R}\rho\right)$$

$$E_t = \frac{ik}{\gamma^2}\nabla_t \psi = \frac{ikE_0}{\gamma^2}\left[\hat{\rho}\frac{x_{01}}{R}J_0'\left(\frac{x_{01}'}{R}\rho\right)\right] = \hat{\rho}\frac{ikE_0 R}{x_{01}}J_0'\left(\frac{x_{01}}{R}\rho\right)$$

$$H_t = \frac{1}{Z}\hat{z}\times E_t = -\hat{\phi}\frac{ikE_0 R}{Zx_{01}}J_0'\left(\frac{x_{01}}{R}\rho\right)$$

using Eq. 8.51 calculating

$$\left|\frac{dP}{dz}\right| = \frac{1}{2\sigma\delta}2\pi R\left|H_t\right|^2$$

and evaluating $\beta(\omega) = \frac{1}{2P}\left|\frac{dP}{dz}\right|$ for a hollow brass guide ($\mu = \mu_c = \mu_0$ and $\varepsilon = \varepsilon_0$), it is found

$$\beta_{E,01}(\omega) = \frac{1}{R}\sqrt{\frac{\varepsilon_0\omega^3}{2\sigma\left(\omega^2 - \omega_{M,01}^2\right)}}$$

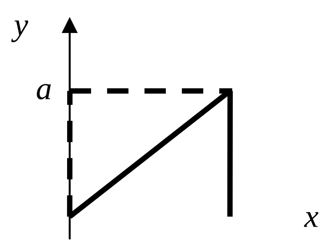
8.5

A waveguide is constructed so that the cross section of the guide forms a right triangle with sides of length a, a, $\sqrt{2}a$, as shown. The medium inside has $\mu_r = \varepsilon_r = 1$.

(a) Assuming infinite conductivity for the walls, determine the possible modes of propagation and their cutoff frequencies.

In general, to solve a problem like this, we need to consider the Dirichlet or Neumann problem for a boundary without any 'standard' (i.e. rectangular or circular) symmetry. In particular, this means there is not natural coordinate system to use for the two-dimensional Helmholtz equation $\left[\nabla_t^2 + \gamma^2\right]\varphi = 0$ that both allows for separation of variables and respects the symmetry of the boundary surface (which would allow a simple specification of the boundary data). A general problem of this form (with no simple boundary symmetry) is quite unpleasant to solve.

In this case, we can think of the triangle as 'half' of a square.



In particular, the key step to this problem is to note that the triangle may be obtained from the square by imposing reflection symmetry along the $x = y$ diagonal. This symmetry is a $\mathbb{Z}_2$ reflection on the coordinates of the form

$$\mathbb{Z}_2: \; x \to y, \; y \to x$$

Eigenfunctions $\varphi(x, y)$ can then be classified as either $\mathbb{Z}_2$-even or $\mathbb{Z}_2$-odd

$$\mathbb{Z}_2: \; \varphi(x, y) \to \pm\varphi(y, x)$$

The odd functions vanish along the diagonal, so they automatically satisfy Dirichlet conditions $\varphi(x = y) = 0$ on the diagonal. Similarly, the even functions have vanishing normal derivative on the diagonal and hence automatically satisfy Neumann conditions. We will use this fact to construct TM and TE modes for the triangle.

We begin with the TM modes. Using rectangular coordinates, it is natural to write solutions of the Helmholtz equation $\left[\partial_x^2 + \partial_y^2 + \gamma^2\right]\varphi = 0$ as $\varphi \sim e^{i(k_x x + k_y y)}$ where $k_x^2 + k_y^2 = \gamma^2$. This means we may expand the eigenfunctions in terms of sines and cosines. For TM modes satisfying the Dirichlet condition $\varphi_S = 0$, we start with eigenfunctions on the square

$$\varphi \sim \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}$$

which automatically satisfy the boundary conditions on the four walls of the square. This gives

$$\gamma_{mn} = \frac{\pi}{a}\sqrt{m^2 + n^2}$$

so the cutoff frequencies are

$$\omega_{mn} = \frac{\pi}{\sqrt{\mu_0\varepsilon_0}\,a}\sqrt{m^2 + n^2} = \frac{\pi c}{a}\sqrt{m^2 + n^2} \qquad (1)$$

In order to satisfy the Dirichlet condition on the diagonal, we take the $\mathbb{Z}_2$-odd combination

(TM) $$\varphi_{mn} = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} - \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}$$

It is simple to verify that $\varphi(x, 0) = \varphi(a, y) = \varphi(x, x) = 0$, so that all boundary conditions on the triangle are indeed satisfied. The cutoff frequencies are given by (1). Note here that the $\mathbb{Z}_2$ projection removes the $m = n$ modes and also antisymmetrizes m with n. As a result, the integer labels m and n may be taken to satisfy the condition $m>n>0$.

The analysis for TE modes is similar. However, for Neumann conditions, we take cosine combinations as well as a $\mathbb{Z}_2$-even eigenfunction. This gives

(TE) $$\varphi_{mn} = \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{a} - \cos\frac{n\pi x}{a}\cos\frac{m\pi y}{a}$$

with identical cutoff frequencies as in (1). This time, however, the labels m and n may be taken to satisfy $m \geq n \geq 0$ (except $m=n=0$ is not allowed).

(b) For the lowest modes of each type calculate the attenuation constant, assuming that the walls have large, but finite, conductivity. Compare the result with that for a square guide of side a made from the same material.

The attenuation coefficients are determined by power and power loss. We begin with TM modes. For the power, we need to compute

$$\int_A |\varphi|^2 da = \int_A \left[\sin k_m x \sin k_n y - \sin k_n x \sin k_m y\right]^2 da \qquad (2)$$

It is perhaps easiest to compute this by integrating over the square and then dividing by two for the triangle. This is because the integration separates into x and y integrals, and we may use orthogonality

$$\int_0^a \sin k_i x \sin k_j x dx = \frac{a}{2}\delta_{i,j}$$

This gives

$$\int_A |\varphi|^2 da = \frac{1}{2}\times 2\left(\frac{a}{2}\right)^2 = \frac{a^2}{4}$$

The factor of 1/2 is for the triangle, while the factor of 2 is because two non-vanishing terms arise when squaring the integrand in (2). (Recall that $m \neq n$ for TM modes.) This gives an expression for the power

$$P = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{1/2}\int_A |\varphi|^2 da = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{1/2}\frac{A}{2}$$

where $A = a^2/2$ is the area of the triangle. Calculating the power loss involves integrating a normal derivative

$$\oint_C \left|\frac{\partial\varphi}{\partial n}\right|^2 dl$$

We break this into three parts: along $y = 0$, along $x = a$ and along the diagonal $x = y$. Along the $y = 0$ wall, we have $\hat{n} = \hat{y}$ and

$$\left.\frac{\partial\varphi}{\partial y}\right|_{y=0} = \frac{\pi}{a}\left[n\sin k_m x - m\sin k_n x\right]$$

As a result

$$\int_0^a \left|\frac{\partial\varphi}{\partial y}\right|^2 dx = \left(\frac{\pi}{a}\right)^2 \frac{a}{2}\left(m^2+n^2\right) = \frac{\pi^2}{2a}\left(m^2+n^2\right) \qquad (3)$$

A similar calculation, or use of symmetry, will result in an identical expression for the integral along the $x = a$ wall. For the diagonal, we use $\hat{n} = \frac{1}{\sqrt{2}}\left(\hat{x}-\hat{y}\right)$ to compute

$$\frac{\partial\varphi}{\partial n} = \frac{1}{\sqrt{2}}\left(\frac{\partial\varphi}{\partial x} - \frac{\partial\varphi}{\partial y}\right)_{y=x} = \sqrt{2}\,\frac{\pi}{a}\left[m\cos k_m x\sin k_n x - n\cos k_n x\sin k_m x\right]$$

$$= \frac{\sqrt{2}}{2}\frac{\pi}{a}\left[(m-n)\sin k_{m+n}x - (m+n)\sin k_{m-n}x\right]$$

This gives

$$\int_0^{\sqrt{a}} \left|\frac{\partial\varphi}{\partial n}\right|^2 dl = \sqrt{2}\int_0^a \left|\frac{\partial\varphi}{\partial n}\right|^2 dx = \sqrt{2}\,\frac{1}{2}\left(\frac{\pi}{a}\right)^2 \frac{a}{2}\left[(m-n)^2+(m+n)^2\right] = \sqrt{2}\,\frac{\pi^2}{2a}\left(m^2+n^2\right)$$

Combining this diagonal with (3) for the sides, we obtain

$$\oint_C \left|\frac{\partial\varphi}{\partial n}\right|^2 dl = C\,\frac{\pi^2}{2a^2}\left(m^2+n^2\right) = \frac{C}{2}\gamma_{mn}^2$$

where $C = a + a + \sqrt{2}a$ is the circumference of the triangle. This gives a TM mode power loss of

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2 \frac{1}{\mu^2\omega_{mn}^2}\overbrace{\oint_C \left|\frac{\partial\varphi}{\partial n}\right|^2 dl}^{\frac{C}{2}\gamma_{mn}^2} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2 \frac{\varepsilon}{\mu}\frac{C}{2}$$

The attenuation coefficient is thus

$$\beta_{mn} = -\frac{1}{2P}\frac{dP}{dz} = \frac{1}{\sigma\delta}\sqrt{\frac{\varepsilon}{\mu}}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{-1/2}\frac{C}{2A}$$

so that the geometrical factor $\xi_{mn} = 1$ is trivial. Note that the energy loss calculation along the diagonal of the triangle gives the same result as along the square edges. As a result, the geometrical factor $\xi_{mn} = 1$ is independent of whether the waveguide is square or right triangular. This is why the triangular TM result is identical to the square TM result, at least up to the ratios $C/A = 2\left(2+\sqrt{2}\right)/a \approx 6.83/a$ for the triangle and $C/A = 4/a$ for the square.

The power loss for the TE modes is somewhat harder to deal with because of the possibility of special cases. Consider

$$\varphi = \cos k_m x \cos k_n y + \cos k_n x \cos k_m y \tag{4}$$

where $m \geq n \geq 0$. If n = 0, we end up with

$$\varphi = \cos k_m x + \cos k_m y \qquad (m>0)$$

In this case

$$\int_A |\varphi|^2 da = \frac{1}{2}\int_0^a dx \int_0^a dy \left[\cos k_m x + \cos k_m y\right]^2 = \frac{1}{2}\times 2\left(\frac{1}{2}a^2\right) = \frac{a^2}{2} = A$$

while the perimeter integrals are

$$\int_0^a dx \left|\varphi(y=0)\right|^2 = \int_0^a dx \left[1+\cos k_m x\right]^2 = a\left(1+\frac{1}{2}\right) = \frac{3a}{2}$$

$$\sqrt{2}\int_0^a dx \left|\varphi(y=x)\right|^2 = \sqrt{2}\int_0^a dx \left[2\cos k_m x\right]^2 = 4\sqrt{2}\left(\frac{1}{2}a\right) = 2\sqrt{2}a$$

which gives

$$\oint_C |\varphi|^2 dl = \left(3+2\sqrt{2}\right)a$$

and

$$\int_0^a dx \left|\hat{n}\times\vec{\nabla}_t\varphi\right|^2 = \int_0^a dx \left|\hat{y}\times\vec{\nabla}_t\varphi\right|^2 = \int_0^a dx \left|-\hat{z}\partial_x\varphi\right|_{y=0}^2$$

$$= \int_0^a dx \frac{\pi^2}{a^2} m^2 \left|\sin k_m x\right|^2 = \frac{\pi^2}{2a} m^2$$

$$\sqrt{2}\int_0^a dx\left|\hat{n}\times\vec{\nabla}_t\varphi\right|^2_{y=x} = \sqrt{2}\int_0^a dx\left|\frac{1}{\sqrt{2}}\hat{z}\left(\partial_y+\partial_x\right)\varphi\right|^2_{y=x}$$

$$= \frac{\sqrt{2}}{2}\int_0^a dx\frac{\pi^2}{a^2}m^2\left|2\sin k_m x\right|^2 = \sqrt{2}\frac{\pi^2}{a}m$$

which gives

$$\oint_C\left|\hat{n}\times\vec{\nabla}_t\varphi\right|^2 dl = \left(1+\sqrt{2}\right)\frac{\pi^2}{a}m^2 = \left(1+\sqrt{2}\right)a\gamma^2_{m0}$$

Using

$$P = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1-\frac{\omega^2_{mn}}{\omega^2}\right)^{1/2}\int_A|\varphi|^2 da$$

and

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2\oint_C\left[\frac{1}{\gamma^2_{mn}}\left(1-\frac{\omega^2_{mn}}{\omega^2}\right)\left|\hat{n}\times\vec{\nabla}_t\varphi\right|^2+\frac{\omega^2_{mn}}{\omega^2}|\varphi|^2\right]dl$$

with the above integrals gives an attenuation coefficient

$$\beta_{m0} = -\frac{1}{2P}\frac{dP}{dz} = \frac{1}{2\sigma\delta}\sqrt{\frac{\varepsilon}{\mu}}\left(1-\frac{\omega^2_{m0}}{\omega^2}\right)^{-1/2}\left[\left(1+\sqrt{2}\right)\left(1-\frac{\omega^2_{m0}}{\omega^2}\right)+\frac{\omega^2_{m0}}{\omega^2}\left(3+2\sqrt{2}\right)\right]\frac{a}{A}$$

$$= \frac{1}{\sigma\delta}\sqrt{\frac{\varepsilon}{\mu}}\left(1-\frac{\omega^2_{m0}}{\omega^2}\right)^{-1/2}\left[\frac{1+\sqrt{2}}{2+\sqrt{2}}+\frac{\omega^2_{m0}}{\omega^2}\right]\frac{C}{2A}$$

where $C = \left(2+\sqrt{2}\right)a$ and $A = a^2/2$. Here the geometrical factors are

$$\xi_{m0} = \frac{1+\sqrt{2}}{2+\sqrt{2}} \quad , \quad \eta_{m0} = 1 \qquad (m>n=0)$$

For the rectangular waveguide, one has instead

$$\xi_{m0} = \frac{a}{a+b}\to\frac{1}{2} \quad , \quad \eta_{m0} = \frac{2b}{a+b}\to 1 \quad \text{when} \quad b\to a$$

This is different because the power loss calculation is no longer universal, giving different coefficients along the diagonal as along the square edges. The remaining TE cases to consider are modes (4) where m=n>0 and m>n>0. Here we simply state the results. For m=n>0 we have

$$\varphi = \cos k_m x\cos k_m y$$

(we have removed an unimportant factor of two) so that

$$\int_A |\varphi|^2 da = \frac{a^2}{8} = \frac{A}{4}$$
$$\oint_C |\varphi|^2 dl = \left(1 + \frac{3\sqrt{2}}{8}\right) a$$
$$\oint_C \left|\hat{n} \times \vec{\nabla}_t \varphi\right|^2 dl = \left(1 + \frac{\sqrt{2}}{4}\right) \frac{\pi^2 m^2}{a} = \left(\frac{1}{2} + \frac{\sqrt{2}}{8}\right) a\gamma_{mm}^2$$

This gives

$$\xi_{mm} = \frac{4+\sqrt{2}}{4+2\sqrt{2}}, \quad \eta_{mm} = 1 \qquad (m=n>0)$$

On the other hand, for the general case $m>n>0$ we find

$$\int_A |\varphi|^2 da = \frac{a^2}{4} = \frac{A}{2}$$
$$\oint_C |\varphi|^2 dl = \left(2+\sqrt{2}\right) a = C$$
$$\oint_C \left|\hat{n} \times \vec{\nabla}_t \varphi\right|^2 dl = \left(2+\sqrt{2}\right) \frac{\pi^2}{2a}\left(m^2+n^2\right) = \frac{C}{2}\gamma_{mn}^2$$

which yields

$$\xi_{mn} = 1 \quad , \quad \eta_{mn} = 1 \qquad (m>n>0)$$

In all cases, $\eta_{mn} = 1$, which is the same for the triangle or the square waveguide. For ξ_{mn}, the factor is essentially a geometric combination of contributions along the perimeter of either 1 or 1/2 depending on the particular mode and its degeneracies.

8.6

A resonant cavity of copper consists of a hollow, right circular cylinder of inner radius R and length L, with flat end faces.

(a) Determine the resonant frequencies of the cavity for all types of waves. With $\left(1/\sqrt{\mu\varepsilon}R\right)$ as a unit of frequency, plot the lowest four resonant frequencies of each type as a function of R/L of $0 < R/L < 2$. Does the same mode have the lowest frequency for all R/L?

(b) If R = 2 cm, L = 3 cm, and the cavity is made of pure copper, what is the numerical value of Q for the lowest resonant mode?

Sol:

(a) From Jackson (8.81) and (8.83),

TM_{mnp}:

$$\omega_{mnp} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\frac{x_{mn}^2}{R^2}+\frac{p^2\pi^2}{L^2}}$$

p = 0, 1, 2…, m = 0, 1, 2…, n = 1, 2, 3…

TE_{mnp}:

$$\omega'_{mnp} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\frac{x'^2_{mn}}{R^2}+\frac{p^2\pi^2}{L^2}}$$

p = 1, 2, 3…, m = 0, 1, 2…, n = 1, 2, 3…

The frequency of TM_{mn0} is independent of L. The foundamental mode is either TE_{111} or TM_{010} dependent on R/L.

(b) $R/L = \frac{2}{3}$, the fundamental mode is TM_{010},

$$E_z = \psi(\rho,\phi)\cos\left(\frac{\rho\pi z}{L}\right) = E_0 J_0\left(\frac{x_{01}\rho}{R}\right)$$

$$E_t = 0$$

$$\vec{H}_t = -\frac{i\varepsilon\omega}{r^2}E_0\frac{x_{01}\rho}{R}J_0'\left(\frac{x_{01}\rho}{R}\right)\hat{\phi} = \frac{i\varepsilon\omega}{r^2}E_0\frac{x_{01}}{R}J_1\left(\frac{x_{01}\rho}{R}\right)\hat{\phi}$$

$$\Rightarrow U = E_0^{\ 2}\int_0^R \rho J_0^{\ 2}\left(\frac{x_{01}\rho}{R}\right)d\rho = \frac{E_0\pi\varepsilon LR^2}{4}J_1^{\ 2}(x_{01})$$

$$P = \frac{1}{2\sigma\delta}\int\left|\vec{H}\right|^2 da = \frac{E_0^{\ 2}\varepsilon^2\omega^2}{2\sigma\delta r^4}\frac{x_{01}^{\ 2}}{R^2}\left\{2\pi RLJ_1^{\ 2}(x_{01}) + 4\pi\int_0^R \rho J_1^{\ 2}\left(\frac{x_{01}\rho}{R}\right)d\rho\right\}$$

$$= \frac{E_0^{\ 2}\varepsilon^2\omega^2}{2\sigma\delta r^4}\frac{x_{01}^{\ 2}}{R^2}2\pi\left\{RLJ_1^{\ 2}(x_{01}) + R^2\left[J_1'^2(x_{01}) + \left(1-\frac{1}{x_{01}^{\ 2}}\right)J_1^{\ 2}(x_{01})\right]\right\}$$

$$\begin{cases} r = \frac{x_{01}}{R} \\ xJ_1'(x) + J_1(x) = xJ_0(x) \\ \frac{J_1'(x_{01})}{J_1(x_{01})} = -\frac{1}{x_{01}} \end{cases}$$

$$Q = \omega\frac{U}{P} = \frac{2L\sigma\delta x_{01}^{\ 2}}{8\varepsilon\omega R\left\{L + R\left[\frac{J_1'(x_{01})}{J_1(x_{01})} + \left(1-\frac{1}{x_{01}^{\ 2}}\right)\right]\right\}}$$

$$= \frac{L\sigma\delta x_{01}^{\ 2}}{4\varepsilon\omega R(L+R)}$$

$\omega^2 = \frac{x_{01}{}^2}{\mu\varepsilon R^2}$ for TM010.

$\Rightarrow Q = \omega \frac{\delta\sigma\mu}{4} \frac{RL}{L+R}$

where $\delta = \sqrt{\frac{2}{\sigma\omega\mu_c}}$

8.18

(a) From the use of Green's theorem in two dimensions show that the TM and TE modes in a waveguide defined by the boundary-value problems (8.34) and (8.36) are orthogonal in the sense that

$$\int_A E_{z,\lambda} E_{z,\mu} da = 0 \quad \text{for} \quad \lambda \neq \mu$$

for TM modes, and a corresponding relation for H_z for TE modes.

Orthogonality is a general property of the eigenfunctions of the wave equation. The general two-dimensional equation is given by

$$\left[\nabla_t^2 + \gamma_\lambda^2\right]\varphi_\lambda = 0$$

where either

$$\varphi_\lambda\big|_S = 0 \qquad \text{TM modes}$$

or

$$\left.\frac{\partial\varphi_\lambda}{\partial n}\right|_S = 0 \qquad \text{TE modes}$$

To prove orthogonality, note that φ_λ and φ_μ satisfy the equations

$$\left[\nabla_t^2 + \gamma_\lambda^2\right]\varphi_\lambda = 0, \qquad \left[\nabla_t^2 + \gamma_\mu^2\right]\varphi_\mu = 0$$

Multiplying the first by φ_μ and the second by φ_λ and subtracting gives

$$\left[\gamma_\mu^2 - \gamma_\lambda^2\right]\varphi_\mu\varphi_\lambda = \varphi_\mu\nabla_t^2\varphi_\lambda - \varphi_\lambda\nabla_t^2\varphi_\mu$$

Integrating this over the cross-sectional area, and using Green’s theorem yields

$$\left(\gamma_\mu^2 - \gamma_\lambda^2\right)\int_A \varphi_\mu\varphi_\lambda da = \int_A\left[\varphi_\mu\nabla_t^2\varphi_\lambda - \varphi_\lambda\nabla_t^2\varphi_\mu\right]da = -\oint_C\left[\varphi_\mu\frac{\partial\varphi_\lambda}{\partial n} - \varphi_\lambda\frac{\partial\varphi_\mu}{\partial n}\right]dl$$

where we have used an inward pointing normal direction. We now note that the right hand side vanishes for either TM or TE boundary conditions. Thus, provided $\gamma_\mu^2 \neq \gamma_\lambda^2$, we end up with

$$\int_A \varphi_\mu \varphi_\lambda da = 0 \qquad (\gamma_\mu^2 \neq \gamma_\lambda^2)$$

For non-degenerate eigenvalues, conclude that

$$\int_A \varphi_\mu \varphi_\lambda da = 0 \qquad \text{for} \quad \mu \neq \lambda$$

so long as φ_μ and φ_λ are both TM modes (or are both TE modes). Note that

$\varphi_\mu = E_{z,\mu}$ for TM modes, while $\varphi_\mu = H_{z,\mu}$ for TE modes.

For degenerate eigenvalues, linearity of the wave equation guarantees that we may find an orthogonal basis using, e.g., a Gram-Schmidt orthogonalization process.

(b) Prove that the relations (8.131)-(8.134) form a consistent set of normalization conditions for the fields, including the circumstances when λ is a TM mode and μ is a TE mode.
We start with relation (8.131), which states

$$\int_A \vec{E}_{t,\lambda} \cdot \vec{E}_{t,\mu} da = \delta_{\lambda,\mu}$$

where $\vec{E}_{t,\lambda}$ may be either a TM or a TE mode. To handle this expression, we note that the transverse fields for TM and TE modes are given by

$$\text{TM:} \qquad \vec{E}_t = \frac{ik}{\gamma^2}\vec{\nabla}_t E_z, \qquad \vec{H}_t = \frac{1}{Z}\hat{z}\times\vec{E}_t, \qquad Z = \frac{k}{\varepsilon\omega}$$

$$(1)$$

$$\text{TE:} \qquad \vec{E}_t = -\frac{i\mu\omega}{\gamma^2}\hat{z}\times\vec{\nabla}_t H_z, \qquad \vec{H}_t = \frac{1}{Z}\hat{z}\times\vec{E}_t, \qquad Z = \frac{\mu\omega}{k}$$

Hence for two TM modes, we end up with

$$\int_A \vec{E}_{t,\lambda} \cdot \vec{E}_{t,\mu} da = -\frac{k^2}{\gamma_\mu^2 \gamma_\lambda^2}\int_A \vec{\nabla}_t E_{z,\lambda} \cdot \vec{\nabla}_t E_{z,\mu} da = -\frac{k^2}{\gamma_\mu^2 \gamma_\lambda^2}\left[-\oint_S E_{z,\lambda}\frac{\partial E_{z,\mu}}{\partial n} dl - \int_A E_{z,\lambda}\nabla_t^2 E_{z,\mu} da\right]$$

The surface term vanishes because of Dirichlet boundary conditions, while the area term may be simplified using $\nabla_t^2 E_{z,\mu} = -\gamma_\mu^2 E_{z,\mu}$. Hence we arrive at

$$\int_A \vec{E}_{t,\lambda} \cdot \vec{E}_{t,\mu} da = -\frac{k^2}{\gamma_\lambda^2} \int_A E_{z,\lambda} E_{z,\mu} da = 0 \qquad \text{for } \lambda \neq \mu \tag{2}$$

When properly normalized for $\lambda = \mu$, this gives (8.131) for two TM modes. The case of two TE modes is similar. We have

$$\int_A \vec{E}_{t,\lambda} \cdot \vec{E}_{t,\mu} da = -\frac{\mu^2 \omega^2}{\gamma_\mu^2 \gamma_\lambda^2} \int_A \left(\hat{z} \times \vec{\nabla}_t H_{z,\lambda} \right) \cdot \left(\hat{z} \times \vec{\nabla}_t H_{z,\mu} \right) da$$

$$= -\frac{\mu^2 \omega^2}{\gamma_\mu^2 \gamma_\lambda^2} \int_A \vec{\nabla}_t H_{z,\lambda} \cdot \vec{\nabla}_t H_{z,\mu} - \left(\hat{z} \cdot \vec{\nabla}_t H_{z,\lambda} \right) \left(\hat{z} \cdot \vec{\nabla}_t H_{z,\mu} \right) da \tag{3}$$

$$= -\frac{\mu^2 \omega^2}{\gamma_\mu^2 \gamma_\lambda^2} \int_A \vec{\nabla}_t H_{z,\lambda} \cdot \vec{\nabla}_t H_{z,\mu} da$$

We have noted that $\hat{z} \cdot \vec{\nabla}_t = 0$ identically (since the transverse gradient is orthogonal to $\hat{z}$). The proof of orthogonality of two TE modes then follows using the same integration method that was used above for the TM modes (but with E_z replaced by H_z, and with $\partial H_z / \partial n$ vanishing on the boundary). Finally, for one TE mode and one TM mode, we have

$$\int_A \vec{E}_{t,\lambda} \cdot \vec{E}_{t,\mu} da = \frac{\mu \omega k}{\gamma_\mu^2 \gamma_\lambda^2} \int_A \left(\vec{\nabla}_t E_{z,\lambda} \right) \cdot \left(\hat{z} \times \vec{\nabla}_t H_{z,\mu} \right) da$$

$$= -\frac{\mu \omega k}{\gamma_\mu^2 \gamma_\lambda^2} \int_A \left[\vec{\nabla}_t E_{z,\lambda} \times \vec{\nabla}_t H_{z,\mu} \right] \cdot \hat{z} da$$

$$= -\frac{\mu \omega k}{\gamma_\mu^2 \gamma_\lambda^2} \int_A \vec{\nabla}_t \times \left(E_{z,\lambda} \vec{\nabla}_t H_{z,\mu} \right) \cdot \hat{z} da$$

$$= -\frac{\mu \omega k}{\gamma_\mu^2 \gamma_\lambda^2} \oint_S E_{z,\lambda} \vec{\nabla}_t H_{z,\mu} \cdot d\vec{l} = 0$$

This integral vanishes because $E_{z,\lambda}$ vanishes on the boundary. As a result, all TE modes are orthogonal to all TM modes. Proper normalization then results in (8.131).

We now turn to relation (8.132), which states

$$\int_A \vec{H}_{t,\lambda} \cdot \vec{H}_{t,\mu} da = \frac{1}{Z_\lambda^2} \delta_{\lambda,\mu}$$

The best way to prove this is to note from (1) that

$$\vec{H}_{t,\lambda} = \frac{1}{Z_\lambda} \hat{z} \times \vec{E}_{t,\lambda}$$

for either TM or TE modes, provided Z_λ is chosen accordingly. In this case

$$\int_A \vec{H}_{t,\lambda} \cdot \vec{H}_{t,\mu} da = \frac{1}{Z_\mu Z_\lambda} \int_A \left(\hat{z} \times \vec{E}_{t,\lambda} \right) \cdot \left(\hat{z} \times \vec{E}_{t,\mu} \right) da$$

$$= \frac{1}{Z_\mu Z_\lambda} \int_A \left[\vec{E}_{t,\lambda} \cdot \vec{E}_{t,\mu} - \left(\hat{z} \cdot \vec{E}_{t,\lambda} \right) \left(\hat{z} \cdot \vec{E}_{t,\mu} \right) \right] da$$

$$= \frac{1}{Z_\mu Z_\lambda} \int_A \vec{E}_{t,\lambda} \cdot \vec{E}_{t,\mu} da = \frac{1}{Z_\mu Z_\lambda} \delta_{\lambda,\mu} = \frac{1}{Z_\lambda^2} \delta_{\lambda,\mu}$$

Here we have made use of the fact that $\hat{z} \cdot \vec{E}_t$ vanishes because $\vec{E}_t$ is transverse to the $\hat{z}$ direction. The last line follows from applying (8.131), which we proved above.

The power flow relation (8.133)

$$\frac{1}{2} \int_A \left(\vec{E}_{t,\lambda} \times \vec{H}_{t,\mu} \right) \cdot \hat{z} da = \frac{1}{2Z_\lambda} \delta_{\lambda,\mu}$$

follows similarly. Specifically, we have

$$\frac{1}{2} \int_A \left(\vec{E}_{t,\lambda} \times \vec{H}_{t,\mu} \right) \cdot \hat{z} da = \frac{1}{2Z_\mu} \int_A \hat{z} \cdot \left[\vec{E}_{t,\lambda} \times \left(\hat{z} \times \vec{E}_{t,\mu} \right) \right] da$$

$$= \frac{1}{2Z_\mu} \int_A \left[\vec{E}_{t,\lambda} \cdot \vec{E}_{t,\mu} - \left(\hat{z} \cdot \vec{E}_{t,\lambda} \right) \left(\hat{z} \cdot \vec{E}_{t,\mu} \right) \right] da$$

$$= \frac{1}{2Z_\mu} \int_A \vec{E}_{t,\lambda} \cdot \vec{E}_{t,\mu} da = \frac{1}{2Z_\mu} \delta_{\lambda,\mu} = \frac{1}{2Z_\lambda} \delta_{\lambda,\mu}$$

The relation (8.134) essentially normalizes the modes for the TM and TE case. Examination of (2) for TM modes and (3) for TE modes indicates that the proper normalization is

TM: $$\int_A E_{z,\lambda} E_{z,\mu} da = -\frac{\gamma_\lambda^2}{k_\lambda^2} \delta_{\lambda,\mu}$$

TE: $$\int_A E_{z,\lambda} E_{z,\mu} da = -\frac{\gamma_\lambda^2}{\mu^2 \omega^2} \delta_{\lambda,\mu} = -\frac{\gamma_\lambda^2}{k_\lambda^2 Z_\lambda^2}$$

8.19

The figure at problem 8.19 shows a cross-sectional view of an infinitely long rectangular waveguide with the center conductor of a coaxial the line extending vertically a distance h into its interior at $z = 0$. The current along the probe oscillates sinusoidally in time with frequency ω, and its variation in space can be approximated as $I(y)=I_0\sin\left[\left(\frac{\omega}{c}\right)(h-y)\right]$. The thickness of the probe can be neglected. The frequency is such that only the TE_{10} mode can propagate in the guide.

(a) Calculate the amplitudes for excitation of both TE and TM modes for all (m,n) and show how the amplitudes depend on m and n for $m,n>>1$ for a fixed frequency ω.

(b) For the propagating mode show that the power radiated in the positive z direction is

$$P=\frac{\mu c^2 I_0^2}{\omega kab}\sin^2\left(\frac{\pi X}{a}\right)\sin^4\left(\frac{\omega h}{2c}\right)$$

with an equal amount in the opposite direction. Here k is the wave number for the TE_{10} mode.

(c) Discuss the modifications that occur if the guide, instead of running off to infinity in both directions, is terminated with a perfectly conducting surface at $z=L$. What values of L will maximize the power flow for a fixed current I_0? What is the radiation resistance of the probe (defined as the ratio of power flow to one-half the square of the current at the base of the probe) at maximum?

Sol:

(a)

$$\vec{J}(y)=I_0\sin\left[\frac{\omega}{c}(k-y)\right]\hat{y}\delta(x)\delta(z)$$

$$a_{TE}^{(\pm)}=-\frac{\omega}{c}\frac{2\pi\mu}{kr^2}\int\frac{\vec{M}^{(\pm)}\cdot\vec{J}}{c}d^3r'$$

$$\vec{M}=\vec{\nabla}\times\hat{z}\psi e^{ikz}$$

$$\psi=\mathbb{C}\frac{\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)}{ab}$$

where

$$\mathbb{C}=\begin{cases}4,\text{both } m,n\neq 0\\ 2,\text{either } m,n=0\\ 1,\text{both } m,n=0\end{cases}$$

$$\vec{M} = -\frac{n\pi}{b}\frac{\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)}{ab}\mathbb{C}e^{ikz}\hat{x} + \frac{m\pi}{a}\frac{\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)}{ab}\mathbb{C}e^{ikz}\hat{y}$$

$$\vec{M}^{(\mp)}\cdot\vec{J} = \frac{I_0 m\pi\mathbb{C}}{a^2 b}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left[\frac{\omega}{c}(h-y)\right]e^{\mp ikz}$$

$$\Rightarrow \int \vec{M}^{(\mp)}\cdot\vec{J}d^3r' = \frac{I_0 m\pi\mathbb{C}}{a^2 b}\sin\left(\frac{m\pi x}{a}\right)e^{\mp ikz}\int_0^h \cos\left(\frac{n\pi y}{b}\right)\sin\left[\frac{\omega}{c}(h-y)\right]dy$$

$$\int_0^h \cos\left(\frac{n\pi y}{b}\right)\sin\left[\frac{\omega}{c}(h-y)\right]dy$$

$$= \sin\left(\frac{\omega}{c}h\right)\int_0^h \cos\left(\frac{\omega}{c}y\right)\cos\left(\frac{n\pi y}{b}\right)dy - \cos\left(\frac{\omega}{c}h\right)\int_0^h \sin\left(\frac{\omega}{c}y\right)\cos\left(\frac{n\pi y}{b}\right)dy$$

$$= \sin\left(\frac{\omega}{c}h\right)\left[\frac{\sin\left(\frac{\omega}{c}-\frac{n\pi}{b}\right)y}{2\left(\frac{\omega}{c}-\frac{n\pi}{b}\right)} + \frac{\sin\left(\frac{\omega}{c}+\frac{n\pi}{b}\right)y}{2\left(\frac{\omega}{c}+\frac{n\pi}{b}\right)}\right]_0^h + \cos\left(\frac{\omega}{c}h\right)\left[\frac{\cos\left(\frac{\omega}{c}-\frac{n\pi}{b}\right)y}{2\left(\frac{\omega}{c}-\frac{n\pi}{b}\right)} + \frac{\cos\left(\frac{\omega}{c}+\frac{n\pi}{b}\right)y}{2\left(\frac{\omega}{c}+\frac{n\pi}{b}\right)}\right]_0^h$$

$$= \frac{\cos\left(\frac{n\pi h}{b}\right) - \cos\left(\frac{\omega}{c}h\right)}{2\left(\frac{\omega}{c}-\frac{n\pi}{b}\right)} + \frac{\cos\left(\frac{n\pi h}{b}\right) - \cos\left(\frac{\omega}{c}h\right)}{2\left(\frac{\omega}{c}+\frac{n\pi}{b}\right)}$$

$$= \frac{4\frac{\omega}{c}\left[\cos\left(\frac{n\pi h}{b}\right) - \cos\left(\frac{\omega}{c}h\right)\right]}{4\left[\left(\frac{\omega}{c}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right]} = \frac{\frac{\omega}{c}}{\left(\frac{\omega}{c}\right)^2 - \left(\frac{n\pi}{b}\right)^2}\left[\cos\left(\frac{n\pi h}{b}\right) - \cos\left(\frac{\omega}{c}h\right)\right]$$

$$a_{TE}^{(\pm)} = \frac{-2\pi I_0 m\pi\mathbb{C}}{a^2 b r^2\sqrt{1-\left(r\frac{c}{\omega}\right)^2}}\frac{\frac{\omega}{c}\mu}{\left(\frac{\omega}{c}\right)^2 - \left(\frac{n\pi}{b}\right)^2}\left[\cos\left(\frac{n\pi h}{b}\right) - \cos\left(\frac{\omega}{c}h\right)\right]\sin\left(\frac{m\pi x}{a}\right)e^{\mp ikz}$$

where

$$r^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

$$a_{TM}^{(\pm)} = \frac{2\pi i\mu}{kr^2}\int\frac{\vec{N}^{(\pm)}\cdot\vec{J}}{c}d^3r'$$

$$\vec{N} = \vec{\nabla}\times\vec{\nabla}\times\left(\hat{z}\chi e^{ikz}\right)$$

$$= \left(r^2\hat{z} + ik\nabla_t\right)\chi e^{ikz}$$

$$\chi = \frac{\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)}{ab}\mathbb{C}$$

$$\vec{N}^{(\pm)} \cdot \vec{J} = \frac{ik\left(\frac{n\pi}{b}\right)I_0}{ab} \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\mathbb{C}e^{\mp ikz}\sin\left[\frac{\omega}{c}(h-y)\right]$$

$$\Rightarrow a_{TM}^{(\pm)} =$$

$$\frac{i2\pi^2 I_0 n\mathbb{C}\mu}{cab^2r^2\sqrt{1-\left(r\frac{c}{\omega}\right)^2}} \frac{1}{\left(\frac{\omega}{c}\right)^2 - \left(\frac{n\pi}{b}\right)^2}\left[\cos\left(\frac{n\pi h}{b}\right) - \cos\left(\frac{\omega}{c}h\right)\right]\sin\left(\frac{m\pi x}{a}\right)e^{\mp ikz}$$

(b) $\mathbb{C} = 2$

$$P^{(+)} = \frac{c}{8\pi\mu}\frac{k}{\left(\frac{\omega}{c}\right)}r^2\left\{\left|a_{TE}^{(+)}\right|^2 + \left|a_{TM}^{(+)}\right|^2\right\}$$

only TE, $m=1$, $n=0$.

$$= \frac{c^2}{8\pi}\frac{k\mu}{c^2\omega}r^2\frac{4\pi^4 I_0^2 \cdot 1 \cdot 4}{a^4b^2r^4\left[1-\left(r\frac{c}{\omega}\right)^2\right]}\frac{\left(\frac{\omega}{c}\right)^2}{\left[\left(\frac{\omega}{c}\right)^2 - 0\right]^2}\left[1-\cos\left(\frac{\omega}{c}h\right)\right]^2\sin^2\left(\frac{\pi x}{a}\right)$$

$$= \frac{\left(\frac{\omega}{c}\right)^3\mu}{\sqrt{1-\left(r\frac{c}{\omega}\right)^2}}\frac{2\pi^3 I_0^2}{\omega r^2a^4b^2}\frac{1}{\left(\frac{\omega}{c}\right)^4}\left[1-\cos\left(\frac{\omega}{c}h\right)\right]^2\sin^2\left(\frac{\pi x}{a}\right)$$

$$\left[1-\cos\left(\frac{\omega}{c}h\right)\right]^2 = 4\sin^4\left(\frac{\omega h}{2c}\right)$$

$$= \frac{8\pi I_0^2\mu}{\sqrt{1-\left(r\frac{c}{\omega}\right)^2}}\frac{c}{\omega r^2a^4b^2}\sin^4\left(\frac{\omega h}{2c}\right)\sin^2\left(\frac{\pi x}{a}\right)$$

$$= \frac{8\pi I_0^2\mu}{ka^2b^2\omega\sqrt{1-\left(r\frac{c}{\omega}\right)^2}}\sin^4\left(\frac{\omega h}{2c}\right)\sin^2\left(\frac{\pi x}{a}\right)$$

$$= \frac{4\pi\mu I_0^2}{\omega kab}\sin^2\left(\frac{\pi x}{a}\right)\sin^4\left(\frac{\omega h}{2c}\right)$$

(c) The original field is not modified. But an additional reflected field with phase difference π is superimposed. Therefore interference will occur. The maximum power will occur at constructive interference.

$$R_{rad} = \frac{P_{\max}}{\frac{1}{2} I_0^2 \sin^2\left(\frac{\omega}{c} h\right)}$$

$E_{\max} = 2E$ in (a), so $P_{\max} = 4P$ in (a).

$$= \frac{8P_{\max}}{I_0^2 \sin^2\left(\frac{\omega}{c} h\right)}$$

$$= \frac{32\pi^4 I_0^2 \cdot 4}{c^2 a^4 b^2 r^4 \left(1 - r\left(\frac{c}{\omega}\right)^2\right)} \frac{1}{\left(\frac{c}{\omega}\right)^2} \frac{\left(1-\cos\left(\frac{\omega}{c} h\right)\right)^2 \sin^2\left(\frac{\pi x}{a}\right)}{I_0^2 \sin^2\left(\frac{\omega}{c} h\right)}$$

where $a^4 b^2 r^4 = a^2 b^2 \pi^2$

$$= \frac{128\pi^2 \sin^2\left(\frac{\omega h}{2c}\right) \sin^2\left(\frac{\pi x}{a}\right)}{a^2 b^2 \left(1 - \left(\frac{\pi}{a}\frac{c}{\omega}\right)^2\right) \omega^2 \cos^2\left(\frac{\omega h}{2c}\right)}$$

$$= \frac{128\pi^2 \tan^2\left(\frac{\omega h}{2c}\right) \sin^2\left(\frac{\pi x}{a}\right)}{a^2 b^2 \left(1 - \left(\frac{\pi}{a}\frac{c}{\omega}\right)^2\right) \omega^2}$$

$$= \frac{32\pi^2}{\omega^2} \tan^2\left(\frac{\omega h}{2c}\right) \sin^2\left(\frac{\pi x}{a}\right)$$

8.20

An infinitely long rectangular waveguide has a coaxial line terminating in the short side of the guide with the thin central conductor forming a semicircular loop of radius R whose center is a height h above the floor of the guide, as shown in the accompanying cross-sectional view. The half-loop is in the plane $z = 0$ and its radius R is sufficiently small that the current can be taken as having a constant value I_0 everywhere on the loop.

(a) Prove that to the extent that the current is constant around the half-loop, the TM modes are not excited. Give a physical explanation of this lack of excitation.

The field in the waveguide can be written as

$$\vec{E}^{(\pm)} = \sum_{\lambda} A_{\lambda}^{(\pm)} \vec{E}_{\lambda}^{(\pm)}$$

where the coefficients $A_{\lambda}^{(\pm)}$ are given by Eq. (8.146):

$$A_{\lambda}^{(\pm)} = -\frac{Z_{\lambda}}{2}\int_V \vec{J}\cdot\vec{E}_{\lambda}^{(\mp)} d^3x = -\frac{Z_{\lambda}}{2}\int \vec{I}\cdot\vec{E}_{\lambda}^{(\mp)} dl$$

Choose the bottom-left corner of the guide as the coordinate origin with the *x*-axis along the edge *a* and the *y*-axis along the edge *b*.

$$\vec{I} = I_0\left(-\sin\phi\hat{x} + \cos\phi\hat{y}\right)$$

Here ϕ is the polar angle with respect to the center-of-the loop. Thus

$$A_{\lambda}^{(\pm)} = -\frac{Z_{\lambda}}{2}\int_{-\pi/2}^{\pi/2} I_0\left(-\sin\phi\hat{x} + \cos\phi\hat{y}\right)\cdot\vec{E}_{\lambda}^{(\mp)}(Rd\phi) = -\frac{1}{2}RI_0Z_{\lambda}\int_{-\pi/2}^{\pi/2}\left\{-\sin\phi\left\{E_{\lambda}^{(\mp)}\right\}_x + \cos\phi\left\{E_{\lambda}^{(\mp)}\right\}_y\right\}d\phi$$

where $\left\{E_{\lambda}^{(\mp)}\right\}_x$ and $\left\{E_{\lambda}^{(\mp)}\right\}_y$ are *x*- and *y*- components of the eigen-field along the loop.

For TM waves, the electric field components are given by Eq. (8.135):

$$\left\{E_{mn}^{(\mp)}\right\}_x = \frac{2\pi m}{\gamma_{mn} a\sqrt{ab}}\cos\left\{\frac{m\pi\left(R\cos\phi\right)}{a}\right\}\sin\left\{\frac{n\pi\left(h + R\sin\phi\right)}{b}\right\}$$

$$\left\{E_{mn}^{(\mp)}\right\}_y = \frac{2\pi n}{\gamma_{mn} b\sqrt{ab}}\sin\left\{\frac{m\pi\left(R\cos\phi\right)}{a}\right\}\cos\left\{\frac{n\pi\left(h + R\sin\phi\right)}{b}\right\}$$

Here

$$\gamma_{mn}^2 = \pi^2\left\{\frac{m^2}{a^2} + \frac{n^2}{b^2}\right\}$$

Therefore,

$$A_{mn}^{(\pm)} = -\frac{\pi R I_0 Z_{mn}}{\gamma_{mn}\sqrt{ab}}\int_{-\pi/2}^{\pi/2}\left\{-\frac{m}{a}\sin\phi\cos\left(\frac{m\pi R\cos\phi}{a}\right)\sin\left(\frac{n\pi\left(h+R\sin\phi\right)}{b}\right) + \frac{n}{b}\cos\phi\sin\left(\frac{m\pi R\cos\phi}{a}\right)\cos\left(\frac{n\pi\left(h+R\sin\phi\right)}{b}\right)\right\}d\phi$$

$$= -\frac{\pi R I_0 Z_{mn}}{\gamma_{mn}\sqrt{ab}}\int_{-\pi/2}^{\pi/2}\left\{\frac{1}{\pi R}\frac{d}{d\phi}\left\{\sin\left(\frac{m\pi R\cos\phi}{a}\right)\right\}\sin\left(\frac{n\pi\left(h+R\sin\phi\right)}{b}\right) + \frac{1}{\pi R}\sin\left(\frac{m\pi R\cos\phi}{a}\right)\frac{d}{d\phi}\left\{\sin\left(\frac{n\pi\left(h+R\sin\phi\right)}{b}\right)\right\}\right\}d\phi$$

$$= -\frac{I_0 Z_{mn}}{\gamma_{mn}\sqrt{ab}}\int_{-\pi/2}^{\pi/2}\frac{d}{d\phi}\left\{\sin\left(\frac{m\pi R\cos\phi}{a}\right)\sin\left(\frac{n\pi\left(h+R\sin\phi\right)}{b}\right)\right\}d\phi$$

$$= -\frac{I_0 Z_{mn}}{\gamma_{mn}\sqrt{ab}}\sin\left(\frac{m\pi R\cos\phi}{a}\right)\sin\left(\frac{n\pi\left(h+R\sin\phi\right)}{b}\right)\Bigg|_{\phi=-\pi/2}^{\phi=\pi/2} = 0$$

Hence, no TM modes are excited. This is because that a circular current in the transverse plane will always result in a non-vanishing longitudinal component of $\vec{H}$, i.e., $H_z \neq 0$.

(b) Determine the amplitude for the lowest TE mode in the guide and show that its value is independent of the height *h*.

For TE waves,

$$\left\{E_{mn}^{(\mp)}\right\}_x = -\frac{2\pi n}{\gamma_{mn} b\sqrt{ab}} \cos\left(\frac{m\pi R\cos\phi}{a}\right) \sin\left(\frac{n\pi\left(h+R sin\phi\right)}{b}\right)$$

$$\left\{E_{mn}^{(\mp)}\right\}_y = \frac{2\pi m}{\gamma_{mn} a\sqrt{ab}} \sin\left(\frac{m\pi R\cos\phi}{a}\right) \cos\left(\frac{n\pi\left(h+R sin\phi\right)}{b}\right)$$

with the normalization reduced by a factor of $\sqrt{2}$ if *m* = 0 or *n* = 0. Thus

$$A_{mn}^{(\pm)} = -\frac{\pi R I_0 Z_{mn}}{\gamma_{mn}\sqrt{ab}} \int_{-\pi/2}^{\pi/2} \left\{ \frac{n}{b}\sin\phi\cos\left(\frac{m\pi R\cos\phi}{a}\right)\sin\left(\frac{n\pi\left(h+R sin\phi\right)}{b}\right) + \frac{m}{a}\cos\phi\sin\left(\frac{m\pi R\cos\phi}{a}\right)\cos\left(\frac{n\pi\left(h+R sin\phi\right)}{b}\right)\right\} d\phi$$

The lowest modes (*m* = 1, *n* = 0):

$$A_{1,0} = -\frac{\pi R I_0 Z_{1,0}}{\gamma_{1,0}\sqrt{2a^3 b}} \int_{-\pi/2}^{\pi/2} \left\{\cos\phi\sin\left(\frac{\pi R\cos\phi}{a}\right)\right\} d\phi = -\frac{\pi R I_0 Z_{1,0}}{\gamma_{1,0}\sqrt{2a^3 b}} \left\{\pi J_1\left(\frac{\pi R}{a}\right)\right\}$$

where $\gamma_{1,0} = \pi / a$. Here we have used the integral representation of Bessel functions:

$$\int_0^{\pi} \sin\theta \sin\left(x\sin\theta\right) d\theta = \int_{-\pi/2}^{\pi/2} \cos\phi \sin\left(x\cos\phi\right) d\phi = J_1\left(x\right)$$

The amplitude is independent of the height *h*. For $R \ll a$,

$$J_1\left(\frac{\pi R}{a}\right) \approx \frac{\pi R}{2a} \Rightarrow A_{1,0} \approx -\frac{\pi^3 R^2 I_0 Z_{1,0}}{\gamma_{1,0}\sqrt{8a^5 b}}$$

(c) Show that the power radiated in either direction in the lowest TE mode is

$$P = \frac{I_0^2}{16} Z \frac{a}{b}\left(\frac{\pi R}{a}\right)^4$$

where Z is the wave impedance of the TE_{10} mode. Here assume $R \ll a, b$.

The average power radiated in either direction

$$P = \frac{1}{2}\int \left(\vec{E}\times\vec{H}^*\right)\cdot \hat{z}da = \frac{1}{2}\int \left\{\left(\sum_{\lambda} A_\lambda \vec{E}_\lambda\right)\times\left(\sum_{\mu} A_\mu^* \vec{H}_\mu^*\right)\right\}\cdot \hat{z}da$$

$$= \frac{1}{2}\sum_{\lambda\mu} A_\lambda A_\mu^* \int \left(\vec{E}_\lambda \times \vec{H}_\mu^*\right)\cdot \hat{z}da = \frac{1}{2}\sum_{\lambda}\frac{\left|A_\lambda\right|^2}{Z_\lambda}$$

In this case,

$$P = \frac{1}{2}\frac{\left|A_{1,0}\right|^2}{Z_{1,0}} \approx \frac{I_0^2}{16} Z_{1,0}\frac{a}{b}\left(\frac{\pi R}{a}\right)^4$$