9.3

Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulating gap. An alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V\cos\omega t$. In the long-wavelength limit, find the radiation fields, the angular distribution of radiated power, and the total radiated power from the sphere.

Sol:

According to Jackson (3.36) and replace $(r/a)^l$ by $(a/r)^{l+1}$, a by R. Therefore

$$\Phi(r,\theta)=V\left[\frac{3}{2}\left(\frac{R}{r}\right)^2 P_1(\cos\theta)-\frac{7}{8}\left(\frac{R}{r}\right)^4 P_3(\cos\theta)+\frac{11}{16}\left(\frac{R}{r}\right)^6 P_5(\cos\theta)+\ldots\right]$$

The potential dominated by the dipole term

$$\Phi(r,\theta)\approx V\frac{3}{2}\left(\frac{R}{r}\right)^2 P_1(\cos\theta)=\frac{3VR^2}{2}\frac{\cos\theta}{r^2}=\frac{1}{4\pi\varepsilon_0}\frac{3V4\pi\varepsilon_0R^2}{2}\frac{\cos\theta}{r^2}$$

Compare with the potential of an electric dipole $\vec{p}=p\hat{z}$

$$\Phi(r,\theta)=\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

we infer the dipole moment of the sphere to be

$$\vec{p}=\frac{3V4\pi\varepsilon_0R^2}{2}\hat{z}=6\pi\varepsilon_0VR^2\hat{z}=6\pi\varepsilon_0VR^2\left(\cos\theta\hat{r}-\sin\theta\hat{\theta}\right)$$

Thus, the radiation fields are given by Jackson (9.19)

$$\vec{H}=\frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r}$$

$$\vec{E}=Z_0\vec{H}\times\hat{n}$$

where $Z_0=\sqrt{\dfrac{\mu_0}{\varepsilon_0}}$

$\Rightarrow$

$$\vec{H}=\frac{ck^2}{4\pi}\left(\hat{r}\times6\pi\varepsilon_0VR^2\hat{z}\right)\frac{e^{ikr}}{r}=-\frac{3}{2}\frac{\omega^2}{c}\varepsilon_0VR^2\sin\theta\frac{1}{r}e^{i\frac{\omega}{c}r}\hat{\phi}$$

$$\vec{E}=Z_0\vec{H}\times\hat{n}=-\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{3}{2}\frac{\omega^2}{c}\varepsilon_0VR^2\frac{1}{r}e^{i\frac{\omega}{c}r}\hat{\phi}\times\hat{r}=-\frac{3}{2}\frac{\omega^2}{c^2}VR^2\sin\theta\frac{1}{r}e^{i\frac{\omega}{c}r}\hat{\theta}$$

$$\frac{dP}{d\Omega}=\frac{c^2Z_0}{32\pi^2}k^4\left|(\hat{n}\times\vec{p})\times\hat{n}\right|^2=\frac{c^2Z_0}{32\pi^2}k^4\left(\left|\vec{p}\right|^2-\left|\vec{p}\cdot\hat{n}\right|^2\right)=\frac{9}{8}\left(\frac{\omega R}{c}\right)^4\frac{V^2}{Z_0}\sin^2\theta$$

The total power $P=\int \frac{dP}{d\Omega}d\Omega = 3\pi\left(\frac{\omega R}{c}\right)^4 \frac{V^2}{Z_0}$

9.6

(a) Starting from the general expression (9.2) for $\vec{A}$ and the corresponding expression for Φ, expand both $R=|\vec{x}-\vec{x}'|$ and $t'=t-R/c$ to first order in $|\vec{x}'|/r$ to obtain the electric dipole potentials for arbitrary time variation

$$\Phi(\vec{x},t)=\frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r^2}\hat{n}\cdot\vec{p}_{ret}+\frac{1}{cr}\hat{n}\cdot\frac{\partial\vec{p}_{ret}}{\partial t}\right]$$
$$\vec{A}(\vec{x},t)=\frac{1}{4\pi\varepsilon_0}\frac{\partial\vec{p}_{ret}}{\partial t}$$

where $\vec{p}_{ret}=\vec{p}(t'=t-r/c)$ is the dipole moment evaluated at the retarded time measured from the origin.

Sol.

The scalar potential is given by

$$\Phi(\vec{x})=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{x}',t-|\vec{x}-\vec{x}'|/c)}{|\vec{x}-\vec{x}'|}d^3x' \tag{1}$$

We use the expansion

$$|\vec{x}-\vec{x}'|\approx r-\hat{n}\cdot\vec{x}'$$

and

$$t'=t-\frac{|\vec{x}-\vec{x}'|}{c}\approx t-\frac{r}{c}+\frac{\hat{n}\cdot\vec{x}'}{c}=t_{ret}+\frac{\hat{n}\cdot\vec{x}'}{c}$$

where $t_{ret}=t-r/c$. Since ρ is a function of t', we make the expansion

$$\rho(\vec{x}',t')=\rho(\vec{x}',t_{ret})+\frac{\hat{n}\cdot\vec{x}'}{c}\frac{\partial\rho(\vec{x}',t_{ret})}{\partial t}+\cdots=\rho_{ret}+\frac{\hat{n}\cdot\vec{x}'}{c}\frac{\partial\rho_{ret}}{\partial t}+\cdots$$

Then (1) is expanded as

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0 r}\int\left[\rho_{ret} + \hat{n}\cdot\vec{x}'\left(\frac{1}{r}\rho_{ret} + \frac{1}{c}\frac{\partial\rho_{ret}}{\partial t}\right)+\cdots\right]d^3x'$$
$$= \frac{1}{4\pi\varepsilon_0 r}\left[Q + \hat{n}\cdot\left(\frac{1}{r}\vec{p}_{ret} + \frac{1}{c}\frac{\partial\vec{p}_{ret}}{\partial t}\right)+\cdots\right]$$

where the expressions for charge and electric dipole moment are

$$Q = \int \rho_{ret} d^3x', \quad \vec{p}_{ret} = \int \vec{x}'\rho_{ret} d^3x'$$

By charge conservation, Q is independent of time, so the subscript Q_{ret} is superfluous. Dropping the static Coulomb potential, which does not radiate, then gives

$$\Phi(\vec{x}) \approx \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r^2}\hat{n}\cdot\vec{p}_{ret} + \frac{1}{cr}\hat{n}\cdot\frac{\partial\vec{p}_{ret}}{\partial t}\right] \tag{2}$$

We only need to keep the lowest order behavior for the vector potential

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}\left(\vec{x}', t - |\vec{x}-\vec{x}'|/c\right)}{|\vec{x}-\vec{x}'|}d^3x' = \frac{\mu_0}{4\pi r}\int\left[\vec{J}_{ret} + \cdots\right]d^3x'$$

Using integration by parts, we have

$$\int J_{ret,i}d^3x' = \int\frac{\partial x_i'}{\partial x_j'}J_{ret,j}d^3x' = -\int x_i'\left(\vec{\nabla}\cdot\vec{J}_{ret}\right)d^3x' = \int x_i'\frac{\partial\rho_{ret}}{\partial t}d^3x' = \frac{\partial\rho_{ret,i}}{\partial t}$$

Therefore

$$\vec{A}(\vec{x}') \approx \frac{\mu_0}{4\pi r}\frac{\partial\vec{p}_{ret}}{\partial t} \tag{3}$$

(b) Calculate the dipole electric and magnetic fields directly from these potentials and show that

$$\vec{B}(\vec{x},t) = \frac{\mu_0}{4\pi}\left[-\frac{1}{cr^2}\hat{n}\times\frac{\partial\vec{p}_{ret}}{\partial t} - \frac{1}{c^2 r}\hat{n}\times\frac{\partial^2\vec{p}_{ret}}{\partial t^2}\right]$$
$$\vec{E}(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\left\{\left(1+\frac{r}{c}\frac{\partial}{\partial t}\right)\left[\frac{3\hat{n}(\hat{n}\cdot\vec{p}_{ret}) - \vec{p}_{ret}}{r^3}\right] + \frac{1}{c^2 r}\hat{n}\times\left(\hat{n}\times\frac{\partial^2\vec{p}_{ret}}{\partial t^2}\right)\right\}$$

Sol.

We use $\vec{B} = \vec{\nabla}\times\vec{A}$ to obtain the magnetic field, where the vector potential is given by (3). Note that the electric dipole $\vec{p}_{ret}$ in (3) is a function of the retarded time

$$\vec{p}_{ret} = \vec{p}\left(t - r/c\right)$$

By the chain rule, we have

$$\frac{\partial \vec{p}_{ret}}{\partial r} = -\frac{1}{c}\frac{\partial \vec{p}_{ret}}{\partial t}$$

Since $\vec{\nabla} r = \hat{n}$, the magnetic field becomes

$$\vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla}\times\left(\frac{1}{r}\frac{\partial \vec{p}_{ret}}{\partial t}\right) = \frac{\mu_0}{4\pi}\hat{n}\times\left(-\frac{1}{r^2}\frac{\partial \vec{p}_{ret}}{\partial t} - \frac{1}{cr}\frac{\partial^2 \vec{p}_{ret}}{\partial t^2}\right) \tag{4}$$

We use (2) and (3) to obtain the expression for the electric field.

$$\begin{aligned}
\vec{E} &= -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \\
&= -\frac{1}{4\pi\varepsilon_0}\vec{\nabla}\left(\frac{\vec{x}}{r^3}\cdot\vec{p}_{ret} + \frac{\vec{x}}{cr^2}\cdot\frac{\partial \vec{p}_{ret}}{\partial t}\right) - \frac{\mu_0}{4\pi r}\frac{\partial^2 \vec{p}_{ret}}{\partial t^2} \\
&= -\frac{1}{4\pi\varepsilon_0}\left[\frac{\vec{p}_{ret}}{r^3} - 3\frac{\vec{x}\left(\vec{x}\cdot\vec{p}_{ret}\right)}{r^5} - \frac{\vec{x}}{cr^4}\left(\vec{x}\cdot\frac{\partial \vec{p}_{ret}}{\partial t}\right) + \frac{1}{cr^2}\frac{\partial \vec{p}_{ret}}{\partial t} - \frac{2\vec{x}}{cr^4}\left(\vec{x}\cdot\frac{\partial \vec{p}_{ret}}{\partial t}\right) - \frac{\vec{x}}{c^2 r^3}\left(\vec{x}\cdot\frac{\partial^2 \vec{p}_{ret}}{\partial t^2}\right)\right] - \frac{1}{4\pi\varepsilon_0}\frac{1}{c^2 r}\frac{\partial^2 \vec{p}_{ret}}{\partial t^2} \\
&= -\frac{1}{4\pi\varepsilon_0}\left[\frac{\vec{p}_{ret} - 3\hat{n}\left(\hat{n}\cdot\vec{p}_{ret}\right)}{r^3} + \frac{1}{cr^2}\frac{\partial}{\partial t}\left(\vec{p}_{ret} - 3\hat{n}\left(\hat{n}\cdot\vec{p}_{ret}\right)\right) + \frac{1}{c^2 r}\frac{\partial^2}{\partial t^2}\left(\vec{p}_{ret} - \hat{n}\left(\hat{n}\cdot\vec{p}_{ret}\right)\right)\right] \\
&= \frac{1}{4\pi\varepsilon_0}\left[\left(1 + \frac{r}{c}\frac{\partial}{\partial t}\right)\frac{3\hat{n}\left(\hat{n}\cdot\vec{p}_{ret}\right) - \vec{p}_{ret}}{r^3} + \frac{1}{c^2 r}\hat{n}\times\left(\hat{n}\times\frac{\partial^2 \vec{p}_{ret}}{\partial t^2}\right)\right]
\end{aligned}$$

(c) Show explicitly how you can go back and forth between these results and the harmonic fields of (9.18) by the substitutions $-i\omega \leftrightarrow \partial/\partial t$ and $\vec{p}e^{ikr-i\omega t} \leftrightarrow \vec{p}_{ret}\left(t'\right)$.

Sol.

With the substitution

$$\vec{p}_{ret} \to \vec{p}e^{ikr} \quad \text{and} \quad \frac{\partial}{\partial t} \to -i\omega$$

The magnetic field (4) becomes

$$\begin{aligned}
\vec{H} &= \frac{1}{4\pi}\hat{n}\times\left(-\frac{1}{r^2}\left(-i\omega\right)\vec{p} - \frac{1}{cr}\left(-\omega^2\right)\vec{p}\right)e^{ikr} \\
&= \frac{\omega^2}{4\pi cr}\left(\hat{n}\times\vec{p}\right)\left(1 - \frac{c}{i\omega r}\right)e^{ikr} = \frac{ck^2}{4\pi}\left(\hat{n}\times\vec{p}\right)\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)
\end{aligned}$$

The electric field (5) becomes

$$\vec{E}=\frac{1}{4\pi\varepsilon_0}\left[\left(1+\frac{r}{c}(-i\omega)\right)\frac{3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}}{r^3}+\frac{1}{c^2r}(-\omega^2)\hat{n}\times(\hat{n}\times\vec{p})\right]e^{ikr}$$
$$=\frac{1}{4\pi\varepsilon_0}\left[\frac{e^{ikr}}{r^3}(1-ikr)\left(3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right)-k^2\frac{e^{ikr}}{r}\hat{n}\times(\hat{n}\times\vec{p})\right]$$

To go in the other direction, we simply read theses equations backwards.

9.14

An antenna consists of a circular loop of wire of radius a located in the $x-y$ plane with its center at the origin. The current in the wire is

$I=I_0\cos\omega t=\operatorname{Re}I_0e^{i\omega t}$

(a) Find the expressions for $\vec{E}$, $\vec{H}$ in the radiation zone without approximations as to the magnitude of ka. Determine the power radiated per unit solid angle.

(b) What is the lowest non-vanishing multipole moment ($Q_{l,m}$ or $M_{l,m}$) ? Evaluate this moment in the limit $ka<<1$.

Sol:

(a) $I(t)=I_0\cos\omega t=\operatorname{Re}\left[I_0E^{-i\omega t}\right]$

$$\vec{J}=\frac{1}{a}I(t)\delta(r-a)\delta(\cos\theta)\hat{\phi}=$$

a is a normalization factor determined by $\int\vec{J}\cdot d\vec{a}=I$

Use Jackson (9.149) and

$$a_E(l,m)=\frac{k^2}{i\sqrt{l(l+1)}}\int Y_l^{m^*}ik(\vec{r}\cdot\vec{J})j_l(kr)d^3x$$
$$a_M(l,m)=\frac{k^2}{i\sqrt{l(l+1)}}\int Y_l^{m^*}\vec{\nabla}\cdot(\vec{r}\times\vec{J})j_l(kr)d^3x$$

Because $\vec{r}\cdot\vec{J}=0$. Therefore, all $a_E=0$.

$$\vec{\nabla}\cdot(\vec{r}\times\vec{J})=\sin\theta\frac{\partial}{\partial\cos\theta}J$$

By azimuthal symmetry, $m=0$.

$$\Rightarrow a_M(l,0)=\frac{k^2}{i\sqrt{l(l+1)}}\int Y_l^{0^*}\left(\sin\theta\frac{\partial}{\partial\cos\theta}J\right)j_l(kr)d^3x$$

$$=\frac{k^2}{i\sqrt{l(l+1)}}\int\left[\frac{\partial}{\partial\cos\theta}\left(\sin\theta Y_l^{0^*}\right)\right]Jj_l(kr)d^3x$$

$$=\frac{i2\pi k^2}{\sqrt{l(l+1)}}\frac{I_0}{a}a^2 j_l(ka)\frac{\partial}{\partial\cos\theta}\left(\sin\theta Y_l^0\right)\Big|_{\cos\theta=0}$$

$$=\frac{i2\pi k^2}{\sqrt{l(l+1)}}\frac{I_0}{a}a^2 j_l(ka)\left(1-x^2\right)^{\frac{1}{2}}\frac{\partial}{\partial x}Y_l^0(x)\Big|_{x=0}$$

$\frac{\partial}{\partial x}Y_l^{0^*}(x)\Big|_{x=0}=0$ when l is even.

Then, from Jackson (8.149) and (9.151) can find $\frac{dP}{d\Omega}$.

(b)

$$a_M(1,0)=\frac{i2\pi k^2 I_0 a}{\sqrt{2}}j_1(ka)\left(1-x^2\right)^{\frac{1}{2}}\frac{\partial}{\partial x}Y_1^0(x)\Big|_{x=0}$$

$$j_1(ka)=\frac{ka}{3}$$

$$\frac{\partial}{\partial x}Y_1^0(x)\Big|_{x=0}=\sqrt{\frac{3}{4\pi}}$$

$$\Rightarrow a_M(1,0)=\frac{i2\pi k^2 I_0 a}{\sqrt{2}}\frac{ka}{3}\sqrt{\frac{3}{4\pi}}=i2\pi k^3 I_0 a^2\sqrt{\frac{1}{24\pi}}=\frac{ik^3}{3}\sqrt{2}M_{10}$$

$$M_{10}=\sqrt{\frac{3}{4\pi}}I_0\pi a^2$$

From Jackson (9.151)

$$\frac{dP}{d\Omega}=\frac{Z_0}{2k^2}\left(2\pi k^3 I_0 a^2\sqrt{\frac{1}{24\pi}}\right)^2\frac{3}{8\pi}\sin^2\theta=\frac{1}{32\pi^2}Z_0 k^4\left(I_0\pi a^2\right)^2\sin^2\theta$$

9.16 A thin linear antenna of length d is excited in such a way that the sinusoidal current makes a full wavelength of oscillation as shown in the figure.

(a) Calculate exactly the power radiated per unit solid angle and plot the angular distribution of radiation.

Sol.

Note that the current flows in opposite directions in the top and bottom half of this antenna. As a result, we may write the source current density as

$$\vec{J}(z)=\hat{z}I\sin(kz)\delta(x)\delta(y)\Theta\left(d/2-|z|\right)$$

where $k = \dfrac{2\pi}{d}$.

In the radiation zone, the vector potential is given by

$$\begin{aligned}\vec{A}(\vec{x}) &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}')e^{-ik\hat{n}\cdot\vec{x}'}d^3x' \\ &= \hat{z}\frac{\mu_0 I}{4\pi}\frac{e^{ikr}}{r}\int_{-d/2}^{d/2}\sin(kz)e^{-ikz\cos\theta}\end{aligned}$$

Since the current source is odd under $z \to -z$, this integral may be written as

$$\begin{aligned}\vec{A} &= -\hat{z}\frac{i\mu_0 I}{4\pi}\frac{e^{ikr}}{r}\int_0^{d/2}2\sin(kz)\sin(kz\cos\theta)dz \\ &= -\hat{z}\frac{i\mu_0 I}{4\pi}\frac{e^{ikr}}{r}\int_0^{d/2}\left[\cos\left((1-\cos\theta)kz\right)-\cos\left((1+\cos\theta)kz\right)\right]dz \\ &= -\hat{z}\frac{i\mu_0 I}{4\pi}\frac{e^{ikr}}{kr}\left[\frac{1}{1-\cos\theta}\sin\left((1-\cos\theta)kz\right)-\frac{1}{1+\cos\theta}\sin\left((1+\cos\theta)kz\right)\right]_0^{d/2} \\ &= -\hat{z}\frac{i\mu_0 I}{2\pi}e^{ikr}kr\frac{\sin(\pi\cos\theta)}{\sin^2\theta}\end{aligned}$$

In the radiation zone, the magnetic field is

$$\vec{H} = \frac{ik}{\mu_0}\hat{n}\times\vec{A} = -\hat{\phi}\frac{I}{2\pi}\frac{e^{ikr}}{r}\frac{\sin(\pi\cos\theta)}{\sin\theta}$$

Where we have used $\hat{n}\times\hat{z} \equiv \hat{r}\times\hat{z} = -\hat{\phi}\sin\theta$. Therefore the radiated power is

$$\frac{dP}{d\Omega} = \frac{Z^2r^2}{2}\left|\vec{H}\right|^2 = \frac{Z_0|I|^2}{8\pi^2}\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}$$

It looks like a quadrupole pattern.

(b) Determine the total power radiated and find a numerical value for the radiation resistance.

Sol.

The total radiated power is obtained by integrating the angular distribution over the solid angle

$$P=\frac{Z_0|I|^2}{8\pi^2}2\pi\int_{-1}^{1}\frac{\sin^2(\pi\cos\theta)}{1-\cos^2\theta}d\cos\theta=\frac{Z_0|I|^2}{4\pi}\int_{-1}^{1}\frac{\sin^2(\pi x)}{1-x^2}dx\approx\frac{Z_0|I|^2}{4\pi}\times1.557$$

Using $P=\frac{1}{2}R_{rad}|I|^2$, the radiation resistance is

$$R_{rad}=\frac{Z_0}{2\pi}\times1.557\approx93.4\Omega$$

9.17

Treat the linear antenna of Problem 9.16 by the multipole expansion method.

(a) Calculate the multipole moments (electric dipole, magnetic dipole, and electric quadrupole) exactly and in the long-wavelength approximation.

(b) Compare the shape of the angular distribution of radiated power for the lowest non-vanishing multipole with the exact distribution of Problem 9.16.

(c) Determine the total power radiated for the lowest multipole and the corresponding radiation resistance using both multipole moments from part a. compare with Problem 9.16b. Is there a paradox here?

Sol:

(a) From Jackson (9.9) and (9.13),

$$\vec{A}=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int\vec{J}d^3x$$

$$=I_0\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int_{-\frac{d}{2}}^{\frac{d}{2}}\sin\left(\frac{z'}{d}2\pi\right)\delta(x')\delta(y')dx'dy'dz'\hat{z}$$

$$=\frac{d}{2\pi}I_0\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left[-\cos\left(\frac{z'}{d}2\pi\right)\right]\Bigg|_{-\frac{d}{2}}^{\frac{d}{2}}\hat{z}$$

$$=0$$

due to electric dipole moment.

From Jackson (9.9) and (9.33)

$$\vec{A}=\frac{ik\mu_0}{4\pi}(\hat{n}\times\vec{m})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)$$

$$\vec{m}=\frac{1}{2}\int(\vec{x}\times\vec{J})d^3x$$

Source $\vec{x}$ only in z direction the same with $\vec{J}$.

$$\vec{m}=0$$

$$\vec{A}=0$$

Due to magnetic dipole.

Then, from Jackson (9.38).

$$\vec{A} = -\frac{\mu_0 ck^2}{8\pi}\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\int \vec{x}'(\hat{n}\cdot\vec{x}')\rho(\vec{x}')d^3x'$$

$$\vec{\nabla}\cdot\vec{J} = \vec{\nabla}\cdot\left(I_0 \sin\left(\frac{z}{d}2\pi\right)\delta(x)\delta(y)\right)$$

$$= \frac{2\pi I_0}{d}\cos\left(\frac{z}{d}2\pi\right)\delta(x)\delta(y) = -\frac{\partial}{\partial t}\rho = +i\omega\rho$$

$$\Rightarrow \rho = \frac{2\pi I_0}{i\omega d}\cos\left(\frac{z}{d}2\pi\right)\delta(x)\delta(y)$$

$$\vec{A} = -\frac{\mu_0 ck^2}{8\pi}\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\int z'(z'\cos\theta)\frac{2\pi I_0}{i\omega d}\cos\left(\frac{z'}{d}2\pi\right)\delta(x')\delta(y')d^3x'\hat{z}$$

$$= -\frac{\mu_0 ck^2}{8\pi}\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\cos\theta\frac{2\pi I_0}{i\omega d}\int_{-\frac{d}{2}}^{\frac{d}{2}} z'^2\cos\left(\frac{z'}{d}2\pi\right)dz'\hat{z}$$

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} z'^2\cos\left(\frac{z'}{d}2\pi\right)dz' = \left(\frac{d}{2\pi}\right)^3\int_{-\frac{d}{2}}^{\frac{d}{2}}\left(\frac{z'}{d}2\pi\right)^2\cos\left(\frac{z'}{d}2\pi\right)d\left(\frac{z'}{d}2\pi\right)$$

$$= \left(\frac{d}{2\pi}\right)^3\int_{-\pi}^{\pi}u^2\cos u\,du$$

$$= \left(\frac{d}{2\pi}\right)^3\left[u^2\sin u - 2u\cdot -\cos u + 2\cdot -\sin u\right]_{-\pi}^{\pi}$$

$$= -4\pi\left(\frac{d}{2\pi}\right)^3 = \frac{-4\pi}{k^3}$$

$$\Rightarrow \vec{A} = \frac{\mu_0 ck^2}{8\pi}\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\cos\theta\frac{2\pi I_0}{i\omega d}\frac{4\pi}{k^3}\hat{z}$$

$$= \frac{I_0\mu_0 c\pi}{i\omega dk}\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\cos\theta\hat{z}$$

due to electric quadrupole.

(b) Electric quadrupole, the lowest non-vanishing multipole.

From Jackson (9.43)

$$\vec{H} = ik\hat{n}\times\frac{\vec{A}}{\mu_0} = -\frac{I_0 c\pi}{\omega d}\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\cos\theta\sin\theta\hat{\phi}$$

From Jackson (9.45)

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{1152\pi^2}k^6\left|\left[\hat{n}\times\vec{Q}(\hat{n})\right]\times\hat{n}\right|^2$$

and Jackson (9.44)

$$\vec{H} = -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\hat{n}\times\vec{Q}$$

$$\frac{dP}{d\Omega}=\frac{c^2Z_0}{1152\pi^2}k^6\left|\left[\frac{\vec{H}}{-\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}}\right]\times\hat{n}\right|^2$$

$$=\frac{c^2Z_0}{1152\pi^2}k^6\left|\left[\frac{-\frac{I_0c\pi}{\omega d}\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\cos\theta\sin\theta\hat{\phi}}{-\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}}\right]\times\hat{r}\right|^2$$

$$=\frac{c^2Z_0}{1152\pi^2}k^6\left|\frac{\frac{24I_0\pi^2}{\omega d}\left(1-\frac{1}{ikr}\right)\cos\theta\sin\theta}{ik^3}\right|^2$$

$$=\frac{c^2Z_0}{1152\pi^2}k^6\frac{\frac{I_0{}^2 576\pi^4}{\omega^2d^2}\left(1-\frac{1}{ikr}\right)^2\cos^2\theta\sin^2\theta}{k^6}$$

$$=\frac{Z_0}{2}\frac{I_0{}^2c^2\pi^2}{\omega^2d^2}\left(1-\frac{1}{ikr}\right)^2\cos^2\theta\sin^2\theta$$

Consider $kr>>1$

$$\frac{dP}{d\Omega}=\frac{Z_0}{2}\frac{I_0{}^2c^2\pi^2}{\omega^2d^2}\cos^2\theta\sin^2\theta$$

(c)

$$\int\frac{dP}{d\Omega}\sin\theta d\theta d\phi$$

$$=\int\frac{Z_0}{2}\frac{I_0{}^2c^2\pi^2}{\omega^2d^2}\cos^2\theta\sin^3\theta d\theta d\phi$$

$$=\frac{Z_0I_0{}^2c^2\pi^3}{\omega^2d^2}\int\cos^2\theta\sin^3\theta d\theta$$

$$=\frac{Z_0I_0{}^2c^2\pi^3}{\omega^2d^2}\left[\frac{1}{5}\cos^5\theta-\frac{1}{3}\cos^3\theta\right]_0^\pi$$

$$=\frac{4}{15}\frac{Z_0I_0{}^2c^2\pi^3}{\omega^2d^2}$$

$$\frac{\omega}{c}d=2\pi$$

$$\Rightarrow\int\frac{dP}{d\Omega}\sin\theta d\theta d\phi=\frac{4}{15}\frac{Z_0I_0{}^2\pi^3}{(2\pi)^2}$$

$$=\frac{Z_0I_0{}^2\pi}{15}$$

9.22 A spherical hole of radius a in a conducting medium can serve as an electromagnetic resonant cavity.

(a) Assuming infinite conductivity, determine the transcendental equations for the characteristic frequencies ω_{lm} of the cavity for TE and TM modes.

Sol.

Because of the spherical symmetry, it is natural to describe the modes of the spherical cavity in terms of a vector spherical wave expansion. These waves fall into either TE or TM modes, depending on whether $\vec{r}\cdot\vec{E}=0$ or $\vec{r}\cdot\vec{H}=0$, respectively. The TE (or magnetic multipole) modes are given by

$$\begin{aligned}\vec{H} &= -\frac{i}{k}\vec{\nabla}\times\left[j_l(kr)\vec{X}_{lm}\right] \\ \vec{E} &= Z_0 j_l(kr)\vec{X}_{lm}\end{aligned} \qquad (1)$$

where we have chosen the spherical Bessel function $j_l(kr)$ since it is regular at $r=0$. For a perfect conductor, we impose the boundary condition $H_\perp=0$ and $E_\parallel=0$ at $r=a$. More precisely, we demand

$$\hat{r}\cdot\vec{H}\Big|_{r=a}=0\,,\quad \hat{r}\times\vec{E}\Big|_{r=a}=0$$

These are equivalent to the condition $j_l(kr)=0$, and leads to the quantization $k_{nlm}=x_{ln}/a$ where x_{ln} is the *n*-th zero of the spherical Bessel function j_l. The TE_{nlm} frequencies are thus

$$\omega_{nlm}=\frac{x_{ln}c}{a},\quad j_l(x_{ln})=0\,,\quad l\ge 1\,,\quad |m|\le l$$

Each frequency specified by *l* and *n* is (2*l*+1)-fold degenerate, with azimuthal quantum number labeled by *m*.

The TM (or electric multipole) modes are similar. The modes themselves are given by

$$\vec{H} = j_l(kr)\vec{X}_{lm}$$
$$\vec{E} = Z_0 \frac{i}{k}\vec{\nabla}\times\left[j_l(kr)\vec{X}_{lm}\right] \tag{2}$$

This time, the $H_\perp = 0$ boundary condition is automatic, while the $E_\parallel = 0$ condition gives

$$\vec{r}\times\left(\vec{\nabla}\times\left[j_l(kr)\vec{X}_{lm}\right]\right)\Big|_{r=a} = 0$$

It may be simplified using

$$\vec{r}\times\left(\vec{\nabla}\times\vec{V}\right) = \vec{\nabla}\left(\vec{r}\cdot\vec{V}\right) - \vec{V} - \left(\vec{r}\cdot\vec{\nabla}\right)\vec{V} = \vec{\nabla}\left(\vec{r}\cdot\vec{V}\right) - \left(1 + r\frac{\partial}{\partial r}\right)\vec{V} = \vec{\nabla}\left(\vec{r}\cdot\vec{V}\right) - \frac{\partial}{\partial r} r\vec{V}$$

With $\vec{V} = j_l(kr)\vec{X}_{lm}$ and $\vec{r}\cdot\vec{X}_{lm} = 0$, we have

$$\vec{r}\times\left(\vec{\nabla}\times\left[j_l(kr)\vec{X}_{lm}\right]\right) = -\frac{\partial}{\partial r}\left(r j_l(kr)\right)\vec{X}_{lm} \tag{3}$$

Therefore the $E_\parallel = 0$ boundary condition leads to the TM_{nlm} frequencies

$$\omega_{nlm} = \frac{y_{ln}c}{a}, \quad \frac{d}{dx}\left[x j_l(x)\right]\Big|_{z=y_{ln}} = 0, \quad l \geq 1, \quad |m| \leq l$$

The y_{ln} correspond to zeros of $\left[x j_l(x)\right]'$ or equivalently $j_l(x) + x j_l'(x)$.

(b) Calculate numerical values for the wavelength λ_{lm} in units of the radius *a* for the four lowest modes for TE and TM waves.

Sol.

The numerical values for the wavelengths are obtained from the zeros x_{ln} and y_{ln}.

For TE_{nlm} modes, the first four zeros of $j_l(x)$ are

$$x_{11} = 4.4934, \quad x_{21} = 5.7635, \quad x_{31} = 6.9879, \quad x_{12} = 7.7253$$

Since $k_{nlm} = x_{ln}/a$ and $\lambda_{nlm} = 2\pi / k_{nlm}$, we have $\lambda_{nlm}/a = 2\pi / x_{ln}$ or

$$\frac{\lambda_{11m}}{a} = 1.398, \quad \frac{\lambda_{12m}}{a} = 1.090, \quad \frac{\lambda_{13m}}{a} = 0.899, \quad \frac{\lambda_{21m}}{a} = 0.813$$

All these modes are (2*l*+1)-fold degenerate.

For TM_{nlm} modes, the first four zeros of $\left[xj_l\left(x\right)\right]'$ are

$$y_{11} = 2.7437\,,\quad y_{21} = 3.8702\,,\quad y_{31} = 4.9734\,,\quad y_{41} = 6.0619$$

With corresponding wavelengths

$$\frac{\lambda_{11m}}{a} = 2.290\,,\quad \frac{\lambda_{12m}}{a} = 1.623\,,\quad \frac{\lambda_{13m}}{a} = 1.263\,,\quad \frac{\lambda_{14m}}{a} = 1.036$$

Note that the next mode, given by $y_{12} = 6.1168$ is nearly degenerate with y_{41}.

(c) Calculate explicitly the electric and magnetic fields inside the cavity for the lowest TE and lowest TM mode.

Sol.

The lowest TE and TM modes both have l = 1. Thus we begin with an overview of l = 1 vector spherical harmonics

$$\vec{X}_{lm} = \frac{1}{\sqrt{2}}\vec{L}Y_{1m}$$

It is natural to write the angular momentum operator $\vec{L}$ in terms of raising and lowering components

$$L_+ = L_x + iL_y\,,\quad L_- = L_x - iL_y\,,\quad L_z$$

Using

$$\begin{aligned} L_+Y_{lm} &= \sqrt{l\left(l+1\right) - m\left(m+1\right)}Y_{l,m+1} \\ L_-Y_{lm} &= \sqrt{l\left(l+1\right) - m\left(m-1\right)}Y_{l,m-1} \\ L_zY_{lm} &= mY_{lm} \end{aligned}$$

for l = 1 gives

$$X_{11}^{+} = 0\,,\quad X_{11}^{z} = \frac{1}{\sqrt{2}}Y_{11}\,,\quad X_{11}^{-} = Y_{10}$$

$$X_{10}^{+} = Y_{11}\,,\quad X_{10}^{z} = 0\,,\quad X_{10}^{-} = Y_{1,-1} \qquad (4)$$

$$X_{1,-1}^{+} = Y_{10}\,,\quad X_{1,-1}^{z} = -\frac{1}{\sqrt{2}}Y_{1,-1}\,,\quad X_{1,-1}^{-} = 0$$

A vector with components $\left(V_+, V_-, V_z\right)$ can be converted to spherical coordinates $\left(V_r, V_\theta, V_\phi\right)$ according to

$$
\begin{aligned}
V_r &= \frac{1}{2}\left(V_+ e^{-i\phi} + V_- e^{i\phi}\right)\sin\theta + V_z \cos\theta \\
V_\theta &= \frac{1}{2}\left(V_+ e^{-i\phi} + V_- e^{i\phi}\right)\cos\theta - V_z \sin\theta \\
V_\phi &= -\frac{i}{2}\left(V_+ e^{-i\phi} - V_- e^{i\phi}\right)
\end{aligned}
$$

Using the explicit form of the spherical harmonics then gives

$$
\begin{aligned}
&X_{11}^r = 0, \quad X_{11}^\theta = \sqrt{\frac{3}{16\pi}} e^{i\phi}, \quad X_{11}^\phi = i\sqrt{\frac{3}{16\pi}}\cos\theta e^{i\phi} \\
&X_{10}^r = 0, \quad X_{10}^\theta = 0, \quad X_{10}^\phi = i\sqrt{\frac{3}{8\pi}}\sin\theta \\
&X_{1,-1}^r = 0, \quad X_{1,-1}^\theta = \sqrt{\frac{3}{16\pi}} e^{-i\phi}, \quad X_{1,-1}^\phi = -i\sqrt{\frac{3}{16\pi}}\cos\theta e^{-i\phi}
\end{aligned}
$$

From (1) for TE_{nlm} modes, we have

$$
\vec{E}_{11m} = Z_0 j_l\left(kr\right)\vec{X}_{1m}, \quad \vec{H}_{11m} = -\frac{i}{Z_0 k}\vec{\nabla}\times\vec{E}_{11m}
$$

The $m = 0$ mode is

$$
\begin{aligned}
\vec{E}_{110} &= iZ_0\sqrt{\frac{3}{8\pi}} j_l\left(kr\right)\sin\theta\hat{\phi} \\
\vec{H}_{110} &= \frac{1}{kr}\sqrt{\frac{3}{8\pi}}\left(2 j_l\left(kr\right)\cos\theta\hat{r} - \left[kr j_0\left(kr\right) - j_1\left(kr\right)\right]\sin\theta\hat{\theta}\right)
\end{aligned}
$$

Note that we have used the spherical Bessel function identity

$$
j_l'\left(\varsigma\right) = j_{l-1}\left(\varsigma\right) - \frac{l+1}{\varsigma} j_l\left(\varsigma\right)
$$

Even more explicitly, we have

$$
j_1\left(\varsigma\right) = \frac{\sin\varsigma}{\varsigma^2} - \frac{\cos\varsigma}{\varsigma}
$$

$$\left[\varsigma j_1(\varsigma)\right]' = \varsigma j_0(\varsigma) - j_1(\varsigma) = -\left(\frac{1}{\varsigma^2} - 1\right)\sin\varsigma + \frac{\cos\varsigma}{\varsigma}$$

The m = 1 mode is given by

$$\begin{aligned}\vec{E}_{111} &= Z_0\sqrt{\frac{3}{16\pi}}\, j_1(kr)e^{i\phi}\left(\hat{\theta} + i\cos\theta\hat{\phi}\right)\\ \vec{H}_{111} &= \frac{1}{kr}\sqrt{\frac{3}{16\pi}}\, e^{i\phi}\left(-2j_1(kr)\sin\theta\hat{r} - \left[krj_0(kr) - j_1(kr)\right]\left(\cos\theta\hat{\theta} + i\hat{\phi}\right)\right)\end{aligned} \tag{6}$$

while the m = -1 mode is given by

$$\begin{aligned}\vec{E}_{11,-1} &= Z_0\sqrt{\frac{3}{16\pi}}\, j_1(kr)e^{-i\phi}\left(\hat{\theta} - i\cos\theta\hat{\phi}\right)\\ \vec{H}_{11,-1} &= \frac{1}{kr}\sqrt{\frac{3}{16\pi}}\, e^{-i\phi}\left(2j_1(kr)\sin\theta\hat{r} + \left[krj_0(kr) - j_1(kr)\right]\left(\cos\theta\hat{\theta} - i\hat{\phi}\right)\right)\end{aligned} \tag{7}$$

We now turn to the lowest TM mode, which is the TM_{11m} mode with fields given by (2)

$$\vec{H}_{11m} = j_l(kr)\vec{X}_{1m}, \quad \vec{E}_{11m} = \frac{iZ_0}{k}\vec{\nabla}\times\vec{H}_{11m}$$

The roles of $\vec{E}$ and $\vec{H}$ are interchanged between the TE and TM modes. In particular, the TM_{11m} fields may be obtained from the TE_{11m} fields of (5), (6) and (7) through the substitution

$$\vec{E} \to Z_0\vec{H}, \quad Z_0\vec{H} \to -\vec{E}$$

This is the action of electric-magnetic duality. The TM_{11m} modes correspond to

$$\vec{H}_{110} = i\sqrt{\frac{3}{8\pi}}\, j_1(kr)\sin\theta\hat{\phi}$$

$$\vec{E}_{110} = \frac{Z_0}{kr}\sqrt{\frac{3}{8\pi}}\left(2j_1(kr)\cos\theta\hat{r} - \left[krj_0(kr) - j_1(kr)\right]\sin\theta\hat{\theta}\right)$$

$$\vec{H}_{111} = \sqrt{\frac{3}{16\pi}}\, j_1(kr)e^{i\phi}\left(\hat{\theta} + i\cos\theta\hat{\phi}\right)$$

$$\vec{E}_{111} = \frac{Z_0}{kr}\sqrt{\frac{3}{16\pi}}\, e^{i\phi}\left(-2j_1(kr)\sin\theta\hat{r} - \left[krj_0(kr) - j_1(kr)\right]\left(\cos\theta\hat{\theta} + i\hat{\phi}\right)\right)$$

$$\vec{H}_{11,-1} = \sqrt{\frac{3}{16\pi}}\, j_1(kr)e^{-i\phi}\left(\hat{\theta} - i\cos\theta\hat{\phi}\right)$$

$$\vec{E}_{11,-1} = \frac{Z_0}{kr}\sqrt{\frac{3}{16\pi}}e^{-i\phi}\left(2j_1(kr)\sin\theta\hat{r} + \left[krj_0(kr) - j_1(kr)\right]\left(\cos\theta\hat{\theta} - i\hat{\phi}\right)\right)$$

The wavenumbers k_{nlm} are quantized differently for the TE versus the TM modes.

9.23

The spherical resonant cavity of Problem 9.22 has non-permeable walls of large, but finite, conductivity. In the approximation that the skin depth δ is small compared to the cavity radius a, show that the Q of the cavity, defined by equation (8.86), is given by

$Q = \frac{a}{\delta}$ for all TE modes

$Q = \frac{a}{\delta}\left(1 - \frac{l(l+1)}{x_{lm}^2}\right)$ for TM modes

where $x_{lm}^2 = (a/c)\omega_{lm}$ for TM modes.

Sol:

Energy density $u = \frac{\varepsilon_0}{4}|\vec{E}|^2 + \frac{\mu_0}{4}|\vec{H}|^2$ and energy is equally distributed between $\vec{E}$ and $\vec{H}$. Thus for TE modes we may immediately write down

$$u = \frac{\varepsilon_0}{2}|\vec{E}|^2 = \frac{\mu_0}{2}j_l(kr)^2|\vec{X}_{lm}|^2$$

The stored energy

$$U = \frac{\mu_0}{2}\int j_l(kr)^2 r^2 dr d\Omega = \frac{\mu_0}{2}\int_0^a j_l(kr)^2 r^2 dr d\Omega$$

$$\int_0^a j_l\left(x_{l,m}\frac{\rho}{a}\right)j_l\left(x_{l,n}\frac{\rho}{a}\right)\rho^2 d\rho = \frac{1}{2}a^3\left[j_l'(x_{l,n})\right]^2\delta_{m,n}$$

$$U_{lmn} = \frac{\mu_0}{4}a^3 j_l'(x_{l,n})^2$$

The power loss is given in terms of the tangential magnetic field at the conducting surface

$$P = \frac{1}{2\sigma\delta}\int|\hat{r}\times\vec{H}|^2 da$$

Using $\vec{h} = -\left(\frac{i}{k}\right)\vec{\nabla}\times j_l(kr)\vec{X}_{lm}$

$$P_{lmn} = \frac{1}{\sigma\delta}\int_{r=a}\left(\frac{1}{kr}\frac{d}{dr}rj_l(kr)\right)^2 \left|\vec{X}_{lm}\right|^2 r^2 d\Omega$$

$$= \frac{1}{2\sigma\delta k^2}\left(rj_l(kr)'\right)^2\Big|_{r=a}$$

$$= \frac{1}{2\sigma\delta k^2}\left(j_l(ka) + kaj_l'(ka)\right)^2$$

$$= \frac{1}{2\sigma\delta} j_l'(x_{l,n})^2$$

where $ka = x_{lm}$ and $j_l(x_{l,n}) = 0$

$\Rightarrow$

$$Q_{lmn} = \omega\frac{U_{lmn}}{P_{lmn}}$$

$$= \frac{\mu_0\sigma\omega\delta a}{2} = \frac{a}{\delta}$$

Where α_n is the n-th positive zero for

$$\left[x^p j_l(x)\right]' = 0$$

Setting $p = 1$ for TM modes, and using the notation $y_{l,n}$ to denote the n-th zero of

$\left[x^p j_l(x)\right]' = 0$, the expression for the stored energy becomes

$$U = \frac{\mu_0}{2}\int_0^a j_l(kr)^2 r^2 dr d\Omega = \frac{\mu_0 a^3}{4}\left(1 - \frac{l(l+1)}{y_{l,n}^2}\right) j_l(y_{l,n})^2$$

Power loss

$$P_{lmn} = \frac{1}{2\sigma\delta}\int\left|\hat{r}\times\vec{H}\right|^2 da = \frac{1}{2\sigma\delta}\int_{r=a} j_l(kr)^2\left|\hat{r}\times\vec{X}_{lm}\right|^2 r^2 d\Omega$$

$$= \frac{a^2}{2\sigma\delta} j_l(y_{mn})^2$$

The Q factor

$$Q_{lmn} = \omega\frac{U_{lmn}}{P_{lmn}} = \frac{\mu_0\sigma\omega\delta a}{2}\left(1 - \frac{l(l+1)}{y_{l,n}^2}\right) = \frac{a}{\delta}\left(1 - \frac{l(l+1)}{y_{l,n}^2}\right)$$