10.2

Electromagnetic radiation with elliptic polarization, descried (in the notation of Section 7.2) by the polarization vector,

$$\vec{\varepsilon} = \frac{1}{\sqrt{1+r^2}}\left(\vec{\varepsilon}_+ + re^{i\alpha}\vec{\varepsilon}_-\right)$$

is scattered by a perfectly conduction sphere of radius a. Generalize the amplitude in the scattering cross section (10.71), which applies for $r=0$ or $r=\infty$, and calculate the cross section for scattering in the long-wavelength limit. Show that

$$\frac{d\sigma}{d\Omega} = k^4a^6\left[\frac{5}{8}\left(1+\cos^2\theta\right) - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi-\alpha)\right]$$

Compare with Problem 10.1.

Sol:

From Jackson (10.71)

$$\frac{d\sigma_{sc}}{d\Omega} \cong \frac{2\pi}{3}a^2(ka)^4\left|\vec{X}_{1,\pm1} \mp 2i\hat{n}\times\vec{X}_{1,\pm1}\right|^2$$

$$= \frac{\frac{dP_{sc}}{d\Omega}}{I_{inc}}$$

$$= \frac{r^2\vec{E}_{sc}\cdot\vec{E}^*{}_{sc}}{\vec{E}_0\cdot\vec{E}_0^*}$$

Because $\vec{\varepsilon} = \frac{1}{\sqrt{1+r^2}}\left(\vec{\varepsilon}_+ + re^{i\alpha}\vec{\varepsilon}_-\right)$

$$\frac{d\sigma_{sc}}{d\Omega} = \frac{2\pi}{3}a^2(ka)^4\frac{1}{1+r^2}\left|\left[\vec{X}_{1,1} - 2i\hat{n}\times\vec{X}_{1,1}\right] + re^{i\alpha}\left[\vec{X}_{1,-1} + 2i\hat{n}\times\vec{X}_{1,-1}\right]\right|^2$$

$$\equiv \frac{2\pi}{3}a^2(ka)^4\frac{1}{1+r^2}|F|^2$$

Using

$$\vec{X}_{l,m} = \frac{1}{\sqrt{l(l+1)}}\hat{L}Y_{lm}$$

$$\hat{L} = \frac{1}{i}\left(\hat{\phi}\partial_\theta - \frac{\hat{\theta}}{\sin\theta}\partial_\phi\right)$$

$$Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$$

$$\Rightarrow \vec{X}_{l,m} = \mp\sqrt{\frac{3}{16\pi}}\left(\hat{\phi}\frac{\cos\theta}{i} \mp \hat{\theta}\right)e^{\pm i\phi}$$

$\hat{n}\times\hat{\theta}=\hat{\phi}$
$\hat{n}\times\hat{\phi}=-\hat{\theta}$

$$\Rightarrow\begin{cases} F_\theta=\sqrt{\dfrac{3}{16\pi}}\left[e^{i\phi}(1-2\cos\theta)+re^{-i\phi+i\alpha}(1-2\cos\theta)\right] \\ F_\phi=\sqrt{\dfrac{3}{16\pi}}\left[ie^{i\phi}(\cos\theta-2)+ire^{-i\phi+i\alpha}(2-\cos\theta)\right] \end{cases}$$

$$\frac{d\sigma_{sc}}{d\Omega}=\frac{2\pi}{3}a^2(ka)^4\frac{1}{1+r^2}|F|^2$$
$$=\frac{2\pi}{3}a^2(ka)^4\left(F_\theta F_\theta^*+F_\phi F_\phi^*\right)$$
$$=\frac{k^4a^6}{8(1+r^2)}\left[(1-2\cos\theta)^2\left(1+r^2+2r\cos(2\pi-\alpha)\right)+(2-\cos\theta)^2\left(1+r^2-2r\cos(2\phi-\alpha)\right)\right]$$
$$=\frac{k^4a^6}{8(1+r^2)}\left[(1+r^2)\left(5(1+\cos^2\theta)-8\cos\theta\right)+2r\cos(2\phi-\alpha)(3\cos^2\theta-3)\right]$$
$$=k^4a^6\left[\frac{5}{8}(1+\cos^2\theta)-\cos\theta-\frac{3r}{4(1+r^2)}\sin^2\theta\cos(2\phi-\alpha)\right]$$

10.3 A solid uniform sphere of radius R and conductivity σ acts as a scatterer of a plane-wave beam of unpolarized radiation of frequency ω, with $\omega R/c\ll 1$. The conductivity is large enough that the skin depth δ is small compared to R.

(a) Justify and use a magnetostatic scalar potential to determine the magnetic field around the sphere, assuming the conductivity is infinite. (Remember that $\omega\neq 0$.)

Sol.
Note that for harmonic fields ($\omega\neq 0$) both the magnetic field and electric field must vanish inside a perfect conductor. There are no source currents outside the solid sphere. As a result of $\vec{J}=0$, and since we are in the long wavelength limit $kR\ll 1$ (so we may work with a quasi-static magnetic field with $\vec{\nabla}\cdot\vec{B}\approx 0$), we may use a magnetostatic scalar potential $\vec{B}=-\vec{\nabla}\Phi_M$, at least in the vicinity (but always outside) of the sphere. Immediately outside the sphere, we take a Legendre expansion

$$\Phi_M=-B_0z+\sum_l\frac{\alpha_l}{r^{l+1}}P_l(\cos\theta)=-B_0rP_1(\cos\theta)+\sum_l\frac{\alpha_l}{r^{l+1}}P_l(\cos\theta)$$

Note that we take the incident magnetic field to point along the *z* direction. Since electromagnetic waves are transverse, this means the incident wave is traveling in the *x-y* plane. Since the perpendicular magnetic field must vanish at the surface $r = R$ of the conducting sphere, we have

$$0 = B_r\big|_{r=R} = -\frac{\partial \Phi_M}{\partial r}\bigg|_{r=R} = B_0 P_1(\cos\theta) + \sum_l \frac{(l+1)\alpha_l}{R^{l+2}} P_l(\cos\theta)$$

Since the Legendre polynomials form an orthogonal set, this implies that all α_l must vanish for $l \neq 1$, while

$$\alpha_1 = -\frac{1}{2} B_0 R^3$$

Therefore

$$\Phi_M = -B_0\left(r + \frac{R^3}{2r^2}\right) P_1(\cos\theta) = -B_0 z\left(r + \frac{R^3}{2r^3}\right)$$

The resulting magnetic field is

$$\vec{B} = -\vec{\nabla}\Phi_M = B_0\left[\hat{z} - \frac{R^3}{2}\frac{3\hat{r}(\hat{r}\cdot\hat{z}) - \hat{z}}{r^3}\right] \qquad (1)$$

The second term is that of a magnetic dipole of strength

$$\vec{m} = -\frac{2\pi R^3}{\mu_0}\vec{B}_0 \quad (\vec{B}_0 = B_0\hat{z})$$

This agrees with the conducting sphere result of (10.13). When combined with the electric dipole term, this gives the long wavelength scattering cross section of (10.14).

(b) Use the technique of Section 8.1 to determine the absorption cross section of the sphere. Show that it varies as $(\omega)^{1/2}$ provided σ is independent of frequency.

Sol.

The power loss $P_{loss} = \frac{1}{2\sigma\delta}\int \left|\hat{n}\times\vec{H}\right|^2 da$

where

$$\hat{n}\times\vec{H} = \frac{1}{\mu_0}\hat{r}\times\vec{B} = \frac{B_0}{\mu_0}\left[1 + \frac{R^3}{2r^3}\right]_{r=R}\hat{r}\times\hat{z} = -\frac{3B_0}{2\mu_0}\sin\theta\hat{\phi}$$

where we use $\vec{B}$ given in (1), and evaluate the field at the surface of the conductor. Integrating this over the sphere gives

$$P_{loss} = \frac{1}{2\sigma\delta}\frac{9|B_0|^2}{4\mu_0^2}\int \sin^2\theta R^2 d\cos\theta d\phi = \frac{3\pi|B_0|^2 R^2}{\sigma\delta\mu_0^2}$$

For normalization, the incident flux is

$$I_0 = \frac{1}{2Z_0}\left|\vec{E}_0\right|^2 = \frac{c^2}{2\sqrt{\mu_0/\varepsilon_0}}\left|\vec{B}_0\right|^2 = \frac{Z_0}{2\mu_0^2}|B_0|^2$$

The absorption cross section

$$\sigma_{abs} = \frac{P_{loss}}{I_0} = \frac{6\pi R^2}{\sigma\delta Z_0}$$

Using $\delta = \sqrt{2/\mu_0\sigma\omega}$ gives

$$\sigma_{abs} = 6\pi R^2\sqrt{\frac{\varepsilon_0\omega}{2\sigma}}$$

which is proportional to $(\omega)^{1/2}$ provided that σ is independent of frequency.

10.7

Discuss the scattering of a plane wave of electromagnetic radiation by a nonpermeable, dielectric sphere of radius a and dielectric const ε_r.

(a) By finding the fields inside the sphere and matching to the incident plus scattered wave outside the sphere, determine without any restriction on ka the multipole coefficients in the scattered wave. Define suit phase shifts for the problem.

(b) Consider the long-wavelength limit ($ka << 1$) and determine explicitly the differential and total scattering cross sections. Compare your results with those of Section 10.1.B.

(c) In the limit $\varepsilon_r \to \infty$ compare your results to those for the perfectly conduction sphere.

Sol:

(a) For the spherical wave analysis, we start with the outside solution. Which is a combination of the incident and scattered wave

$$\vec{E} = \sum_l i^l\sqrt{4\pi(2l+1)}\left[\left(j_l(kr)+\frac{1}{2}\alpha_\pm(l)h_l^{(1)}(kr)\right)\vec{X}_{l,\pm1} \pm \frac{1}{k}\vec{\nabla}\times\left(j_l(kr)+\frac{1}{2}\beta_\pm(l)h_l^{(2)}(kr)\right)\vec{X}_{l,\pm1}\right]$$

$$\vec{H}=\frac{1}{Z_0}\sum_l i^l\sqrt{4\pi(2l+1)}\left[-\frac{i}{k}\vec{\nabla}\times\left(j_l(kr)+\frac{1}{2}\alpha_\pm(l)h_l^{(1)}(kr)\right)\vec{X}_{l,\pm1}\mp i\left(j_l(kr)+\frac{1}{2}\beta_\pm(l)h_l^{(1)}(kr)\right)\vec{X}_{l,\pm1}\right]$$

Inside the dielectric sphere, we have no sources, and only a modified dielectric constant ε_r. As a result, the waves inside the sphere must be ordinary spherical waves, however with modified wave number

$$k'=\omega\sqrt{\mu_0\varepsilon}=\left(\omega\sqrt{\mu_0\varepsilon_0}\right)\sqrt{\varepsilon_r}=k\sqrt{\varepsilon_r}$$

Defining also

$$Z=\sqrt{\frac{\mu_0}{\varepsilon}}=\frac{Z_0}{\sqrt{\varepsilon_r}}$$

the spherical waves inside the dielectric sphere may be parameterized by

$$\vec{E}=\sum_l i^l\sqrt{4\pi(2l+1)}\left[a_{M,\pm}(l)j_l(k'r)\vec{X}_{l,\pm1}\pm\frac{1}{k'}a_{E,\pm}(l)\vec{\nabla}\times j_l(k'r)\vec{X}_{l,\pm1}\right]$$

$$\vec{H}=\frac{1}{Z}\sum_l i^l\sqrt{4\pi(2l+1)}\left[a_{M,\pm}(l)j_l(k'r)\vec{X}_{l,\pm1}\pm\frac{1}{k'}a_{E,\pm}(l)\vec{\nabla}\times j_l(k'r)\vec{X}_{l,\pm1}\right]$$

For the perpendicular fields,

$$\hat{r}\cdot\vec{X}_{lm}=0$$

while

$$\begin{aligned}&\hat{r}\cdot\vec{\nabla}\times f_l(kr)\vec{X}_{lm}\\&=\hat{r}\times\vec{\nabla}\cdot f_l(kr)\vec{X}_{lm}\\&=i\vec{L}\cdot f_l(kr)\vec{X}_{lm}\\&=if_l(kr)\vec{L}\cdot\vec{X}_{lm}\\&=i\sqrt{l(l+1)}f_l(kr)Y_{lm}\end{aligned}$$

This indicates that only the curl terms in $\vec{E}$ and $\vec{H}$ survive in the perpendicular direction. For the parallel fields, on the other hand, both terms contribute. In particular

$$\hat{r}\times\vec{X}_{lm}\neq 0$$

and

$$\hat{r}\times\left(\vec{\nabla}\times f_l(kr)\vec{X}_{lm}\right)$$
$$=\vec{\nabla}\left(f_l(kr)\hat{r}\cdot\vec{X}_{lm}\right)-\frac{1}{r}f_l(kr)\left(\vec{X}_{lm}-\hat{r}\left(\hat{r}\cdot\vec{X}_{lm}\right)\right)-\left(\hat{r}\cdot\vec{\nabla}\right)f_l(kr)\vec{X}_{lm}$$
$$=-\frac{1}{r}\frac{d}{dr}\left(rf_l(kr)\right)\vec{X}_{lm}$$

where we have used $\hat{r}\cdot\vec{X}_{lm}=0$. Matching linearly independent terms in the inside (5) and outside (4) solutions gives

$B_\perp$:
$$a_{M,\pm}(l)j_l(x')=j_l(x)+\frac{1}{2}\alpha_\pm(l)h_l^{(1)}(x)$$
$H_{//}$:
$$\sqrt{\varepsilon_r}a_{E,\pm}(l)j_l(x')=j_l(x)+\frac{1}{2}\beta_\pm(l)h_l^{(1)}(x)$$
$$a_{M,\pm}(l)\frac{d}{dx'}x'j_l(x')=\frac{d}{dx}x\left(j_l(x)+\frac{1}{2}\alpha_\pm(l)h_l^{(1)}(x)\right)$$
$D_\perp$:
$$\sqrt{\varepsilon_r}a_{E,\pm}(l)j_l(x')=j_l(x)+\frac{1}{2}\beta_\pm(l)h_l^{(1)}(x)$$
$E_{//}$:
$$a_{M,\pm}(l)j_l(x')=j_l(x)+\frac{1}{2}\alpha_\pm(l)h_l^{(1)}(x)$$
$$a_{E,\pm}(l)\frac{d}{dx'}x'j_l(x')=\sqrt{\varepsilon_r}\frac{d}{dx}x\left(j_l(x)+\frac{1}{2}\beta_\pm(l)h_l^{(1)}(x)\right)$$

where we have defined

$$x=ka$$
$$x'=k'a$$
$$=x\sqrt{\varepsilon_r}$$

We note that two of the six equations are redundant (this also happened in the case of plane waves reflecting and refracting off of a plane dielectric boundary). This allows us to solve four equations for four unknowns α, β, a_E and a_M. Since we are only directly interested in the multipole coefficients α and β, we eliminate a_E and a_M from the above to obtain the solution

$$\alpha_\pm(l)+1=-\frac{h_l^{(2)}(x)\frac{d}{dx'}x'j_l(x')-j_l(x')\frac{d}{dx}xh_l^{(2)}(x)}{h_l^{(1)}(x)\frac{d}{dx'}x'j_l(x')-j_l(x')\frac{d}{dx}xh_l^{(1)}(x)}$$

$$\beta_\pm(l)+1=-\frac{h_l^{(2)}(x)\frac{d}{dx'}x'j_l(x')-\varepsilon_r j_l(x')\frac{d}{dx}xh_l^{(2)}(x)}{h_l^{(1)}(x)\frac{d}{dx'}x'j_l(x')-\varepsilon_r j_l(x')\frac{d}{dx}xh_l^{(1)}(x)}$$

We now note that (at least for real ε_r) the above expressions are of the form of a ratio of a complex quantity divided by its complex conjugate. This indicates that the fractions have unit magnitude, and can be written in terms of real phase shifts

$$\alpha_\pm(l)=1=e^{2i\delta_l}$$
$$\beta_\pm(l)+1=e^{2i\delta_l'}$$

Noting that

$$e^{2i\delta_l}=-\frac{a-ib}{a+ib}$$
$$\tan\delta_l=\frac{a}{b}$$

gives

$$\tan\delta_l=\frac{j_l(x)\frac{d}{dx'}x'j_l(x')-j_l(x')\frac{d}{dx}xj_l(x)}{n_l(x)\frac{d}{dx'}x'j_l(x')-j_l(x')\frac{d}{dx}xn_l(x)}$$
$$\tan\delta_l'=\frac{j_l(x)\frac{d}{dx'}x'j_l(x')-\varepsilon_r j_l(x')\frac{d}{dx}xj_l(x)}{n_l(x)\frac{d}{dx'}x'j_l(x')-\varepsilon_r j_l(x')\frac{d}{dx}xn_l(x)}$$

With a bit of simplification, these can be rewritten in the form

$$\tan\delta_l=\frac{xj_l(x)-B_l j_l(x)}{xn_l(x)-B_l j_l(x)}$$
$$\tan\delta_l'=\frac{xj_l(x)-B_l' j_l(x)}{xn_l(x)-B_l' j_l(x)}$$

where the coefficients B_l and B_l' are

$$B_l=x'\frac{j_l'(x')}{j_l(x')}$$
$$B_l'=\frac{1}{\varepsilon_r}\left(x'\frac{j_l'(x')}{j_l(x')}+1-\varepsilon_r\right)$$

and may be thought of as parametrizing the matching conditions at the boundary of dielectric sphere. Note that the expressions for $\tan\delta_l$ and B_l are indentical to that from the quantum mechanical scattering problem. The presence of the primed quantities is the result of vector waves as opposed to scalar waves.

(b) For $ka << 1$ only the lowest ($l = 1$) phase shift is important. In this case, we may approximate the spherical Bessel functions

$$j_1(x) = \frac{1}{45}x^3\left(x'^2 - x^2\right) = \frac{1}{45}(\varepsilon_r - 1)(ka)^5$$

$$n_1(x) = -\frac{1}{x^2}\left(1 + \frac{x^2}{2} + \cdots\right)$$

Then

$$\tan\delta_1 = \frac{1}{45}x^3\left(x'^2 - x^2\right) = \frac{1}{45}(\varepsilon_r - 1)(ka)^5$$

$$\tan\delta_1' = \frac{2}{3}\frac{\varepsilon_r - 1}{\varepsilon_r + 2}x^3 = \frac{2}{3}\frac{\varepsilon_r - 1}{\varepsilon_r + 2}(ka)^3$$

The multipole expansions are then approximated by

$$\alpha_\pm(1) = e^{2i\delta_1} - 1 \approx 2i\delta_1 = \frac{2i}{45}(\varepsilon_r - 1)(ka)^2$$

$$\beta_\pm(1) = e^{2i\delta_1'} - 1 \approx 2i\delta_1' = \frac{4i}{3}\frac{\varepsilon_r - 1}{\varepsilon_r + 2}(ka)^3$$

We see that only the β_1 (electric dipole) coefficient dominates at low energies. The scattering cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\pi}{2k^2}\left|\sum_l \sqrt{(2l+1)}\left[\alpha_\pm(l)\vec{X}_{l,\pm1} \pm i\beta_\pm(l)\hat{n}\times\vec{X}_{l,\pm1}\right]\right|^2$$

$$\approx \frac{\pi}{2k^2}\left|\sqrt{3}\frac{4i}{3}\frac{\varepsilon_r - 1}{\varepsilon_r + 2}(ka)^3\hat{n}\times\vec{X}_{l,\pm1}\right|^2$$

$$= \frac{16\pi}{6k^2}\frac{4i}{3}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)^2(ka)^6\left|\hat{n}\times\vec{X}_{l,\pm1}\right|^2$$

$$= \frac{1}{2}k^4a^6\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)^2\left(1 + \cos^2\theta\right)$$

(c) In the limit $\varepsilon_r \to \infty$, $x' = \sqrt{\varepsilon_r}\,x \to \infty$

$$j_l(x') \sim \frac{1}{x'}\sin\left(x' - \frac{l\pi}{2}\right)$$

$$\Rightarrow B_l \sim x'\cot\left(x' - \frac{l\pi}{2}\right) - 1 \to \infty$$

$$B_l' \sim \frac{1}{\sqrt{\varepsilon_r}}x'\cot\left(x' - \frac{l\pi}{2}\right) - 1 \to -1$$

$$\tan\delta_l = \frac{j_l(x)}{n_l(x)}$$

$$\tan\delta_l' = \frac{xj_l'(x) - j_l(x)}{xn_l'(x) - n_l(x)} = \frac{\frac{d}{dx}xj_l(x)}{\frac{d}{dx}xn_l(x)}$$